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Fourier coefficients of modular forms of half integral weight, periods of modular forms and the special values of zeta functions

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ABSTRACT. The purpose of this note is to investigate relations between Fourier coefficients of modular forms of half integral weight and the special values of zeta functions of modular forms of integral weight. We shall derive that Fourier coefficients of modular forms of half integral weight at every non-square positive integer is explicitly expressed by means of the special values of zeta functions associated with modular forms determined by the image of Shimura correspondence of modular forms of half integral weight.

Introduction

Let $S_{k+1/2}(4N, \chi)$ and $S_{2k}(M, \psi)$ be the space of cusp forms of Neben-type χ and ψ and of weight $k+1/2$ and $2k$ with respect to $\Gamma_0(4N)$ and $\Gamma_0(M)$, respectively. In [4], Shimura showed the existence of a correspondence $\Psi^t_{N,k,\chi}$ between the space $S_{k+1/2}\left(4N, \chi\left(\frac{-1}{*}\right)^k\right)$ and the space $S_{2k}(2N, \chi^2)$ and he proved that $\Psi^t_{N,k,\chi}$ commutes with the action of Hecke operators. More precisely, let $f(z) = \sum_{n=1}^{\infty} a(n)e[nz]$ be a cusp form of $S_{k+1/2}\left(4N, \chi\left(\frac{-1}{*}\right)^k\right)$. If $f(z)$ is an eigenfunction of all Hecke operators, then there is a modular form $F(z) = \sum_{n=1}^{\infty} A(n)e[nz]$ of $S_{2k}(2N, \chi^2)$ such that

$$a(t) \sum_{n=1}^{\infty} A(n)n^{-s} = L(s-k+1, \chi\omega_t) \sum_{n=1}^{\infty} a(tn^2)n^{-s}$$

for every square-free positive integer t, where $L(*, \chi\omega_t)$ is the Dirichlet L function with a character $\chi\omega_t$ (cf. [4]).

In [7], Shintani established a lifting from a cusp form F of integral weight $2k$ to a cusp form f of integral weight $k+1/2$ and gave an expression

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of Fourier coefficients of f by means of certain integrals of $F(z)z^m dz$ for $m(0 \leqq m \leqq 2k-2)$. Shimura [5] purposed a question whether there is an interesting relation between $a(t)$ and the critical values of the zeta function associated with $F(z)$ for every square-free integer t and he pointed out that it is a most intriguing theme to investigate the Shintani's integral further in connection with periods of modular forms and the special values of zeta functions of modular forms. Kohnen-Zagier [2] verified that $a(D)$ can be determined by critical values of zeta functions associated with $F(z)$ for every discriminant $D(>0)$ in the case where $f(z)$ belongs to the Kohnen's subspace $S^{+}_{k+1/2}(4)$ of $S_{k+1/2}(4, \chi_0)$ with the trivial character χ_0 modulo 4 (cf. [8]).

The purpose of this note is to derive that the coefficients $a^*(n)$ of the Fourier expansion of the Shintani lifting $\Psi^*_{N,k,\chi}(F)(\tau) = \sum_{n=1}^{\infty} a^*(n)e[n\tau]$ of F can be explicitly expressed in terms of the special values of zeta functions of F for every positive integer n with an arbitrary integer N, where F belongs to $S_{2k}(2N, \chi^2)$.

In Section 1, we shall summarize some results about the Shimura correspondence and the Shintani integral. In Section 2, we shall express periods of a modular form of $F(z) = \sum_{n=1}^{\infty} A(n)e[nz]$ belonging to the space $S_l(L, \xi)$ as a finite sum of special values of the modified zeta function $L\left(s, F, \frac{d}{c}\right) = \sum_{n=1}^{\infty} A(n)e^{2\pi i n d/c} n^{-s}$. Moreover, adapting a computation of the Gauss sum and the property of the orthogonality of Dirichlet characters, we may derive an expression of special values of $L\left(s, F, \frac{d}{c}\right)$ in terms of special values of zeta functions $L(s, F, \psi) = \sum_{n=1}^{\infty} A(n)\psi(n)n^{-s}$ with Dirichlet characters ψ. Using these results, we can verify that the periods of modular forms are explicitly expressed as a finite sum of special values of zeta functions associated with modular forms. As an application of this result, we may derive that $a^*(n)$ can be explicitly expressed in terms of special values of several zeta functions $L(s, F, \psi)$. Furthermore, applying the result on periods of modular forms and Shimura's theorem, we obtain a result concerning the arithmeticity of the fundamental periods of modular forms. This yields a reformulation of Theorem 4.7 in [6].

