HIROSHIMA MATH. J.
27 (1997), 47–75

A family of Yang-Mills connections on 4-dimensional pseudo-Riemannian spaces

Dedicated to Professor Kiyosato Okamoto for his 60th birthday

Hiroshi KAJIMOTO

(Received September 6, 1995)

ABSTRACT. A family of Yang-Mills connections on 4-dimensional pseudo-Riemannian spaces S^4, $S^1 \times S^3$, $S^2 \times S^2$ of respective indices are constructed by a group theoretic method. The index and the nullity of their second variations are calculated.

1. Introduction—A Review of Riemannian case

In this article we construct a family of Yang-Mills connections on 4-dimensional pseudo-Riemannian spaces S^4, $S^1 \times S^3$, $S^2 \times S^2$ equipped with the indefinite Riemannian metrics of the index $(4, 0)$, $(1, 3)$, $(2, 2)$ respectively by a unified method. And then we study the index and nullity of their second variations at the canonical connection. We are interested especially in the compactified Minkowski space $S^1 \times S^3$. On the Riemannian space S^4 our connection is the BPST-instanton of the Hopf fibering $S^7 \to S^4$ (see Atiyah [1], Chapter II and Chapter III, 2). We review this case first from a group theoretic view point.

The BPST-instanton whose instanton number equals one can be constructed on Euclidean 4-space $\boldsymbol{R}^4$ (identified with the set $\boldsymbol{H}$ of quaternions) according to the following diagram:

$$\begin{tikzcd}
 & G = SL_2(\boldsymbol{H}) \arrow[r, "\kappa"] \arrow[dd] & K \arrow[dd, "\theta"', "M"] & = & Sp_2 \arrow[d] & \\
x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} & & & & Sp_2/Sp_1 \arrow[d] & = S^7 \arrow[d, "Sp_1"] \\
\boldsymbol{R}^4 = \boldsymbol{H} = \overline{N} \arrow[uur, "i"] \arrow[r, "j", "\text{open dense}"'] & G/P & = K/M & = & Sp_2/Sp_1 \times Sp_1 & = S^4
\end{tikzcd}$$

where we use the following notation. $G = SL_2(\boldsymbol{H}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \boldsymbol{H}\right.$

1991 *Mathematics Subject Classification.* 53C07, 53C35, 58E15

Key words and phrases. Yang-Mills connection, Semisimple Lie groups, Second variation, index, nullity.

and the corresponding complex 4×4-matrix has determinant $1\}$. The element $\alpha + \beta j \in \boldsymbol{H}$ ($\alpha, \beta \in \boldsymbol{C}$) corresponds to a complex 2×2-matrix $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in M_2(\boldsymbol{C})$. The Iwasawa decomposition and its K-projection is denoted by $G = KAN \xrightarrow{\kappa} K$ where $K = Sp_2$, $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \boldsymbol{R},\ a > 0 \right\}$ and $N = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in \boldsymbol{H} \right\}$. $P = MAN = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in G \right\}$ is a parabolic subgroup and $M = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a, b \in \boldsymbol{H},\ |a| = |b| = 1 \right\} = Sp_1 \times Sp_1$ is the centralizer of A in K. We use the decomposition $\bar{N}P \subset G$ (open dense). And Θ is the K-invariant canonical connection of the bundle $K \to K/M$. i.e., Θ is a K-invariant, $\mathfrak{m}$-valued 1-form on K, whose value at identity is the orthogonal projection relative to the Killing form of $\mathfrak{k}$, $\Theta_e \colon \mathfrak{k} = \mathfrak{m} + \mathfrak{s} \to \mathfrak{m}$, where $\mathfrak{k}$, $\mathfrak{m}$ are the Lie algebras of K, M and $\mathfrak{s}$ is the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$.

We equip the base space K/M with a positive definite K-invariant Riemannian metric h induced by the negative of the Killing form. And we equip the Lie algebra $\mathfrak{m}$ of the structure group with a positive definite invariant inner product. Laquer [10] considers the Yang-Mills functional associated with principal bundles over homogeneous spaces and shows that the canonical connection Θ in $K \to K/M$ is a Yang-Mills connection relative to the metric h. The embedding j is easily checked to be a conformal map from the standard Euclidean 4-space $\boldsymbol{R}^4 = \boldsymbol{H}$ to an open dense subset of the compact Riemannian space $K/M = S^4$. The Yang-Mills functional or the Hodge star operator $*$ on 2-forms is conformally invariant over 4-dimensional spaces. Therefore we get the finite-action Yang-Mills connection in $\boldsymbol{R}^4 \times M \to \boldsymbol{R}^4$ of the pull-back bundle. Put $u = \kappa \circ i$ and compute the pull back $u^*\Theta$. Then we have a $\mathfrak{m}$-valued 1-form on $\boldsymbol{H}$

$$(u^*\Theta)(x) = \text{the orthogonal projection of } (u^{-1}du)(x) \text{ to } \mathfrak{m}$$

$$= \left[\begin{array}{c|c} \operatorname{Im} \dfrac{x d\bar{x}}{1 + |x|^2} & 0 \\ \hline 0 & \operatorname{Im} \dfrac{\bar{x} dx}{1 + |x|^2} \end{array} \right]$$

Its projection to the second $\mathfrak{sp}_1$:

$$A(x) = \operatorname{Im} \frac{\bar{x} dx}{1 + |x|^2}$$

is an anti-self-dual connection which is a local form of the so called BPST anti-instanton. This projection corresponds to a projection of the structure group $M = Sp_1 \times Sp_1 \to Sp_1$, which reduces to the Hopf bundle $S^7 \to S^4$ as above. (The projection to the first $\mathfrak{sp}_1$ is a self-dual connection).

In the above diagram the group $G = SL_2(\boldsymbol{H})$ acts naturally on K, $K/M = S^4$ and $\bar{N}$ as follows. For $g \in G$, $k \in K$ and $\bar{n} \in \bar{N}$, we put

$$\tau_g(k) = \kappa(gk), \qquad \tau_g(kM) = \kappa(gk)M \qquad \text{and} \qquad \tau_g(\bar{n}) = \bar{n}(g\bar{n})$$

where $g = \bar{n}(g)m(g)a(g)n(g) \in \bar{N}MAN$ is the decomposition of open dense subset $\bar{N}MAN$ of G. Identifying $\boldsymbol{H} = \bar{N}$ by $x \mapsto \bar{n}_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, G-action on $\boldsymbol{H}$ is

$$\tau_g(x) = (ax + b)(cx + d)^{-1}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad x \in \boldsymbol{H}.$$

G-actions on $K/M = S^4$ and $\bar{N} = \boldsymbol{H} = \boldsymbol{R}^4$ are conformal transformations. Hence G acts on the space $\mathscr{Y}$ of all the Yang-Mills connections and the space $\mathscr{Y}^-$ of all the anti-self-dual connections in $K \to K/M$. Its action is pulled back on $\boldsymbol{H}$ as

$$A^g(x) = u^*(\tau_g^* \Theta)(x) = J_g(x)^{-1}(\tau_g^* A)(x) J_g(x) + J_g(x)^{-1} dJ_g$$
$$\sim (\tau_g^* A)(x) \quad \text{modulo gauge group } \mathscr{G},$$

where $J_g(x) = m(g\bar{n}_x) = (cx + d)/|cx + d| \in M = Sp_1$. The gauge equivalence is valid at least on $\boldsymbol{H} \backslash \{-c^{-1}d\}$. If $g \in K = Sp_2$ then $\tau_g^* \Theta = \Theta$ by the K-invariance and hence the map $g \mapsto \tau(g^{-1})^* \Theta$ induces the smooth map

$$SL_2(\boldsymbol{H})/Sp_2 \to \mathscr{M}^- = \mathscr{Y}^-/\mathscr{G}$$

where the right hand side is the moduli space of anti-instantons. Atiyah, Hitchin, Singer [3] show that $G = SL_2(\boldsymbol{H})$ acts transitively on the moduli space $\mathscr{M}^-$ and hence that the above map is an onto diffeomorphism. In other words the Moduli space is a single G-orbit: $\mathscr{M} = G[\Theta]$.

This group theoretic method using the structure of a semisimple conformal transformation group G is available for the construction of Yang-Mills connections on the other pseudo-Riemannian 4-spaces $\boldsymbol{R}^{1,3}$, $\boldsymbol{R}^{2,2}$ or rather their compactifications $S^1 \times S^3$, $S^2 \times S^2$, which are identified with appropriate homogeneous spaces $G/P = K/K \cap M$ of the conformal transformation group G. For the group G we take $SU(2, 2)$, $SL_4(\boldsymbol{R})$ and pick up a suitable parabolic subgroup $P = MAN$ whose N-part is of dimension 4. Note that $SL_2(\boldsymbol{H})$, $SU(2, 2)$ and $SL_4(\boldsymbol{R})$ are real forms of the complex Lie group $SL_4(\boldsymbol{C})$. In

§2 and §3 we give detailed constructions for $\boldsymbol{R}^{1,3}$ and $\boldsymbol{R}^{2,2}$ respectively and study some related properties of G-orbit $G[\Theta]$. In §4 we study their second variations at the canonical connection Θ by a method of homogeneous vector bundle and representation theory of compact Lie groups.

2. The case of $\boldsymbol{R}^{1,3}$—Minkowski space

Take $G = SU(2,2) = \left\{g \in SL_4(\boldsymbol{C}) \,\middle|\, g^*\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\right\}$ and the parabolic subgroup $P = \left\{\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in G\right\}$, where g^* is the transposed conjugate matrix of g. The Langlands decomposition is $P = MAN$ where $A = \left\{\begin{pmatrix} aI & 0 \\ 0 & a^{-1}I \end{pmatrix} \,\middle|\, a > 0,\ a \in \boldsymbol{R}\right\}$, $\mathrm{M} = \left\{\begin{pmatrix} g^{*-1} & 0 \\ 0 & g \end{pmatrix} \,\middle|\, g \in M_2(\boldsymbol{C}), \det g = \pm 1\right\} \simeq SL_2^{\pm}(\boldsymbol{C})$ and $N = \left\{\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \,\middle|\, X \in M_2(\boldsymbol{C}), X = X^*\right\}$. Let $K = G \cap SU_4 = \left\{k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\middle|\, A^*A + B^*B = I, A^*B - B^*A = 0,\ \det k = 1\right\} = S(U_2 \times U_2)$ be the maximal compact subgroup of G. Then $H = K \cap M = \left\{\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \,\middle|\, g^*g = I, \det g = \pm 1\right\} \simeq SU_2^{\pm}$. In the subsequent sections we also use another realization of $G = SU(2,2)$ i.e., $G = \left\{g \in SL_4(\boldsymbol{C}) \,\middle|\, g^*\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} g = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\right\}$. Two realizations of $SU(2,2)$ are connected by a Cayley transformation $C = \mathrm{Ad}\dfrac{1}{\sqrt{2}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}$, e.g. the maximal compact subgroup $K = S(U_2 \times U_2)$ is transformed by $C \cdot \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} A - iB & 0 \\ 0 & A + iB \end{pmatrix}$, $A \pm iB \in U_2$, $\det(A - iB) \cdot (A + iB) = 1$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{h}$ be the Lie algebras of G, K, m, A, N and H respectively and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Then we have the decomposition: $G = K \times \exp(\mathfrak{m} \cap \mathfrak{p}) \times A \times N$ (cf. [14], 1.2.4.11). Put $\kappa\colon G \to KM = K \times \exp(\mathfrak{m} \cap \mathfrak{p})$ and $k\colon G \to K$ be the corresponding projections. Let $\bar{N} = \left\{\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \,\middle|\, X^* = X\right\}$. Then we also have the decomposition of an open dense subset $\bar{N}P$ of G: $\bar{N}P = \bar{N} \times M \times A \times N$ denoted by $g = \bar{n}(g)m(g)a(g)n(g)$.

