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On the existence of Feller semigroups with boundary conditions III

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ABSTRACT. This paper is devoted to the functional analytic approach to the problem of construction of Feller semigroups with Ventcel' (Wentzell) boundary conditions in probability theory, generalizing the previous work. In this paper we construct a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space until it "dies" at the time when it reaches the set where the particle is definitely absorbed.

0. Introduction and results

Let D be a bounded domain of Euclidean space $\boldsymbol{R}^N$ with smooth boundary ∂D, and let $C(\bar{D})$ be the space of real-valued, continuous functions on the closure $\bar{D} = D \cup \partial D$. We equip the space $C(\bar{D})$ with the topology of uniform convergence on the whole $\bar{D}$; hence it is a Banach space with the maximum norm

$$\|f\| = \max_{x \in \bar{D}} |f(x)|.$$

A strongly continuous semigroup $\{T_t\}_{t \geq 0}$ on the space $C(\bar{D})$ is called a *Feller semigroup* on $\bar{D}$ if it is non-negative and contractive on $C(\bar{D})$:

$$f \in C(\bar{D}), \qquad 0 \leq f \leq 1 \qquad \text{on } \bar{D} \Rightarrow 0 \leq T_t f \leq 1 \qquad \text{on } \bar{D}.$$

It is known (cf. [8]) that if T_t is a Feller semigroup on $\bar{D}$, then there exists a unique Markov transition function p_t on $\bar{D}$ such that

$$T_t f(x) = \int_{\bar{D}} p_t(x, dy) f(y), \qquad f \in C(\bar{D}).$$

It can be shown that the function p_t is the transition function of some strong *Markov process*; hence the value $p_t(x, E)$ expresses the transition probability that a Markovian particle starting at position x will be found in the set E at time t.

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Furthermore it is known (cf. [1], [8]) that the infinitesimal generator $\mathfrak{A}$ of a Feller semigroup $\{T_t\}_{t\geq 0}$ is described analytically by a Waldenfels operator W and a Ventcel' boundary condition L, which we formulate precisely.

Let W be a second-order *elliptic* integro-differential operator with real coefficients such that

$$\begin{aligned} Wu(x) &= Pu(x) + S_r u(x) \\ &:= \sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) \\ &\quad + \int_D s(x, y)\left[u(y) - \sigma(x, y)\left(u(x) + \sum_{j=1}^{N} (y_j - x_j) \frac{\partial u}{\partial x_j}(x)\right)\right] dy, \end{aligned}$$

where

(1) $a^{ij} \in C^\infty(\boldsymbol{R}^N)$, $a^{ij} = a^{ji}$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i \xi_j \geq a_0 |\xi|^2, \qquad x \in \boldsymbol{R}^N, \ \xi \in \boldsymbol{R}^N.$$

(2) $b^i \in C^\infty(\boldsymbol{R}^N)$.

(3) $c \in C^\infty(\boldsymbol{R}^N)$ and $c \leq 0$ in D.

(4) The integral kernel $s(x, y)$ is the distribution kernel of a properly supported pseudo-differential operator $S \in L^{2-\kappa}_{1,0}(\boldsymbol{R}^N)$, $\kappa > 0$, which has the *transmission property* with respect to the boundary ∂D (see Subsection 2.2), and $s(x, y) \geq 0$ off the diagonal $\{(x, x) : x \in \boldsymbol{R}^N\}$ in $\boldsymbol{R}^N \times \boldsymbol{R}^N$. The measure dy is the Lebesgue measure on $\boldsymbol{R}^N$.

(5) The function $\sigma(x, y)$ is a non-negative smooth function on $\bar{D} \times \bar{D}$ such that $\sigma(x, y) = 1$ in a neighborhood of the diagonal $\{(x, x) : x \in \bar{D}\}$ in $\bar{D} \times \bar{D}$. The function $\sigma(x, y)$ depends on the shape of the domain D. More precisely it depends on a family of local charts on $\bar{D}$ in each of which the Taylor expansion is valid for functions u. For example, if D is convex, one may take $\sigma(x, y) \equiv 1$ on $\bar{D} \times \bar{D}$.

(6) $W1(x) = c(x) + \int_D s(x, y)[1 - \sigma(x, y)] dy \leq 0$ in D.

The operator W is called a second-order *Waldenfels operator*. The differential operator P describes analytically a strong Markov process with continuous paths (diffusion process) in the interior D. The integro-differential operator S_r is supposed to correspond to the jump phenomenon in the interior D. Therefore the Waldenfels operator W is supposed to correspond to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space D.

We remark that the integro-differential operator S_r is a "regularization" of S, since the integrand is absolutely convergent. Indeed it suffices to note (see [5, Chapitre IV, Proposition 1]) that, for any compact $K \subset \boldsymbol{R}^N$, there exists a constant $C_K > 0$ such that the distribution kernel $s(x, y)$ of S satisfies the estimate

$$|s(x, y)| \leq \frac{C_K}{|x - y|^{N+2-\kappa}}, \qquad x, \ y \in K, \ x \neq y.$$

The intuitive meaning of condition (6) is that the jump phenomenon from a point $x \in D$ to the outside of a neighborhood of x in D is "dominated" by the absorption phenomenon at x. In particular, if $c(x) \equiv 0$ in D, condition (6) implies that any Markovian particle does not move by jumps from $x \in D$ to the outside of a neighborhood $V(x)$ of x in the interior D, since we have

$$\int_D s(x, y)[1 - \sigma(x, y)]dy = 0,$$

and so by conditions (4) and (5)

$$s(x, y) = 0, \qquad y \in D \setminus V(x).$$

Let L be a second-order boundary condition such that in local coordinates $(x_1, x_2, \ldots, x_{N-1})$ on ∂D

$$\begin{aligned} Lu(x') &= Qu(x') + \mu(x')\frac{\partial u}{\partial \boldsymbol{n}}(x') - \delta(x')Wu(x') + \Gamma u(x') \\ &:= \sum_{i,j=1}^{N-1} \alpha^{ij}(x')\frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x')\frac{\partial u}{\partial x_i}(x') + \gamma(x')u(x') \\ &\quad + \mu(x')\frac{\partial u}{\partial \boldsymbol{n}}(x') - \delta(x')Wu(x') \\ &\quad + \int_{\partial D} r(x', y')\left[u(y') - \sigma(x', y')\left(u(x') + \sum_{j=1}^{N-1} (y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right)\right]dy' \\ &\quad + \int_D t(x', y)\left[u(y) - \sigma(x', y)\left(u(x') + \sum_{j=1}^{N-1} (y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right)\right]dy, \end{aligned}$$

where:

(1) The operator Q is a second-order *degenerate* elliptic differential operator on ∂D with non-positive principal symbol. In other words the α^{ij} are the components of a smooth symmetric contravariant tensor of type $\binom{2}{0}$ on ∂D satisfying

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j \geq 0, \qquad x' \in \partial D, \qquad \xi' = \sum_{j=1}^{N-1} \xi_j dx_j \in T_{x'}^*(\partial D).$$

Here $T_{x'}^*(\partial D)$ is the cotangent space of ∂D at x'.

(2) $Q1 = \gamma \in C^\infty(\partial D)$ and $\gamma \leq 0$ on ∂D.

(3) $\mu \in C^\infty(\partial D)$ and $\mu \geq 0$ on ∂D.

(4) $\delta \in C^\infty(\partial D)$ and $\delta \geq 0$ on ∂D.

(5) $\boldsymbol{n} = (n_1, n_2, \dots, n_N)$ is the unit interior normal to the boundary ∂D.

(6) The integral kernel $r(x', y')$ is the distribution kernel of a pseudo-differential operator $R \in L_{1,0}^{2-\kappa_1}(\partial D)$, $\kappa_1 > 0$ and $r(x', y') \geq 0$ off the diagonal $\Delta_{\partial D} = \{(x', x') : x' \in \partial D\}$ in $\partial D \times \partial D$. The density dy' is a strictly positive density on ∂D.

(7) The integral kernel $t(x, y)$ is the distribution kernel of a properly supported, pseudo-differential operator $T \in L_{1,0}^{2-\kappa_2}(\boldsymbol{R}^N)$, $\kappa_2 > 0$, which has the transmission property with respect to the boundary ∂D, and $t(x, y) \geq 0$ off the diagonal $\{(x, x) : x \in \boldsymbol{R}^N\}$ in $\boldsymbol{R}^N \times \boldsymbol{R}^N$.

