HIROSHIMA MATH. J.
27 (1997), 271–293

Nonexistence of positive solutions of Neumann problems for elliptic inequalities of the mean curvature type

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(Received December 5, 1995)
(Revised April 8, 1996)

ABSTRACT. Exterior Neumann problems for quasilinear elliptic inequalities are considered. The leading terms of operators under consideration are the mean curvature type and generalized mean curvature types. Sufficient conditions are given for some Neumann problems to have no positive solutions.

0. Introduction

This paper concerns to elliptic boundary value problems of the form

$$\begin{cases} Mu \equiv \operatorname{div}\left[\dfrac{Du}{(1+|Du|^2)^{\alpha}}\right] \geq p(x)f(u), & x \in \Omega, \\ D_\nu u \leq 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $x = (x_i)$, $Du = (D_i u)$, $D_i u = \partial u/\partial x_i$ for $i = 1, 2, \ldots, N$, $N \geq 2$, $\Omega \subset \boldsymbol{R}^N$ is an exterior domain whose boundary $\partial\Omega$ is of class C^2, $\nu : \partial\Omega \to \boldsymbol{R}^N$, is a vector field pointing outward with respect to Ω, and $D_\nu u$ denotes the derivative of u along the vector ν. Throughout the paper we always assume the following without further mention:

(A_1) $0 \leq \alpha \leq 1/2$;

(A_2) $p : \bar{\Omega} \to (0, \infty)$ is continuous;

(A_3) $f : (0, \infty) \to (0, \infty)$ is locally Lipschitz continuous and strictly increasing with $\lim_{u\to\infty} f(u) = \infty$.

A typical example of f satisfying (A_3) is the function $f(u) = u^\sigma$, $\sigma > 0$. In this case we shall refer to (P) as (P_σ):

$$\begin{cases} Mu \geq p(x)u^\sigma, & x \in \Omega, \\ D_\nu u \leq 0, & x \in \partial\Omega. \end{cases} \tag{P_σ}$$

As is well known, when $\alpha = 1/2$, the operator M is called the mean

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1991 *Mathematics Subject Classification.* 35B40, 35J60, 53A10

Key words and phrases. positive solution, Neumann problem, mean curvature.

curvature operator. When $0 < \alpha < 1/2$, M is often referred to as the generalized mean curvature operator. The capillarity equation and the equation of prescribed mean curvature are important special cases of (P) for which $\alpha = 1/2$ and p and f are suitably specialized. In recent years many authors recognize widely the importance of these quasilinear operators in differential geometry and mathematical sciences [1, 2, 3, 4, 9]. Especially, we can find various results in [2] for the case $\alpha = 1/2$ that have interest of theoretical as well as numerical. According to [4], the case of $0 < \alpha < 1/2$ appears in describing some chemical reactions.

A positive solution u of problem (P) is defined to be a positive function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ which satisfies (P). Of course, if $\Omega = \boldsymbol{R}^N$, it is supposed that the boundary condition is void, and in such a case, we shall call positive solutions of (P) as positive entire solutions.

For quasilinear elliptic problems like (P), we know numerous existence theorems of positive solutions equipped with various supplementary conditions. In particular, asymptotic properties of positive entire solutions have been investigated in full detail. On the other hand, we have not found so many studies in which the nonexistence properties of positive solutions of (P) were discussed. Motivated by this point, in the earlier paper [11] the author considered the same problem as (P) (with $\Omega = \boldsymbol{R}^N$), and tried to give nonexistence criteria of positive solutions to (P). Our results in [11] require, besides (A_1)–(A_3), some additional assumptions on the nonlinearity f. More precisely, the following are imposed on f:

$$\int_1^\infty \left(\int_0^u f(s)ds \right)^{-1/2} du < \infty \qquad \text{and} \qquad \int_0^1 \left(\int_0^u f(s)ds \right)^{-1/2} du = \infty. \quad (0.1)$$

This integral condition is a kind of superlinearity. When $f(u) = u^\sigma$, $\sigma > 0$, (0.1) hold if and only if $\sigma > 1$. (This means that in [11] the case $\sigma \le 1$ for problem (P_σ) was automatically excluded from our consideration.) It seemed for the author that by [11] in this case the nonexistence property of positive solutions is completely characterized. However, as soon as [11] was finished, some questions occurred to the author:

(I) When $0 \le \alpha < 1/2$, is it possible to weaken the superlinear condition (0.1) so as to extend the results in [11]?; and,

(II) When $\alpha = 1/2$, are the superlinear conditions like (0.1) unnecessary to guarantee the nonexistence of positive solutions of (P) provided that p is sufficiently large?

In the present paper, we intend to give several answers to these questions affirmatively.

The paper is organized as follows. In Section 1 we prepare basic results mainly concerning to local solutions of inequalities of the form appearing in

(P). The results in Section 1 play crucial roles to analyze our main problems. In Section 2 nonexistence criteria of positive solutions for (P) are established for the case of $0 \leq \alpha < 1/2$. As mentioned above, we can improve the nonexistence criteria in [11] considerably, that is, answers to the problem (I) are given. In Section 3 we consider the case of $\alpha = 1/2$, and we shall show that our conjecture (II) can be settled affirmatively in some cases. When $f(u)$ is specialized to the function u^σ, $0 < \sigma \leq 1$, one can obtain refinements of the results in Section 3. Section 4 is devoted to this subject. Other related results are found in [1–5, 7–10].

Finally, we introduce notation by means of which our results are formulated. Let $r_0 > 0$ be an arbitrarily fixed number through the paper such that $\{x : |x| \geq r_0\} \subset \Omega$. We denote by p_* and m, respectively, continuous functions satisfying

$$0 < p_*(r) \leq \min_{|x|=r} p(x), \qquad 0 < m(r) \leq \min_{r/2 \leq |x| \leq 3r/2} p(x)$$

for $r \geq 2r_0$. For example, if $p(x)$ behaves like a constant multiple of $|x|^\ell$, $\ell \in \mathbf{R}$, as $|x| \to \infty$, then we may take $p_*(r) = c_1 r^\ell$ and $m(r) = c_2 r^\ell$ for some c_1, $c_2 > 0$.

1. Preliminaries

We give some auxiliary observation which will frequently be used later.

1.1. It is to be noted that the operator Mu can be rewritten in the form

$$Mu = \sum_{i,j=1}^{N} a_{ij}(Du(x))D_{ij}u, \qquad x \in \Omega, \tag{1.1}$$

where $D_{ij} = D_i D_j$, the symmetric matrix $(a_{ij}(z))$, $z = (z_i)$, is given by

$$a_{ij}(z) = (1 + |z|^2)^{-\alpha}\left[\delta_{ij} - \frac{2\alpha z_i z_j}{1 + |z|^2}\right].$$

We therefore find that the matrix $(a_{ij}(z))$ has two distinct positive eigenvalues $(1 + |z|^2)^{-\alpha}$ and $(1 + |z|^2)^{-\alpha-1}[1 + (1 - 2\alpha)|z|^2]$, and hence M is uniformly elliptic on each compact subset in Ω with respect to any C^2-function u.

Let $k > 0$ be a constant, and f a positive function satisfying (A_3). The following simple comparison principle is often employed in the paper. The proof is similar to that of [11, Lemma 1].

LEMMA 1.1. *Let $B \subset \Omega$ be a bounded domain with smooth boundary ∂B. Suppose that there are positive functions $u \in C^2(\overline{B})$ and $v \in C^2(B)$ satisfying*

$$Mu \geq kf(u) \ \text{ on } \overline{B}, \qquad Mv \leq kf(v) \ \text{ in } B, \qquad \text{and} \qquad D_n v = \infty \ \text{ on } \partial B,$$

where n is the outward normal to ∂B. Then, $u \leq v$ in B.