LEMMA 2.1. *The decomposition* $G = K \times \exp(\mathfrak{m} \cap \mathfrak{p}) \times A \times N = KM \times A \times N$ *is given by the following: For*

$$g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G;\quad \det g=1,\quad A^*D-C^*B=I,$$

$$A^*C-C^*A=B^*D-D^*B=0,$$

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}E & F\\ -F & E\end{pmatrix}\begin{pmatrix}h^{-1} & 0\\ 0 & h\end{pmatrix}\begin{pmatrix}sI & 0\\ 0 & s^{-1}I\end{pmatrix}\begin{pmatrix}I & 0\\ X & I\end{pmatrix}$$

$$=\begin{pmatrix}s^{-1}(A-BX) & sB\\ s^{-1}(C-DX) & sD\end{pmatrix}\begin{pmatrix}sI & 0\\ 0 & s^{-1}I\end{pmatrix}\begin{pmatrix}I & 0\\ X & I\end{pmatrix}$$

where

$$s=\det(B^*B+D^*D)^{-1/4}\in \boldsymbol{R},\qquad h=s(B^*B+D^*D)^{1/2}\in SL_2^{\pm}(\boldsymbol{C})$$

$$E=D(B^*B+D^*D)^{-1/2},\qquad F=B(B^*D+D^*D)^{-1/2},$$

$$X=(A^*B+C^*D)(B^*B+D^*D)^{-1}.$$

And the decomposition: $\overline{N}P=\overline{N}\times M\times A\times N$ *is given by*

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}I & BD^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix}D^{*-1} & 0\\ C & D\end{pmatrix}$$

$$=\begin{pmatrix}I & BD^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix}s^{-1}D^{*-1} & 0\\ 0 & sD\end{pmatrix}\begin{pmatrix}sI & 0\\ 0 & s^{-1}I\end{pmatrix}\begin{pmatrix}I & 0\\ C^*D^{*-1} & I\end{pmatrix}$$

where $s=\det(D^*D)^{-1/4}\in\boldsymbol{R}$. □

We consider the principal M-bundle $KM\to KM/M=K/K\cap M=K/H$ and the canonical K-invariant connection Θ. Θ is by definition a K-invariant $\mathfrak{m}$-valued 1-form on KM whose value at identity is the projection $\Theta_e\colon \mathfrak{s}+\mathfrak{m}\to\mathfrak{m}$ where the tangent space $T_e(KM)$ is identified with $\mathfrak{k}+\mathfrak{m}=\mathfrak{s}+\mathfrak{m}$ and $\mathfrak{s}=\left\{\begin{pmatrix}0 & B\\ -B & 0\end{pmatrix}\middle| B=B^*\right\}$ is the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}+\mathfrak{m}$ relative to the Killing form of $\mathfrak{g}$. In other words, $\mathfrak{s}$ and its left K and right M-translates are horizontal subspaces in KM. The base space $K/H=U_2$ is homeomorphic to $S^1\times S^3$. The isometry $K/H=U_2$ is given by $\begin{pmatrix}A & B\\ -B & A\end{pmatrix}H\mapsto (A-iB)(A+iB)^{-1}$. We give a K-invariant (1, 3)-metric h as follows. $\mathfrak{k}=\left\{\begin{pmatrix}A & B\\ -B & A\end{pmatrix}\middle| A+A^*=0,\ \operatorname{tr}A=0,\ B^*=B\right\}=\mathfrak{s}+\mathfrak{h}$ is isomorphic to $\mathfrak{s}(\mathfrak{u}_2\times\mathfrak{u}_2)$. $\mathfrak{h}=\mathfrak{k}\cap\mathfrak{m}\simeq\mathfrak{su}_2$. We see $\mathfrak{k}=\mathfrak{z}+\mathfrak{k}_1$ where $\mathfrak{z}=\boldsymbol{R}Z$ $\left(Z=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\right)$ is the 1-dimensional center of $\mathfrak{k}$ which is contained in $\mathfrak{s}$ and $\mathfrak{k}_1=[\mathfrak{k},\mathfrak{k}]=\mathfrak{h}+\mathfrak{s}\cap\mathfrak{k}_1\simeq$

$\mathfrak{su}_2 \times \mathfrak{su}_2$. Hence $\mathfrak{s} = \mathfrak{z} + \mathfrak{s} \cap \mathfrak{k}_1$ is isomorphic to $\mathfrak{u}_2$ as $\mathfrak{h}$-module. The Killing form of $\mathfrak{k}_1 \simeq \mathfrak{su}_2 \times \mathfrak{su}_2$ is $B(X, Y) = 4\,\mathrm{tr}(XY)$ for $X, Y \in \mathfrak{k}_1$. Define h on $\mathfrak{s} = T_o(K/H)$ as $h(X, Y) = 4\,\mathrm{tr}(XY)$, $h(X, Z) = 0$ for $X, Y \in \mathfrak{s} \cap \mathfrak{k}_1$ and $Z \in \mathfrak{z}$ with $h(Z, Z) = -4\,\mathrm{tr}\, Z^2 = 16$. h is $(\mathrm{Ad}\, H)$-invariant, so that h is extended to a K-invariant $(1, 3)$ metric on K/H. Under the isometry $K/H = U_2$ this metric corresponds to $h = -4\det(g^{-1}dg)$, $g \in U_2$. We give the Lie algebra $\mathfrak{m} = \mathfrak{sl}_2(\boldsymbol{C})$ of the structure group M an indefinite inner product $\langle X, Y\rangle = \mathrm{Re}\,\mathrm{tr}(XY)$. We can prove in the same way as Laquer [10]'s Riemannian case that the canonical connection Θ is a Yang-Mills connection relative to the invariant metric h (see [10], Theorem 3.1). Put $H(2) =$ the space of all 2×2-Hermitian symmetric matrices. Take the coordinates on $H(2)$ by $X = \begin{pmatrix} x_1 - x_2 & ix_3 - x_4 \\ -ix_3 - x_4 & x_1 + x_2 \end{pmatrix}$. We give $H(2)$ the $(1, 3)$-metric $\det dX = dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2$ and regard it as the standard Minkowski space $\boldsymbol{R}^{1,3}$. Identify $H(2) = \bar{N}$ by the map $X \mapsto \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$.

Consider a diagram:

$$\begin{tikzcd} & G = SU(2,2) \arrow[r, "\kappa"] \arrow[d] & KM \arrow[d, "\Theta"', "M = SL_2^{\pm}(C)"] \\ \boldsymbol{R}^{1,3} = H(2) = \bar{N} \arrow[ur, "i"] \arrow[r, "j"] & G/P \arrow[r, phantom, "="] & K/H = U_2 \simeq S^1 \times S^3 \end{tikzcd}$$

$$X \longmapsto \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \longmapsto (I - iX)(I + iX)^{-1}.$$

The embedding $j\colon \bar{N} \to K/H = U_2$ has an open dense image and $j(X) = (I - iX)(I + iX)^{-1}$, $j^*h = -4\det(j^{-1}dj) = 16\det(I + X^2)^{-1}(\det dX)$. Since $16\det(I + X^2)^{-1} > 0$ on $H(2)$ we might say that the map j is conformal. Hence we get a finite-action Yang-Mills connection on $\boldsymbol{R}^{1,3}$. Put $u = \kappa \circ i$ and compute the pull-back $u^*\Theta$ on $\boldsymbol{R}^{1,3} = H(2)$. Then

PROPOSITION 2.2. *The local form B of the canonical connection Θ is given by the following $\mathfrak{m}$-valued 1-form on $H(2)$.*

$$B(X) = (u^*\Theta)(X) = \mathrm{Im}(I + X^2)^{-1}XdX, \qquad x \in H(2),$$

where Im-*part of a matrix means* $\mathrm{Im}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (a-d)/2 & b \\ c & (d-a)/2 \end{pmatrix}$. *The curvature form $F_B = dB + B \wedge B$ is*

$$F_B(X) = (I + X^2)^{-1}dX(I + X^2)^{-1}dX.$$

PROOF. The canonical connection Θ in the bundle $KM \to K/H$ is expressed as in the Riemannian case by

$$\Theta = \mathrm{proj}(u^{-1}du), \qquad u \in KM,$$

where proj: $\mathfrak{g} \to M_2(\boldsymbol{C})$ is $\mathrm{proj}\begin{pmatrix} -A^* & B \\ C & A \end{pmatrix} = A$, $\mathfrak{g} = \left\{\begin{pmatrix} -A^* & B \\ C & A \end{pmatrix} \middle| \mathrm{tr}(A - A^*) = 0, C^* = C, B^* = B\right\}$, $\mathfrak{m} = \left\{\begin{pmatrix} -A^* & 0 \\ 0 & A \end{pmatrix} \middle| \mathrm{tr}\, A = 0\right\} \simeq \mathfrak{sl}_2(\boldsymbol{C})$ by $\begin{pmatrix} -A^* & 0 \\ 0 & A \end{pmatrix} \mapsto A$. In fact let $X \in T_u(KM)$, $u = km \in KM$ and let $u(t) = k(t)m(t)$ ($k(0) = k$, $m(0) = m$) be a C^∞-curve in KM tangent to X at u. Then

$$X = \left(\frac{d}{dt}\right)_{t=0} u(t) = \left(\frac{dk}{dt}\right)_0 m + k\left(\frac{dm}{dt}\right)_0,$$

in the matrix algebra. The value of the connection $\Theta(X)$ at u is in $\mathfrak{m}$ and given by

$$\begin{aligned}\Theta_u(X) &= \Theta_{km}\left(\left(\frac{dk}{dt}\right)_0 m\right) + \Theta_{km}\left(k\left(\frac{dm}{dt}\right)_0\right) = m^{-1}\Theta_k\left(\left(\frac{dk}{dt}\right)_0\right)m + \Theta_m\left(\left(\frac{dm}{dt}\right)_0\right) \\ &= m^{-1}\Theta_e\left(k^{-1}\left(\frac{dk}{dt}\right)_0\right)m + m^{-1}\left(\frac{dm}{dt}\right)_0,\end{aligned}$$

due to the left K-invariance and right M-equivariance of our connection Θ. We note that $k^{-1}\left(\frac{dk}{dt}\right)_0 \in \mathfrak{k}$, $m^{-1}\left(\frac{dm}{dt}\right)_0 \in \mathfrak{m}$, $\mathfrak{k} + \mathfrak{m} = \left\{\begin{pmatrix} -A^* & B \\ -B & A \end{pmatrix} \middle| \mathrm{tr}\, A = 0, B^* = B\right\}$ and $\Theta_e\begin{pmatrix} -A^* & B \\ -B & A \end{pmatrix} = A$. If we extend Θ_e to the map proj on $\mathfrak{g}$ as above, we know that $m^{-1}\Theta_e\left(k^{-1}\left(\frac{dk}{dt}\right)_0\right)m = \mathrm{proj}\left(m^{-1}k^{-1}\left(\frac{dk}{dt}\right)_0 m\right)$. Hence

$$\begin{aligned}\Theta_u(X) &= \mathrm{proj}\left(m^{-1}k^{-1}\left(\frac{dk}{dt}\right)_0 m + m^{-1}\left(\frac{dm}{dt}\right)_0\right) \\ &= \mathrm{proj}\left(u^{-1}\left(\frac{du}{dt}\right)(0)\right) = \mathrm{proj}(u^{-1}du(X)).\end{aligned}$$

Now by Lemma 2.1 we know that if $u = \kappa \circ i$ and $X \in H(2)$,

$$u(X) = \kappa\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$
$$= \begin{pmatrix} (I+X^2)^{-1/2} & X(I+X^2)^{-1/2} \\ -X(I+X^2)^{-1/2} & (I+X^2)^{-1/2} \end{pmatrix}\begin{pmatrix} s^{-1}(I+X^2)^{-1/2} & 0 \\ 0 & s(I+X^2)^{1/2} \end{pmatrix}$$
$$= \begin{pmatrix} s^{-1}(I+X^2)^{-1} & sX \\ -s^{-1}X(I+X^2)^{-1} & sI \end{pmatrix} \quad \text{where} \quad s = \det(I+X^2)^{-1/4}.$$

Therefore

$$u(X)^{-1} = \begin{pmatrix} sI & -sX \\ s^{-1}X(I+X^2)^{-1} & s^{-1}(I+X^2)^{-1} \end{pmatrix}, \qquad du(X) = \begin{pmatrix} * & Xds + sdX \\ * & dsI \end{pmatrix}.$$

Hence we get the pull-back connection on $H(2)$:

$$B(X) = (u^*\Theta)(X) = \operatorname{proj}\begin{pmatrix} * & * \\ s^{-1}X(I+X^2)^{-1} & s^{-1}(I+X^2)^{-1} \end{pmatrix}\begin{pmatrix} * & Xds + sdX \\ * & dsI \end{pmatrix}$$
$$= s^{-1}dsI + (I+X^2)^{-1}XdX$$
$$= \operatorname{Im}(I+X^2)^{-1}XdX.$$

The curvature form $F_B = dB + B \wedge B$ is calculated by using the expression $B(X) = s^{-1}dsI + (I+X^2)^{-1}XdX$. □

The Yang-Mills equation $d_B * F_B = 0$ is checked also by a direct calculation on $\boldsymbol{R}^{1,3} = H(2)$. We remark that F_B does not satisfy the self-duality condition: $*F_B = \pm iF_B$ (note $*^2 = -1$ on $\boldsymbol{R}^{1,3}$). Its density function $D(B)(X) = \langle F_B, F_B\rangle(X)$ is computed from $\langle F_B, F_B\rangle dv = \langle F_B \wedge *F_B\rangle$, $dv = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ by a lengthy computation as follows:

$$D(B)(X) = \frac{-24}{\det(I+X^2)^2}.$$

The group $G = SU(2,2)$ acts naturally on KM, $K/H = S^1 \times S^3$ and $\bar{N} = H(2)$ as follows. For $g \in G$, $k \in K$, $m \in M$ and $\bar{n} \in \bar{N}$, define

$$\bar{\tau}_g(km) = \kappa(gkm), \quad \tau_g(kH) = \kappa(gk)M = k(gk)H \quad \text{and} \quad \tau_g(\bar{n}) = \bar{n}(g\bar{n}).$$

Then we know that $\tau_g(kH) = \bar{\tau}_g(k)H$, $j \circ \tau_g(\bar{n}) = \tau_g \circ j(\bar{n})$ and $\bar{\tau}_g \circ u(\bar{n}) = (u \circ \tau_g)(\bar{n})m(g\bar{n})$ hold. Identifying $H(2) = \bar{N}$, G-action on $H(2)$ is given by

$$\tau_g(X) = (AX+B)(CX+D)^{-1}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G, \quad X \in H(2).$$

G-actions on $K/H = S^1 \times S^3$ and $H(2) = \boldsymbol{R}^{1,3}$ are checked to be conformal transformations, i.e. $\tau_g^*h = f_g h$, $f_g > 0$, $\in C^\infty(K/H)$. Hence we get a G-action on the space $\mathscr{Y}$ of all Yang-Mills connections in $KM \to K/H$. The G-action can be computed by pulling-back to $H(2)$ as

PROPOSITION 2.3. *Let ω be a connection in $KM \to K/H$ and let $A = u^*\omega$ be a local form of ω which is a $\mathfrak{m}$-valued 1-form on $H(2)$. If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $X \in H(2)$ then*

$$u^*(\bar{\tau}_g^*\omega)(X) = J_g(X)^{-1}(\tau_g^*A)(X)J_g(X) + J_g(X)^{-1}dJ_g,$$

where $J_g(X) = m(g\bar{n}_X) = s^{-1}(CX + D) \in M = SL_2^{\pm}(C)$, $\bar{n}_X = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \in \bar{N}$ and $s = (\det(CX + D)^*(CX + D))^{1/4} \in \boldsymbol{R}$. Hence τ_g^*A is gauge equivalent to a local form of $\bar{\tau}_g^*\omega$ on the open dense subset $H(2)\backslash\{X \mid \det(CX + D) = 0\}$ of $H(2)$.