Finally, we mention that our results give an expression of Fourier coefficients of Jacobi forms in terms of special values of zeta functions associated with modular forms of integral weight (cf. [1]).

Notation

We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For $z \in \boldsymbol{C}$, we put $e[z] = \exp(2\pi i z)$ with $i = \sqrt{-1}$, and define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg z^{1/2} \leqq \pi/2$. Furthermore, we put $z^{\kappa/2} = (\sqrt{z})^{\kappa}$ for every $\kappa \in \boldsymbol{Z}$. Let $GL^{+}(2, \boldsymbol{R})$ (resp. $SL(2, \boldsymbol{R})$) denote the group of all real matrices of degree 2 with positive determinant (resp. determinant one) and $\mathfrak{H}$ the complex upper half plane, i.e.,

$$\mathfrak{H} = \{z \in \boldsymbol{C} \mid \Im z > 0\}.$$

Define an action of $GL^{+}(2, \boldsymbol{R})$ on $\mathfrak{H}$ by

$$z \to \sigma(z) = (az + b)(cz + d)^{-1}$$

for every $z \in \mathfrak{H}$ and for every $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^{+}(2, \boldsymbol{R})$. For a positive integer M, put

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{R}) \,\middle|\, a, b, c, d \in \boldsymbol{Z},\ c \equiv 0 \pmod{M} \right\},$$

$$\Gamma_1(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M) \,\middle|\, a \equiv d \equiv 1 \pmod{M} \right\} \quad \text{and} \quad SL(2, \boldsymbol{Z}) = \Gamma_0(1).$$

Furthermore, we also set

$$\theta(z) = \sum_{n=-\infty}^{\infty} e[n^2 z] \qquad \text{and} \qquad j(\gamma, z) = \theta(\gamma(z))/\theta(z)$$

for every $z \in \mathfrak{H}$ and for every $\gamma \in \Gamma_0(4)$.

1. Shimura correspondence and Shintani integral

Let M, N and k be positive integers and χ (resp. ψ) a Dirichlet character modulo M (resp. $4N$). We denote by $S_{k+1/2}(4N, \chi)$ (resp. $S_{2k}(M, \psi)$) the space of cusp forms of Neben-type χ (resp. ψ) and of weight $k + 1/2$ (resp. $2k$) with respect to $\Gamma_0(4N)$ (resp. $\Gamma_0(M)$). For τ, $z \in \mathfrak{H}$, we define a function $\Omega_{N,\chi}(\tau, -2N\bar{z})$ on $\mathfrak{H} \times \mathfrak{H}$ by

$$(1.1) \qquad \Omega_{N,\chi}(\tau, -2N\bar{z}) = (-1)^k \sum_{n=1}^{\infty} \sum_{(a,b,c)} \chi(c)\overline{(az^2 + bz + c)}^{-k} n^{k-1/2} e[n\tau]$$

for every positive integer $k > 1$, where (a, b, c) runs over $\boldsymbol{Z}^3$ under the conditions that $4N^2 | a$, $4N | b$ and $b^2 - 4ac = 16N^2 n$. For a positive integer n, put

$$L_{N,n} = \left\{ Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \,\middle|\, 4N^2 | a,\ 4N | b,\ b^2 - 4ac = 16N^2 n \right\}.$$

The group $\Gamma_0(2N)$ acts on $L_{N,n}$ as follows:

$$Q \to Q \circ g = {}^t g Q g$$

for every $Q \in L_{N,n}$ and for every $g \in \Gamma_0(2N)$. For $z \in \mathfrak{H}$, put

$$\omega_{k,N}(z; \chi, n) = \sum_{Q \in L_{N,n}} \bar{\chi}(Q) Q(z, 1)^{-k}, \tag{1.2}$$

where $\chi(Q) = \chi(c)$ and $Q(z, 1) = az^2 + bz + c$ with $Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$. We may easily check that

$$\omega_{k,N}(g(z); \chi, n) = \chi(d)^2 J(g, z)^{2k} \omega_{k,N}(z; \chi, n) \tag{1.3}$$

for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$ with $J(g, z) = (cz + d)$. By our definition, we see

$$\Omega_{N,\chi}(\tau, -2N\bar{z}) = (-1)^k \sum_{n=1}^{\infty} \overline{\omega_{k,N}(z; \chi, n)} n^{k-1/2} e[n\tau]. \tag{1.4}$$