(8) The operator Γ is a boundary operator of order $2 - \min(\kappa_1, \kappa_2)$, and satisfies the condition

$$\begin{aligned} \Gamma 1(x') = &\int_{\partial D} r(x', y')[1 - \sigma(x', y')]dy' \\ &+ \int_D t(x', y)[1 - \sigma(x', y)]dy = 0, \qquad x' \in \partial D. \end{aligned}$$

The boundary condition L is called a second-order *Ventcel' boundary condition.* The six terms of L

$$\sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial u}{\partial x_i}(x'),$$

$$\gamma(x')u(x'), \qquad \mu(x')\frac{\partial u}{\partial \boldsymbol{n}}(x'), \qquad \delta(x')Wu(x'),$$

$$\int_{\partial D} r(x', y')\left[u(y') - \sigma(x', y')\left(u(x') + \sum_{j=1}^{N-1} (y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right)\right]dy',$$

$$\int_D t(x', y)\left[u(y) - \sigma(x', y)\left(u(x') + \sum_{j=1}^{N-1} (y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right)\right]dy$$

are supposed to correspond to the diffusion along the boundary, the absorption phenomenon, the reflection phenomenon, the viscosity phenomenon and the jump phenomenon on the boundary, and the inward jump phenomenon from the boundary, respectively.

The intuitive meaning of condition (8) is that any Markovian particle does not move by jumps from $x' \in \partial D$ to the outside of a neighborhood $W(x')$ of x' in the closure $\bar{D} = D \cup \partial D$, since we have

$$\begin{cases} r(x', y') = 0, & y' \in \partial D \backslash (W(x') \cap \partial D); \\ t(x', y) = 0, & y \in D \backslash (W(x') \cap D). \end{cases}$$

This paper is devoted to the functional analytic approach to the problem of construction of Feller semigroups with Ventcel' boundary conditions. More precisely we consider the following problem:

PROBLEM. *Conversely, given analytic data* (W, L), *can we construct a Feller semigroup* $\{T_t\}_{t \geq 0}$ *whose infinitesimal generator* $\mathfrak{A}$ *is characterized by* (W, L)?

We say that the boundary condition L is *transversal* on the boundary ∂D if it satisfies the condition

$$\int_D t(x', y) dy = +\infty \qquad \text{if } \mu(x') = \delta(x') = 0.$$

Intuitively the transversality condition implies that a Markovian particle jumps away "instantaneously" from the points $x' \in \partial D$ where neither reflection nor viscosity phenomenon occurs.

The next theorem asserts that there exists a Feller semigroup on $\bar{D}$ corresponding to such a diffusion phenomenon that one of the reflection phenomenon, the viscosity phenomenon and the inward instantaneously jump phenomenon from the boundary occurs at each point of the boundary:

THEOREM 1. *We define a linear operator* $\mathfrak{A}$ *from the space* $C(\bar{D})$ *into itself as follows*:

(a) *The domain of definition* $D(\mathfrak{A})$ *of* $\mathfrak{A}$ *is the set*

$$D(\mathfrak{A}) = \{u \in C(\bar{D}) : Wu \in C(\bar{D}), Lu = 0\}.$$

(b) $\mathfrak{A}u = Wu, \ u \in D(\mathfrak{A})$.

Here Wu *and* Lu *are taken in the sense of* distributions.

Assume that the boundary condition L *is* transversal *on the boundary* ∂D. *Then the operator* $\mathfrak{A}$ *generates a Feller semigroup* $\{T_t\}_{t \geq 0}$ *on* $\bar{D}$.

Theorem 1 was proved by [9, Theorem 1], and also by [3, Théorème 3.2]. We remark that Takanobu–Watanabe [11] proved a probabilistic version of Theorem 1 in the case where the domain D is the half space $\boldsymbol{R}^N_+$. On the other hand, Taira [10] studied the case where the differential operator P is *degenerate* and the integro-differential operator S_r vanishes identically in D.

The purpose of this paper is to generalize Theorem 1 to the *non-transversal* case, improving [9, Theorem 2], which we state precisely.

First we assume that

(H) there exists a second-order Ventcel' boundary condition L_v such that

$$Lu = mL_v u + \gamma u, \tag{0.1}$$

where

(3′) $m \in C^\infty(\partial D)$ and $m \geq 0$ on ∂D,

and the boundary condition L_v is given in local coordinates $(x_1, x_2, \dots, x_{N-1})$ by the formula

$$\begin{aligned} L_v u(x') &= \bar{Q}u(x') + \bar{\mu}(x')\frac{\partial u}{\partial \boldsymbol{n}}(x') - \bar{\delta}(x')Wu(x') + \bar{\Gamma}u(x') \\ &:= \sum_{i,j=1}^{N-1} \bar{\alpha}^{ij}(x')\frac{\partial^2 u}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \bar{\beta}^i(x')\frac{\partial u}{\partial x_i}(x') \\ &\quad + \bar{\mu}(x')\frac{\partial u}{\partial \boldsymbol{n}}(x') - \bar{\delta}(x')Wu(x') \\ &\quad + \int_{\partial D} \bar{r}(x', y')\left[u(y') - \sigma(x', y')\left(u(x') + \sum_{j=1}^{N-1}(y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right)\right]dy' \\ &\quad + \int_D \bar{t}(x', y)\left[u(y) - \sigma(x', y)\left(u(x') + \sum_{j=1}^{N-1}(y_j - x_j)\frac{\partial u}{\partial x_j}(x')\right)\right]dy, \end{aligned}$$

and satisfies the conditions

$$\begin{aligned} \bar{\Gamma}1(x') = \int_{\partial D} \bar{r}(x', y')[1 - \sigma(x', y')]dy' \\ + \int_D \bar{t}(x', y)[1 - \sigma(x', y)]dy = 0, \qquad x' \in \partial D, \end{aligned} \tag{0.2}$$

and

$$\int_D \bar{t}(x', y)dy = +\infty \qquad \text{if } \bar{\mu}(x') = \bar{\delta}(x') = 0. \tag{0.3}$$

We let

$$M = \left\{x' \in \partial D : \mu(x') = \delta(x') = 0, \int_D t(x', y)dy < \infty\right\}.$$

Then, by condition (0.3) it follows that

$$M = \{x' \in \partial D : m(x') = 0\},$$

since we have $\mu(x') = m(x')\bar{\mu}(x')$, $\delta(x') = m(x')\bar{\delta}(x')$, and $t(x', y) = m(x')\bar{t}(x', y)$. Hence we find that the boundary condition L is *not* transversal on ∂D.

Furthermore we assume that

(A) $m(x') - \gamma(x') > 0$ on ∂D.

Intuitively conditions (H) and (A) imply that a Markovian particle does not stay on ∂D for any period of time until it "dies" at the time when it reaches the set M where the particle is definitely absorbed.

Now we introduce a subspace of $C(\bar{D})$ which is associated with the boundary condition L.

By condition (A), we find that the boundary condition

$$Lu = mL_\nu u + \gamma u = 0 \quad \text{on } \partial D$$

includes the condition

$$u = 0 \quad \text{on } M.$$

With this fact in mind, we let

$$C_0(\bar{D}\backslash M) = \{u \in C(\bar{D}) : u = 0 \text{ on } M\}.$$

The space $C_0(\bar{D}\backslash M)$ is a closed subspace of $C(\bar{D})$; hence it is a Banach space.

A strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on the space $C_0(\bar{D}\backslash M)$ is called a *Feller semigroup* on $\bar{D}\backslash M$ if it is non-negative and contractive on $C_0(\bar{D}\backslash M)$:

$$f \in C_0(\bar{D}\backslash M),\ 0 \leq f \leq 1 \quad \text{on } \bar{D}\backslash M \Rightarrow 0 \leq T_t f \leq 1 \quad \text{on } \bar{D}\backslash M.$$

We define a linear operator $\mathfrak{M}$ from $C_0(\bar{D}\backslash M)$ into itself as follows:

(a) The domain of definition $D(\mathfrak{M})$ of $\mathfrak{M}$ is the set

$$D(\mathfrak{M}) = \{u \in C_0(\bar{D}\backslash M) : Wu \in C_0(\bar{D}\backslash M),\ Lu = 0\}. \tag{0.4}$$

(b) $\mathfrak{M}u = Wu$, $u \in D(\mathfrak{M})$.

The next theorem is a generalization of Theorem 1 to the non-transversal case, and is an improvement on [9, Theorem 2] which only treated the case where $\bar{\mu} \equiv 1$ on ∂D:

THEOREM 2. *If conditions (H) and (A) are satisfied, then the operator $\mathfrak{M}$ defined by formula* (0.4) *generates a Feller semigroup $\{T_t\}_{t\geq 0}$ on $\bar{D}\backslash M$.*

Theorem 2 asserts that there exists a Feller semigroup on $\bar{D}\backslash M$ corresponding to such a diffusion phenomenon that a Markovian particle moves both by jumps and continuously in the state space $\bar{D}\backslash M$ until it "dies" at the time when it reaches the set M where the particle is definitely absorbed.