PROOF. The formula: $(\bar{\tau}_g \circ u)(\bar{n}) = (u \circ \tau_g)(\bar{n})J_g(\bar{n})$ $(J_g(\bar{n}) = m(g\bar{n}))$ says that the difference between two maps $\bar{\tau}_g \circ u$ and $u \circ \tau_g$ is the right action of the structure group given by $J_g: \bar{N} \to M$. Let $Y \in T_X\bar{N}$ and let n_t $(n_0 = \bar{n}_X = X)$ be a C^∞-curve in $\bar{N}$ tangent to Y at X. Then we have

$$\begin{aligned} d(\bar{\tau}_g \circ u)(Y) &= \left(\frac{d}{dt}\right)_{t=0}(\bar{\tau}_g \circ u)(n_t) = \left(\frac{d}{dt}\right)_{t=0}(u \circ \tau_g)(n_t)J_g(n_t) \\ &= d(u \circ \tau_g)(Y)J_g(X) + (u \circ \tau_g)(X)dJ_g(Y) \\ &= d(u \circ \tau_g)(Y)J_g(X) + (\bar{\tau}_g \circ u)(X)J_g(X)^{-1}dJ_g(Y), \quad J_g(X)^{-1}dJ_g(Y) \in \mathfrak{m}. \end{aligned}$$

Hence

$$\begin{aligned} u^*(\bar{\tau}_g^*\omega)_X(Y) &= ((\bar{\tau}_g \circ u)^*\omega)_X(Y) = \omega((\bar{\tau}_g \circ u)_*(Y)) \\ &= \omega(d(u \circ \tau_g)(Y)J_g(X)) + \omega((\bar{\tau}_g \circ u)(X)J_g(X)^{-1}dJ_g(Y)) \\ &= J_g(X)^{-1}\omega((u \circ \tau_g)_*(Y))J_g(X) + J_g(X)^{-1}dJ_g(Y) \\ &= (J_g(X)^{-1}(\tau_g^*A)_X(Y)J_g(X) + J_g(X)^{-1}dJ_g(Y), \end{aligned}$$

by the definition of the connection form. □

Since Θ is a K-invariant Yang-Mills connection we have a map:

$$\varphi: G/K = SU(2, 2)/S(U_2 \times U_2) \to \mathscr{M} = \mathscr{Y}/\mathscr{G}, \qquad \varphi(gK) = [\tau(g^{-1})^*\Theta]$$

where $\mathscr{G}$ denotes the gauge group, $\mathscr{M}$ is the moduli space of all Yang-Mills connections and $[\omega]$ denotes the class of a connection ω modulo $\mathscr{G}$. Since $\dim G/K = 8$ we obtain an 8-dimensional family $G[\Theta]$ of Yang-Mills connections as seen in the following:

PROPOSITION 2.4. *The map φ is injective.*

PROOF. If two connections coincide in $\mathscr{M}$ then the corresponding density functions coincide. It suffices to show that if two density functions coincide: $D(\tau(g^{-1})^*B) = D(B)$ on $H(2)$ then $g \in K$. According to the decomposition $G = \overline{N} \times A \times \exp(\mathfrak{m} \cap \mathfrak{p}) \times K$, set $g = \bar{p}k$ and $\bar{p} = \bar{n}am$. Then $\tau(g^{-1})^*B = \tau(k^{-1}\bar{p}^{-1})^*B = \tau(\bar{p}^{-1})^*\tau(k^{-1})^*B \sim \tau(\bar{p}^{-1})^*B \bmod \mathscr{G}$, by the K-invariance of Θ. We may put

$$m = \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix}; \qquad A = A^*, \quad \det A = 1 \text{ and } A \text{ is positive definite}, \tag{2.1}$$

$$a = \begin{pmatrix} t^{-1}I & 0 \\ 0 & tI \end{pmatrix}; \qquad t \in \boldsymbol{R},\ t > 0 \quad \text{and} \tag{2.2}$$

$$\bar{n} = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix};\ Y \in H(2). \tag{2.3}$$

Then $\bar{p}^{-1} = (ma\bar{n})^{-1} = \begin{pmatrix} tA & -t^{-1}YA^{-1} \\ 0 & t^{-1}A^{-1} \end{pmatrix}$. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, we have $\tau_g(X) = (AX + B)(CX + D)^{-1}$ $(X \in H(2))$ and $\tau_g^*(\det dX) = h_g(X) \det dX$ where $h_g(X) = \det(A - \tau_g(X)C) \det(CX + D)^{-1}$. We also have $F_{\tau(g)^*B} = \tau_g^* F_B$ and $D(\tau_g^* B)(X) = \langle \tau_g^* F_B, \tau_g^* F_B \rangle(X) = \langle F_B, F_B \rangle(\tau_g(X)) h_g(X)^2 = D(B)(\tau_g(X)) h_g(X)^2$. Using these formula we get

$$D(\tau(g^{-1})^*B)(X) = -24t^8/\det\{I + (t^2AXA - Y)^2\}^2.$$

We note that a positive function $\det(I + X^2) = 1 + \operatorname{tr} X^2 + \det X^2$ on $H(2)$ attains its absolute minimum only at $X = 0$.

Suppose that $D(B) = D(\tau(g^{-1})^*B)$. Then comparing absolute minima we get $-24 = -24t^8$, hence by (2.2) $t = 1$, i.e. $a = I$. We then have $\det\{I + (AXA - Y)^2\} = \det(I + X^2)$. Put $X = 0$. Then $\det(I + Y^2) = 1$, hence $Y = 0$, i.e. $\bar{n} = I$. The identity reduces to $\det\{I + (AXA)^2\} = \det(I + X^2)$, hence $\operatorname{tr}(AXA)^2 = \operatorname{tr}(A^2X)^2 = \operatorname{tr} X^2$ for all $X \in H(2)$. From this and the form of matrix A (2.1), a direct calculation shows that $A = I$, i.e. $m = I$. □

We shall compute the value of the Yang-Mills functional:

$$\mathscr{YM}(B) = \int_{\overline{N}} D(B) dv.$$

Since $\overline{N} \subset K/N$ is open dense the above integral gives the value of Yang-Mills functional at Θ.

PROPOSITION 2.5. *The value of $\mathscr{YM}$ is equal to $-3\pi^3$ on the G-orbit $G[\Theta]$.*

PROOF. Just a calculation. For the coordinates $X = \begin{pmatrix} x_1 - x_2 & ix_3 - x_4 \\ -ix_3 - x_4 & x_1 + x_2 \end{pmatrix}$ in $H(2)$, $dv = d^4x = dx_1dx_2dx_3dx_4$ and $\det(I + X^2) = (1 - x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 + 4x_1^2$. Put $x = x_1$ and $y^2 = x_2^2 + x_3^2 + x_4^2$. Then

$$D(B)(X) = -24/\{(1 + y^2 - x^2)^2 + 4x^2\}^2.$$

Hence,

$$\begin{aligned}
\mathscr{YM}(B) &= -24 \int_{\boldsymbol{R}^4} \frac{d^4x}{\{(1 + y^2 - x^2)^2 + 4x^2\}^2} \\
&= -192\pi \int_0^\infty \int_0^\infty \frac{y^2 dxdy}{\{(1 + y^2 - x^2)^2 + 4x^2\}^2} \qquad (d^4x = y^2 dxdyd\sigma) \\
&= -192\pi \int_0^\infty \int_0^{\pi/2} \frac{r^2 \sin^2 \theta r dr d\theta}{(1 + 2r^2 + 4^4 \cos^2 2\theta)^2} \qquad (x = r\cos\theta, y = r\sin\theta) \\
&= -96\pi \int_0^\infty s ds \int_0^{\pi/2} \frac{\sin^2 \theta d\theta}{(1 + 2s + s^2 \cos^2 2\theta)^2} \qquad (s = r^2)
\end{aligned}$$

Put

$$\begin{aligned}
I(s) &= \int_0^{\pi/2} \frac{\sin^2 \theta d\theta}{(1 + 2s + s^2 \cos^2 2\theta)^2} \\
&= \frac{1}{2} \int_0^{\pi/2} \frac{(1 - \cos 2\theta) d\theta}{(1 + 2s + s^2 \cos^2 2\theta)^2} = \frac{1}{2} \int_0^{\pi/2} \frac{d\theta}{(1 + 2s + s^2 \cos^2 2\theta)^2}.
\end{aligned}$$

Putting $t = \tan\theta$, we then have

$$\begin{aligned}
I(s) &= \frac{1}{2(1 + 2s)^2} \int_0^\infty \frac{(t^2 + 1)dt}{(t^2 + a^2)^2} \qquad \left(\text{where } a = \frac{s + 1}{\sqrt{1 + 2s}}\right) \\
&= \frac{1}{2(1 + 2s)^2} \int_0^\infty \left(\frac{1}{t^2 + a^2} - \frac{b}{(t^2 + a^2)^2}\right) dt \qquad \left(b = \frac{s^2}{1 + 2s}\right) \\
&= \frac{1}{2(1 + 2s)^2} \left(\frac{\pi}{2a} - \frac{b\pi}{4a^3}\right) = \frac{\pi}{8} \frac{s^2 + 4s + 2}{(1 + 2s)^{3/2}(s + 1)^3}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathscr{YM}(B) &= -12\pi^2 \int_0^\infty \frac{s(s^2 + 4s + 2)}{(1 + 2s)^{3/2}(s + 1)^3} ds \\
&= -12\pi^2 \int_1^\infty \frac{(u^2 - 1)(u^4 + 6u^2 + 1)}{u^2(u^2 + 1)^3} du \qquad (u^2 = 1 + 2s)
\end{aligned}$$

$$= -12\pi^2 \int_1^\infty \left(-\frac{1}{u^2} + \frac{2}{u^2+1} + \frac{4}{(u^2+1)^2} - \frac{8}{(u^2+1)^3} \right) du$$

$$= (-12\pi^2) \times \frac{\pi}{4} = -3\pi^3.$$

Conformal transformations $\tau(g)$ $(g \in G)$ leave the functional $\mathscr{YM}$ invariant, thus the value of $\mathscr{YM}$ is $-3\pi^3$ on our G-orbit $G[\Theta]$. □