Lion and Vergne [3] verified the following transformation formula

$$\Omega_{N,\chi}(\gamma(\tau), -2N\bar{z}) = \left(\frac{-1}{d}\right)^k \chi(d) j(\gamma, \tau)^{2k+1} \Omega_{N,\chi}(\tau, -2N\bar{z}) \tag{1.5}$$

for every $\gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(4N)$. Define a function $\Psi_{N,k,\chi}(f)(z)$ on $\mathfrak{H}$ by

$$\Psi_{N,k,\chi}(f)(z) = \int_{\Gamma_0(4N)\backslash \mathfrak{H}} f(\tau) \overline{\Omega_{N,\chi}(\tau, -2N\bar{z})} (\Im\tau)^{k-3/2} |d\tau d\bar{\tau}| \tag{1.6}$$

for every $f(\tau) \in S_{k+1/2}\left(4N, \chi\left(\frac{-1}{*}\right)^k\right)$. The following theorem was first proved by Shimura [4] and it was reformulated by Lion and Vergne [3].

THEOREM A. *Suppose that* $k > 1$ *and* $f \in S_{k+1/2}\left(4N, \chi\left(\frac{-1}{*}\right)^k\right)$. *Then* $\Psi_{N,k,\chi}(f)$ *belongs to* $S_{2k}(2N, \chi^2)$. *Moreover,*

$$\Psi_{N,k,\chi}(f)(z) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} n^{k-1} \chi(n) a(m^2) \right) e[mnz].$$

Define the adjoint mapping $\Psi^*_{N,k,\chi}$ of $\Psi_{N,k,\chi}$ by

$$\Psi^*_{N,k,\chi}(F)(\tau) = \int_{\Gamma_0(2N)\backslash \mathfrak{H}} F(z) \Omega_{N,\chi}(\tau, -2N\bar{z}) y^{2k-2} dx dy \tag{1.7}$$

for every $F \in S_{2k}(2N, \chi^2)$. We reduce the above integral to the integral on a certain cycle. We say two elements Q' and Q in $L_{N,n}$ to be equivalent, if $Q' = Q \circ \gamma = {}^t\gamma Q \gamma$ for some $\gamma \in \Gamma_1(2N)$. We denote by $[Q]$ the equivalence class containing Q. Moreover, we denote by $L_{N,n}/\sim$ the set of all the equivalence classes in $L_{N,n}$. From this definition, we easily see that $\omega_{k,N}(z; \chi, n)$ may be rewritten as follows:

$$(1.8) \qquad \omega_{k,N}(z; \chi, n) = \sum_{[Q] \in L_{N,n}/\sim} \sum_{\gamma \in \Gamma_1(2N)_Q \backslash \Gamma_1(2N)} \bar{\chi}(Q \circ \gamma)(Q \circ \gamma)(z, 1)^{-k},$$

where $\Gamma_1(2N)_Q = \{\gamma \in \Gamma_1(2N) | Q \circ \gamma = Q\}$. By (1.4) and (1.7), we deduce

(1.9)

$$\Psi^*_{N,k,\chi}(F)(\tau) = i_{N,k} \sum_{n=1}^{\infty} n^{k-1/2} \sum_{[Q]} \left(\int_{\Gamma_1(2N)_Q \backslash \mathfrak{H}} F(z)\chi(Q)\overline{Q(z, 1)}^{-k} y^{2k-2} dxdy \right) e[n\tau],$$

where $i_{N,k} = (-1)^k [\Gamma_0(2N)/\{\pm 1\} : \{\pm 1\}\Gamma_1(2N)/\{\pm 1\}]$ and $[Q]$ runs over all elements in $L_{N,n}/\sim$. For $Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in L_{N,n}$ with a non-square n, put

$$SL(2, \mathbf{Z})_Q = \{\gamma \in SL(2, \mathbf{Z}) | Q \circ \gamma = Q\} \quad \text{and} \quad A_Q = \begin{pmatrix} (t - b'u)/2 & -c'u \\ a'u & (t + b'u)/2 \end{pmatrix},$$