1.2. Occasionally, we are required to find suitable positive radial solutions (related to some point in Ω, say x^0) $v = v(r)$, $r = |x - x^0|$, of equations of the form

$$Mv = g(r, v).$$

In such case, it is convenient to employ the polar form of the operator M; that is, we use the equivalent form

$$Mv \equiv r^{1-N}(r^{N-1}\psi(v'))' = g(r, v),$$

where $\psi \in C^1(\boldsymbol{R}; \boldsymbol{R})$ is the function given by

$$\psi(s) = \frac{s}{(1 + s^2)^\alpha}, \qquad s \in \boldsymbol{R}.$$

It is easily seen that ψ has the inverse function on $\boldsymbol{R}$ or on the interval $(-1, 1)$ according as $0 \leq \alpha < 1/2$ or $\alpha = 1/2$. In this paper we denote the inverse function of ψ by $\phi : \phi = \psi^{-1}$. When $\alpha = 1/2$, ϕ is explicitly given by

$$\phi(s) = \frac{s}{(1 - s^2)^{1/2}} \qquad \text{for } s \in (-1, 1).$$

When $0 < \alpha < 1/2$, we cannot give the explicit formula for ϕ.

1.3. Consider the initial value problem for the quasilinear ordinary differential equation

$$\begin{cases} Mv \equiv r^{1-N}(r^{N-1}\psi(v'))' = kf(v), & r > 0; \quad (1.2) \\ v(0) = \lambda, \quad v'(0) = 0, & \quad (1.3) \end{cases}$$

where $k > 0$, and $\lambda > 0$ is a parameter. It can be shown that this problem admits a unique positive solution on some neighborhood of zero. We denote this solution by v_λ, and the maximal interval of existence of v_λ by I_λ in the sequel. The following lemmas concerning to IVP (1.2)–(1.3) play important roles in proving our main results later.

LEMMA 1.2. *Let $\hat{r}_\lambda$ be the right end point of I_λ, $\lambda > 0$, and suppose that $\hat{r}_\lambda < \infty$.*

(i) *If $0 \leq \alpha < 1/2$, then $v_\lambda(\hat{r}_\lambda - 0) = v'_\lambda(\hat{r}_\lambda - 0) = \infty$.*

(ii) *If $\alpha = 1/2$, then $v_\lambda(\hat{r}_\lambda - 0) < \infty$, $v'_\lambda(\hat{r}_\lambda - 0) = \infty$.*

LEMMA 1.3. (i) *Let $\mu > \lambda > 0$. Then, $v_\mu(r) > v_\lambda(r)$ and $v'_\mu(r) \geq v'_\lambda(r)$ whenever they both exist.*

(ii) *On each compact subset of I_λ, uniformly $\lim_{\mu \to \lambda} v_\mu(r) = v_\lambda(r)$ and $\lim_{\mu \to \lambda} v'_\mu(r) = v'_\lambda(r)$.*

PROOF OF LEMMA 1.2. Since the proof of (i) is standard, we prove (ii) only here.

We denote v_λ by v, I_λ by I, and $\hat{r}_\lambda$ by $\hat{r}$ for simplicity. It is easily seen that $v'(r) \geq 0$, and hence an integration of (1.2) shows that

$$\begin{aligned}\psi(v'(r)) &= k\int_0^r \left(\frac{s}{r}\right)^{N-1} f(v(s))ds \\ &\leq kf(v(r))\int_0^r \left(\frac{s}{r}\right)^{N-1} ds = \frac{k}{N} rf(v(r)), \qquad r \in I.\end{aligned}$$

This yields

$$\frac{1}{r}\psi(v') \leq \frac{k}{N}f(v), \qquad r \in I. \tag{1.4}$$

Since (1.2) can be rewritten as

$$\psi'(v')v'' + \frac{N-1}{r}\psi(v') = kf(v), \qquad r \in I,$$

from (1.4) we have

$$\psi'(v')v'' \geq \frac{k}{N}f(v), \qquad r \in I, \tag{1.5}$$

namely,

$$\frac{v''}{(1+(v')^2)^{3/2}} \geq \frac{kf(v)}{N}, \qquad r \in I.$$

Note that this inequality is equivalent to

$$\frac{v'v''}{(1+(v')^2)^{3/2}} \geq \frac{k}{N}(F(v))', \qquad r \in I,$$

where $F(v) = \int_0^v f(s)ds$. An integration gives

$$1 - \frac{1}{(1+(v')^2)^{1/2}} \geq \frac{k}{N}(F(v) - F(\lambda)), \qquad r \in I.$$

Obviously, this inequality shows that $v(\hat{r} - 0) < \infty$. Since $v'' \geq 0$ by (1.5), we must have $v'(\hat{r} - 0) = \infty$ by the standard argument. The proof is complete.

The proof of (i) of Lemma 1.3 is easy; hence the proof is left to the reader. The assertion of (ii) of Lemma 1.3 seems very natural. However, we cannot find the proof in the literature, so we give it for completeness and for future reference.

PROOF OF (ii) OF LEMMA 1.3. We shall give the proof for the case $\alpha = 1/2$, since a parallel argument holds for the case $0 \le \alpha < 1/2$. Let $[0, R]$ be a compact interval included in I_λ.

First we show that $\lim_{\mu \uparrow \lambda} v_\mu^{(i)}(r) = v_\lambda^{(i)}(r)$, $i = 0, 1$. Since

$$k \int_0^r \left(\frac{s}{r}\right)^{N-1} f(v_\lambda(s))ds < 1, \qquad r \in [0, R],$$

there is a small $\delta > 0$ such that

$$k \int_0^r \left(\frac{s}{r}\right)^{N-1} f(v_\lambda(s))ds \le 1 - \delta, \qquad r \in [0, R].$$

By (i), for $\mu \in (0, \lambda]$, v_μ exists at least on $[0, R]$ and obviously satisfies

$$k \int_0^r \left(\frac{s}{r}\right)^{N-1} f(v_\mu(s))ds \le 1 - \delta, \qquad r \in [0, R].$$

Let $C = \max_{0 \le s \le 1-\delta} \phi'(s)$. From the above observation and the mean value theorem we have

$$0 \le v'_\lambda(r) - v'_\mu(r) \le kC \int_0^r \left(\frac{s}{r}\right)^{N-1} [f(v_\lambda(s)) - f(v_\mu(s))]ds,$$

$$0 \le v_\lambda(r) - v_\mu(r) \le \lambda - \mu + kC \int_0^r \int_0^s \left(\frac{t}{s}\right)^{N-1} [f(v_\lambda(t)) - f(v_\mu(t))]dtds$$

for $r \in [0, R]$. Since f is locally Lipschitz continuous, a standard argument leads to the desired conclusion.