3. The case of $\boldsymbol{R}^{2,2}$

Take the group $G = SL_4(\boldsymbol{R})$. Let $P = MAN = \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in G \right\}$ be a parabolic subgroup, where $A = \left\{ \begin{pmatrix} aI & 0 \\ 0 & a^{-1}I \end{pmatrix} \middle| a \in \boldsymbol{R},\ a > 0 \right\}$, $M = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \det A = \det B = \pm 1 \right\}$ and $N = \left\{ \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \middle| X \in M_2(\boldsymbol{R}) \right\}$. We take the identity component $P_0 = M_0AN$ instead of P itself, for $G/P_0 = SO_4/SO_2 \times SO_2 = S^2 \times S^2$. Let $K = SO_4$ be a maximal compact subgroup of G. Then the diagram is

$$\begin{array}{ccccc}
 & & G = SL_4(\boldsymbol{R}) & \xrightarrow{\ \kappa\ } & KM_0 = KM \\
 & \nearrow^{i} & \big\downarrow & & \big\downarrow \Theta \ \ M_0 = SL_2(\boldsymbol{R}) \times SL_2(\boldsymbol{R}) \\
\boldsymbol{R}^{2,2} = M_2(\boldsymbol{R}) = \overline{N} & \xrightarrow{\ j\ } & G/P_0 & = & K/K \cap M_0 = SO_4/SO_2 \times SO_2 = S^2 \times S^2 \\
X \longmapsto \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} & & & &
\end{array}$$

where $M_2(\boldsymbol{R}) = \overline{N}$ is identified with $\boldsymbol{R}^{2,2}$ by the metric $\det dX$. Let $X = \begin{pmatrix} x_1 - x_2 & x_3 - x_4 \\ -x_3 - x_4 & x_1 + x_2 \end{pmatrix} \in M_2(\boldsymbol{R})$. Then $\det dX = dx_1^2 - dx_2^2 + dx_3^2 - dx_4^2$. Put $u = \kappa \circ i$. Then a Yang-Mills connection on $\boldsymbol{R}^{2,2}$ is obtained by the pull-back:

$$(u^*\Theta)(X) = \left(\begin{array}{c|c} -\mathrm{Im}\, dX(I + {}^tXX)^{-1}\,{}^tX & 0 \\ \hline 0 & \mathrm{Im}(I + {}^tXX)^{-1}\,{}^tXdX \end{array} \right) \in \mathfrak{m}$$

where $\mathfrak{m} = \mathfrak{sl}_2(\boldsymbol{R}) + \mathfrak{sl}_2(\boldsymbol{R})$ and ${}^tX$ is the transposed matrix of X. The projection to the second $\mathfrak{sl}_2(\boldsymbol{R})$:

$$B(X) = \mathrm{Im}(I + {}^tXX)^{-1}\,{}^tXdX$$

is a self-dual connection ($*F_B = F_B$ on $\boldsymbol{R}^{2,2}$) and that to the first $\mathfrak{sl}_2(\boldsymbol{R})$ is anti-self-dual. The curvature form and its density function are

$$F_B(X) = (I + {}^tXX)^{-1}d^tX(I + X^tX)^{-1}dX \qquad \text{and}$$

$$D(B)(X) = \frac{16}{\det(I + {}^tXX)^2}.$$

G-actions on $\overline{N}$ and $K/K \cap M_0$ are conformal transformations. Hence we get a map:

$$G/K = SL_4(\boldsymbol{R})/SO_4 \to \mathscr{M}^+ = \mathscr{Y}^+/\mathscr{G} \qquad \text{by} \quad gK \mapsto [\tau(g^{-1})^*\Theta]$$

where $\mathscr{M}^+$ is the moduli space of self-dual connections. This map is proved to be injective by the investigation of the density function $D(\tau_g^*B)(X)$. Since $\dim G/K = 9$ we then obtain a 9-dimensional family $G[\Theta]$ of self-dual Yang-Mills connections on $S^2 \times S^2$. The value of the functional $\mathscr{YM}$ on this $G[\Theta]$ is computed to be $8\pi^2$.

4. Nullity and index of the second variation

In this section we compute the nullity and the index of the second variation at the canonical connection of the pseudo-Riemannian spaces. We recall general facts on the second variation of a Yang-Mills functional (see [2], 3 and 4). Let $P \to M$ be the principal bundle with structure group G and compact oriented pseudo-Riemannian base space M. This adjoint bundle $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$ is by definition the bundle associated with P via the adjoint action of the structure group G on its Lie algebra $\mathfrak{g}$. A fixed pseudo-Riemannian metric and a fixed orientation on M define the Hodge $*$-operator in the space $\Omega^p(M)$ of p-forms by

$$\theta \wedge *\eta = \langle \theta, \eta \rangle dv \qquad \text{for } \theta, \ \eta \in \Omega^p,$$

where $\langle \ , \ \rangle$ denotes the natural pseudo-Riemannian structure on $\Omega^p(M)$, and dv is the volume form of M. Then the inner product $\langle \ , \ \rangle$ on $\mathfrak{g}$ and the Riemannian metric of M combine to give the space $\Omega^*(\mathfrak{g}_P) = \Gamma^\infty(\wedge T^*M \otimes \mathfrak{g}_P)$ of $\mathfrak{g}_P$-valued forms a natural inner product $(\ , \ )$ which possibly has an indefinite index:

$$(\theta, \varphi) = \int_M \langle \theta \wedge *\varphi \rangle.$$

We shall equip the space $\Omega^*(\mathfrak{g}_P) = \Gamma^\infty(\wedge T^*M \otimes \mathfrak{g}_P)$ with the Frechet space $\mathscr{E}$ topology which is defined by the collection of supremum norms of a section and each of its derivatives measured by a fixed norm in fibers of $\wedge T^*M \otimes \mathfrak{g}_P$. The convergence $\theta_n \to \theta$ in $\mathscr{E}$-topology implies that $(\theta_n, \varphi) \to (\theta, \varphi)$ for any φ and $(\theta_n, \theta_n) \to (\theta, \theta)$.

The space of connections $\mathscr{A}_P$ has an affine structure with associated vector space $\Omega^1(\mathfrak{g}_P)$. The Yang-Mills equation is the first variational equation of the Yang-Mills functional on $\mathscr{A}_P$:

$$\mathscr{Y}\mathscr{M}(A) = (F_A, F_A) = \int_M D(A)dv,$$

where $F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(\mathfrak{g}_P)$ is the curvature form of the connection A and $D(A)(X) = \langle F_A, F_A\rangle(X)$ is the density function. $D(A)$ and $\mathscr{Y}\mathscr{M}(A)$ are not necessarily positive on pseudo-Riemannian manifolds. Because $\mathscr{A}_P$ is an affine space we can vary A along lines

$$A_t = A + t\eta, \qquad \eta \in \Omega^1(\mathfrak{g}_P).$$

Then the curvature F_t of A_t is given by

$$F_t = F_A + td_A\eta + \tfrac{1}{2}t^2[\eta \wedge \eta],$$

where d_A is the covariant exterior derivative of the connection A. Taking the inner product in $\Omega^2(\mathfrak{g}_P)$,

$$(F_t, F_t) = (F_A, F_A) + 2t(d_A\eta, F_A) + t^2\{(d_A\eta, d_A\eta) + (F_A, [\eta \wedge \eta])\} + O(t^3).$$

Hence at an extremum $(d_A\eta, F_A) = 0$, or equivalently $(\eta, \delta_A F_A) = 0$, for all $\eta \in \Omega^1(\mathfrak{g}_P)$. Hence at an extremum $\delta_A F_A = 0$. Here δ_A is the adjoint of d_A relative to the inner product on $\Omega^*(\mathfrak{g}_P)$ and just as in the usual Hodge theory $\delta_A = \pm * d_A *$. Therefore the first variational equation of $\mathscr{Y}\mathscr{M}$ is

$$d_A * F_A = 0.$$

The above expansion also yields the Hessian Q of $\mathscr{Y}\mathscr{M}$ at an extremum A. Q is a quadratic form on the tangent space to $\mathscr{A}_P$ at A, which is presicely $\Omega^1(\mathfrak{g}_P)$. Thus the second variational formula of $\mathscr{Y}\mathscr{M}$ is

$$Q(\eta, \eta) = (\delta_A d_A\eta + \mathscr{F}_A\eta, \eta).$$

Here we write $\mathscr{F}_A$ for the endomorphism of $\Omega^1(\mathfrak{g}_P)$

$$\mathscr{F}_A\colon \eta \mapsto (-1)^q * [*F_A \wedge \eta] \tag{4.1}$$

where (p, q) is the index of the metric on M. $\mathscr{F}_A$ is also characterized by

$$(\mathscr{F}_A\eta, \xi) = (F_A, [\eta \wedge \xi])$$

and is therefore self-adjoint. The functional $\mathscr{Y}\mathscr{M}$ is invariant under the action of the group $\mathscr{G}$ of gauge transformations. The tangent space $T_A\mathscr{G}$ is identified with $\operatorname{Im} d_A$ of $\Omega^0(\mathfrak{g}_P)$ in $T_A\mathscr{A}_P = \Omega^1(\mathfrak{g}_P)$. Hence $\delta_A d_A\eta + \mathscr{F}_A\eta = 0$ if $\eta \in \operatorname{Im} d_A$. We shall define the nullity and the index of A as follows.

DEFINITION. The *nullity* of a Yang-Mills connection A is the dimension of the null space of the operator $\delta_A d_A + \mathscr{F}_A$ in the quotient space $\Omega^1(\mathfrak{g}_P)/\mathrm{Im}\, d_A$. The *index* of A is the dimension of the negative eigenspaces of the operator $\delta_A d_A + \mathscr{F}_A$ in $\Omega^1(\mathfrak{g}_P)/\mathrm{Im}\, d_A$.

When the Riemannian metric on the base space and the inner product on $\mathfrak{g}$ are positive definite, and the space $(\ ,\ )$ in $\Omega^*(\mathfrak{g}_P)$ is also positive definite and the space $\Omega^1(\mathfrak{g}_P)$ has the orthogonal decomposition:

$$\Omega^1(\mathfrak{g}_P) = \mathrm{Ker}\, \delta_A \oplus \mathrm{Im}\, d_A, \tag{4.2}$$

relative to the inner product (cf. [13] 5.8.10 for the case where $\mathfrak{g}_P$ is a homogeneous vector bundle). This implies that the spaces $\mathrm{Ker}\, \delta_A$ and $\mathrm{Im}\, d_A$ are closed in $\Omega^1(\mathfrak{g}_P)$ with respect to the topology defined by the inner product $(\ ,\ )$. Hence they are also closed with respect to our $\mathscr{E}$-topology. Thus $\Omega^1(\mathfrak{g}_P)/\mathrm{Im}\, d_A$ is identified with $\mathrm{Ker}\, \delta_A$. Define the Jacobi operator S_A and the quadratic form Q by

$$S_A = \Delta_A + \mathscr{F}_A = d_A\delta_A + \delta_A d_A + \mathscr{F}_A \qquad \text{and} \qquad Q(\eta, \eta) = (S_A\eta, \eta).$$

Then $S_A = \delta_A d_A + \mathscr{F}_A$ on $\mathrm{Ker}\, \delta_A$, $S_A = d_A\delta_A$ on $\mathrm{Im}\, d_A$, and hence Q is strictly positive definite on $\mathrm{Im}\, d_A$. Thus in the Riemannian case we may define the nullity of A as the nullity of S_A on $\mathrm{Ker}\, \delta_A$ and define the index of A as the index of the quadratic form Q on $\mathrm{Ker}\, \delta_A$, i.e. the dimension of a maximal subspace on which Q is strictly negative definite. And they are equal to the index and the nullity of S_A on all of $\Omega^1(\mathfrak{g}_P)$. The ellipticity of the operator $S_A = \Delta_A + \mathscr{F}_A$ implies that they are finite. When the metric on base space and the inner product on $\mathfrak{g}$ are indefinite, these nullity and index are possibly infinite. Furthermore the above decomposition (4.2) does not hold in general (see Lemma 4.4 below), but the spaces $\mathrm{Ker}\, \delta_A$ and $\mathrm{Im}\, d_A$ are closed with respect to $\mathscr{E}$-topology because the space $\mathrm{Im}\, d_A$ is invariant under the change of metric. Hence the quotient space $\Omega^1(\mathfrak{g}_P)/\mathrm{Im}\, d_A$ is a Hausdorff, locally convex, topological vector space.

We are concerned with the canonical connection Θ for principal M-bundles of the form $P\colon KM \to KM/M = K/H$ where K, M and $H = K \cap H$ are subgroups of a larger group G and $KM = \{km \mid k \in K,\ m \in M\}$ is a submanifold of G. This bundle is an associated principal bundle of the principal bundle $K \to K/H$ via the inclusion homomorphism $H = K \cap M \to M$, i.e.

$$P \simeq K \times_H M = K \times M/(kh, m) \sim (k, hm), \qquad k \in K, \quad h \in H, \quad m \in M.$$

This construction effects a change of the structure group. The adjoint bundle $\mathfrak{m}_P$ is a homogeneous vector bundle:

$$\mathfrak{m}_P \simeq K \underset{H}{\times} \mathfrak{m} \to K/H,$$

via the adjoint representation of H on $\mathfrak{m}$. The space $\Omega^p(\mathfrak{m}_P) = \Gamma^\infty(K \underset{H}{\times} (\wedge^p \mathfrak{s}^* \otimes \mathfrak{m}))$ of $\mathfrak{m}_P$-valued p-forms is naturally isomorphic to a K-module of the induced representation $C_H^\infty(K, \wedge^p \mathfrak{s}^* \otimes \mathfrak{m})$.