Theorems 1 and 2 solve from the viewpoint of functional analysis the problem of construction of Feller semigroups with Ventcel' boundary conditions for elliptic Waldenfels operators.

The rest of this paper is organized as follows.

Sections 1 and 2 provide a review of basic results about Feller semigroups and pseudo-differential operators which will be used in the subsequent sections.

In Section 3 we give a general existence theorem for Feller semigroups in terms of boundary value problems. We reduce the problem of construction of Feller semigroups to the problem of *unique solvability* for the boundary value problem

$$\begin{cases}(\alpha - W)u = f & \text{in } D,\\ Lu = \varphi & \text{on } \partial D,\end{cases}$$

where α is a positive parameter.

Section 4 is devoted to the proof of Theorem 2. The idea of our approach is essentially the same as that of [9], although the proof is a little more complicated and difficult.

First, applying Theorem 1 to the transversal boundary condition L_ν we can solve uniquely the following boundary value problem:

$$\begin{cases}(\alpha - W)v = f & \text{in } D,\\ L_\nu v = 0 & \text{on } \partial D.\end{cases}$$

We let

$$v = G_\alpha^\nu f.$$

The operator G_α^ν is a generalization of the classical Green operator. Then it follows that a function u is a solution of the problem

$$\begin{cases}(\alpha - W)u = f & \text{in } D,\\ Lu = mL_\nu u + \gamma u = 0 & \text{on } \partial D\end{cases} \tag{$*$}$$

if and only if the function $w = u - v$ is a solution of the problem

$$\begin{cases}(\alpha - W)w = 0 & \text{in } D,\\ Lw = -Lv = -\gamma v & \text{on } \partial D.\end{cases}$$

However we know that every solution w of the homogeneous equation $(\alpha - W)w = 0$ in D can be expressed as follows:

$$w = H_\alpha \psi.$$

The operator H_α is a generalization of the classical harmonic (Poisson) operator. Thus, by using the operators G_α^ν and H_α one can reduce the study of problem $(*)$ to that of the equation

$$LH_\alpha \psi = -LG_\alpha^\nu f = -\gamma v \quad \text{on } \partial D.$$

We find that if $S \in L^{2-\kappa}_{1,0}(\mathbf{R}^N)$ and $T \in L^{2-\kappa_2}_{1,0}(\mathbf{R}^N)$ have the transmission property, then the operator LH_α is a pseudo-differential operator of second order on the boundary ∂D. Thus we can reduce the study of problem (*) to that of the pseudo-differential equation

$$LH_\alpha\psi = -\gamma G^\nu_\alpha f \quad \text{on } \partial D.$$

By using the Hölder space theory of psuedo-differential operators, we can show that if condition (A) is satisfied, then the operator LH_α is *bijective* in the framework of Hölder spaces.

Therefore we find that a unique solution u of problem (*) can be expressed in the following form (see formula (4.5)):

$$u = G^\nu_\alpha f - H_\alpha((LH_\alpha)^{-1}(LG^\nu_\alpha f)).$$

This formula allows us to verify all the conditions of the generation theorem of Feller semigroups.

The author would like to thank Kôhei Uchiyama for his helpful suggestions on the formulation of conditions (H) and (A) in Theorem 2 from the viewpoint of probability theory.

1. Theory of Feller semigroups

This section provides a brief description of basic results about Feller semigroups, which forms a functional analytic background for the proof of Theorem 2.

If K is a locally compact, separable metric space, then we add a point ∂ to K as the point at infinity if K is not compact, and as an isolated point if K is compact; so the space $K_\partial = K \cup \{\partial\}$ is compact.

Let $C(K)$ be the space of real-valued, bounded continuous functions on K. The space $C(K)$ is a Banach space with the supremum norm

$$\|f\| = \sup_{x \in K} |f(x)|.$$

We say that a function $f \in C(K)$ converges to zero as $x \to \partial$ if, for each $\varepsilon > 0$, there exists a compact subset E of K such that $|f(x)| < \varepsilon$, $x \in K \setminus E$; we then write $\lim_{x \to \partial} f(x) = 0$. It is easy to see that

$$C_0(K) := \left\{ f \in C(K) : \lim_{x \to \partial} f(x) = 0 \right\} = \{ f \in C(K_\partial) : f(\partial) = 0 \}.$$

The space $C_0(K)$ is a closed subspace of $C(K)$; hence it is a Banach space.

A family $\{T_t\}_{t \geq 0}$ of bounded linear operators acting on $C_0(K)$ is called a *Feller semigroup* on K if it satisfies the following three conditions:

(i) $T_{t+s} = T_t \cdot T_s$, $t, s \geq 0$; $T_0 = I$.

(ii) The family $\{T_t\}$ is strongly continuous in t for $t \geq 0$:

$$\lim_{s \downarrow 0} \| T_{t+s} f - T_t f \| = 0, \qquad f \in C_0(K).$$

(iii) The family $\{T_t\}$ is non-negative and contractive on $C_0(K)$:

$$f \in C_0(K),\ 0 \leq f \leq 1 \quad \text{on } K \Rightarrow 0 \leq T_t f \leq 1 \quad \text{on } K.$$

If $\{T_t\}_{t \geq 0}$ is a Feller semigroup on K, we define its *infinitesimal generator* $\mathfrak{A}$ by the formula

$$\mathfrak{A} u = \lim_{t \downarrow 0} \frac{T_t u - u}{t}, \tag{1.1}$$

provided that the limit (1.1) exists in the space $C_0(K)$.

The next theorem is a version of the Hille–Yosida theorem adapted to the present context (cf. [8, Theorem 9.3.1 and Corollary 9.3.2]):

THEOREM 1.1. (i) *Let $\{T_t\}_{t \geq 0}$ be a Feller semigroup on K and $\mathfrak{A}$ its infinitesimal generator. Then we have the following*:

(a) *The domain $D(\mathfrak{A})$ is everywhere dense in the space $C_0(K)$.*

(b) *For each $\alpha > 0$, the equation $(\alpha I - \mathfrak{A})u = f$ has a unique solution u in $D(\mathfrak{A})$ for any $f \in C_0(K)$. Hence, for each $\alpha > 0$, the Green operator $(\alpha I - \mathfrak{A})^{-1}: C_0(K) \to C_0(K)$ can be defined by the formula*

$$u = (\alpha I - \mathfrak{A})^{-1} f, \qquad f \in C_0(K).$$

(c) *For each $\alpha > 0$, the operator $(\alpha I - \mathfrak{A})^{-1}$ is non-negative on the space $C_0(K)$:*

$$f \in C_0(K),\ f \geq 0 \quad \text{on } K \Rightarrow (\alpha I - \mathfrak{A})^{-1} f \geq 0 \quad \text{on } K.$$

(d) *For each $\alpha > 0$, the operator $(\alpha I - \mathfrak{A})^{-1}$ is bounded on the space $C_0(K)$ with norm*

$$\|(\alpha I - \mathfrak{A})^{-1}\| \leq \frac{1}{\alpha}.$$

(ii) *Conversely, if $\mathfrak{A}$ is a linear operator from the space $C_0(K)$ into itself satisfying condition* (a) *and if there exists a constant $\alpha_0 \geq 0$ such that, for all $\alpha > \alpha_0$, conditions* (b) *through* (d) *are satisfied, then $\mathfrak{A}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t \geq 0}$ on K.*

COROLLARY 1.2. *Let K be a compact metric space and let $\mathfrak{A}$ be the infinitesimal generator of a Feller semigroup on K. Assume that the constant function* 1 *belongs to the domain $D(\mathfrak{A})$ of $\mathfrak{A}$ and that we have for some con-*

stant c

$$\mathfrak{A}1 \leq -c \quad on\ K.$$

Then the operator $\mathfrak{A}' = \mathfrak{A} + cI$ *is the infinitesimal generator of some Feller semigroup on* K.

2. Theory of pseudo-differential operators

In this section we present a brief description of basic concepts and results of the Hölder space theory of pseudo-differential operators which will be used in the subsequent sections. For detailed studies of pseudo-differential operators, the reader is referred to Chazarain–Piriou [4], Hörmander [6] and Rempel–Schulze [7].