Secondly we show that $\lim_{\mu \downarrow \lambda} v_\mu^{(i)}(r) = v_\lambda^{(i)}(r)$, $i = 0, 1$. The proof seems somewhat complicated. We can find small positive constants ε and $\delta < 1$ satisfying

$$k \int_0^r \left(\frac{s}{r}\right)^{N-1} f(v_\lambda(s) + 2\varepsilon R)ds \le 1 - \delta \qquad \text{for } r \in [0, R]. \tag{1.6}$$

As a first step, we prove that v_μ exists on $[0, R]$ if $\mu > \lambda$ is sufficiently close to λ. For this purpose, we assert that $v'_\mu(r) < v'_\lambda(r) + 2\varepsilon$ as long as v_μ exists if μ is sufficiently close to λ. Suppose to the contrary. Then by (i) of Lemma 1.3, we can find sequences $\{\mu_n\}$ and $\{R_n\}$ satisfying

$$\mu_1 > \mu_2 > \cdots > \mu_n > \cdots \ge \lambda, \qquad \lim_{n \to \infty} \mu_n = \lambda;$$

$$R_1 < R_2 < \cdots < R_n < \cdots < R;$$

and

$$v'_{\mu_n}(r) < v'_\lambda(r) + 2\varepsilon, \qquad 0 \le r < R_n; \qquad v'_{\mu_n}(R_n) = v'_\lambda(R_n) + 2\varepsilon$$

for $n \in \boldsymbol{N}$. Let us introduce the function

$$\hat{f}(r, v) = \begin{cases} f(v) & \text{for } r \in [0, R],\ 0 < v \le v_\lambda(r) + 2\varepsilon R; \\ f(v_\lambda(r) + 2\varepsilon R) & \text{for } r \in [0, R],\ v \ge v_\lambda(r) + 2\varepsilon R, \end{cases}$$

and consider the initial value problem

$$\begin{cases} Mz = k\hat{f}(r, z), & r > 0; \\ z(0) = \mu_n > 0, & z'(0) = 0. \end{cases}$$

By noting the obvious inequality $\hat{f}(r, v) \le f(v_\lambda(r) + 2\varepsilon R)$ holding on $[0, R] \times (0, \infty)$, we can show that this problem has a unique positive solution z_n on $[0, R]$. In fact, it is easily seen that the integral equation

$$z_n(r) = \mu_n + \int_0^r \phi\left(k \int_0^s \left(\frac{t}{s}\right)^{N-1} \hat{f}(t, z_n(t))dt\right) ds, \qquad 0 \le r \le R,$$

has a unique positive solution z_n satisfying

$$\mu_n \le z_n(r) \le \mu_n + R\phi(1 - \delta), \qquad 0 \le r \le R.$$

Moreover, we find by the monotonicity of f that the sequence $\{z_n\}$ is nonincreasing. Hence, the Ascoli-Arzelà theorem implies that $\{z_n\}$ has the uniform limit $\hat{z} \in C^2[0, R]$, which satisfies

$$\begin{cases} M\hat{z} = k\hat{f}(r, \hat{z}), & 0 < r \le R; \\ \hat{z}(0) = \lambda, & \hat{z}'(0) = 0. \end{cases}$$

We claim that $\hat{z}'(r) < v_\lambda'(r) + \varepsilon$, $0 \le r \le R$. Indeed, if this is not the case, we can find $\bar{r} \in (0, R]$ such that

$$\hat{z}'(r) < v_\lambda'(r) + \varepsilon, \qquad 0 \le r < \bar{r}; \qquad \hat{z}'(\bar{r}) = v_\lambda'(\bar{r}) + \varepsilon.$$

Accordingly, $\hat{z}(r) \le v_\lambda(r) + \varepsilon R$ on $[0, \bar{r}]$. This implies that $\hat{z} \equiv v_\lambda$ on $[0, \bar{r}]$ by the definition of $\hat{f}(r, v)$. However, this gives a contradictory equality $\hat{z}'(\bar{r}) = v_\lambda'(\bar{r})$. Therefore, we have $\hat{z}'(r) < v_\lambda'(r) + \varepsilon$ on $[0, R]$, as stated above. Consequently, it follows that $\hat{z}(r) \le v_\lambda(r) + \varepsilon R$ on $[0, R]$, and hence $\hat{z} \equiv v_\lambda$ on $[0, R]$ by the definition of $\hat{f}$.

On the other hand, since $\lim_{n \to \infty} z_n' = \hat{z}' \equiv v_\lambda'$ uniformly on $[0, R]$, we have $z_n'(r) \le v_\lambda'(r) + \varepsilon$ on $[0, R]$ if n is sufficiently large, say, $n \ge n_0$. Therefore, $z_n(r) \le \mu_n - \lambda + v_\lambda(r) + \varepsilon R \le v_\lambda(r) + 2\varepsilon R$ on $[0, R]$ if $n \ge n_1$ for sufficiently large $n_1 \ge n_0$. This implies that $z_n \equiv v_{\mu_n}$ on $[0, R_n]$ if $n \ge n_1$. However, this gives a contradiction at $r = R_n$, for $z_n'(R_n) \le v_\lambda'(R_n) + \varepsilon < v_\lambda'(R_n) + 2\varepsilon = v_{\mu_n}'(R_n) = z_n'(R_n)$.

From the preceding observation we conclude that, for some $\lambda_0 > \lambda$, v_μ

with $\lambda \leq \mu \leq \lambda_0$ exists on $[0, R]$ and satisfies

$$k \int_0^r \left(\frac{s}{r}\right)^{N-1} f(v_\mu(s))ds \leq 1 - \delta, \qquad 0 \leq r \leq R.$$

Here δ is the number appearing in (1.6). The rest of the proof proceeds as in the case that $\mu \uparrow \lambda$. The proof of (ii) of Lemma 1.3 is complete.

1.4. Finally, to show the sharpness of our nonexistence criteria in the forthcoming sections, we present here a simple existence theorem of positive solutions for (P) with $\Omega = \boldsymbol{R}^N$:

PROPOSITION 1.4. *Let f satisfy* (A_3), *and $q \in C([0, \infty); [0, \infty))$ be such that*

$$\limsup_{r \to \infty} \int^r \left(\frac{s}{r}\right)^{N-1} q(s)f(as)ds < \infty \qquad \text{for some } a > 0. \tag{1.7}$$

Then, the equation

$$Mu = cq(|x|)f(u), \qquad x \in \boldsymbol{R}^N, \tag{1.8}$$

has positive radial entire solutions for sufficiently small $c > 0$.

PROOF. Let $b \in (0, 1)$ be a constant satisfying $\phi(b) < a$. Then by (1.7), we can find a sufficiently small $c > 0$ so that

$$c \int_0^r \left(\frac{s}{r}\right)^{N-1} q(s)f(1 + \phi(b)s)ds < b \qquad \text{for } r \geq 0.$$

We shall show the existence of a global solution v of the initial value problem

$$\begin{cases} Mv \equiv r^{1-N}(r^{N-1}\psi(v'))' = cq(r)f(v), & r > 0; \\ v(0) = 1, \quad v'(0) = 0. \end{cases}$$

The unique solution of this problem surely exists near zero. We prove that this solution can be continued to ∞. For this purpose it suffices to prove that $\psi(v'(r)) < b$ (< 1) as long as it exists. If this is not the case, then there is an $R > 0$ satisfying

$$\psi(v'(r)) < b, \qquad 0 \leq r < R; \qquad \psi(v'(R)) = b.$$

This implies moreover that $v(r) \leq \phi(b)r + 1$, $0 \leq r \leq R$. However, this also gives a contradictory inequality:

$$\begin{aligned} b = \psi(v'(R)) &= c \int_0^R \left(\frac{s}{R}\right)^{N-1} q(s)f(v(s))ds \\ &\leq c \int_0^R \left(\frac{s}{R}\right)^{N-1} q(s)f(\phi(b)s + 1)ds < b. \end{aligned}$$

The proof is complete.

When $f(u) = u^\sigma$, $\sigma > 0$, and $q(r)$ is asymptotic to a constant multiple of the function $r^{-\ell}$, $\ell > 0$, as $r \to \infty$, (1.7) is equivalent to the condition $\ell \geq 1 + \sigma$. Other existence theorems for (1.8) with different feature are found in [6].

