Hiroshima Math. J.
27 (1997), 189–207

The Poincaré duality and the Gysin homomorphism for flag manifolds

Dedicated to Professor K. Okamoto for his 60th birthday

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(Received November 6, 1995)

Abstract. The Poincaré duality for a partial flag manifold G/P is described in terms of the Weyl group of G. The Gysin homomorphism for natural projections between partial flag manifolds is calculated by using it. We investigate the case of complex flag manifolds and Grassmannians in C^n and show the relation to the Chern classes.

0. Introduction

Let M be an m-dimensional connected compact oriented manifold without boundary which has the fundamental homology class μ_M of the orientation. Then the Poincaré duality $\mathscr{P}_M$ for M is an isomorphism defined by a cap product:

$$\mathscr{P}_M = \mu_M \cap \colon H^p(M) \simeq H_{m-p}(M), \qquad \mathscr{P}_M \alpha = \mu_M \cap \alpha,$$

between homology and cohomology of M. Let M and N be connected compact oriented manifolds without boundary and let $f\colon M \to N$ be a continuous map. Then the Gysin homomorphism $f_!$ associated to f is by definition the homomorphism $f_! = \mathscr{P}_N^{-1} \circ f_* \circ \mathscr{P}_M$ between their cohomology modules, i.e., given by the following commutative diagram:

$$\begin{array}{ccc} H^*(M) & \xrightarrow{f_!} & H^*(N) \\ {\scriptstyle \mathscr{P}_M}\downarrow & & \wr\downarrow{\scriptstyle \mathscr{P}_N} \\ H_*(M) & \xrightarrow{f_*} & H_*(N). \end{array}$$

In the case that M and N are complex flag manifolds and f is a natural projection between them, the Gysin homomorphism is investigated by many

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1991 *Mathematics Subject Classification.* 57T15, 22E46.

Key words and phrases. flag manifolds G/P, Poincaré duality, Gysin homomorphism, Bruhat decomposition, Weyl group, symmetric function.

authors from a variety of view points. In particular J. Damon [4] determined $f_!$ by using the higher dimensional residue symbol in the context of algebraic geometry and T. Sugawara [13] determined $f_!$ by using "integration over the fiber" of fiber bundles in the context of algebraic topology. On the other hand Bernstein-Gel'fand-Gel'fand [1] investigated the connection between homology and cohomology of the flag manifold G/B where G is a complex semisimple Lie group and B is a Borel subgroup of G, and constructed a basis of cohomology dual to the Schubert basis of homology by introducing a divided difference operator. In their course of study they also determined the Poincaré duality on G/B. Therefore it seems natural to determine the Poincaré duality on other partial flag manifolds G/P and calculate the Gysin homomorphism by using it. The purpose of this paper is a report of the results (Theorem 2.1, 2.3 and 3.2). We use the Bruhat-Schubert cell decomposition and describe the homology and cohomology in terms of the Weyl group of G.

A brief account of contents of this article: We heavily depend upon the formalism of B.G.G. [1], so in §1 we review their formulations and results on homology and cohomology structure of a partial flag manifold G/P and fix the notation. In §2 we determine the Poincaré duality on G/P in terms of the Weyl group action and give a description of the Gysin homomorphism between them. We then specify the classical case of complex flag manifolds and Grassmannians in §3. We give their Bruhat-Schubert cell decomposition and describe the Gysin homomorphism in terms of the Chern classes.

The author would like to express his gratitude to Professor T. Sugawara for a kind guidance to the subject and valuable discussions.

1. Preliminaries, homology and cohomology of G/P

We begin by introducing the notation that is used throughout.

G is a connected complex reductive Lie group, that is, its Lie algebra $\mathfrak{g}$ is a reductive Lie algebra over C, $\mathfrak{g} = \mathfrak{c} + [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{c}$ is the center of $\mathfrak{g}$, $[\mathfrak{g}, \mathfrak{g}]$ is a semisimple ideal (cf. [14, 1.1.5]). We will specify $G = GL_n(C)$ in §3. We henceforth give $\mathfrak{g}$ an invariant non-degenerate bilinear form (,).

B is a fixed Borel subgroup of G.

G/B is a (full) flag manifold of G. In case of $G = GL_n(C)$ and $B =$ the large upper triangular matrix subgroup, $G/B = Fl_n(C)$ is the manifold of full flags in C^n.

N is the unipotent radical of B and H is a maximal algebraic torus of G such that $H \subset B$. $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{h}$ are the Lie subalgebras of $\mathfrak{g}$ corresponding B, N and H respectively. Then $B = HN$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, C)$ is the dual vector space of $\mathfrak{h}$.

$\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ is the root system of $(\mathfrak{g}, \mathfrak{h})$. Δ^+ is the set of positive roots corresponding to $\mathfrak{n}$ i.e. $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X$ for $H \in \mathfrak{h}\}$ is the α-root space of $\mathfrak{g}$.

$\Sigma \subset \Delta^+$ is the set of simple roots and $\Delta^- = -\Delta^+$.