2.1. Hölder spaces

Let Ω be an open subset of Euclidean space $\boldsymbol{R}^n$. If m is a non-negative integer, we let

$$C^m(\Omega) = \text{the space of functions of class } C^m \text{ in } \Omega,$$

and

$$C^m(\bar{\Omega}) = \text{the space of functions in } C^m(\Omega) \text{ all of whose derivatives of order } \leq m \text{ have continuous extensions to the closure } \bar{\Omega}.$$

If Ω is bounded, then $C^m(\bar{\Omega})$ is a Banach space with the norm

$$\|u\|_{C^m(\bar{\Omega})} = \max_{\substack{x \in \bar{\Omega} \\ |\alpha| \leq m}} |\partial^\alpha u(x)|.$$

If m is a non-negative integer and $0 < \theta < 1$, we define the Hölder spaces

$$C^{m+\theta}(\Omega) = \text{the space of functions in } C^m(\Omega) \text{ all of whose } m\text{-th order derivatives are locally Hölder continuous with exponent } \theta \text{ in } \Omega,$$

and

$$C^{m+\theta}(\bar{\Omega}) = \text{the space of functions in } C^m(\bar{\Omega}) \text{ all of whose } m\text{-th order derivatives are Hölder continuous with exponent } \theta \text{ on } \bar{\Omega}.$$

If Ω is bounded, then the Hölder space $C^{m+\theta}(\bar{\Omega})$ is a Banach space with the norm

$$\|u\|_{C^{m+\theta}(\bar{\Omega})} = \|u\|_{C^m(\bar{\Omega})} + \max_{|\alpha|=m} \sup_{\substack{x,y\in\bar{\Omega} \\ x\neq y}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\theta}.$$

If M is an n-dimensional compact smooth manifold without boundary, then the Hölder space $C^{m+\theta}(M)$ is defined to be locally the Hölder space $C^{m+\theta}(\boldsymbol{R}^n)$, upon using local coordinate systems flattening out M, together with a partition of unity. The norm of $C^{m+\theta}(M)$ will be denoted by $\|\cdot\|_{C^{m+\theta}(M)}$.

2.2. Pseudo-differential operators

If $m \in \boldsymbol{R}$ and $0 \leq \delta < \rho \leq 1$, we let

$S^m_{\rho,\delta}(\Omega \times \boldsymbol{R}^N)$ = the set of all functions $a \in C^\infty(\Omega \times \boldsymbol{R}^N)$ with the property that, for any compact $K \subset \Omega$ and any multi-indices α, β, there exists a constant $C_{K,\alpha,\beta} > 0$ such that we have for all $x \in K$ and $\theta \in \boldsymbol{R}^N$

$$|\partial_\theta^\alpha \partial_x^\beta a(x, \theta)| \leq C_{K,\alpha,\beta}(1 + |\theta|)^{m-\rho|\alpha|+\delta|\beta|}.$$

The elements of $S^m_{\rho,\delta}(\Omega \times \boldsymbol{R}^N)$ are called *symbols* of order m.

A *psuedo-differential operator* of order m on Ω is a Fourier integral operator of the form

$$Au(x) = \iint_{\Omega\times\boldsymbol{R}^n} e^{i(x-y)\cdot\xi} a(x, y, \xi)u(y)dyd\xi, \qquad u \in C_0^\infty(\Omega),$$

with some $a \in S^m_{\rho,\delta}(\Omega \times \Omega \times \boldsymbol{R}^n)$. Here the integral is taken in the sense of *oscillatory integrals.*

We let

$L^m_{\rho,\delta}(\Omega)$ = the set of all pseudo-differential operators of order m on Ω.

Now we formulate the notion of transmission property essentially due to Boutet de Monvel [2], which is a condition about symbols in the normal direction at the boundary.

We let

$L^m_{1,0}(\overline{\boldsymbol{R}^n_+})$ = the space of pseudo-differential operators in $L^m_{1,0}(\boldsymbol{R}^n_+)$ which can be extended to a pseudo-differential operator in $L^m_{1,0}(\boldsymbol{R}^n)$.

A pseudo-differential operator $A \in L^m_{1,0}(\overline{\boldsymbol{R}^n_+})$ is said to have the *transmission property* with respect to the boundary $\boldsymbol{R}^{n-1}$ if the restriction of $A(u^0)$ to $\boldsymbol{R}^n_+$ has a C^∞ extension to $\overline{\boldsymbol{R}^n_+}$ for every $u \in C_0^\infty(\overline{\boldsymbol{R}^n_+})$, where u^0 is the extension of u to $\boldsymbol{R}^n$ by 0 outside $\overline{\boldsymbol{R}^n_+}$.

We remark that the notion of transmission property may be transferred to manifolds. Indeed, if Ω is a relatively compact open subset of an n-dimensional paracompact smooth manifold M without boundary, then the notion of transmission property can be extended to the class $L^m_{1,0}(M)$, upon using local coordinate systems flattening out the boundary $\partial\Omega$.

Then we have the following results (cf. [2], [7]):

(I) If a pseudo-differential operator $A \in L^m_{1,0}(M)$ has the transmission property with respect to the boundary $\partial\Omega$, then the operator

$$A_\Omega\colon C^\infty(\bar{\Omega}) \to C^\infty(\Omega)$$
$$u \mapsto A(u^0)|_\Omega$$

maps $C^\infty(\bar{\Omega})$ continuously into itself, where u^0 is the extension of u to M by 0 outside $\bar{\Omega}$.

(II) If a pseudo-differential operator $A \in L^m_{1,0}(M)$ has the transmission property, then the operator A_Ω maps $C^{k+\theta}(\bar{\Omega})$ continuously into $C^{k-m+\theta}(\bar{\Omega})$ for any integer $k \geq m$ and $0 < \theta < 1$.

2.3. A unique solvability theorem for pseudo-differential operators

The next result will play an essential role in the construction of Feller semigroups in Section 4 (cf. [9, Theorem 2.1]):

THEOREM 2.1. *Let T be a classical pseudo-differential operator of second order on an n-dimensional compact smooth manifold M without boundary such that*

$$T = P + S,$$

where:

(a) *The operator P is a second-order degenerate* elliptic differential operator *on M with non-positive principal symbol, and $P1 \leq 0$ on M.*

(b) *The operator S is a classical pseudo-differential operator of order $2-\kappa$, $\kappa > 0$, on M and its distribution kernel $s(x, y)$ is* non-negative *off the diagonal $\Delta_M = \{(x, x) : x \in M\}$ in $M \times M$.*

(c) $T1 = P1 + S1 \leq 0$ *on M.*

Then, for each integer $k \geq 1$, there exists a constant $\lambda = \lambda(k) > 0$ such that, for any $f \in C^{k+\theta}(M)$, one can find a function $\varphi \in C^{k+\theta}(M)$ satisfying

$$(T - \lambda I)\varphi = f \quad \text{on } M,$$

and

$$\|\varphi\|_{C^{k+\theta}(M)} \leq C_{k+\theta}(\lambda)\,\|f\|_{C^{k+\theta}(M)}.$$

Here $C_{k+\theta}(\lambda) > 0$ is a constant independent of f.

3. A general existence theorem for Feller semigroups

The purpose of this section is to give a general existence theorem for Feller semigroups in terms of boundary value problems (cf. [1], [8]).

First we consider the following Dirichlet problem:

$$\begin{cases}(\alpha - W)u = f & \text{in } D, \\ u = \varphi & \text{on } \partial D,\end{cases} \tag{D}$$

where α is a positive parameter.

The next theorem summarizes the basic facts about problem (D) in the framework of *Hölder spaces* (see [1, Théorème XV]):

THEOREM 3.1. *Let k be an arbitrary non-negative integer and $0 < \theta < 1$. For any $f \in C^{k+\theta}(\bar{D})$ and any $\varphi \in C^{k+2+\theta}(\partial D)$, problem* (D) *has a unique solution u in $C^{k+2+\theta}(\bar{D})$.*

Theorem 3.1 with $k = 0$ tells us that problem (D) has a unique solution $u \in C^{2+\theta}(\bar{D})$ for any $f \in C^{\theta}(\bar{D})$ and any $\varphi \in C^{2+\theta}(\partial D)$ with $0 < \theta < 1$. Therefore we can introduce linear operators

$$G_\alpha^0 \colon C^{\theta}(\bar{D}) \to C^{2+\theta}(\bar{D}), \qquad \alpha > 0,$$

and

$$H_\alpha \colon C^{2+\theta}(\partial D) \to C^{2+\theta}(\bar{D}), \qquad \alpha > 0,$$

as follows.