2. The case of $0 \leq \alpha < 1/2$: generalized mean curvature operators

We begin with the case of $0 \leq \alpha < 1/2$. In addition to (A_1)–(A_3), we always assume the following throughout the section:

(B) The function $F(u) \equiv \int_0^u f(s)ds$, $u > 0$, satisfies

$$\int_1^\infty [F(u)]^{-1/(2-2\alpha)}du < \infty.$$

An important example of f satisfying (B) (and (A_3)) is $f(u) = u^\sigma$, $\sigma > 1 - 2\alpha$. If the first condition in (0.1) holds, then (B) also holds; however, the converse does not necessarily. Moreover, unlike the earlier paper [11], no restrictions are imposed on the behavior of F near origin. Thus less assumptions are imposed on f here than in [11].

Condition (B) enable us to introduce the function $G: (0, \infty) \to (0, \infty)$ given by

$$G(u) = \int_u^\infty [F(z) - F(u)]^{-1/(2-2\alpha)}dz, \qquad u > 0.$$

Noting that G can be rewritten as

$$G(u) = \int_0^\infty \left(\int_0^z f(u + s)ds\right)^{-1/(2-2\alpha)} dz, \qquad u > 0,$$

we find that G is strictly decreasing on $(0, \infty)$ and $G(+\infty) = 0$. (A similar function was employed in [10]; see [10, Lemma 2.1].) Hence G has the inverse function G^{-1} on $(0, G(+0))$. In what follows we put $G^{-1} = H$ for simplicity. When $f(u) = u^\sigma$, $\sigma > 1 - 2\alpha$, for example, we have $H(u) = Cu^{(2-2\alpha)/(1-2\alpha-\sigma)}$, $C = \text{const} > 0$.

The aim of the section is to answer question (I) in the Introduction. Our nonexistence results for (P) are as follows:

THEOREM 2.1. *Problem* (P) *has no positive solutions if*

$$\liminf_{r\to\infty} H(C(N, \alpha)r[m(r)]^{1/(2-2\alpha)})\left[\int^r \int^s \left(\frac{t}{s}\right)^{N-1} p_*(t)dtds\right]^{-1} = 0, \tag{2.1}$$

where

$$C(N, \alpha) = \frac{1}{2}\left(\frac{2 - 2\alpha}{N}\right)^{1/(2-2\alpha)}.$$

A simple corollary to Theorem 2.1 follows.

COROLLARY 2.2. *Problem* (P) *has no positive solutions if*

$$\liminf_{|x|\to\infty} |x|^{2-2\alpha} p(x) > 0.$$

This corollary gives a generalization of [10, Corollary 3.1], in which only the case of $\alpha = 0$ was treated.

REMARK 2.3. When it happens that

$$C(N, \alpha) r [m(r)]^{1/(2-2\alpha)} \notin \operatorname{dom} H \equiv (0, G(+0)) \qquad \text{for some } r \geq 2r_0,$$

r_0 being the number introduced in the Introduction, it is understood that problem (P) has no positive solutions. This fact follows from a close look at the proofs of Theorem 2.1 and Proposition 2.5 below. Such a situation may occur if

$$G(+0) \equiv \int_0^\infty [F(z)]^{-1/(2-2\alpha)} dz < \infty.$$

EXAMPLE 2.4. Let $0 < \alpha < 1/2$ and $\sigma > 1 - 2\alpha$. Consider problem (P_σ). Theorem 2.1 states that, if

$$\liminf_{|x|\to\infty} |x|^{2-2\alpha+\beta} p(x) > 0 \qquad \text{for some } \beta \in \left(0, (\sigma + 2\alpha - 1)\left(1 - \frac{\sigma}{\sigma + 2\alpha}\right)\right),$$

then, (P_σ) does not possess positive solutions. On the other hand, if $\Omega = \boldsymbol{R}^N$, p has radial symmetry, and

$$\limsup_{|x|\to\infty} |x|^{2-2\alpha+\beta} p(x) < \infty \qquad \text{for some } \beta \geq \sigma + 2\alpha - 1,$$

then, the equation

$$Mu = cp(x)u^\sigma, \qquad x \in \boldsymbol{R}^N,$$

has positive radial entire solutions for sufficiently small $c > 0$. This is a direct consequence of Proposition 1.4. Noting that the difference of two quantities $(\sigma + 2\alpha - 1)\left(1 - \dfrac{\sigma}{\sigma + 2\alpha}\right)$ and $\sigma + 2\alpha - 1$ can be made as close to zero as desired by choosing suitable σ and α (with keeping the relations $0 < \alpha < 1/2$ and $\sigma > 1 - 2\alpha$), we can conclude that Theorem 2.1 is best possible in some sense.

To prove Theorem 2.1 we need the following, whose proof is given at the end of this section. It is due to this proposition that we can improve the results in [11].

Proposition 2.5. *Let $x^0 \in \boldsymbol{R}^N$, k, $R > 0$ be constants, and u be a positive C^2-function satisfying*

$$Mu \geq kf(u) \qquad \text{in } |x - x^0| \leq R.$$

Then, we have

$$u(x^0) \leq H\left(\left[\frac{k(2-2\alpha)}{N}\right]^{1/(2-2\alpha)} R\right). \tag{2.2}$$

Proof of Theorem 2.1. This theorem is proved by the same method as was used in [11, Proofs of Theorems 1 and 2].

Suppose to the contrary that (P) admits a positive solution u. Let $x^0 \in \Omega$ be fixed so that $|x^0| = r$ is sufficiently large. Since u satisfies

$$Mu \geq m(r)f(u), \qquad |x - x^0| \leq r/2,$$

Proposition 2.5 implies that

$$u(x^0) \leq H\left(\frac{r}{2}\left[\frac{(2-2\alpha)m(r)}{N}\right]^{1/(2-2\alpha)}\right).$$

Hence we have

$$u(x) \leq H(C(N, \alpha)|x|[m(|x|)]^{1/(2-2\alpha)}) \qquad \text{for large } |x|. \tag{2.3}$$

Choose a sufficiently small $\delta > 0$ so that $f(\max_{|x|=r_0} u) \geq \delta > 0$. Define $w(r)$, $r \geq r_0$, by

$$w(r) = \frac{1}{\delta}\int_{r_0}^{r} \phi\left(\frac{\delta}{2}\int_{r_0}^{s}\left(\frac{t}{s}\right)^{N-1} p_*(t)dt\right)ds, \qquad r \geq r_0, \tag{2.4}$$

where $\phi = \psi^{-1}$. We find that

$$w(r_0) = w'(r_0) = 0; \tag{2.5}$$

$$w(r) > 0, \qquad w'(r) > 0 \qquad \text{for } t > r_0; \tag{2.6}$$

$$M(\delta w)(|x|) = \frac{\delta}{2}p_*(|x|) < \delta p(x), \qquad |x| \geq r_0, \tag{2.7}$$

and

$$w(r) \geq C\int_{r_0}^{r}\int_{r_0}^{s}\left(\frac{t}{s}\right)^{N-1} p_*(t)dtds, \qquad r \geq r_0 \tag{2.8}$$

for some constant $C = C(\delta) > 0$.

From (2.3), (2.8) and assumption (2.1) we have $\liminf_{|x|\to\infty}(u - \delta w) < 0$. This means that $u - \delta w < 0$ on $|x| = r_1$ for some sufficienty large $r_1 > r_0$. Let

$\hat{x}$ be a point such that

$$(u - \delta w)(\hat{x}) = \max_{r_0 \le |x| \le r_1} (u - \delta w).$$

Since $u - \delta w > 0$ on $|x| = r_0$, and $u - \delta w < 0$ on $|x| = r_1$, $\hat{x}$ satisfies $r_0 \le |\hat{x}| < r_1$. We claim more precisely that $r_0 < |\hat{x}| < r_1$. To see this suppose the contrary that $|\hat{x}| = r_0$ for a moment. Since w is radial, obviously $u(\hat{x}) = \max_{|x|=r_0} u$. Noting that $u \not\equiv$ const by the positivity of p and the boundary condition $D_\nu u \le 0$ on Ω, we find that $u(\hat{x}) = \max \{u(x): x \in \Omega$ and $|x| \le r_0\}$, and hence $D_n u(\hat{x}) > 0$, where n denotes the outward normal on $|x| = r_0$. Accordingly, $D_n(u - \delta w)(\hat{x}) > 0$ because $w'(r_0) = 0$. This contradicts the fact that $u - \delta w$ attains the maximum on the set $r_0 \le |x| \le r_1$ at $\hat{x}$. Therefore $|\hat{x}| > r_0$, as stated above. Note that we have automatically $u(\hat{x}) \ge \max_{|x|=r_0} u$.