where $(a', b', c') = \frac{1}{l}(a, b, c)$, (t, u) is the smallest positive solution of Pell's equation $t^2 - 16N^2 nu^2/l^2 = 4$ and l is the greatest common divisor of a, b, c. Then $SL(2, \mathbf{Z})_Q/\{\pm 1\}$ is generated by A_Q. Since A_Q is hyperbolic and $[SL(2, \mathbf{Z})_Q : \Gamma_1(2N)_Q] < \infty$, we may check that $\{\pm 1\}\Gamma_1(2N)_Q/\{\pm 1\}$ is an infinite cyclic group and $\{\pm 1\}\Gamma_1(2N)_Q = \{\pm A'_Q\}$ for some A'_Q. By a method similar to that of Kohnen-Zagier [2], we may derive

$$(1.10) \quad \int_{\Gamma_1(2N)_Q \backslash \mathfrak{H}} F(z)\chi(Q)\overline{Q(z, 1)}^{-k} y^{2k-2} dxdy$$
$$= 2^{-2k+3} \chi(Q) \pi (16N^2 n)^{1/2-k} \binom{2k-3}{k-1} \int_{z_0}^{A'_Q(z_0)} F(z) Q(z, 1)^{k-1} dz$$

with any $z_0 \in \mathfrak{H}$. Note that the integral of the right hand side is independent of the choice of z_0. For $Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in L_{N,n}$ with a square n, we also have

$$(1.11) \quad \int_{\Gamma_1(2N)_Q \backslash \mathfrak{H}} F(z)\chi(Q)\overline{Q(z, 1)}^{-k} y^{2k-2} dxdy$$
$$= 2^{-2k+3} \chi(Q) \pi (16N^2 n)^{1/2-k} \binom{2k-3}{k-1} \int_{C(Q)} F(z) Q(z, 1)^{k-1} dz,$$

where $C(Q)$ is the geodesic line from $(-b+4N\sqrt{n})/2c$ to $(-b-4N\sqrt{n})/2c$ for $c \neq 0$ and if $c=0$, $C(Q)$ is the geodesic line from $+i\infty$ (resp. a/b) to a/b (resp. $+i\infty$) for $b>0$ (resp. $b<0$). The integral of this type was first introduced by Shintani [7].

2. Periods of modular forms and the special values of zeta functions

The purpose of this section is to express periods of modular forms as a finite sum of special values of zeta functions associated with modular forms. Let L and l be arbitrary positive numbers and ξ a Dirichlet character modulo L. Assume that $F(z)=\sum_{n=1}^{\infty} A(n)e[nz]$ belongs to $S_l(L, \xi)$. For $\sigma \in \Gamma_0(L)$, we put

$$Z_i(\sigma) = \int_0^{\sigma(0)} F(z)z^i dz \qquad (0 \leqq i \leqq l-2). \tag{2.1}$$

We may easily check

$$Z_i(\sigma\tau) = \int_0^{\sigma(0)} F(z)z^i dz + \int_{\sigma(0)}^{\sigma\tau(0)} F(z)z^i dz \tag{2.2}$$

for every σ, $\tau \in \Gamma_0(L)$. Making the substitution, $z=\sigma(w)$, we obtain

$$\int_{\sigma(0)}^{\sigma\tau(0)} F(z)z^i dz = \sum_{j=0}^{l-2} \alpha(i,j) \int_0^{\tau(0)} F(w)w^j dw \tag{2.3}$$

for some $\alpha(i,j) \in \boldsymbol{Z}[\xi]$ $(0 \leqq i, j \leqq l-2)$, where $\boldsymbol{Z}[\xi]$ denotes the ring generated by the values ξ over $\boldsymbol{Z}$. Hence we have

$$Z_i(\sigma\tau) = Z_i(\sigma) + \sum_{j=0}^{l-2} \alpha(i,j)Z_j(\tau). \tag{2.4}$$

Let $\{a_1, a_2, \dots, a_n\}$ be a system of generators of $\Gamma_0(L)$. Then we have

$$Z_i(\sigma) = \sum_{s=0}^{n} \sum_{j=0}^{l-2} \alpha(i,j,s) \int_0^{\sqrt{-1}\infty} F\left(z+\frac{\alpha_s}{M_s}\right) z^j dz \tag{2.5}$$

for some $\alpha(i,j,s) \in \boldsymbol{Z}[\xi]$, where α_s and M_s are integers satisfying $\alpha_0=0$, $M_0=1$ and $a_s(0)=\dfrac{\alpha_s}{M_s}$ $(1 \leqq s \leqq n)$. For $\sigma \in SL(2, \boldsymbol{Z})$, we also put

$$Z_i(\sigma) = \int_0^{\sigma(0)} F(z)z^i dz \qquad (0 \leqq i \leqq l-2). \tag{2.6}$$