(a) For any $f \in C^{\theta}(\bar{D})$, the function $G_\alpha^0 f \in C^{2+\theta}(\bar{D})$ is the unique solution of the problem

$$\begin{cases}(\alpha - W)G_\alpha^0 f = f & \text{in } D, \\ G_\alpha^0 f = 0 & \text{on } \partial D.\end{cases} \tag{3.1}$$

(b) For any $\varphi \in C^{2+\theta}(\partial D)$, the function $H_\alpha \varphi \in C^{2+\theta}(\bar{D})$ is the unique solution of the problem

$$\begin{cases}(\alpha - W)H_\alpha \varphi = 0 & \text{in } D, \\ H_\alpha \varphi = \varphi & \text{on } \partial D.\end{cases} \tag{3.2}$$

Then we have the following results (see [1, Proposition III.1.6]):

THEOREM 3.2. (i) *The operator G_α^0 can be uniquely extended to a non-negative, bounded linear operator on $C(\bar{D})$ into itself, denoted again G_α^0, with norm $\|G_\alpha^0\| \leq 1/\alpha$ for any $\alpha > 0$.*

(ii) *The operator H_α can be uniquely extended to a non-negative, bounded linear operator on $C(\partial D)$ into $C(\bar{D})$, denoted again H_α, with norm $\|H_\alpha\| \leq 1$ for any $\alpha > 0$.*

Now we consider the boundary value problem (*) in the framework of the spaces of *continuous functions*:

$$\begin{cases}(\alpha - W)u = f & \text{in } D, \\ Lu = 0 & \text{on } \partial D.\end{cases} \tag{*}$$

To do so, we introduce three operators associated with problem (*).

(I) First we introduce a linear operator

$$W: C(\bar{D}) \to C(\bar{D})$$

as follows.

(a) The domain $D(W)$ of W is the space $C^{2+\theta}(\bar{D})$.

(b) $Wu = Pu + S_r u$, $u \in D(W)$.

Then we have the following (cf. [8, Lemma 9.6.5]):

LEMMA 3.3. *The operator W has its minimal closed extension $\overline{W}$ in the space $C(\bar{D})$.*

The extended operators $G_\alpha^0: C(\bar{D}) \to C(\bar{D})$ and $H_\alpha: C(\partial D) \to C(\bar{D})$ still satisfy formulas (3.1) and (3.2) respectively in the following sense (cf. [8, Lemma 9.6.7 and Corollary 9.6.8]):

LEMMA 3.4. (i) *For any $f \in C(\bar{D})$, we have*

$$\begin{cases}G_\alpha^0 f \in D(\overline{W}), \\ (\alpha I - \overline{W})G_\alpha^0 f = f & \text{in } D.\end{cases}$$

(ii) *For any $\varphi \in C(\partial D)$, we have*

$$\begin{cases}H_\alpha \varphi \in D(\overline{W}), \\ (\alpha I - \overline{W})H_\alpha \varphi = 0 & \text{in } D.\end{cases}$$

Here $D(\overline{W})$ is the domain of the closed extension $\overline{W}$.

COROLLARY 3.5. *Every function $u \in D(\overline{W})$ can be written in the following form*:

$$u = G_\alpha^0(\alpha I - \overline{W})u) + H_\alpha(u|_{\partial D}), \qquad \alpha > 0.$$

(II) Secondly we introduce a linear operator

$$LG_\alpha^0: C(\bar{D}) \to C(\partial D), \qquad \alpha > 0,$$

as follows.

(a) The domain $D(LG_\alpha^0)$ of LG_α^0 is the space $C^\theta(\bar{D})$.

(b) $LG_\alpha^0 f = L(G_\alpha^0 f)$, $f \in D(LG_\alpha^0)$.

Then we have the following (cf. [8, Lemma 9.6.9]):

LEMMA 3.6. *The operator* LG_α^0 *can be uniquely extended to a non-negative, bounded linear operator* $\overline{LG_\alpha^0}: C(\overline{D}) \to C(\partial D)$ *for any* $\alpha > 0$.

(III) Finally we introduce a linear operator

$$LH_\alpha: C(\partial D) \to C(\partial D), \qquad \alpha > 0,$$

as follows.

(a) The domain $D(LH_\alpha)$ of LH_α is the space $C^{2+\theta}(\partial D)$.

(b) $LH_\alpha\psi = L(H_\alpha\psi)$, $\psi \in D(LH_\alpha)$.

Then we have the following (cf. [8, Lemmas 9.6.11 and 9.6.13]):

LEMMA 3.7. *The operator* LH_α *has its minimal closed extension* $\overline{LH_\alpha}$ *in the space* $C(\partial D)$ *for any* $\alpha > 0$. *Moreover the domain* $D(\overline{LH_\alpha})$ *of* $\overline{LH_\alpha}$ *does not depend on* $\alpha > 0$.

Now we can give a general existence theorem for Feller semigroups on ∂D in terms of boundary value problems (cf. [8, Theorem 9.6.15]):

THEOREM 3.8. (i) *If the operator* $\overline{LH_\alpha}$ *is the infinitesimal generator of a Feller semigroup on* ∂D, *then, for each constant* $\lambda > 0$, *the boundary value problem*

$$\begin{cases} (\alpha - W)u = 0 & \text{in } D, \\ (\lambda - L)u = \varphi & \text{on } \partial D \end{cases} \tag{$*'$}$$

has a solution u in $C^{2+\theta}(\overline{D})$, $0 < \theta < 1$, *for any* φ *in some* dense *subset of* $C(\partial D)$.

(ii) *Conversely, if, for some constant* $\lambda \geq 0$, *problem* ($*'$) *has a solution u in* $C^{2+\theta}(\overline{D})$ *for any* φ *in some* dense *subset of* $C(\partial D)$, *then the operator* $\overline{LH_\alpha}$ *is the infinitesimal generator of some Feller semigroup on* ∂D.

We conclude this section by giving a precise meaning to the boundary conditions Lu for functions $u \in D(\overline{W})$.

We let

$$D(L) = \{u \in D(\overline{W}) : u|_{\partial D} \in \mathscr{D}\},$$

where $\mathscr{D}$ is the common domain of the operators $\overline{LH_\alpha}$, $\alpha > 0$. Corollary 3.5 tells us that every function $u \in D(L) \subset D(\overline{W})$ can be written in the form

$$u = G_\alpha^0((\alpha I - \overline{W})u) + H_\alpha(u|_{\partial D}), \qquad \alpha > 0. \tag{3.3}$$

Then we define

$$Lu = \overline{LG_\alpha^0}(\alpha I - \overline{W})u) + \overline{LH_\alpha}(u|_{\partial D}). \tag{3.4}$$

The next lemma justifies definition (3.4) of Lu for all $u \in D(L)$ (cf. [8, Lemma 9.6.16]):

LEMMA 3.9. *The right-hand side of formula* (3.4) *depends only on* u, *not on the choice of expression* (3.3).

4. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. In the proof we make essential use of the unique solvability theorem for pseudo-differential operators (Theorem 2.1) and the general existence theorem for Feller semigroups in terms of boundary value problems (Theorem 3.8).

4.1. The space $C_0(\overline{D}\backslash M)$

First we consider a one-point compactification $K_\partial = K \cup \{\partial\}$ of the space $K = \overline{D}\backslash M$, where

$$M = \left\{x' \in \partial D : \mu(x') = \delta(x') = 0, \int_D t(x', y)dy < \infty\right\}$$
$$= \{x' \in \partial D : m(x') = 0\}.$$

We say that two points x and y of $\overline{D}$ are equivalent modulo M if $x = y$ or $x, y \in M$; we then write $x \sim y$. We denote by $\overline{D}/M$ the totality of equivalence classes modulo M. On the set $\overline{D}/M$, we define the quotient topology induced by the projection $q: \overline{D} \to \overline{D}/M$. It is easy to see that the topological space $\overline{D}/M$ is a *one-point compactification* of the space $\overline{D}\backslash M$ and that the *point at infinity* ∂ corresponds to the set M:

$$K_\partial = K \cup \{\partial\} = \overline{D}/M, \qquad \partial = M.$$

Furthermore we have the following isomorphism:

$$C(K_\partial) \cong \{u \in C(\overline{D}) : u \text{ is constant on } M\}. \tag{4.1}$$

Now we introduce a closed subspace of $C(K_\partial)$ as in Section 1:

$$C_0(K) = \{u \in C(K_\partial) : u(\partial) = 0\}.$$

Then we have by assertion (4.1) the isomorphism

$$C_0(K) \cong C_0(\overline{D}\backslash M) = \{u \in C(\overline{D}) : u = 0 \text{ on } M\}. \tag{4.2}$$

4.2. Proof of Theorem 2

We shall apply part (ii) of Theorem 1.1 to the operator $\mathfrak{M}$ defined by formula (0.4).