Consider the linear elliptic operator with constant coefficients L defined by

$$L \equiv \sum_{i,j} a_{ij}(Du(\hat{x}))D_{ij} = \sum_{i,j} a_{ij}(\delta Dw(|\hat{x}|))D_{ij},$$

where a_{ij}'s are introduced in (1.1). Obviously we have $L(u - \delta w)(\hat{x}) \le 0$. From (P) and (2.7), however, we have

$$L(u - \delta w)(\hat{x}) \ge Mu(\hat{x}) - M(\delta w(|\hat{x}|)) \ge p(\hat{x})[f(\max_{|x|=r_0} u) - \delta] > 0,$$

which is a contradiction. The proof is completed.

The remaining work for us is to establish Proposition 2.5. The rest of the section is devoted to this purpose. We need the following.

LEMMA 2.6. *Let $k > 0$, $0 < R_1 < R_2$, and $v_0 > 0$ be given constants. Then, we can find a number $c = c(k, R_1, R_2, v_0) > 0$ having the property that the solutions v of the problem*

$$Mv = kf(v), \qquad 0 < r \le R_1,$$

$$v'(0) = 0, \qquad v(R_1) \ge v_0, \qquad v'(R_1) \ge c,$$

blow up before R_2.

PROOF. Let $c(k, R_1, R_2, v_0) > 0$ be a sufficiently large number such that

$$R_2 - R_1 > \int_{v_0}^{\infty} \left[\frac{k(2-2\alpha)}{N}(F(z) - F(v_0)) + c^{2-2\alpha}\right]^{-1/(2-2\alpha)} dz. \tag{2.9}$$

The existence of such a c is guaranteed by the fact that the right hand side of the above tends to zero as $c \to \infty$ by our assumptions and the monotone convergence theorem. From (i) of Lemma 1.3, it is sufficient to show that v satisfying

$$Mv = kf(v), \qquad 0 < r \le R_1,$$
$$v'(0) = 0, \qquad v(R_1) = v_0, \qquad v'(R_1) \ge c,$$

blows up before R_2.

To this end, suppose to the contrary that v can be prolonged to the interval $[0, R_2)$. Then, $v'(r) \ge 0$, and as in the proof of (ii) of Lemma 1.2, we have

$$\psi'(v'(r))v''(r) \ge \frac{k}{N} f(v(r)), \qquad 0 < r < R_2.$$

Using the simple inequality $\psi'(s) \le s^{-2\alpha}$ for $s \ge 0$, we have

$$v''(r) \ge \frac{k}{N}[v'(r)]^{2\alpha} f(v(r)), \qquad 0 < r < R_2,$$

which is equivalent to

$$\frac{1}{2-2\alpha}([v'(r)]^{2-2\alpha})' \ge \frac{k}{N}(F(v(r)))', \qquad 0 < r < R_2.$$

An integration from R_1 to $r < R_2$ shows

$$v'(r) \ge \left[\frac{k(2-2\alpha)}{N}(F(v(r)) - F(v_0)) + c^{2-2\alpha}\right]^{1/(2-2\alpha)}, \qquad R_1 \le r < R_2.$$

One more integration of this inequality gives for $r \in [R_1, R_2)$

$$\int_{v_0}^{v(r)} \left[\frac{k(2-2\alpha)}{N}(F(z) - F(v_0)) + c^{2-2\alpha}\right]^{-1/(2-2\alpha)} dz \ge r - R_1.$$

Letting $r \to R_2 - 0$, we have a contradiction to (2.9). This contradiction completes the proof.

PROOF OF PROPOSITION 2.5. To see this proposition as a first step we establish the existence of auxiliary function $v \in C^2[0, R)$ such that

$$\begin{gathered} Mv = kf(v), \qquad 0 < r < R, \qquad v'(0) = 0, \\ v(R-0) = v'(R-0) = \infty. \end{gathered} \tag{2.10}$$

We show that for suitable λ, v_λ gives the desired function v by means of a simple shooting method. Here, of course, v_λ denotes the unique solution of IVP (1.2)–(1.3) introduced in the Introduction. The proof is decomposed into three steps.

Step 1. Let us introduce the set S by

$$S = \{\lambda > 0 : v_\lambda \text{ blows up at some } R_\lambda < R\}.$$

We show that a neighborhood of infinity is included in S. Let λ be so large that

$$\int_\lambda^\infty [F(z) - F(\lambda)]^{-1/(2-2\alpha)} dz < \left[\frac{k(2-2\alpha)}{N}\right]^{1/(2-2\alpha)} R. \tag{2.11}$$

Then, v_λ with such a λ, blows up before R. In fact, suppose to the contrary that v_λ exists on $[0, R)$. Then, as in the proof of Lemma 2.5, we reach the inequality

$$\int_\lambda^{v_\lambda(r)} [F(z) - F(\lambda)]^{-1/(2-2\alpha)} dz \geq \left[\frac{k(2-2\alpha)}{N}\right]^{1/(2-2\alpha)} r, \qquad 0 \leq r < R.$$

Letting $r \to R - 0$, we have a contradiction to (2.11). Therefore, v_λ blows up before R if λ is sufficiently large, as stated above.

Next we prove that, conversely, if λ is sufficiently small, then v_λ exists at least on $[0, R]$, that is, some neighborhood of zero is not included in S. To prove this suppose the contrary. Then, we find from (i) of Lemma 1.3 that, for all $\lambda > 0$, v_λ blows up at some $R_\lambda \leq R$. Since $v'_\lambda(R_\lambda - 0) = \infty$, Lemma 1.1 shows that $u(x) \leq v_\lambda(|x - x^0|)$ in $|x - x^0| < R_\lambda$, from which we obtain $u(x^0) \leq v_\lambda(0) = \lambda$. However, it is impossible since $\lambda > 0$ is arbitrary.

From the above observation and (i) of Lemma 1.3, we find that $\inf S \equiv \lambda_*$ is finite and positive.

Step 2. As for v_{λ_*} there are three possibilities:

(a) v_{λ_*} exists at least on $[0, R]$;

(b) v_{λ_*} blows up just at R;

(c) v_{λ_*} blows up before R.

We will show that actually case (b) occurs.

Suppose that case (a) occurs. By (ii) of Lemma 1.3 if $\mu > \lambda_*$ is sufficiently close to λ_*, v_μ also exists on $[0, R]$. This contradicts the definition of λ_*.

Suppose next that case (c) occurs. Let $\hat{R} \in [0, R)$ be the point at which v_{λ_*} blows up, and $c(k, \hat{R}, R, 1) > 0$ be the number introduced in the statement of Lemma 2.6. Since $v_{\lambda_*}(\hat{R} - 0) = v'_{\lambda_*}(\hat{R} - 0) = \infty$, one can find a sufficiently small $\rho > 0$ satisfying $v_{\lambda_*}(\hat{R} - \rho) > 1$ and $v'_{\lambda_*}(\hat{R} - \rho) > c(k, \hat{R}, R, 1)$. By (ii) of Lemma 1.3, there is a $\mu \in (0, \lambda_*)$ satisfying $v_\mu(\hat{R} - \rho) > 1$ and $v'_\mu(\hat{R} - \rho) > c(k, \hat{R}, R, 1)$. Then, v_μ blows up before R by the definition of $c(k, \hat{R}, R, 1)$. This contradicts the definition of λ_*, again.