For any $\sigma \in SL(2, \boldsymbol{Z})$, put $\sigma = \tau\tau'$ with $\tau \in \Gamma_0(L)$ and $\tau' \in \Gamma_0(L)\backslash SL(2, \boldsymbol{Z})$. Then

$$Z_i(\sigma) = Z_i(\tau) + \sum_{j=0}^{l-2} \beta(i, j) Z_j(\tau') \tag{2.7}$$

for some $\beta(i, j) \in \boldsymbol{Z}[\xi]$ $(0 \leqq i, j \leqq l-2)$. Therefore, for our further calculation of (2.6), it is enough to investigate the integral

$$\int_0^{i\infty} F\left(z + \frac{\alpha}{M}\right) z^j dz.$$

For a positive integer M and for an integer $x \in \boldsymbol{Z}$, put

$$U_s\left(\frac{x}{M}, F\right) = \int_0^{i\infty} F\left(z + \frac{x}{M}\right) z^s dz \tag{2.8}$$

with a positive $s \gg 0$. Moreover, for a Dirichlet character ψ modulo M, put

$$H_s(F, \psi) = \sum_{x=1}^{M} \psi(x) U_s\left(\frac{x}{M}, F\right). \tag{2.9}$$

By our definition, we easily check

$$\sum_{x=1}^{M} \psi(x) F\left(z + \frac{x}{M}\right) = \sum_{n=1}^{\infty} A(n) G(n, \psi) e[nz],$$

where $G(n, \psi) = \sum_{x=1}^{M} \psi(x) e^{2\pi i n x/M}$ is the Gauss sum associated with a Dirichlet character ψ. Consider the factorization of M such that $M = \prod_{i=1}^{r} p_i^{m_i}$ and put $M_i = p_i^{m_i}$. Then the character ψ is decomposed to the form $\psi = \prod_{i=1}^{r} \psi_i$ according as the decomposition $M = \prod_{i=1}^{r} M_i$. Let ψ_i^* be the primitive character associated with ψ_i whose conductor is $p_i^{n_i}$. We may derive the following lemma.

LEMMA 2.1. *For a positive integer n, put $n = n' \prod_{i=1}^{r} p_i^{\alpha_i}$ with $(n', M) = 1$. Then*

$$G(n, \psi) = \bar{\psi}(n') \prod_{i=1}^{r} \psi_i\left(\frac{M}{M_i}\right) \bar{\psi}_i\left(\prod_{j \neq i} p_j^{\alpha_j}\right) G(p_i^{\alpha_i}, \psi_i)$$

with

$$G(p_i^{\alpha_i}, \psi_i) = \begin{cases} p_i^{m_i - 1}(p_i - 1) & \text{if } \alpha_i \geqq m_i,\ n_i = 0, \\ -p_i^{m_i - 1} & \text{if } \alpha_i = m_i - 1,\ n_i = 0, \\ 0 & \text{if } \alpha_i \leqq m_i - 2,\ n_i = 0, \\ G(1, \psi_i^*) p_i^{\alpha_i} & \text{if } \alpha_i = m_i - n_i,\ 0 < n_i < m_i, \\ \bar{\psi}_i(p_i^{\alpha_i}) G(1, \psi_i) & \text{if } n_i = m_i, \\ 0 & \text{if } 0 < n_i < m_i,\ \alpha_i \neq m_i - n_i. \end{cases}$$

Let $F(z)=\sum_{n=1}^{\infty}A(n)e[nz]$ be an eigenfunction of all Hecke operators of $S_l(L,\xi)$. Then we obtain

$$(2.10)\quad \sum_{x=1}^{M}\psi(x)F\left(z+\frac{x}{M}\right)$$
$$=\sum_{(n',M)=1}\sum_{\alpha_1=0}^{\infty}\sum_{\alpha_2=0}^{\infty}\cdots\sum_{\alpha_r=0}^{\infty}A\left(n'\prod_{i=1}^{r}p_i^{\alpha_i}\right)G\left(n'\prod_{i=1}^{r}p_i^{\alpha_i},\psi\right)e\left[n'\prod_{i=1}^{r}p_i^{\alpha_i}z\right].$$