First we simplify the boundary condition L given by formula (0.1). If conditions (A) and (H) are satisfied, then one may assume that the boundary condition L is of the form

$$Lu = mL_\nu u + (m - 1)u = 0 \quad \text{on } \partial D, \tag{4.3}$$

with

$$0 \leq m \leq 1 \quad \text{on } \partial D.$$

Indeed it suffices to note that the boundary condition (0.1) is equivalent to the boundary condition

$$\left(\frac{m}{m - \gamma}\right) L_\nu u + \left(\frac{\gamma}{m - \gamma}\right) u = 0 \quad \text{on } \partial D.$$

Therefore the next theorem proves Theorem 2:

THEOREM 4.1. *We define a linear operator*

$$\mathfrak{M}: C_0(\bar{D}\backslash M) \to C_0(\bar{D}\backslash M)$$

as follows.

(a) *The domain* $D(\mathfrak{M})$ *of* $\mathfrak{M}$ *is the set*

$$D(\mathfrak{M}) = \{u \in C_0(\bar{D}\backslash M) : \overline{W}u \in C_0(\bar{D}\backslash M), Lu = 0\}. \tag{4.4}$$

(b) $\mathfrak{M}u = \overline{W}u, \ u \in D(\mathfrak{M})$.

Assume that the following condition (A′) *is satisfied*:

(A′) $$0 \leq m(x') \leq 1 \quad on \ \partial D.$$

Then the operator $\mathfrak{M}$ *is the infinitesimal generator of some Feller semigroup* $\{T_t\}_{t \geq 0}$ *on* $\bar{D}\backslash M$, *and the Green operator* $G_\alpha = (\alpha I - \mathfrak{M})^{-1}$, $\alpha > 0$, *is given by the following formula*:

$$G_\alpha f = G_\alpha^\nu f - H_\alpha(\overline{LH_\alpha}^{-1}(LG_\alpha^\nu f)), \qquad f \in C_0(\bar{D}\backslash M). \tag{4.5}$$

Here G_α^ν *is the Green operator for the boundary condition* L_ν *given by the formula*

$$G_\alpha^\nu f = G_\alpha^0 f - H_\alpha(\overline{L_\nu H_\alpha}^{-1}(\overline{L_\nu G_\alpha^0} f)), \qquad f \in C(\bar{D}). \tag{4.6}$$

PROOF. We apply part (ii) of Theorem 1.1 to the operator $\mathfrak{M}$ defined by formula (4.4). The proof is divided into several steps.

(1) First we prove that

for *all* $\alpha > 0$, the operator $\overline{LH_\alpha}$ generates a Feller semigroup on ∂D.

By virtue of the transmission property of $S \in L^{2-\kappa}_{1,0}(\boldsymbol{R}^N)$ and $T \in L^{2-\kappa_2}_{1,0}(\boldsymbol{R}^N)$, we find (cf. [2], [7, Chapter 3]) that the operator LH_α is the sum of a degenerate elliptic differential operator of second order and a classical pseudo-differential operator of order $2 - \min(\kappa_1, \kappa_2)$:

$$
\begin{aligned}
&LH_\alpha\varphi(x') \\
&\quad = m(x')L_\nu H_\alpha\varphi(x') + (m(x') - 1)\varphi(x') \\
&\quad = m(x')\left(\sum_{i,j=1}^{N-1} \bar{\alpha}^{ij}(x')\frac{\partial^2\varphi}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{N-1} \bar{\beta}^i(x')\frac{\partial\varphi}{\partial x_i}(x')\right) \\
&\qquad + ((m(x') - 1) - \alpha m(x')\bar{\delta}(x'))\varphi(x') \\
&\qquad + m(x')\bar{\mu}(x')\frac{\partial}{\partial \boldsymbol{n}}(H_\alpha\varphi)(x') \\
&\qquad + \int_{\partial D} m(x')\bar{r}(x', y')\left[\varphi(y') - \sigma(x', y')\left(\varphi(x') + \sum_{j=1}^{N-1}(y_j - x_j)\frac{\partial\varphi}{\partial x_j}(x')\right)\right]dy' \\
&\qquad + \int_D m(x')\bar{t}(x', y)\left[H_\alpha\varphi(y) - \sigma(x', y)\left(\varphi(x') + \sum_{j=1}^{N-1}(y_j - x_j)\frac{\partial\varphi}{\partial x_j}(x')\right)\right]dy.
\end{aligned}
$$

Furthermore it follows from an application of the boundary point lemma (see Appendix, Theorem A.2) that

$$\frac{\partial}{\partial \boldsymbol{n}}(H_\alpha 1) < 0 \quad \text{on } \partial D. \tag{4.7}$$

This implies that

$$LH_\alpha 1(x') \leq 0 \quad \text{on } \partial D.$$

Thus, applying Theorem 2.1 to the operator LH_α we obtain that

> if $\lambda > 0$ is sufficiently large, then the range $R(LH_\alpha - \lambda I)$ contains the space $C^{2+\theta}(\partial D)$. (4.8)

This implies that the range $R(LH_\alpha - \lambda I)$ is a *dense* subset of $C(\partial D)$. Therefore, applying part (ii) of Theorem 3.8 to the operator L we obtain that the operator $\overline{LH_\alpha}$ is the infinitesimal generator of some Feller semigroup on ∂D, for all $\alpha > 0$.

(2) Next we prove that

if condition (A′) is satisfied, then the equation

$$\overline{LH_\alpha}\psi = \varphi$$

has a unique solution ψ in $D(\overline{LH_\alpha})$ for any $\varphi \in C(\partial D)$; hence the inverse $\overline{LH_\alpha}^{-1}$ of $\overline{LH_\alpha}$ can be defined on the whole space $C(\partial D)$.

Further the operator $-\overline{LH_\alpha}^{-1}$ is non-negative and bounded on $C(\partial D)$. (4.9)

By inequality (4.7) and conditions (0.2) and (0.3), it follows that

$$\begin{aligned} LH_\alpha 1(x') &= m(x')L_\nu H_\alpha 1(x') + (m(x') - 1) \\ &= m(x')\left(\bar{\mu}(x')\frac{\partial}{\partial \boldsymbol{n}}(H_\alpha 1)(x') - \alpha\bar{\delta}(x')\right) + (m(x') - 1) \\ &\quad + m(x')\left(\int_{\partial D} \bar{r}(x', y')[1 - \sigma(x', y')]dy' + \int_D \bar{t}(x', y)[1 - \sigma(x', y)]dy\right) \\ &\quad + m(x')\int_D \bar{t}(x', y)[H_\alpha 1(y) - 1]dy \\ &= m(x')\left(\bar{\mu}(x')\frac{\partial}{\partial \boldsymbol{n}}(H_\alpha 1)(x') - \alpha\bar{\delta}(x')\right) + (m(x') - 1) \\ &\quad + m(x')\int_D \bar{t}(x', y)[H_\alpha 1(y) - 1]dy \\ &< 0 \quad \text{on } \partial D, \end{aligned}$$

so that

$$k_\alpha = -\sup_{x' \in \partial D} LH_\alpha 1(x') > 0.$$

Here we remark that the constants k_α are increasing in $\alpha > 0$:

$$\alpha \geq \beta > 0 \Rightarrow k_\alpha \geq k_\beta.$$

Moreover, using Corollary 1.2 with $K = \partial D$, $\mathfrak{A} = \overline{L\ H_\alpha}$ and $c = k_\alpha$ we obtain that the operator $\overline{LH_\alpha} + k_\alpha I$ is the infinitesimal generator of some Feller semigroup on ∂D. Therefore, since $k_\alpha > 0$, it follows from an application of part (i) of Theorem 1.1 with $\mathfrak{A} = \overline{LH_\alpha} + k_\alpha I$ that the equation

$$-\overline{LH_\alpha}\psi = (k_\alpha I - (\overline{LH_\alpha} + k_\alpha I))\psi = \varphi$$

has a unique solution $\psi \in D(\overline{LH_\alpha})$ for any $\varphi \in C(\partial D)$, and further that the operator $-\overline{LH_\alpha}^{-1} = (k_\alpha I - (\overline{LH_\alpha} + k_\alpha I))^{-1}$ is non-negative and bounded on the space $C(\partial D)$ with norm

$$\|-\overline{LH_\alpha}^{-1}\| = \|(k_\alpha I - (\overline{LH_\alpha} + k_\alpha I))^{-1}\| \leq \frac{1}{k_\alpha}. \tag{4.10}$$