We review basic facts on induced representations and homogeneous differential operators (cf. [4] or [13], 5.3 and 5.4). Let H be a closed subgroup of a compact group K and let (σ, V) be a finite dimensional complex representation of H. Suppose that V has an H-invariant, nondegenerate, Hermitian inner product $\langle\ ,\ \rangle$ which possibly has an indefinite index. The induced representation $\rho = \mathrm{Ind}_H^K V$ is the function space:

$$C_H^\infty(K, V) = \{f\colon K \overset{C^\infty}{\to} V \,|\, f(gh) = \sigma(h)^{-1} f(g),\ g \in K,\ h \in H\}$$

with the K-representation ρ:

$$(\rho(g)f)(x) = f(g^{-1}x).$$

This space has a K-invariant Hermitian inner product:

$$(f_1, f_2) = \int_K \langle f_1(g), f_2(g)\rangle dg, \qquad dg \text{ is a Haar measure on } K,$$

where $f_1, f_2 \in C_H^\infty(K, V)$. $(\ ,\ )$ is nondegenerate and possibly has an indefinite index. Nondegeneracy of $(\ ,\ )$ is easily seen from the eigenspace decomposition of V with respect to the Hermitian form $\langle\ ,\ \rangle$. We shall equip the function space $C_H^\infty(K, V)$ with the Frechet space $\mathscr{E}$ topology which is defined by the collection of supremum norms of a function and each of its derivatives measured by a fixed norm in V. Frobenius reciprocity theorem states that

$$\mathrm{Hom}_K(W, \mathrm{Ind}_H^K V) = \mathrm{Hom}_H(W|_H, V),$$

where W is a K-representation and $W|_H$ is its restriction to H. Let $U(\mathfrak{k})$ be the universal envelopping algebra of $\mathfrak{k}^C$, the complexification of $\mathfrak{k}$. Then $U(\mathfrak{k})$ is identified with the ring of left invariant differential operators on K. Let (σ, V) and (τ, W) be finite dimensional complex representations of H. If $X \in U(\mathfrak{k})$ and if $L\colon V \to W$ is a linear map then define a differential operator $X \otimes L\colon C^\infty(K, V) \to C^\infty(K, W)$ by

$$(X \otimes L)f(g) = L(Xf(g)) \qquad f \in C^\infty(K, V), \quad g \in K.$$

The product of such operators is given by $(X_1 \otimes L_1)(X_2 \otimes L_2) = X_1X_2 \otimes L_1L_2$. The differential operator D is called a homogeneous differential operator if it commutes with K-actions. We know that a differential operator $D\colon C_H^\infty(K, V) \to C_H^\infty(K, W)$ is homogeneous if and only if D has an expression as above such that $D \in (U(\mathfrak{k}) \otimes \mathrm{Hom}(V, W))^H$.

In the case of the Riemannian base space S^4 the index and the nullity of the canonical connection Θ are, respectively, the number of negative and zero eigenvalues of the Jacobi operator S_Θ of Θ on $\Omega^1(\mathfrak{m}_P)$. T. Laquer [10] computes these numbers for all compact irreducible symmetric space. For $S^4 = Sp_2/Sp_1 \times Sp_1$ (or SO_5/SO_4) the index is equal to 0 and the nullity is 10 (see [10], 5, Table II). The index and the nullity of the (anti-)self-dual part of the canonical connection are 0 and 5, respectively.

For the (1, 3)-Riemannian space $S^1 \times S^3$ we determine the nullity and the index of the canonical connection Θ. We treat the space of $\mathfrak{m}_P$-valued p-forms:

$$\Omega^p(\mathfrak{m}_P) = \Gamma^\infty(K \underset{H}{\times} (\wedge^p \mathfrak{s}^* \otimes \mathfrak{m})) = C_H^\infty(K, \wedge^p \mathfrak{s}^* \otimes \mathfrak{m}), \qquad p = 0,\ 1,\ 2.$$

We first note that the vector space $\wedge^p \mathfrak{s}^* \otimes \mathfrak{m}$ has a complex structure induced from that of $\mathfrak{m} = \mathfrak{sl}_2(C)$ and has an $(\mathrm{Ad}\, H)$-invariant indefinite Hermitian inner product induced from h on $\mathfrak{s} = T_o(K/H)$ and $(X, Y) = \mathrm{tr}(XY^*)$ on $\mathfrak{m}$. So $\Omega^p(\mathfrak{m}_P)$ is a complex vector space on which K acts naturally by the induced representation. The covariant exterior derivatives of invariant connection Θ are denoted by

$$\Omega^0(\mathfrak{m}_P) \xrightarrow{d} \Omega^1(\mathfrak{m}_P) \xrightarrow{d} \Omega^2(\mathfrak{m}_P).$$

(Hereafter all covariant operators are relative to the connection Θ, so we omit subscripts in the symbols). They commute with the K-action, hence homogeneous differential operators. Hence their formal adjoint δ, the operators $\mathscr{F}$ of (4.1), $\delta d + \mathscr{F}$ and S are also homogeneous. We find the explicit form of these operators as follows. We realize the Lie algebras of $K = S(U_2 \times U_2)$ and $H = SU_2^\pm$ as

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle|\, A, B \in \mathfrak{u}_2,\ \mathrm{tr}\, A + \mathrm{tr}\, B = 0 \right\} = \mathfrak{h} + \mathfrak{s},$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle|\, A \in \mathfrak{su}_2 \right\} \simeq \mathfrak{su}_2, \qquad \mathfrak{s} = \left\{ \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} \middle|\, B \in \mathfrak{u}_2 \right\}.$$

We take the orthonormal basis $(U_\alpha)_{\alpha=0}^6$ of $\mathfrak{k}$ relative to the inner product h as follows:

$$U_0 = Z/4,\quad U_1 = \frac{1}{4}\begin{pmatrix} u_1 & 0 \\ 0 & -u_1 \end{pmatrix},\quad U_2 = \frac{1}{4}\begin{pmatrix} u_2 & 0 \\ 0 & -u_2 \end{pmatrix},\quad U_3 = \frac{1}{4}\begin{pmatrix} u_3 & 0 \\ 0 & -u_3 \end{pmatrix},$$

$$U_4 = \frac{1}{4}\begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix},\quad U_5 = \frac{1}{4}\begin{pmatrix} u_2 & 0 \\ 0 & u_2 \end{pmatrix},\quad U_6 = \frac{1}{4}\begin{pmatrix} u_3 & 0 \\ 0 & u_3 \end{pmatrix},$$

where

$$Z = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}, \quad u_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad u_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.3}$$

Then $\boldsymbol{R}U_0$ is the center of $\mathfrak{k}$ which is contained in $\mathfrak{s}$, (U_0, U_1, U_2, U_3) is a basis of $\mathfrak{s}$, (U_4, U_5, U_6) is that of $\mathfrak{h}$. We have $h(U_0, U_0) = 1$, $h(U_0, U_i) = 0$, $h(U_i, U_j) = -\delta_{ij}$ $(1 \le i, j \le 6)$. Let $(\mu^\alpha)_{\alpha=0}^6$ be the dual basis of $(U_\alpha)_{\alpha=0}^6$. Then the covariant exterior derivative d and its formal adjoint δ are given by

$$d = \sum_{\alpha=0}^{3} U_\alpha \otimes \varepsilon(\mu^\alpha), \qquad \delta = -U_0 \otimes i(U_0) + \sum_{\beta=1}^{3} U_\beta \otimes i(U_\beta) \tag{4.4}$$

where $\varepsilon(\mu)$: $\wedge^p \mathfrak{s}^* \otimes \mathfrak{m} \to \wedge^{p+1}\mathfrak{s}^* \otimes \mathfrak{m}$, $i(U)$: $\wedge^p \mathfrak{s}^* \otimes \mathfrak{m} \to \wedge^{p-1}\mathfrak{s}^* \otimes \mathfrak{m}$ $(\mu \in \mathfrak{s}^*$, $U \in \mathfrak{s})$ are, respectively, the exterior and the interior products on $\wedge \mathfrak{s}^*$ tensoring with identities on $\mathfrak{m}$ (cf. [10], 3). Let $C = \sum_{\alpha=1}^6 U_\alpha^2$ be the negative of Casimir element of $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{su}_2 \times \mathfrak{su}_2$. Then we have

LEMMA 4.1. (1) *For* $\varphi \in \Omega^0(\mathfrak{m}_P)$ *which is a* $\mathfrak{m}$*-valued function on* K, *we have*

$$\delta d\varphi = \left(-U_0^2 + C + \frac{1}{2}\right)\varphi, \tag{4.5}$$

i.e. $\delta d = (-U_0^2 + C + \frac{1}{2}) \otimes I \in U(\mathfrak{k}) \otimes \operatorname{End}\mathfrak{m}$, on $\Omega^0(\mathfrak{m}_P)$.

(2) *For* $\eta \in \Omega^1(\mathfrak{m}_P)$, *we have*

$$S\eta = \left(-U_0^2 + C + \frac{1}{2}\right)\eta, \tag{4.6}$$

i.e. $S = (-U_0^2 + C + \frac{1}{2}) \otimes I \in U(\mathfrak{k}) \otimes \operatorname{End}(s^* \otimes \mathfrak{m})$, on $\Omega^1(\mathfrak{m}_P)$.

PROOF. We follow the arguments in Appendix of [10] adapted to our (1, 3)-Riemannian case. We first note three facts. If $U \in \mathfrak{s}^*$ and $\mu \in \mathfrak{s}^*$ we have $\varepsilon(\mu)i(U) + i(U)\varepsilon(\mu) = \langle \mu, U \rangle I$ on $\wedge \mathfrak{s}^* \otimes \mathfrak{m}$. If (σ, V) is a H-representation and if $U \in \mathfrak{h}$, $L \in \operatorname{Hom}(V, W)$ then for $f \in C_H^\infty(K, V)$, $g \in K$,

$$\begin{aligned}(U \otimes L)f(g) &= L\left(\left(\frac{d}{dt}\right)_{t=0} f(g \operatorname{ext} tU)\right) = L\left(\left(\frac{d}{dt}\right)_{t=0} \sigma(\exp -tU)f(g)\right) \\ &= -L(\sigma(U)f(g)) = -(I \otimes L\sigma(U))f(g).\end{aligned}$$

We know that a value of the Casimir element $\sum_{k=4}^6 U_k^2$ of $\mathfrak{h}$ is $\operatorname{ad}(\sum_{k=4}^6 U_k^2) = (-1/2)I$ on $\mathfrak{m}$ and on $[\mathfrak{k}, \mathfrak{k}]$.

For the identity (1), the operator δd has the following form on $\Omega^0(\mathfrak{m}_P) = C_H^\infty(K, \mathfrak{m})$.

$$\begin{aligned}\delta d &= \left\{-U_0 \otimes i(U_0) + \sum_{\beta=1}^{3} U_\beta \otimes i(U_\beta)\right\} \sum_{\alpha=0}^{3} U_\alpha \otimes \varepsilon(\mu^\alpha) \\ &= \sum_{\alpha=0}^{3} \left\{-U_0 U_\alpha \otimes i(U_0)\varepsilon(\mu^\alpha) + \sum_{\beta=1}^{3} U_\beta U_\alpha \otimes i(U_\beta)\varepsilon(\mu^\alpha)\right\} \\ &= \left(-U_0^2 + \sum_{\beta=1}^{3} U_\beta^2\right) \otimes I = \left(-U_0^2 + C - \sum_{k=4}^{6} U_k^2\right) \otimes I \\ &= \left(-U_0^2 + C + \frac{1}{2}\right) \otimes I,\end{aligned}$$

because $i(U_\beta)\varepsilon(\mu^\alpha) = \delta_\beta^\alpha I$ on $\mathfrak{m}$ and $-\sum_{k=4}^{6} U_k^2 \otimes I = -I \otimes \mathrm{ad}\left(\sum_{k=4}^{6} U_k^2\right) = (1/2)I$.

For the identity (2), we know that the curvature form F of the canonical connection Θ is

$$F = -\frac{1}{2} \sum_{\alpha,\beta=1}^{3} [U_\alpha, U_\beta] \mu^\alpha \wedge \mu^\beta.$$

See [10], (3.3) or if we use the Ricci formula: $d_A d_A \omega = [F_A \wedge \omega]$ for definition of curvature, we have

$$\begin{aligned}dd &= \sum_{\alpha=0}^{3} U_\alpha \otimes \varepsilon(\mu^\alpha) \sum_{\beta=0}^{3} U_\beta \otimes \varepsilon(\mu^\beta) = \sum_{\alpha,\beta=0}^{3} U_\alpha U_\beta \otimes \varepsilon(\mu^\alpha)\varepsilon(\mu^\beta) \\ &= \sum_{\alpha<\beta} [U_\alpha, U_\beta] \otimes \varepsilon(\mu^\alpha \wedge \mu^\beta) = I \otimes \left(-\sum_{\alpha<\beta} \varepsilon(\mu^\alpha \wedge \mu^\beta)\,\mathrm{ad}[U_\alpha, U_\beta]\right) \\ &= I \otimes \left(-\frac{1}{2} \sum_{\alpha,\beta=1}^{3} \varepsilon(\mu^\alpha \wedge \mu^\beta)\,\mathrm{ad}[U_\alpha, U_\beta]\right),\end{aligned}$$

since $\varepsilon(\mu^\beta)\varepsilon(\mu^\alpha) = -\varepsilon(\mu^\alpha)\varepsilon(\mu^\beta) = -\varepsilon(\mu^\alpha \wedge \mu^\beta)$ and $[U_\alpha, U_\beta] \in \mathfrak{h}$, $[U_0, U_\alpha] = 0$. This shows that $F = -\frac{1}{2}\sum_{\alpha,\beta=1}^{3} [U_\alpha, U_\beta]\mu^\alpha \wedge \mu^\beta$. From (4.1) we get a formula for the operator $\mathscr{F}$:

$$\mathscr{F} = I \otimes \left(-\sum_{\alpha,\beta=1}^{3} \varepsilon(\mu^\alpha)\,\mathrm{ad}[U_\alpha, U_\beta] i(U_\beta)\right).$$

The operator $d\delta + \delta d$ has the following form on $\Omega^1(\mathfrak{m}_P) = C_H^\infty(K, \mathfrak{s}^* \otimes \mathfrak{m})$.