Since $A(n'\prod_{i=1}^{r}p_i^{\alpha_i})=A(n')\prod_{i=1}^{r}A(p_i^{\alpha_i})$ and

$$G\left(n'\prod_{i=1}^{r}p_i^{\alpha_i},\psi\right)=\bar{\psi}(n')\prod_{i=1}^{r}\psi_i\left(\frac{M}{M_i}\right)\bar{\psi}_i\left(\prod_{j\neq i}p_j^{\alpha_j}\right)G(p_i^{\alpha_i},\psi_i),$$

We deduce that

$$(2.11)\quad H_s(F,\psi)=(2\pi i^{-1})^{-(s+1)}\Gamma(s+1)\sum_{(n',M)=1}A(n')\bar{\psi}(n')n'^{-(s+1)}\left(\prod_{i=1}^{r}\psi_i\left(\frac{M}{M_i}\right)\right.$$
$$\left.\times\sum_{\alpha_1=0}^{\infty}\sum_{\alpha_2=0}^{\infty}\cdots\sum_{\alpha_r=0}^{\infty}\prod_{i=1}^{r}A(p_i^{\alpha_i})(p_i^{\alpha_i})^{-(s+1)}\bar{\psi}_i\left(\prod_{j\neq i}p_j^{\alpha_j}\right)G(p_i^{\alpha_i},\psi_i)\right)$$
$$=(2\pi i^{-1})^{-(s+1)}\Gamma(s+1)L(s+1,F,\bar{\psi})\prod_{i=1}^{r}\psi_i\left(\frac{M}{M_i}\right)$$
$$\times\prod_{i=1}^{r}\left(\sum_{\alpha_i=0}^{\infty}A(p_i^{\alpha_i})(p_i^{\alpha_i})^{-(s+1)}\left(\prod_{j\neq i}^{r}\bar{\psi}_j\right)(p_i^{\alpha_i})G(p_i^{\alpha_i},\psi_i)\right),$$

where $L(s,F,\bar{\psi})=\sum_{n=1}^{\infty}A(n)\bar{\psi}(n)n^{-s}$. Notice that F is an eigenfunction of all Hecke operators of $S_l(L,\xi)$. Combining this with Lemma 2.1, we may derive that

$$\sum_{\alpha_i=0}^{\infty}A(p_i^{\alpha_i})(p_i^{\alpha_i})^{-(s+1)}\left(\prod_{j\neq i}^{r}\bar{\psi}_j\right)(p_i^{\alpha_i})G(p_i^{\alpha_i},\psi_i)$$

belongs to $\boldsymbol{Q}(\psi)(A(p_i))$ for every non-negative integer s, where $\boldsymbol{Q}(\psi)$ is the field generated by the values of ψ over $\boldsymbol{Q}$ and $\boldsymbol{Q}(\psi)(A(p_i))$ is the field generated by $A(p_i)$ over $\boldsymbol{Q}(\psi)$. Consequently, we conclude

LEMMA 2.2. *Let j be an integer such that $0\leqq j\leqq l-2$. Then*

$$H_j(F,\psi)=\sqrt{-1}^{j+1}j!(2\pi)^{-(j+1)}K_j(\psi)(A(p_1),\dots,A(p_r))L(j+1,F,\bar{\psi}),$$

where $p_1,\dots,p_r$ are all prime factors of M and $K_j(\psi)(X_1,\dots,X_r)$ is a rational function of variables $X_1,\dots,X_r$ over $\boldsymbol{Q}(\psi)$.