(3) By assertion (4.9), we can define the operator G_α by formula (4.5) for all $\alpha > 0$. We prove that

$$G_\alpha = (\alpha I - \mathfrak{M})^{-1}, \qquad \alpha > 0. \tag{4.11}$$

By formulas (4.5) and (4.6), it follows that we have for all $f \in C_0(\bar{D} \backslash M)$

$$G_\alpha f \in D(\overline{W}),$$

and

$$\overline{W} G_\alpha f = \alpha G_\alpha f - f.$$

Furthermore we have

$$LG_\alpha f = LG_\alpha^\nu f - \overline{LH_\alpha}(\overline{LH_\alpha}^{-1}(LG_\alpha^\nu f)) = 0 \quad \text{on } \partial D, \tag{4.12}$$

or equivalently,

$$mL_\nu(G_\alpha f) + (m-1)G_\alpha f = 0 \quad \text{on } \partial D. \tag{4.12$'$}$$

This implies that

$$G_\alpha f = 0 \quad \text{on } M = \{x' \in \partial D : m(x') = 0\},$$

and so

$$\overline{W} G_\alpha f = \alpha G_\alpha f - f = 0 \quad \text{on } M.$$

Summing up, we have proved that

$$G_\alpha f \in D(\mathfrak{M}) = \{u \in C_0(\bar{D} \backslash M) : \overline{W} u \in C_0(\bar{D} \backslash M), Lu = 0\},$$

and

$$(\alpha I - \mathfrak{M}) G_\alpha f = f, \qquad f \in C_0(\bar{D} \backslash M),$$

that is,

$$(\alpha I - \mathfrak{M}) G_\alpha = I \quad \text{on } C_0(\bar{D} \backslash M).$$

Therefore, in order to prove formula (4.11) it suffices to show the injectivity of the operator $\alpha I - \mathfrak{M}$ for $\alpha > 0$.

Assume that

$$u \in D(\mathfrak{M}) \qquad \text{and} \qquad (\alpha I - \mathfrak{M})u = 0.$$

Then, by formula (3.3) it follows that the function u can be written as

$$u = H_\alpha(u|_{\partial D}), \qquad u|_{\partial D} \in \mathscr{D} = D(\overline{LH_\alpha}).$$

Thus we have by definition (3.4)

$$\overline{LH_\alpha}(u|_{\partial D}) = Lu = 0.$$

In view of assertion (4.9), this implies that

$$u|_{\partial D} = 0,$$

so that

$$u = H_\alpha(u|_{\partial D}) = 0 \quad \text{in } D.$$

(4) Now we prove the following three assertions:

(i) The operator G_α is non-negative on the space $C_0(\overline{D}\backslash M)$:

$$f \in C_0(\overline{D}\backslash M),\ f \geq 0 \quad \text{on } \overline{D}\backslash M \Rightarrow G_\alpha f \geq 0 \quad \text{on } \overline{D}\backslash M.$$

(ii) The operator G_α is bounded on the space $C_0(\overline{D}\backslash M)$ with norm

$$\|G_\alpha\| \leq \frac{1}{\alpha}, \qquad \alpha > 0. \tag{4.13}$$

(iii) The domain $D(\mathfrak{M})$ is everywhere dense in the space $C_0(\overline{D}\backslash M)$.

PROOF OF ASSERTION (i). Recall that the Dirichlet problem

$$\begin{cases} (\alpha - W)u = f & \text{in } D, \\ u = \varphi & \text{on } \partial D \end{cases}$$

is uniquely solvable in the framework of the spaces of continuous functions. Hence it follows that

$$G_\alpha^\nu f = H_\alpha(G_\alpha^\nu f|_{\partial D}) + G_\alpha^0 f \quad \text{on } \overline{D}. \tag{4.14}$$

Indeed it suffices to note that the both sides have the same boundary values $G_\alpha^\nu f|_{\partial D}$ and satisfy the same equation: $(\alpha - W)u = f$ in D.

Thus, applying the boundary operator L to the both sides of formula (4.14) we obtain that

$$LG_\alpha^\nu f = \overline{LH_\alpha}(G_\alpha^\nu f|_{\partial D}) + \overline{LG_\alpha^0} f.$$

Since the operators $-\overline{LH_\alpha}^{-1}$ and $\overline{LG_\alpha^0}$ are non-negative, it follows that

$$\begin{aligned}(-\overline{LH_\alpha}^{-1})(LG_\alpha^\nu f) &= -G_\alpha^\nu f|_{\partial D} + (-\overline{LH_\alpha}^{-1})(\overline{LG_\alpha^0} f) \\ &\geq -G_\alpha^\nu f|_{\partial D} \quad \text{on } \partial D.\end{aligned}$$

Therefore, by the non-negativity of H_α and G_α^0 we find that

$$\begin{aligned}G_\alpha f &= G_\alpha^\nu f + H_\alpha(-\overline{LH_\alpha}^{-1}(LG_\alpha^\nu f)) \\ &\geq G_\alpha^\nu f - H_\alpha(G_\alpha^\nu f|_{\partial D}) \\ &= G_\alpha^0 f \geq 0 \quad \text{on } \overline{D}.\end{aligned}$$

PROOF OF ASSERTION (ii). It suffices to show that

$$f \in C_0(\overline{D}\backslash M),\ f \geq 0 \quad \text{on } \overline{D} \Rightarrow \alpha G_\alpha f \leq \max_{\overline{D}} f \quad \text{on } \overline{D}, \tag{4.13'}$$

since G_α is non-negative on the space $C(\overline{D})$.

We remark (see formula (4.3)) that

$$LG_\alpha^\nu f = mL_\nu G_\alpha^\nu f + (m-1)G_\alpha^\nu f = (m-1)G_\alpha^\nu f \quad \text{on } \partial D,$$

so that

$$\begin{aligned} G_\alpha f &= G_\alpha^\nu f - H_\alpha(\overline{LH_\alpha}^{-1}(LG_\alpha^\nu f)) \\ &= G_\alpha^\nu f + H_\alpha(-\overline{LH_\alpha}^{-1}((m-1)G_\alpha^\nu f|_{\partial D})). \end{aligned}$$

Therefore, by the non-negativity of H_α and $-\overline{LH_\alpha}^{-1}$ it follows that

$$G_\alpha f = G_\alpha^\nu f + H_\alpha(-\overline{LH_\alpha}^{-1}((m-1)G_\alpha^\nu f|_{\partial D})) \leq G_\alpha^\nu f \leq \frac{1}{\alpha}\max_{\overline{D}} f \quad \text{on } \overline{D},$$

since $(m-1)G_\alpha^\nu f \leq 0$ on ∂D and $\|G_\alpha^\nu\| \leq 1/\alpha$. This proves assertion (4.13') and hence assertion (4.13).

PROOF OF ASSERTION (iii). In view of formula (4.11), it suffices to show that

$$\lim_{\alpha \to +\infty} \|\alpha G_\alpha f - f\| = 0, \qquad f \in C_0(\overline{D}\backslash M) \cap C^\infty(\overline{D}). \tag{4.15}$$

We remark that

$$\begin{aligned} \alpha G_\alpha f - f &= \alpha G_\alpha^\nu f - f - \alpha H_\alpha(\overline{LH_\alpha}^{-1}(LG_\alpha^\nu f)) \\ &= (\alpha G_\alpha^\nu f - f) + H_\alpha(\overline{LH_\alpha}^{-1}(\alpha(1-m)G_\alpha^\nu f|_{\partial D})). \end{aligned} \tag{4.16}$$

We estimate the two terms in the last line of formula (4.16).