$$\begin{aligned}d\delta + \delta d &= \sum_{\alpha=0}^{3} U_\alpha \otimes \varepsilon(\mu^\alpha)\left\{-U_0 \otimes i(U_0) + \sum_{\beta=1}^{3} U_\beta \otimes i(U_\beta)\right\} \\ &\quad + \left\{-U_0 \otimes i(U_0) + \sum_{\beta=1}^{3} U_\beta \otimes i(U_\beta)\right\} \sum_{\alpha=0}^{3} U_\alpha \otimes \varepsilon(\mu^\alpha)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha=0}^{3} \left\{ -U_\alpha U_0 \otimes \varepsilon(\mu^\alpha) i(U_0) + \sum_{\beta=1}^{3} U_\alpha U_\beta \otimes \varepsilon(\mu^\alpha) i(U_\beta) \right\} \\
&\quad + \sum_{\alpha=0}^{3} \left\{ -U_0 U_\alpha \otimes i(U_0)\varepsilon(\mu^\alpha) + \sum_{\beta=1}^{3} U_\beta U_\alpha \otimes i(U_\beta)\varepsilon(\mu^\alpha) \right\} \\
&= \sum_{\alpha=0}^{3} \left\{ -U_0 U_\alpha \otimes (\varepsilon(\mu^\alpha) i(U_0) + i(U_0)\varepsilon(\mu^\alpha)) \right. \\
&\quad \left. + \sum_{\beta=1}^{3} U_\beta U_\alpha \otimes (\varepsilon(\mu^\alpha) i(U_\beta) + i(U_\beta)\varepsilon(\mu^\alpha)) \right\} \\
&\quad + \sum_{\alpha,\beta=1}^{3} [U_\alpha, U_\beta] \otimes \varepsilon(\mu^\alpha) i(U_\beta) \\
&= \left(-U_0^2 + \sum_{\beta=1}^{3} U_\beta^2 \right) \otimes I + \sum_{\alpha,\beta=1}^{3} [U_\alpha, U_\beta] \otimes \varepsilon(\mu^\alpha) i(U_\beta) \\
&= \left(-U_0^2 + C - \sum_{k=4}^{6} U_4^2 \right) \otimes I + \sum_{\alpha,\beta=1}^{3} [U_\alpha, U_\beta] \otimes \varepsilon(\mu^\alpha) i(U_\beta) \\
&= \left(-U_0^2 + C \right) \otimes I - I \otimes \operatorname{ad}\left(\sum_{k=4}^{6} U_k^2 \right) \\
&\quad - I \otimes \left(\sum_{\alpha,\beta=1}^{3} \varepsilon(\mu^\alpha) i(U_\beta) \operatorname{ad}[U_\alpha, U_\beta] \right),
\end{aligned}
$$

since $U_\alpha U_\beta = U_\beta U_\alpha + [U_\alpha, U_\beta]$ and $\varepsilon(\mu^\alpha) i(U_\beta) + i(U_\beta)\varepsilon(\mu^\alpha) = \delta_\beta^\alpha$. Here we note that operators $-\operatorname{ad}(\sum_4^6 U_k^2)$, $-\sum_{\alpha,\beta=1}^3 \varepsilon(\mu^\alpha) i(U_\beta) \operatorname{ad}[U_\alpha, U_\beta]$ and $\mathscr{F} = -\sum_{\alpha,\beta=1}^3 \varepsilon(\mu^\alpha) \operatorname{ad}[U_\alpha, U_\beta] i(U_\beta)$ act on $\mathfrak{s}^* \otimes \mathfrak{m}$. If $v \otimes X \in \mathfrak{s}^* \otimes \mathfrak{m}$,

$$
\mathscr{F}(v \otimes X) = -\sum_{\alpha,\beta=1}^{3} \langle v, U_\beta \rangle \mu^\alpha \otimes [[U_\alpha, U_\beta], X].
$$

$$
\begin{aligned}
&-\left(\sum_{\alpha,\beta=1}^{3} \varepsilon(\mu^\alpha) i(U_\beta) \operatorname{ad}[U_\alpha, U_\beta] \right)(v \otimes X) \\
&= -\sum_{\alpha,\beta=1}^{3} \mu^\alpha \otimes (\langle \operatorname{ad}[U_\alpha, U_\beta] v, U_\beta \rangle X + \langle v, U_\beta \rangle [[U_\alpha, U_\beta], X]) \\
&= \sum_{\alpha} \left\langle v, \left(\sum_{\beta} \operatorname{ad} U_\beta^2 \right) U_\alpha \right\rangle \mu^\alpha \otimes X + \mathscr{F}(v \otimes X) \\
&= -\frac{1}{2} \sum_{\alpha=1}^{3} \langle v, U_\alpha \rangle \mu^\alpha \otimes X + \mathscr{F}(v \otimes X),
\end{aligned}
$$

since $\sum_{\beta=1}^3 \operatorname{ad} U_\beta^2 = -(1/2)I$ on $[\mathfrak{k}, \mathfrak{k}]$. And we have

$$-\operatorname{ad}\left(\sum_{k=4}^{6} U_k^2\right)(v \otimes X)$$

$$= -\sum_{k=4}^{6} \{(\operatorname{ad} U_k^2)v \otimes X + 2(\operatorname{ad} U_k)v \otimes [U_k, X] + v \otimes (\operatorname{ad} U_k^2)X\}$$

$$= \frac{1}{2}\sum_{\alpha=1}^{3} \langle v, U_\alpha\rangle \mu^\alpha \otimes X - 2\sum_{k=4}^{6} (\operatorname{ad} U_k)v \otimes [U_k, X] + \frac{1}{2}v \otimes X.$$

Since

$$(\operatorname{ad} U_k)v = \sum_{\alpha=0}^{3} \langle (\operatorname{ad} U_k)v, U_\alpha\rangle \mu^\alpha = -\sum_{\alpha=1}^{3} \langle v, [U_k, U_\alpha]\rangle \mu^\alpha$$

$$= -\sum_{\alpha,\beta=1}^{3} \langle v, U_\beta\rangle\langle \mu^\beta, [U_k, U_\alpha]\rangle \mu^\alpha = -\sum_{\alpha,\beta=1}^{3} \langle v, U_\beta\rangle\langle \mu^k, [U_\alpha, U_\beta]\rangle \mu^\alpha,$$

$$\langle \mu^\beta, [U_k, U_\alpha]\rangle = -B(U_\beta, [U_k, U_\alpha]) = -B([U_\alpha, U_\beta], U_k) = \langle \mu^k, [U_\alpha, U_\beta]\rangle$$

where B is the Killing form, we have

$$-\sum_{k=4}^{6} (\operatorname{ad} U_k)v \otimes [U_k, X] = \sum_{\alpha,\beta=1}^{3} \langle v, U_\beta\rangle \mu^\alpha \otimes \left[\sum_k \langle \mu^k, [U_\alpha, U_\beta]\rangle U_k, X\right]$$

$$= \sum_{\alpha,\beta=1}^{3} \langle v, U_\beta\rangle \mu^\alpha \otimes [[U_\alpha, U_\beta], X] = -\mathscr{F}(v \otimes X).$$

Hence $-\operatorname{ad}(\sum_{k=4}^{6} U_k^2)(v \otimes X) = \frac{1}{2}\sum_{\alpha=1}^{3} \langle v, U_\alpha\rangle \mu^\alpha \otimes X + 2\mathscr{F}(v \otimes X) + \frac{1}{2}v \otimes X$. These show that $d\delta + \delta d = (-U_0^2 + C) \otimes I + (1/2)I - \mathscr{F}$. Therefore

$$S = d\delta + \delta d + \mathscr{F} = \left(-U_0^2 + C + \frac{1}{2}\right) \otimes I. \qquad \square$$

Let $\hat{K}$ be the set of all equivalence classes of irreducible, finite dimensional unitary representations of K. Decompose the induced representations:

$$\Omega^0(\mathfrak{m}_P) = \sum_{\lambda \in \hat{K}} \oplus W_\lambda^0, \qquad \Omega^1(\mathfrak{m}_P) = \sum_{\lambda \in \hat{K}} \oplus W_\lambda^1 \quad \text{and} \quad \Omega^2(\mathfrak{m}_P) = \sum_{\lambda \in \hat{K}} \oplus W_\lambda^2,$$

to K-primary components. Here λ-component W_λ^p of $\Omega^p(\mathfrak{m}_P)$ is precisely the subspace under which K transforms according to the irreducible representation $\lambda \in \hat{K}$ and by the notation $\sum_\lambda \oplus W_\lambda^p$ we means that the algebraic sum $\sum_\lambda W_\lambda^p$ is direct and dense in $\Omega^p(\mathfrak{m}_P)$ with respect to the Frechet $\mathscr{E}$ topology (cf. [6], pp. 553, Theorem 3.5(iii)). Each primary component is finite dimensional and has an indefinite, nondegenerate inner product induced from (,) and distinct components are mutually orthogonal with respect to (,). We see that homogeneous differential operators d and $\delta d + \mathscr{F}$ are continuous and

$$W_\lambda^0 \xrightarrow{d} W_\lambda^1 \xrightarrow{d} W_\lambda^2, \qquad \delta d + \mathscr{F}: W_\lambda^1 \to W_\lambda^1.$$

Hence we get a decomposition of locally convex spaces:

$$\Omega^1(\mathfrak{m}_P)/\mathrm{Im}\, d = \sum_{\lambda \in \hat{K}} \oplus W_\lambda^1/\mathrm{Im}\, d \cap W_\lambda^1.$$

We shall determine the nullity N_λ on each W_λ^1 i.e.

$$N_\lambda = \dim_{\boldsymbol{R}} \mathrm{Ker}(\delta d + \mathscr{F} \,|\, W_\lambda^1/\mathrm{Im}\, d \cap W_\lambda^1).$$

Then the whole nullity N is the sum of N_λ if it is finite: $N = \sum_{\lambda \in \hat{K}} N_\lambda$. From lemma 4.1 we know that δd and $S = d\delta + \delta d + \mathscr{F}$ are the same scalar operators on λ-primary component, i.e.

$$\delta d = c(\lambda) I \quad \text{on } W_\lambda^0 \qquad \text{and} \qquad S = c(\lambda) I \quad \text{on } W_\lambda^1.$$

Here the scalar constant $c(\lambda)$ is evaluated below. Put $\mathrm{Ker}_\lambda\, \delta = \mathrm{Ker}\, \delta \cap W_\lambda^1$ and $\mathrm{Im}_\lambda\, d = \mathrm{Im}\, d \cap W_\lambda^1$. Then we have

LEMMA 4.2. (1) *If* $c(\lambda) \neq 0$ *then* $W_\lambda^1 = \mathrm{Ker}_\lambda\, \delta \oplus \mathrm{Im}_\lambda\, d$ (*orthogonal direct sum*) *and the nullity* $N_\lambda = 0$. (2) *If* $c(\lambda) = 0$ *then* $\mathrm{Im}_\lambda\, d \subset \mathrm{Ker}_\lambda\, \delta$ *and* $N_\lambda \geq \dim_{\boldsymbol{R}} W_\lambda^1 - 2 \dim_{\boldsymbol{R}} W_\lambda^0$. (3) *If* $c(\lambda) = 0$ *and furthermore the map* d: $W_\lambda^0 \to W_\lambda^1$ *is injective, then* $N_\lambda = \dim_{\boldsymbol{R}} W_\lambda^1 - 2 \dim_{\boldsymbol{R}} W_\lambda^0$.

PROOF. W_λ^1 is a finite dimensional complex vector space equipped with a nondegenerate Hermitian inner product (,). With respect to (,) we see that

$$\mathrm{Ker}_\lambda\, \delta = (\mathrm{Im}_\lambda\, d)^\perp \qquad \text{and} \qquad \dim W_\lambda^1 = \dim \mathrm{Ker}_\lambda\, \delta + \dim \mathrm{Im}_\lambda\, d.$$

For the claim (1), suppose that $\beta \in \mathrm{Ker}_\lambda\, \delta \cap \mathrm{Im}_\lambda\, d$. Then $S\beta = d\delta\beta + (\delta d + \mathscr{F})\beta = 0$ since $\delta d + \mathscr{F} = 0$ on $\mathrm{Im}\, d$. On the other hand $S\beta = c(\lambda)\beta$ and $c(\lambda) \neq 0$. Hence $\beta = 0$. So $\mathrm{Ker}_\lambda\, \delta \cap \mathrm{Im}_\lambda\, d = 0$. By dimension counting we know that

$$W_\lambda^1 = \mathrm{Ker}_\lambda\, \delta \oplus \mathrm{Im}_\lambda\, d \qquad \text{(orthogonal direct sum).}$$

Let $\beta \in W_\lambda^1 \backslash \mathrm{Im}\, d$ and $(\delta d + \mathscr{F})\beta = 0$. Then β is decomposed as $\beta = \beta_1 + \beta_2$ ($\beta_1 \in \mathrm{Ker}_\lambda\, \delta$, $\beta_2 \in \mathrm{Im}_\lambda\, d$). We have $0 = (\delta d + \mathscr{F})\beta = (\delta d + \mathscr{F})\beta_1 = S\beta_1 = c(\lambda)\beta_1$. Hence $\beta_1 = 0$: a contradiction.