Hence, v_{λ_*} blows up at R, and so, v_{λ_*} gives a desired function v.

Step 3. We are now ready for proving the Proposition. Since (2.10) is equivalent to

$$Mv = kf(v) \quad \text{in } |x - x^0| < R; \qquad \text{and} \qquad D_n v = \infty \quad \text{on } |x - x^0| = R,$$

where $r = |x - x^0|$ and n denote the unit outward normal to $|x - x^0| = R$, Lemma 1.1 implies that $u \le v$ in $|x - x^0| < R$. This gives especially $u(x^0) \le v(0)$. On the other hand, arguing as in Step 1, we can easily see that

$$v(0) \le H\left(\left[\frac{k(2-2\alpha)}{N}\right]^{1/(2-2\alpha)} R\right).$$

Hence, the validity of (2.2) is established. The proof of Proposition 2.5 is completed.

3. The case of $\alpha = 1/2$: the mean curvature operator

In this section we treat the case of $\alpha = 1/2$. Therefore the operator M is the mean curvature operator. Note that no additional conditions are needed here other than (A_1)–(A_3). Answers to question (II) are given. Our nonexistence results are as follows:

THEOREM 3.1. *Suppose that p_* is of class C^1 near infinity, and $(rp_*(r))' \le 0$ for all large r. Then, problem* (P) *has no positive solutions provided*

$$\liminf_{r\to\infty} f^{-1}\left(\frac{2N}{rm(r)}\right)\left(\int^r sp_*(s)ds\right)^{-1} = 0, \tag{3.1}$$

where f^{-1} denotes the inverse function of f.

THEOREM 3.2. *Problem* (P) *has no positive solutions provided*

$$\limsup_{r\to\infty} \int^r \left(\frac{s}{r}\right)^{N-1} p_*(s)ds < \infty \tag{3.2}$$

and

$$\liminf_{r\to\infty} f^{-1}\left(\frac{2N}{rm(r)}\right)\left[\int^r \int^s \left(\frac{t}{s}\right)^{N-1} p_*(t)dtds\right]^{-1} = 0.$$

COROLLARY 3.3. *Problem* (P) *has no positive solutions provided*

$$\liminf_{|x|\to\infty} |x|\, p(x) > 0.$$

REMARK 3.4. As in Remark 2.3, when it happens that

$$\frac{2N}{rm(r)} \notin \operatorname{dom} f^{-1} = (f(+0), \infty) \qquad \text{for some } r \ge 2r_0,$$

it is understood that problem (P) has no positive solutions. Such a situation may occur if $f(+0) > 0$.

Example 3.5. Let $\bar{\Omega} \subset \{x : |x| > e\}$, and consider the problem

$$\begin{cases} Mu \geq \dfrac{c \log(1+u)}{|x| \log\log|x|}, & x \in \Omega, \\ D_\nu u \leq 0, & x \in \partial\Omega, \end{cases}$$

where $c > 0$ is a constant. We may choose

$$p_*(r) = \frac{c}{r \log\log r}, \quad \text{and} \quad m(r) = \frac{2c}{3r \log\log(3r/2)}$$

for $r \geq r_0$ sufficiently large. Applying Theorem 3.1, we find that this problem has no positive solutions.

Example 3.6. Let $\Omega = \boldsymbol{R}^N$ and $\varepsilon > 0$. Then, Proposition 1.4 asserts that the equation

$$Mu = \frac{cu^\varepsilon}{(1+|x|)^{1+\varepsilon}}, \quad x \in \boldsymbol{R}^N,$$

has positive radial entire solutions for sufficiently small constant $c > 0$. Hence under our standing assumptions (A_1)–(A_3), the decaying order of p indicated in Corollary 3.3 seems to be best possible in some sense.

As in the previous section, the following is crucial to prove our results. The proof of this proposition will be given later.

Proposition 3.7. *Let $x^0 \in \boldsymbol{R}^N$, k, $R > 0$ be constants, and u be a positive C^2-function satisfying*

$$Mu \geq kf(u) \quad \text{in } |x - x^0| \leq R.$$

Then, we have

$$u(x^0) \leq f^{-1}\left(\frac{N}{kR}\right). \tag{3.3}$$

The proof of Theorems 3.1 and 3.2 relies on the argument already used for Theorem 2.1. It is in turn based on the fact that we can construct a useful comparison function like w defined by (2.4), that is, a C^2-function w satisfying (2.5)–(2.7). Now, we employ different comparison functions as seen below.

Proof of Theorem 3.1. Suppose to the contrary that (P) has a positive solution u. We find by Proposition 3.7 that

$$u(x) \leq f^{-1}\left(\frac{2N}{|x|m(|x|)}\right) \qquad \text{for large } |x|. \tag{3.4}$$

Put

$$w(r) = \frac{1}{2(N-1)} \int_{r_0}^{r} s p_*(s) ds, \qquad r \geq r_0.$$

That w satisfies (2.5)–(2.7) for any $\delta > 0$ was proved in [11]. Since (3.1) and (3.4) imply that $\liminf_{|x| \to \infty} (u - \delta w) < 0$ for any $\delta > 0$, the function $u - \delta w$ takes a local maximum at some $\hat{x}$, $|\hat{x}| > r_0$ for some $\delta > 0$. Arguing as before, we can easily get a contradiction. We refer the reader to [11] for the detailed argument. The proof is complete.

Proof of Theorem 3.2. Suppose that (P) admits a positive solution u. Of course, the inequality (3.4) is still valid. By (3.2) we can find a small $\delta > 0$ having the properties

$$\delta \sup_{r \geq r_0} \int_{r_0}^{r} \left(\frac{s}{r}\right)^{N-1} p_*(s) ds < 1 \qquad \text{and} \qquad f(\max_{|x|=r_0} u) \geq \delta > 0.$$

Hence the function

$$w(r) = \frac{1}{\delta} \int_{r_0}^{r} \phi\left(\frac{\delta}{2} \int_{r_0}^{s} \left(\frac{t}{s}\right)^{N-1} p_*(t) dt\right) ds, \qquad r \geq r_0, \tag{2.4}$$

is well-defined, and satisfies (2.5)–(2.8). Exactly as in the proof of Theorem 2.1, we can easily get a contradiction. The proof is complete.

Our final task in this section is to prove Proposition 3.7. For this purpose we need the following.

Lemma 3.8. *Let $k > 0$ and let v_λ and $\hat{r}_\lambda$ be as in Lemma 1.2. Then, we have*

$$v_\lambda(\hat{r}_\lambda) > f^{-1}\left(\frac{N}{k\hat{r}_\lambda}\right).$$

Proof. The proof is by contradiction. For simplicity we write v for v_λ and $\hat{r}$ for $\hat{r}_\lambda$. Suppose to the contrary that $v(\hat{r}) \leq f^{-1}\left(\frac{N}{k\hat{r}}\right)$, namely, $f(v(\hat{r})) \leq \frac{N}{k\hat{r}}$. We then have $f(v(r)) \leq \frac{N}{k\hat{r}}$, $0 \leq r \leq \hat{r}$, by the monotonicity of v. On the other hand, v satisfies

$$\psi(v'(r)) = k\int_0^r \left(\frac{s}{r}\right)^{N-1} f(v(s))ds, \qquad 0 \le r < \hat{r}.$$

Letting $r \to \hat{r} - 0$ in this equation, we obtain a contradiction because

$$1 = k\int_0^{\hat{r}} \left(\frac{s}{\hat{r}}\right)^{N-1} f(v(s))ds < k\cdot\frac{N}{k\hat{r}}\int_0^{\hat{r}} \left(\frac{s}{\hat{r}}\right)^{N-1} ds = 1.$$

Proof of Proposition 3.7. The proof is analogous to that of Proposition 2.5. Accordingly, we must construct a comparison function $v \in C^2[0, R)$ such that

$$\begin{aligned} &Mv = kf(v), \qquad 0 < r < R, \qquad v'(0) = 0; \\ &v(R-0) < \infty, \qquad v'(R-0) = \infty. \end{aligned} \tag{3.5}$$

We again employ the family of solutions $\{v_\lambda\}$ of VIP (1.2)–(1.3).