(iii-1) First, applying Theorem 1 to the boundary condition L_ν we find that the first term tends to 0 (cf. [9, assertion (3.22)]):

$$\lim_{\alpha \to +\infty} \|\alpha G_\alpha^\nu f - f\| = 0. \tag{4.17}$$

(iii-2) To estimate the second term, we decompose it into the following form:

$$\begin{aligned} H_\alpha(\overline{LH_\alpha}^{-1}(\alpha(1-m)G_\alpha^\nu f|_{\partial D})) &= H_\alpha(\overline{LH_\alpha}^{-1}((1-m)f|_{\partial D})) \\ &\quad + H_\alpha(\overline{LH_\alpha}^{-1}((1-m)(\alpha G_\alpha^\nu f - f)|_{\partial D})). \end{aligned}$$

However we have by assertions (4.10) and (4.17)

$$\begin{aligned}\|H_\alpha(\overline{LH_\alpha}^{-1}((1-m)(\alpha G_\alpha^\nu f - f)|_{\partial D}))\| &\leq \|-\overline{LH_\alpha}^{-1}\| \cdot \|(1-m)(\alpha G_\alpha^\nu f - f)|_{\partial D}\| \\ &\leq \frac{1}{k_\alpha}\|(1-m)(\alpha G_\alpha^\nu f - f)|_{\partial D}\| \\ &\leq \frac{1}{k_1}\|\alpha G_\alpha^\nu f - f\| \to 0 \quad \text{as } \alpha \to +\infty. \end{aligned} \tag{4.18}$$

Here we have used the fact that

$$k_1 = -\sup_{x' \in \partial D} LH_1 1(x') \leq k_\alpha = -\sup_{x' \in \partial D} LH_\alpha 1(x') \quad \text{for all } \alpha \geq 1.$$

Thus we are reduced to the study of the term

$$H_\alpha(\overline{LH_\alpha}^{-1}((1-m)f|_{\partial D})).$$

To do so, we need a lemma on the behavior of the function $1/(-L_\nu H_\alpha 1)$ as $\alpha \to +\infty$ (cf. [1, Proposition III.1.6]):

LEMMA 4.2. *If the boundary condition L_ν is* transversal *on the boundary ∂D, then we have*

$$\lim_{\alpha \to +\infty} \left\| \frac{1}{-L_\nu H_\alpha 1} \right\| = 0.$$

Now, for any given $\varepsilon > 0$, one can find a function $h \in C^\infty(\partial D)$ such that

$$\begin{cases} h = 0 & \text{near } M = \{x' \in \partial D : m(x') = 0\}, \\ \|(1-m)f|_{\partial D} - h\| < \varepsilon. \end{cases}$$

Then we have for all $\alpha \geq 1$

$$\begin{aligned}\|H_\alpha(\overline{LH_\alpha}^{-1}((1-m)f|_{\partial D})) - H_\alpha(\overline{LH_\alpha}^{-1}h)\| &\leq \|-\overline{LH_\alpha}^{-1}\| \cdot \|(1-m)f|_{\partial D} - h\| \\ &\leq \frac{\varepsilon}{k_\alpha} \\ &\leq \frac{\varepsilon}{k_1}. \end{aligned}$$

Since ε is arbitrary, this proves that

$$\lim_{\alpha \to +\infty} \|H_\alpha(\overline{LH_\alpha}^{-1}((1-m)f|_{\partial D}) - H_\alpha(\overline{LH_\alpha}^{-1}h)\| = 0. \tag{4.19}$$

On the other hand one can find a function $\chi \in C_0^\infty(\partial D)$ such that

$$\begin{cases} \chi = 1 & \text{near } M, \\ (1-\chi)h = h & \text{on } \partial D. \end{cases}$$

Then we have

$$\begin{aligned} h(x') &= (1 - \chi(x'))h(x') \\ &= (-LH_\alpha 1(x'))\left(\frac{1 - \chi(x')}{-LH_\alpha 1(x')}\right)h(x') \\ &\le \left[\sup_{x' \in \partial D}\left(\frac{1 - \chi(x')}{-LH_\alpha 1(x')}\right)\right]\|h\|(-LH_\alpha 1(x')). \end{aligned}$$

Since the operator $-\overline{LH_\alpha}^{-1}$ is non-negative on $C(\partial D)$, it follows that

$$-\overline{LH_\alpha}^{-1}h \le \sup_{x' \in \partial D}\left(\frac{1 - \chi(x')}{-LH_\alpha 1(x')}\right)\cdot\|h\| \quad \text{on } \partial D,$$

and so

$$\|H_\alpha(\overline{LH_\alpha}^{-1}h)\| \le \|-\overline{LH_\alpha}^{-1}h\| \le \sup_{x' \in \partial D}\left(\frac{1 - \chi(x')}{-LH_\alpha 1(x')}\right)\cdot\|h\|. \tag{4.20}$$

However, there exists a constant $c_0 > 0$ such that

$$0 \le \frac{1 - \chi(x')}{m(x')} \le c_0, \qquad x' \in \partial D,$$

so that

$$\begin{aligned} \frac{1 - \chi(x')}{-LH_\alpha 1(x')} &\le \left(\frac{1 - \chi(x')}{m(x')(-L_\nu H_\alpha 1(x')) + (1 - m(x'))}\right) \\ &\le c_0\left\|\frac{1}{-L_\nu H_\alpha 1}\right\|. \end{aligned}$$

In view of Lemma 4.2, this implies that

$$\lim_{\alpha\to+\infty}\left[\sup_{x' \in \partial D}\left(\frac{1 - \chi(x')}{-LH_\alpha 1(x')}\right)\right] = 0.$$

Hence we have by inequality (4.20)

$$\lim_{\alpha\to+\infty}\|H_\alpha(\overline{LH_\alpha}^{-1}h)\| = 0. \tag{4.21}$$

Therefore we obtain from inequalities (4.19) and (4.21) that

$$\limsup_{\alpha\to+\infty}\|H_\alpha(\overline{LH_\alpha}^{-1}((1 - m)f|_{\partial D}))\| = 0. \tag{4.22}$$

Finally, combining assertions (4.18) and (4.22) we find that the second term in the last line of formula (4.16) also tends to 0:

$$\lim_{\alpha \to +\infty} \|H_\alpha(\overline{LH_\alpha}^{-1}(\alpha(1-m)G_\alpha^\nu f|_{\partial D}))\| = 0.$$

This completes the proof of assertion (4.15) and hence that of assertion (iii).

(5) Summing up, we have proved that the operator $\mathfrak{M}$, defined by formula (4.4), satisfies conditions (a) through (d) in Theorem 1.1. Hence, in view of assertion (4.2) it follows from an application of part (ii) of the same theorem that the operator $\mathfrak{M}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D}\backslash M$.

The proof of Theorem 4.1 and hence that of Theorem 2 is now complete.

□

Appendix: The maximum principle

In this appendix, following [1] we formulate two useful maximum principles for second-order elliptic Waldenfels operators.

First we state the strong maximum principle (see [1, Théorème VII]):

THEOREM A.1. *Let W be a second-order elliptic Waldenfels operator. Assume that a function $u \in C^2(\overline{D})$ satisfies $Wu \geq 0$ in D and $C = \max_{\overline{D}} u \geq 0$. If the function u takes its maximum C at some point $x_0 \in D$, then $u \equiv C$ in the connected component containing x_0.*

Next we consider the interior normal derivative $(\partial u)/(\partial \boldsymbol{n})$ at a boundary point where the function $u \in C^2(\overline{D})$ takes its non-negative maximum.

The boundary point lemma reads as follows (see [1, Théorème VIII]):

THEOREM A.2 (The boundary point lemma). *Let W be a second-order elliptic Waldenfels operator. Assume that a function $u \in C^2(\overline{D})$ satisfies $Wu \geq 0$ in D, and that there exists a point x_0' of ∂D such that*

$$\begin{cases} u(x_0') = \max_{\overline{D}} u(x) \geq 0, \\ u(x) < u(x_0'), \qquad x \in D. \end{cases}$$

Then we have

$$\frac{\partial u}{\partial \boldsymbol{n}}(x_0') < 0.$$

References

[1] J.-M. Bony, P. Courrège et P. Priouret, Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum, Ann. Inst. Fourier (Grenoble) **18** (1968), 369–521.

[2] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, Acta Math. **126** (1971), 11–51.

[3] C. Cancelier, Problèmes aux limites psuedo-différentiels donnant lieu au principe du maximum, Comm. Partial Differential Equations **11** (1986), 1677–1726.

[4] Chazarain, J. et A. Piriou, Introduction à la théorie des équations aux dérivées partielles linéaires, Gauthier-Villars, Paris, 1981.

[5] R. R. Coifman et Y. Meyer, Au-delà des opérateurs pseudo-différentiels, Astérisque No. 57, Soc. Math. France, Paris, 1978.

[6] L. Hörmander, The analysis of linear partial differential operators III, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1985.

[7] S. Rempel and B.-W. Schulze, Index theory of elliptic boundary problems, Akademie-Verlag, Berlin, 1982.

[8] K. Taira, Diffusion processes and partial differential equations, Academic Press, San Diego New York London Tokyo, 1988.

[9] K. Taira, On the existence of Feller semigroups with boundary conditions, Memoirs Amer. Math. Soc. No. 475 (1992).

[10] K. Taira, On the existence of Feller semigroups with boundary conditions II, J. Functional Analysis **129** (1995), 108–131.

[11] S. Takanobu and S. Watanabe, On the existence and uniqueness of diffusion processes with Wentzell's boundary conditions, J. Math. Kyoto Univ. **28** (1988), 71–80.

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