For the claim (2), let $c(\lambda) = 0$. Then we know that $\delta d = 0$ on W_λ^0 and hence $\mathrm{Im}_\lambda\, d \subset \mathrm{Ker}_\lambda\, \delta$. We know that if $\beta \in W_\lambda^1$ then

$$0 = S\beta = d\delta\beta + (\delta d + \mathscr{F})\beta.$$

Hence, that $\beta \in \mathrm{Ker}_\lambda\, \delta$ implies that $(\delta d + \mathscr{F})\beta = 0$. Thus we know that $N_\lambda \geq \dim_{\boldsymbol{R}} \mathrm{Ker}_\lambda\, \delta - \dim_{\boldsymbol{R}} \mathrm{Im}_\lambda\, d$, $\dim_{\boldsymbol{R}} W_\lambda^0 \geq \dim_{\boldsymbol{R}} \mathrm{Im}_\lambda\, d$ and $\dim_{\boldsymbol{R}} W_\lambda^1 = \dim_{\boldsymbol{R}} \mathrm{Ker}_\lambda\, \delta + \dim_{\boldsymbol{R}} \mathrm{Im}_\lambda\, d$. From these we obtain that $N_\lambda \geq \dim_{\boldsymbol{R}} W_\lambda^1 - 2 \dim_{\boldsymbol{R}} W_\lambda^0$.

For (3), we know that $\beta \in \mathrm{Ker}_\lambda \delta$ if and only if $(\delta d + \mathscr{F})\beta = 0$, since d is injective. Hence that $N_\lambda = \dim_{\boldsymbol{R}} \mathrm{Ker}_\lambda \delta - \dim_{\boldsymbol{R}} \mathrm{Im}_\lambda d$, $\dim_{\boldsymbol{R}} W_\lambda = \dim_{\boldsymbol{R}} \mathrm{Im}_\lambda d$ and $\dim_{\boldsymbol{R}} W_\lambda^1 = \dim_{\boldsymbol{R}} \mathrm{Ker}_\lambda \delta + \dim_{\boldsymbol{R}} \mathrm{Im}_\lambda d$. These imply that $N_\lambda = \dim_{\boldsymbol{R}} W_\lambda^1 - 2 \dim_{\boldsymbol{R}} W_\lambda^0$. □

To decompose the induced representation we study irreducible representations of $K = S(U_2 \times U_2)$ and $H = SU_2^\pm$ (cf. [12] and [13] 4.6.12 for example). We realize both groups as

$$K = S(U_2 \times U_2) = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \middle| g, h \in U_2, \det gh = 1 \right\} \quad \text{and}$$

$$H = SU_2^\pm = \left\{ \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \middle| h \in U_2, \det h = \pm 1 \right\}.$$

Both representations are constructed from irreducible representations of SU_2. Let (π_n, V_n) be the irreducible unitary representation of SU_2 of dimension $n + 1$, $(n \geq 0)$. Then we know that the Casimir element $\Omega = -(u_1^2 + u_2^2 + u_3^2)/8$ of SU_2 acts on V_n as the scalar operator: $\pi_n(\Omega) = n(n+2)/8$. Let $Z_0 = \exp \boldsymbol{R}Z$ $\left(Z = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}\right)$ be the identity component of the center of K. Then we know that $K = Z_0(SU_2 \times SU_2)$. Equivalence classes $\hat{K}$ of irreducible unitary representations of K are parametrized by

$$\hat{K} = \{\lambda = (k, l, m) \in \boldsymbol{Z} \times \boldsymbol{N} \times \boldsymbol{N} | k - (l + m) \in 2\boldsymbol{Z}),$$

where $\boldsymbol{Z}$ denotes the set of integers and $\boldsymbol{N}$ denotes the set of nonnegative integers. The representation $(\pi(\lambda), V(\lambda)) = (\pi(k, l, m), V(k, l, m))$ corresponding to $\lambda = (k, l, m)$ is determined by the following three conditions:

$$V(k, l, m) = V_1 \otimes V_m. \tag{4.8}$$

$$\pi(k, l, m) | SU_2 \times SU_2 = \pi_1 \boxtimes \pi_m \qquad \text{i.e. for } g,\ h \in SU_2,$$

$$\pi(k, l, m) \begin{pmatrix} g & 0 \\ 0 & \mathrm{h} \end{pmatrix} = \pi_1(g) \otimes \pi_m(h) \qquad \text{on } V_1 \otimes V_m. \tag{4.9}$$

$$\pi(k, l, m) | Z_0 \qquad \text{are scalar operators with the differential}$$

$$d\pi(k, l, m)(Z) = (ik)I. \tag{4.10}$$

Hence for $\lambda = (k, l, m) \in \hat{K}$ we get

$$d\pi(\lambda)(U_0^2) = \frac{1}{18} d\pi(\lambda)(Z^2) = -\frac{k^2}{16},$$

$$d\pi(\lambda)(C) = -\frac{1}{8}\{l(l+2) + m(m+2)\}.$$

Thus by lemma 4.1 we have the value of $c(\lambda)$:

LEMMA 4.3. $c(\lambda) = \dfrac{k^2}{16} - \dfrac{1}{8}\{l(l+2) + m(m+2)\} + \dfrac{1}{2}.$

On the other hand H has two connected components: $H = SU_2^{\pm} = SU_2 \cup \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} SU_2$. If ρ is an irreducible unitary representation of H its restriction to the identity component $\rho|SU_2$ is also irreducible, hence coincides with some π_n. The value of $\rho\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ which is a scalar by Schur's lemma is sufficient to determine the representation ρ. Equivalence classes $\hat{H}$ of irreducible unitary representations of H are

$$\hat{H} = \{(\pi_n^+, V_n^+) | n \in N\} \cup \{(\pi_n^-, V_n^-) | n \in N\}.$$

Here the representation $(\pi_n^{\pm}, V_n^{\pm})$ is determined by the following:

$$V_n^{\pm} = V_n \tag{4.11}$$

$$\pi_n^{\pm}|SU_2 = \pi_n \tag{4.12}$$

$$\pi_n^{\pm}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \begin{cases} \pm I & \text{if } n \in 2\boldsymbol{Z} \\ \pm iI & \text{if } n \in 2\boldsymbol{Z} + 1. \end{cases} \tag{4.13}$$

The inclusion map $H \to K$ is $h \mapsto \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$. Thus the restriction of $\pi(k, l, m)$ to H is given by the Clebsch-Gordan formula:

$$\pi(k, l, m)|H = (\pi_1 \otimes \pi_m)^{\pm} = (\pi_{|l-m|} + \pi_{|l-m|+2} + \cdots + \pi_{l+m-2} + \pi_{l+m})^{\pm}$$

where the sign $\pm$ is determined by (k, l, m);

$$(\pi(k, l, m)|H)\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \pi(k, l, m)\begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix} = e^{(2l-k)\pi i/2} I.$$

Here note that $\begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \exp(-\pi \boldsymbol{Z}/2).$

LEMMA 4.4. *The map $d: \Omega^0(\mathfrak{m}_P) \to \Omega^1(\mathfrak{m}_P)$ is injective.*

PROOF. We know that $\operatorname{Ker} d = \sum \oplus_{\lambda \in \hat{K}} W_\lambda^0 \cap \operatorname{Ker} d$. Let $W_\lambda^0 \cap \operatorname{Ker} d \neq 0$ for $\lambda = (k, l, m)$ and take $\varphi \in W_\lambda^0 \cap \operatorname{Ker} d$, $\varphi \neq 0$. We get that $U_\alpha \varphi = d\pi(\lambda)(U_\alpha)\varphi = 0$ $(0 \leq \alpha \leq 3)$ by (4.4). Hence $d\pi(\lambda)(C)\varphi = d\pi(\lambda)(\sum_4^6 U_k^2)\varphi =$

$\sum_4^6 (\mathrm{ad}\, U_k)^2 \varphi = -(1/2)\varphi$. On the other hand we know $d\pi(\lambda)(C) = -\{l(l+2) + m(m+2)\}/8$. Hence we get that $l(l+2) + m(m+2) = 4$, but there is no such $(l, m) \in \boldsymbol{N} \times \boldsymbol{N}$. $\square$

To determine the nullity we have to search for $\lambda = (k, l, m) \in \hat{K}$ such that $c(\lambda) = 0$ and to count $\dim_{\boldsymbol{R}} W_\lambda^1 - 2 \dim_{\boldsymbol{R}} W_\lambda^0$. Note that as H-representation: $\mathfrak{m} \otimes \mathfrak{s}^* = \mathfrak{sl}_2(\boldsymbol{C}) \otimes (\boldsymbol{C}Z + \mathfrak{sl}_2(\boldsymbol{C})) = V_2^+ \otimes (V_0^+ + V_2^+) = V_0^+ + 2V_2^+ + V_4^+$. By the Frobenius reciprocity theorem $V(k, l, m)$ occurs in $\Omega^1(\mathfrak{m}_P)$ if and only if $|l - m| = 0$, 2, 4, k is even and $2l - k \equiv 0 \pmod 4$. Among these λ's only two representations $\lambda = (\pm 2, 1, 1)$ satisfy $c(\lambda) = 0$. For $\lambda = (\pm 2, 1, 1)$, $\dim_{\boldsymbol{R}} V(\lambda) = 2 \dim_{\boldsymbol{C}} V(\lambda) = 8$ and $V(\lambda)|H = V_0^+ + V_2^+$. By the Frobenius reciprocity again we see that each $V(\lambda)$ occurs 3-times in $\Omega^1(\mathfrak{m}_P)$ i.e. $W_\lambda^1 = 3V(\lambda)$. We know similarly that $W_\lambda^0 = V(\lambda)$ for $\lambda = (\pm 2, 1, 1)$. By Lemma 4.2(3), we get $N_\lambda = (3 - 2) \times 8 = 8$ for $\lambda = (\pm 2, 1, 1)$. Thus the whole nullity N of Θ is counted to be $2 \times 8 = 16$. We next show that the index of Θ is infinite. We know by Lemma 4.1(1) that if $c(\lambda) < 0$ then $W_\lambda^1/\mathrm{Im}_\lambda\, d = \mathrm{Ker}_\lambda\, \delta$ and $S = \delta d + \mathscr{F} = c(\lambda) I$ on this space. So for each $\lambda \in \hat{K}$ such that $c(\lambda) < 0$ we have a negative eigenspace of $\delta d + \mathscr{F}$ of dimension: $\dim_{\boldsymbol{R}} W_\lambda^1/\mathrm{Im}_\lambda\, d = \dim_{\boldsymbol{R}} W_\lambda^1 - \dim_{\boldsymbol{R}} W_\lambda^0$. It suffices to consider a series $\lambda = (0, l, l)$, $l \geq 2$. Then $c(\lambda) < 0$ and $\dim_{\boldsymbol{R}} W_\lambda^1 - \dim_{\boldsymbol{R}} W_\lambda^0 = 6(l+1)^2$, $l \geq 2$. Thus we see that the index of Θ is infinite.

Finally for the (2, 2)-Riemannian space $S^2 \times S^2$ we show that the nullity and the index of self-dual part of the canonical connection are both infinite. For a technical reason we use a local isomorphism: $\mathfrak{o}_4 = \mathfrak{o}_3 \times \mathfrak{o}_3 = \mathfrak{su}_2 \times \mathfrak{su}_2$.

$$\begin{array}{ccc}
SU_2 \times SU_2/\boldsymbol{Z}_2 & \longrightarrow & SO_4 \\
\Big\downarrow {\scriptstyle T_- \times T_+/\boldsymbol{Z}_2} & & \Big\downarrow {\scriptstyle SO_2 \times SO_2 \to SL_2(\boldsymbol{R}) \times SL_2(\boldsymbol{R})} \\
S^2 \times S^2 = SU_2/T_- \times SU_2/T_+ & \longrightarrow & SO_4/SO_2 \times SO_2
\end{array}$$

where $T_\pm = \left\{ t_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \theta \in \boldsymbol{R} \right\}$ is a maximal torus of SU_2 and $\boldsymbol{Z}_2 = \{\pm(I, I)\}$. This local isomorphism is given by

$$SU_2 \times SU_2 \longleftarrow Sp_1 \times Sp_1 \longrightarrow SO_4 = SO(\boldsymbol{H})$$

$$\left(\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \begin{pmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{pmatrix} \right) \longleftarrow\!\shortmid (x, y) \longmapsto L_x \circ R_{y^{-1}}$$

hence,

$$\begin{array}{ccccc} T_- \times T_+ & \longrightarrow & & & SO_2 \times SO_2 \\ (t_\theta, t_\eta) & \longmapsto & (e^{i\theta}, e^{i\eta}) & \longmapsto & \begin{pmatrix} R(\theta-\eta) & 0 \\ 0 & R(\theta+\eta) \end{pmatrix} \end{array}$$

where $x = \alpha + \beta j$, $y = \gamma + \delta j \in \boldsymbol{H}$, L_x and R_x are left and right multiplications by x in $\boldsymbol{H}$ and $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. In the Lie algebra level,

$$\mathfrak{su}_2 \times \mathfrak{su}_2 \longrightarrow \mathfrak{o}_4$$

$$\left(\begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix}, \begin{pmatrix} iy_1 & y_2 + iy_3 \\ -y_2 + iy_3 & -iy_1 \end{pmatrix} \right)$$

$$\longmapsto \left[\begin{array}{cc|cc} 0 & -x_1 + y_1 & -x_2 + y_2 & -x_3 + y_3 \\ x_1 - y_1 & 0 & -x_3 - y_3 & x_2 + y_2 \\ \hline x_2 - y_2 & x_3 + y_3 & 0 & -x_1 - y_1 \\ x_3 - y_3 & -x_2 - y_2 & x_1 + y_1 & 0 \end{array} \right]$$

and hence,

$$\mathfrak{t}_- \times \mathfrak{t}_+ \longrightarrow \mathfrak{o}_2 \times \mathfrak{o}_2$$

$$\left(\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix}, \begin{pmatrix} iy & 0 \\ 0 & -iy \end{pmatrix} \right) \longmapsto \left[\begin{array}{cc|cc} 0 & -x + y & & \\ x - y & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -x - y \\ \multicolumn{2}{c|}{0} & x + y & 0 \end{array} \right]. \tag{4.14}$$