Step 1. Define the subset $S \subset (0, \infty)$ by

$$S = \{\lambda > 0 : v_\lambda \text{ blows up at some } R_\lambda < R\}.$$

We claim that λ's satisfying

$$f(\lambda) > \frac{N}{kR}$$

are members of S. (Since $f(\infty) = \infty$, such a λ surely exists.) To see this, suppose that $\lambda \notin S$. Then, v_λ exists on $[0, R)$, and satisfies

$$\psi(v_\lambda'(r)) = k\int_0^r \left(\frac{s}{r}\right)^{N-1} f(v_\lambda(s))ds, \qquad 0 \le r < R.$$

Letting $r \to R$, we reach a contradiction because

$$1 \ge \psi(v_\lambda'(R)) = k\int_0^R \left(\frac{s}{R}\right)^{N-1} f(v_\lambda(s))ds \ge kR^{1-N}f(\lambda)\int_0^R s^{N-1}ds > 1.$$

This contradiction shows that $\lambda \in S$. On the other hand, exactly as in the proof of Proposition 2.5, we know that sufficiently small λ's do not belong to S. From these considerations and (i) of Lemma 1.3, we find that $\inf S = \lambda_*$ is finite and positive.

Step 2. Let us consider the solution v_{λ_*}. That v_{λ_*} does blow up at some $\hat{R} \in (0, R]$ is easily proved by (ii) of Lemma 1.3. We show that actually $\hat{R} = R$. To this end suppose to the contrary that $\hat{R} < R$. We first note that

$$k\int_0^{\hat{R}} \left(\frac{s}{\hat{R}}\right)^{N-1} f(v_{\lambda_*}(s))ds = 1 \qquad \text{and} \qquad f(v_{\lambda_*}(\hat{R})) > \frac{N}{k\hat{R}}.$$

Recall that the later inequality is given by Lemma 3.8. We can find a sufficiently small number $c \in (0, 1)$ such that

$$\left(\frac{\hat{R}}{R}\right)^{N-1}(1-c) + \frac{R}{\hat{R}} - \left(\frac{\hat{R}}{R}\right)^{N-1} > 1,$$

because the left hand side of this inequality tends to $R/\hat{R} > 1$ as $c \to 0$. With this choice of c, we next choose $\mu = \mu_c < \lambda_*$ sufficiently close to λ_* so that

$$k\int_0^{\hat{R}} \left(\frac{s}{\hat{R}}\right)^{N-1} f(v_\mu(s))ds > 1 - c \qquad \text{and} \qquad f(v_\mu(\hat{R})) > \frac{N}{k\hat{R}}.$$

This is possible because of (ii) of Lemma 1.3 and the fact that v_μ exists at least on $[0, \hat{R}]$. Then, we can show that v_μ never exists on $[0, R)$. In fact, supposing to the contrary that v_μ exists on $[0, R)$, we have

$$\begin{aligned} 1 \geq \psi(v'(R-0)) &= k\int_0^R \left(\frac{s}{R}\right)^{N-1} f(v_\mu(s))ds \\ &= k\left(\frac{\hat{R}}{R}\right)^{N-1}\int_0^{\hat{R}}\left(\frac{s}{\hat{R}}\right)^{N-1} f(v_\mu(s))ds + k\int_{\hat{R}}^R \left(\frac{s}{R}\right)^{N-1} f(v_\mu(s))ds \\ &> \left(\frac{\hat{R}}{R}\right)^{N-1}(1-c) + k\cdot\frac{N}{k\hat{R}}\int_{\hat{R}}^R\left(\frac{s}{R}\right)^{N-1} ds > 1. \end{aligned}$$

This contradiction implies that v_μ blows up before R. However, this fact also contradicts the definition of $\lambda_* = \inf S$, and hence $\hat{R} = R$.

Step 3. By Step 2, we can construct a function v satisfying (3.5). As in the proof of Proposition 2.5, we have $u(x^0) \leq v(0)$. On the other hand, from

$$1 = \psi(v'(R-0)) = k\int_0^R \left(\frac{s}{R}\right)^{N-1} f(v(s))ds \geq kf(v(0))\int_0^R\left(\frac{s}{R}\right)^{N-1} ds$$

we find that $1 \geq kRf(v(0))/N$. Combining these inequalities, we can obtain (3.3). The proof of Proposition 3.7 is complete.

REMARK 3.9. (Cf. Corollary 3.3 and Remark 3.4) When $f(+0) > 0$, a simple observation yields some nonexistence criteria related to Corollary 3.3 and Remark 3.4.

(i) Consider the special case that ν is parallel to the outward normal n on $\partial\Omega$. Then (P) has no positive solutions provided that

$$r^{1-N}\int_{\Omega_r} p(x)dx > \frac{\omega_N}{f(+0)} \qquad \text{for some } r > r_0, \tag{3.6}$$

where $\Omega_r = \Omega \cap \{x : |x| \leq r\}$, and $\omega_N = \int_{|x|=1} dS$.

To prove this fact let u be a positive solution of (P). Integrating the inequality in (P) on Ω_r, and employing the divergence theorem, we have

$$\frac{1}{r}\int_{|x|=r}\frac{Du\cdot x}{\sqrt{1+|Du|^2}}dS+\int_{\partial\Omega}\frac{D_n u}{\sqrt{1+|Du|^2}}dS\geq f(+0)\int_{\Omega_r}p(x)dx.$$

Since $D_n u\leq 0$ on $\partial\Omega$ and the first integral in the left hand side is bounded above by $\int_{|x|=r}dS=\omega_N r^{N-1}$, we reach a contradiction to (3.6).

(ii) Generally, similar results hold. For example, if

$$\limsup_{r\to\infty} r^{1-N}\int_{\Omega_r}p(x)dx>\frac{\omega_N}{f(+0)},$$

then (P) has no positive solutions. The proof is as in (i), and hence is left to the reader.

These criteria do not apply in the case $f(+0)=0$, which we are mainly interested in.

4. The mean curvature operator with the nonlinearity $f(u)=u^\sigma$

The last section is devoted to the case that $\alpha=1/2$ and $f(u)=u^\sigma$, $\sigma>0$ in (P):

$$\begin{cases} Mu\geq p(x)u^\sigma, & x\in\Omega,\\ D_\nu u\leq 0, & x\in\partial\Omega. \end{cases} \tag{P_σ}$$

Applying Theorem 3.1 (or 3.2), we find that, if

$$\liminf_{|x|\to\infty}|x|^{(1+2\sigma)/(1+\sigma)-\varepsilon}p(x)>0 \qquad \text{for some } \varepsilon>0,$$

then, (P_σ) has no positive solutions. However, as seen from Proposition 1.4 this criterion is not effective. We shall show that the nonexistence criteria of positive solutions for (P) can be considerably improved if (P) is specialized to (P_σ), $0<\sigma\leq 1$. Note that the case of $\sigma>1$ was discussed in [11].