Observe that $K_j(\psi)(X_1,\dots,X_r)$ can be explicitly calculated. By our definition, for an integer y which is prime to M, we have

$$H_j(F, \psi) = \sum_{x=1}^{M} \psi(xy) U_j\left(\frac{xy}{M}, F\right),$$

which yields

$$\bar{\psi}(y) H_j(F, \psi) = \sum_{x=1}^{M} \psi(x) U_j\left(\frac{xy}{M}, F\right).$$

Hence we obtain

$$\sum_{\psi} \bar{\psi}(y) H_j(F, \psi) = \sum_{\psi} \sum_{x=1}^{M} \psi(x) U_j\left(\frac{xy}{M}, F\right), \tag{2.12}$$

where ψ runs over all Dirichlet characters modulo M, which implies that

$$\sum_{\psi} \bar{\psi}(y) H_j(F, \psi) = \sum_{x=1}^{M} \sum_{\psi} \psi(x) U_j\left(\frac{xy}{M}, F\right). \tag{2.13}$$

Notice that

$$\sum_{\psi} \psi(x) = \begin{cases} \varphi(M) & \text{if } x \equiv 1 \pmod{M}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

where $\varphi(M)$ is the number of $(\boldsymbol{Z}/M\boldsymbol{Z})^{\times}$. Consequently, we conclude

THEOREM 1. *Suppose that F is an eigenfunction of all Hecke operators of $S_l(L, \xi)$. Then*

$$U_j\left(\frac{x}{M}, F\right) = \frac{1}{\varphi(M)} \sum_{\psi} \bar{\psi}(y) H_j(F, \psi)$$

for every j $(0 \leqq j \leqq l-2)$, where the sum $\sum_{\psi}$ is taken over all Dirichlet characters ψ modulo M and $H_j(F, \psi)$ is equal to $\sqrt{-1}^{j+1} j!(2\pi)^{-(j+1)} K_j(\psi)(A(p_1), \ldots, A(p_r)) L(j+1, F, \bar{\psi})$ given in lemma 2.2.

The following theorem was proved by Shimura [6].

THEOREM B. *Suppose that $F(z)$ is a primitive form of level L in $S_l(L, \xi)$. Then there are two constants $v_+(F)$ and $v_-(F)$ with the following properties: for $j \in \boldsymbol{Z}$, $0 \leqq j \leqq l-2$, we have*

$$(\sqrt{-1}\pi)^{-(j+1)} G(1, \psi)^{-1} L(j+1, F, \psi) \in \begin{cases} v_+(F) K_F \cdot K_{\psi} & \text{if } \psi(-1) = (-1)^{j+1}, \\ v_-(F) K_F \cdot K_{\psi} & \text{if } \psi(-1) = (-1)^{j}, \end{cases}$$

and

$$\sqrt{-1}^{1-l} \pi G(1, \xi) \langle F, F \rangle \in v_+(F) v_-(F) \cdot K_F,$$

where K_F and K_ψ denote the fields generated over $\boldsymbol{Q}$ by the coefficients $A(n)$ and the values $\psi(n)$, respectively and $\langle F, F\rangle$ is the normalized Petersson inner product.

Combining Theorem 1 with Theorem B, we may derive the following corollary.

COROLLARY. *Suppose that $F(z)$ is a primitive form of level L in $S_l(L, \xi)$. Then the periods of $F(z)$ belong to $\bar{\boldsymbol{Q}}v_+(F) + \bar{\boldsymbol{Q}}v_-(F)$, where $\bar{\boldsymbol{Q}}$ means the algebraic closure of $\boldsymbol{Q}$.*

This corollary was first proved by Shimura [5] in the case of the modular forms on indefinite quaternion algebra over $\boldsymbol{Q}$ under the assumption that $L(z, F) \neq 0$, where $L(z, F)$ is the Shintani lifting of F (cf. [6]). By Theorem 1, we have the following theorem.

THEOREM 2. *Suppose that $k > 1$ and $F(z) = \sum_{n=1}^{\infty} A(n)e[nz]$ is an eigenfunction of all Hecke operators of $S_{2k}(2N, \chi^2)$. Put $\Psi^*_{N,k,\chi}(F)(\tau) = \sum_{n=1}^{\infty} a^*(n)e[nz]$. Then*

$$a^*(n) = (-1)^k 2^{-2k+3}(16N^2)^{1/2-k}\pi\binom{2k-3}{k-1}\sum_{\psi}\sum_{j=0}^{2k-2} c_j(\psi, F)L(j+1, F, \psi)$$

for every integer n, where ψ runs over certain finite Dirichlet characters modulo M, M is an integer determined by N not depending of the choice of F and $c_j(\psi, F)$ is an explicitly calculable quantity by ψ, j and finite Fourier coefficients $A(n)$ of F.

Here we mention that Theorem 1 also implies that Fourier coefficients of Jacobi forms is explicitly expressed by means of special values of zeta functions associated with modular forms (cf. [1]).

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