Put $K = K_- \times K_+/\boldsymbol{Z}_2$, $K_\pm = SU_2$ and $H = T_- \times T_+/\boldsymbol{Z}_2$. Their Lie algebras are $\mathfrak{k} = \mathfrak{k}_- \times \mathfrak{k}_+ = \mathfrak{su}_2 \times \mathfrak{su}_2$ and $\mathfrak{h} = \mathfrak{t}_- \times \mathfrak{t}_+$. Then $\mathfrak{k}_\pm = \mathfrak{t}_\pm + \mathfrak{s}_\pm$ where $\mathfrak{s}_\pm = \left\{ \begin{pmatrix} 0 & x_2 + ix_3 \\ -x_2 + ix_3 & 0 \end{pmatrix} \right\}$. We give $\mathfrak{k}_\pm$ an invariant bilinear form $h_\pm$ such that $h_\pm(X, Y) = \mp(\text{the Killing form}) = \mp 4\,\mathrm{tr}(XY)$ for X, $Y \in \mathfrak{k}_\pm$ and give $\mathfrak{s}_- \times \mathfrak{s}_+ = T_o(K/H)$ the restricted (2, 2)-inner product. Then $S^2 \times S^2 = K/H$ has the invariant (2, 2)-Riemannian metric. And we give the Lie algebra $\mathfrak{m} = \mathfrak{m}^1 + \mathfrak{m}^2$ of the structure group $M_0 = SL_2(\boldsymbol{R}) \times SL_2(\boldsymbol{R})$ an indefinite inner product: $\langle X, Y \rangle = \mathrm{tr}(XY)$. We proceed as in the (1, 3)-Riemannian case and use the notations in (4.3). Let $(U_1^\pm, U_2^\pm, U_3^\pm) = (u_1/\sqrt{8}, u_2/\sqrt{8}, u_3/\sqrt{8})$ be the orthonormal basis of $\mathfrak{k}_\pm = \mathfrak{su}_2$ such that $U_j^+ \in \mathfrak{k}_+$ and $U_j^- \in \mathfrak{k}_-$. Then we define negative of the Casimir element $C^\pm$ of $\mathfrak{k}_\pm$ by

$$C^\pm = (U_1^\pm)^2 + (U_2^\pm)^2 + (U_3^\pm)^2.$$

Given an H-representation (ρ, V), we let $\Gamma_V^\pm$ be the operator:

$$\Gamma_V^{\pm} = \rho(U_1^{\pm})^2,$$

and let $1_{\pm}\colon \mathfrak{s} = \mathfrak{s}_- \times \mathfrak{s}_+ \to \mathfrak{s}_{\pm}$ be the projection. Then we have

LEMMA 4.5. $\delta d = (-C^+ + C^-)\otimes I + I\otimes(\Gamma_{\mathfrak{m}}^+ - \Gamma_{\mathfrak{m}}^-)$ on $\Omega^0(\mathfrak{m}_P)$

$$S = (-C^+ + C^-)\otimes I + I\otimes(\Gamma_{\mathfrak{m}}^+ - \Gamma_{\mathfrak{m}}^- + 1_+ + 2\Gamma_{\mathfrak{s}}^+ - 1_- - 2\Gamma_{\mathfrak{s}}^-) \text{ on } \Omega^1(\mathfrak{m}_P).$$

The set $\hat{K}$ of equivalence classes of irreducible unitary representations of $K = SU_2 \times SU_2/\mathbf{Z}_2$ is parametrized by

$$\hat{K} = \{\lambda = (k, l) \in \mathbf{N}\times\mathbf{N} \mid k + l \in 2\mathbf{Z}\}.$$

For $\lambda = (k, l)\in\hat{K}$, the corresponding representation is given by the exterior tensor product of two representations of SU_2: $(\pi(k, l), V(k, l)) = (\pi_k \boxtimes \pi_1, V_k\otimes V_l)$. The irreducible unitary representations of an Abelian group $H = T_- \times T_+/\mathbf{Z}_2$ are all one dimensional and parametrized by

$$\hat{H} = \{(n, m)\in\mathbf{Z}\times\mathbf{Z} \mid n + m \in 2\mathbf{Z}\}.$$

Let $(\chi_n, \mathbf{C}_n)$ be the one dimensional representation of the torus T given by $\chi_n(t_\theta) = e^{in\theta}$. Then for $(n, m)\in\hat{H}$, the corresponding representation is $(\chi_{n,m}, \mathbf{C}_{n,m}) = (\chi_n\otimes\chi_m, \mathbf{C}_n\otimes\mathbf{C}_m)$, i.e. $\chi_{n,m}(t_\theta, t_\eta) = e^{in\theta}e^{im\eta}$.

We have $\Omega^p(\mathfrak{m}_P) = \Omega^p(\mathfrak{m}_P^1) + \Omega^p(\mathfrak{m}_P^2)$. We prefer to treat its complexification: $\Omega^p(\mathfrak{m}_P)^{\mathbf{C}} = \Omega^p(\mathfrak{m}_P^1)^{\mathbf{C}} + \Omega^p(\mathfrak{m}_P^2)^{\mathbf{C}}$ and decompose $\Omega^p(\mathfrak{m}_P^2)^{\mathbf{C}}$. As H-representation, $(\mathfrak{m}^1)^{\mathbf{C}} = \mathfrak{sl}_2(\mathbf{C}) = \mathbf{C}_{0,0} + \mathbf{C}_{2,-2} + \mathbf{C}_{-2,2}$, $(\mathfrak{m}^2)^{\mathbf{C}} = \mathbf{C}_{0,0} + \mathbf{C}_{2,2} + \mathbf{C}_{-2,-2}$ by (4.14) and $\mathfrak{s}^{\mathbf{C}} = \mathfrak{s}_-^{\mathbf{C}}\times\mathfrak{s}_+^{\mathbf{C}} = (\mathbf{C}_{2,0} + \mathbf{C}_{-2,0}) + (\mathbf{C}_{0,2} + \mathbf{C}_{0,-2})$. Hence $(\mathfrak{m}^2\otimes\mathfrak{s}^*)^{\mathbf{C}} = 2(\mathbf{C}_{2,0} + \mathbf{C}_{0,2} + \mathbf{C}_{-2,0} + \mathbf{C}_{0,-2}) + \mathbf{C}_{4,2} + \mathbf{C}_{2,4} + \mathbf{C}_{-4,-2} + \mathbf{C}_{-2,-4}$. We know that $V(k, l)|H = \bigoplus_{n,m}\mathbf{C}_{n,m}$ where (n, m) runs over $|n|\le k$, $|m|\le l$ satisfying $n\equiv k$, $m\equiv 1 \pmod 2$ and that $\dim V(k, l) = (k + l)(l + 1)$. Hence we know that $V(k, l)$ (k and l are odd) cannot occur in $\Omega^1(\mathfrak{m}_P^2)^{\mathbf{C}}$. And we see that $V(0, 0)$ occurs once in $\Omega^0(\mathfrak{m}_P^2)^{\mathbf{C}}$ but it cannot occur in $\Omega^1(\mathfrak{m}_P^2)^{\mathbf{C}}$. Hence the map $d\colon \Omega^0(\mathfrak{m}_P^2)^{\mathbf{C}}\to\Omega^1(\mathfrak{m}_P^2)^{\mathbf{C}}$ is not injective. We estimate the nullity from below. Put $\Gamma_{n,m}^{\pm} = \Gamma_{\mathbf{C}_{n,m}}^{\pm}$. Then $\Gamma_{n,m}^+ = \chi_{n,m}(U_1^+)^2/8 = -m^2/8$ and $\Gamma_{n,m}^- = -n^2/8$. So we get that $\Gamma_{\mathfrak{m}}^+ - \Gamma_{\mathfrak{m}}^- = 0$ and $(1 + 2\Gamma_{\mathfrak{s}})_{\pm} = 0$. Hence we have $\delta d = -C^+ + C^- = c(\lambda)I$ on W_λ^0 and $S = -C^+ + C^- = c(\lambda)I$ on W_λ^1 where $c(\lambda) = \{-k(k + 2) + l(l + 2)\}/8$ for $\lambda = (k, l)$. Thus $c(k, l) = 0$ if and only if $k = l$. Hence $V(2k, 2k)$ $(k\ge 1)$ occur in $\Omega^1(\mathfrak{m}_P^2)^{\mathbf{C}}$ and satisfy that $c(2k, 2k) = 0$. Reciprocity shows that for $k\ge 2$, $V(2k, 2k)$ occurs 12-times in $\Omega^1(\mathfrak{m}_P^2)^{\mathbf{C}}$ and 3-times in $\Omega^0(\mathfrak{m}_P^2)^{\mathbf{C}}$. Hence by Lemma 4.2(2), we have for $k\ge 2$,

$$\begin{aligned}\text{the nullity in } W^1(2k, 2k) &\ge (12 - 2\times 3)\times(2k + 1)^2\\ &= 6(2k + 1)^2.\end{aligned}$$

This shows that the nullity of Θ and its self-dual part are infinite. The index of Θ is checked to be infinite by a similar argument involving Lemma 4.2(1).

The following table summarizes the local form B of canonical connections or its (anti-)self-dual parts pulled back on 4-dimensional flat spaces, the value $\mathcal{YM}(B)$ of their Yang-Mills functionals, their nullities and indices, on these pseudo-Riemannian spaces S^4, $S^1 \times S^3$ and $S^2 \times S^2$.

THEOREM.

space	B	$\mathcal{YM}(B)$	nullity	index
S^4	$\mathrm{Im}\dfrac{\bar{x}dx}{1+\|x\|^2}$	$\pi^2/6$	5	0
$S^1 \times S^3$	$\mathrm{Im}(I+X^2)^{-1}XdX$	$-3\pi^3$	16	infinite
$S^2 \times S^2$	$\mathrm{Im}(I+{}^tXX)^{-1}\,{}^tXdX$	$8\pi^2$	infinite	infinite

Acknowledgement

The author wishes to express his hearty thanks to Prof. K. Okamoto for his helpful discussion and encouragement. In fact main idea of this construction is due to him. The author wishes to express his deep appreciation to H. Doi and X. Y. Chen for their stimulating conversations and valuable comments. He is also indebted to Prof. T. Tanaka for his kind assistance in this work.

References

[1] M. F. Atiyah, Geometry of Yang-Mills Fields, Scuola Normal Superiore Pisa, Pisa, 1979.

[2] M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A **308** (1982), 523–615.

[3] M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in four dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461.

[4] R. Bott, The index theorem for homogeneous differential operators, Differential and combinatorial topology, 167–186, Princeton Univ. Press, Princeton, N.J., 1965.

[5] J. P. Bourguignon and H. B. Lawson, Jr, Stability and isolation phenomena for Yang-Mills fields, Commun. Math. Phys. **79** (1981), 189–230.

[6] S. Helgason, Groups and Geometric Analysis. Integral geometry, Invariant Differential Operators and Spherical Functions, Academic Press, New York, 1984.

[7] J. E. Humpherys, Introduction to Lie algebras and representation theory, Springer, New York, 1972.

[8] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. I. and Vol. II, Interscience, New York, 1963 and 1969.

[9] S. Kobayashi, Y. Ohnita, and M. Takegoshi, On Inatability of Yang-Mills Connections, Math. Z. **193** (1986), 165–189.

[10] H. T. Laquer, Stability properties of the Yang-Mills functional near the canonical connection, Mich. Math. J. **31** (1984), 139–159.

[11] J. Nowakowski and A. Trautman, Natural connections on Stiefel bundles are sourceless gauge fields, J. Math. Phys. **19**, (1978), 1100.

[12] M. Sugiura, Unitary Representations and Harmonic Analysis: An Introduction, Wiley, New York, 1975.

[13] N. R. Wallach, Harmonic analysis on homogeneous spaces, Marcel Dekker, New York, 1973.

[14] G. Warner, Harmonic analysis on semisimple Lie groups I, Springer-Verlag, Berlin, 1972.

Department of Mathematics
Faculty of Education
Nagasaki University