Theorem 4.1. *Let $0<\sigma\leq 1$. Suppose that p_* is of class C^1 near infinity, and $(rp_*(r))'\leq 0$ for large r. Then, problem (P_σ) has no positive solutions if*

$$\limsup_{r\to\infty} r^{1+\sigma-\varepsilon}m(r)\left(\int^r sp_*(s)ds\right)^\varepsilon=\infty \qquad \textit{for some } \varepsilon\in(0,\sigma).$$

Theorem 4.2. *Let $0<\sigma\leq 1$. Then, problem (P_σ) has no positive solutions provided*

$$\limsup_{r\to\infty}\int^r\left(\frac{s}{r}\right)^{N-1}p_*(s)ds<\infty$$

and

$$\limsup_{r\to\infty} r^{1+\sigma-\varepsilon}m(r)\left[\int^r\int^s\left(\frac{t}{s}\right)^{N-1}p_*(t)dtds\right]^\varepsilon = \infty \qquad \text{for some } \varepsilon\in(0,\sigma).$$

COROLLARY 4.3. *Let* $0<\sigma\le 1$. *Then, problem* (P_σ) *has no positive solutions if*

$$\liminf_{|x|\to\infty}|x|^{1+\sigma-\varepsilon}p(x)>0 \qquad \text{for some } \varepsilon>0.$$

EXAMPLE 4.4. Let $\sigma>0$ be *arbitrary* and consider the problem

$$Mu\ge\frac{cu^\sigma}{(1+|x|)^\ell} \qquad x\in\boldsymbol{R}^N, \tag{4.1}$$

where $c>0$ is a parameter, and $\ell>0$ is a constant. We can completely characterize the existence of positive entire solutions of this inequality by virtue of Proposition 1.4, Corollary 4.3 and some results in [6, 11]:

(i) Let $\sigma>1$ and $\ell>2$. Then, (4.1) has positive radial entire solutions.

(ii) Let $\sigma>1$ and $\ell\le 2$. Then, (4.1) has no positive entire solutions for all $c>0$.

(iii) Let $0<\sigma\le 1$ and $\ell\ge 1+\sigma$. Then, (4.1) has positive radial entire solutions for sufficiently small $c>0$.

(iv) Let $0<\sigma\le 1$ and $\ell<1+\sigma$. Then, (4.1) has no positive entire solutions.

The key point of the proof of Theorems 4.1 and 4.2 is to obtain more precise estimates for the function $v(r)$ satisfying

$$\begin{gathered} Mv=kv^\sigma, \qquad 0<r<R, \qquad v'(0)=0;\\ v(R-0)<\infty, \qquad v'(R-0)=\infty, \end{gathered}\tag{4.2}$$

whose existence has been established in the proof of Proposition 3.7. Such estimates are described in Proposition 4.5 below. We omit the proof of Theorems 4.1 and 4.2, since it can be carried out as before.

PROPOSITION 4.5. *Let* $0<\sigma\le 1$. *Let moreover* $x^0\in\boldsymbol{R}^N$, k, $R>0$ *be constants, and* u *be a positive* C^2*-function satisfying*

$$Mu\ge ku^\sigma \qquad \text{in } |x-x^0|\le R.$$

Then, for each $\varepsilon\in(0,\sigma)$ *we have*

$$u(x^0)\le\frac{C(N,\sigma,\varepsilon)}{(kR^{1+\sigma-\varepsilon})^{1/\varepsilon}}, \tag{4.3}$$

where $C(N,\sigma,\varepsilon)$ *is a constant not depending on* k *nor* R.

PROOF. Let v be a C^2-function satisfying (4.2). It suffices to show the validity of (4.3) with $u(x^0)$ replaced by $v(0)$. We shall elaborate the proof of Proposition 3.7.

Arguing as in the proof of Lemma 1.2-(ii), we have

$$\frac{v''(r)}{(1+[v'(r)]^2)^{3/2}} \geq \frac{k}{N}[v(r)]^{\sigma}, \qquad 0 \leq r < R.$$

Since $v(r) \geq v(0)$, the above implies that

$$\frac{v''(r)}{(1+[v'(r)]^2)^{3/2}} \geq \hat{k}[v(r)]^{\hat{\sigma}}, \qquad 0 \leq r < R,$$

where we put $k[v(0)]^{\varepsilon}/N = \hat{k}$ and $\sigma - \varepsilon = \hat{\sigma}$ for simplicity. Multiplying $v'(r) \geq 0$ and integrating, we have

$$1 - \frac{1}{(1+[v'(r)]^2)^{1/2}} \geq \frac{\hat{k}}{\hat{\sigma}+1}([v(r)]^{\hat{\sigma}+1} - [v(0)]^{\hat{\sigma}+1}), \qquad 0 \leq r < R. \tag{4.4}$$

Letting $r \to R-0$ in the above, by virtue of $v'(R-0) = \infty$, we obtain

$$1 \geq \frac{\hat{k}}{\hat{\sigma}+1}([v(R)]^{\hat{\sigma}+1} - [v(0)]^{\hat{\sigma}+1}),$$

which is equivalent to

$$v(R) \leq \left(\frac{\hat{\sigma}+1}{\hat{k}} + [v(0)]^{\hat{\sigma}+1}\right)^{1/(\hat{\sigma}+1)} \equiv A(\hat{\sigma}, \hat{k}). \tag{4.5}$$

On the other hand, quadrature gives from (4.4)

$$\int_{v(0)}^{v(R)} \frac{1 - \dfrac{\hat{k}}{\hat{\sigma}+1}(z^{\hat{\sigma}+1} - [v(0)]^{\hat{\sigma}+1})}{\left(1 - \left[1 - \dfrac{\hat{k}}{\hat{\sigma}+1}(z^{\hat{\sigma}+1} - [v(0)]^{\hat{\sigma}+1})\right]^2\right)^{1/2}} dz \geq R.$$

Replacing the upper limit $v(R)$ of the above integral by $A(\hat{\sigma}, \hat{k})$ (given by (4.5)), and performing the change of variable $w = \hat{k}(\hat{\sigma}+1)^{-1}(z^{\hat{\sigma}+1} - [v(0)]^{\hat{\sigma}+1})$ in the resulting inequality, we get

$$\frac{1}{\hat{k}}\int_0^1 \frac{1-w}{\left(\dfrac{\hat{\sigma}+1}{\hat{k}}w + [v(0)]^{\hat{\sigma}+1}\right)^{\hat{\sigma}/(\hat{\sigma}+1)} (1-(1-w)^2)^{1/2}} dw \geq R.$$

Consequently, we find

$$\frac{1}{\hat{k}}\int_0^1 \frac{1-w}{\left(\dfrac{\hat{\sigma}+1}{\hat{k}}w\right)^{\hat{\sigma}/(\hat{\sigma}+1)} (1-(1-w)^2)^{1/2}} dw \geq R,$$

from which it is easily seen the validity of the inequality

$$v(0) \leq \frac{C(N, \sigma, \varepsilon)}{(kR^{1+\sigma-\varepsilon})^{1/\varepsilon}}$$

with $C(N, \sigma, \varepsilon)$ given by

$$C(N, \sigma, \varepsilon) = N^{1/\varepsilon}(\sigma - \varepsilon + 1)^{-(\sigma-\varepsilon)/\varepsilon} \cdot \left[\int_0^1 \frac{1 - w}{w^{(\sigma-\varepsilon)/(\sigma-\varepsilon+1)}(1 - (1 - w)^2)^{1/2}} dw\right]^{(\sigma-\varepsilon+1)/\varepsilon}.$$

The proof is complete.

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