$W = N_G(H)/H$ is the Weyl group of (G, H) where $N_G(H)$ is the normalizer of H in G. W acts on H, $\mathfrak{h}$ and $\mathfrak{h}^*$ naturally. W is determined only by $(\mathfrak{g}, \mathfrak{h})$ or Δ and if $s_\alpha\colon \mathfrak{h}^* \to \mathfrak{h}^*$ is a reflection in the hyperplane orthogonal to $\alpha \in \Delta$:

$$s_\alpha(\chi) = \chi - (\chi, \alpha^\vee)\alpha \qquad \text{where } \alpha^\vee = 2\alpha/(\alpha, \alpha) \text{ is the coroot of } \alpha,$$

then $(W, \tilde{\Sigma} = \{s_\alpha | \alpha \in \Sigma\})$ is a Coxeter system. For each $w \in W = N_G(H)/H$ the same letter w is used to denote its representative in $N_G(H) \subset G$. We know that the triple $(G, B, N_G(H))$ is a Tits system (cf. [3, Ch. IV]).

$\ell(w)$ is the length of $w \in W$ relative to the generators $\tilde{\Sigma} = \{s_\alpha | \alpha \in \Sigma\}$ of W, that is the least number of factors in the decomposition

$$w = s_1 s_2 \cdots s_l \qquad \text{where } s_i = s_{\alpha_i} \in \tilde{\Sigma}.$$

This expression is said to be reduced if $l = \ell(w)$.

$s_0 \in W$ is the unique element of maximal length r in W. We have $s_0 \Sigma = -\Sigma$, $r = \ell(s_0) = |\Delta^+|$, $s_0^2 = 1$ and $\ell(ws_0) = \ell(s_0) - \ell(w)$ (cf. [3, Ch. VI, §1, no. 6, Cor. 3 of Prop. 17]). Notice also $r = \dim \mathfrak{n} = \dim_C G/B$.

$\bar{N} = s_0 N s_0^{-1}$ is an analytic subgroup of G with Lie algebra $\bar{\mathfrak{n}} = \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. For $w \in W$ put $N_w = w\bar{N}w^{-1} \cap N$ and $N'_w = wNw^{-1} \cap N$. Then N_w and N'_w are unipotent subgroups of G with Lie algebras $\mathfrak{n}_w = (\mathrm{Ad}\, w)\bar{\mathfrak{n}} \cap \mathfrak{n} = \sum_{\alpha \in w\Delta^- \cap \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}'_w = (\mathrm{Ad}\, w)\mathfrak{n} \cap \mathfrak{n}$ of complex dimensions $\ell(w)$ and $\ell(s_0) - \ell(w)$ respectively. We have $N = N_w N'_w$ and $\mathfrak{n} = \mathfrak{n}_w + \mathfrak{n}'_w$.

1.1 (*Bruhat decomposition*). *Under the above notation we have the double coset decomposition $B \backslash G/B$ as follows*:

$$G = \bigcup_{w \in W} BwB \quad (\textit{disjoint union}), \qquad \textit{and hence}$$

$$G/B = \bigcup_{w \in W} Bw \cdot B \quad (\textit{disjoint union}),$$

where the notation $Bw \cdot B = BwB/B \subset G/B$ indicates the subset of the coset space G/B. Each $Bw \cdot B$ is a cell of complex dimension $\ell(w)$ in the space G/B, that is

$$\mathfrak{n}_w \xrightarrow{\exp} N_w \xrightarrow{\text{natural}} N_w w \cdot B = Nw \cdot B = Bw \cdot B \subset G/B$$

are onto analytic diffeomorphisms and $\mathfrak{n}_w \simeq C^{\ell(w)}$ is an affine space.

For proof see [3, Ch. IV], or [14, 1.2] for example. We will see later that in case of $G = GL_n(C)$ the Bruhat decomposition corresponds exactly to the classical Schubert cell decomposition.

We collect elementary properties of $(W, \tilde{\Sigma})$ and parabolic subgroups of $(G, B, N_G(H))$.

1.2 LEMMA (cf. [8, 1.6], [3, Ch. VI, §1]). *$(W, \tilde{\Sigma})$ has the following properties. For $w \in W$, $\alpha \in \Sigma$ simple,*

(1) *If $w = s_1 s_2 \cdots s_k$ ($s_i = s_{\alpha_i}, \alpha_i \in \Sigma$) is a reduced expression put $\theta_i = s_1 s_2 \cdots s_{i-1}(\alpha_i)$ ($1 \leq i \leq k$). Then θ_i are all distinct positive roots and*

$$\{\theta_i | 1 \leq i \leq k\} = \Delta^+ \cap w\Delta^-.$$

(2) $\ell(w) = |\Delta^+ \cap w^{-1}\Delta^-|$, *and so* $\ell(w^{-1}) = \ell(w)$

(3) $\ell(ws_\alpha) = \ell(w) + 1$ *iff* $w\alpha > 0$

$\ell(ws_\alpha) = \ell(w) - 1$ *iff* $w\alpha < 0$

From this, $\dim_C \mathfrak{n}_w = |w\Delta^- \cap \Delta^+| = \ell(w^{-1}) = \ell(w)$ follows.

For each subset $\Theta \subset \Sigma$ of simple system, define

$$W_\Theta = \langle s_\alpha | \alpha \in \Theta \rangle = \textit{the subgroup of } W \textit{ generated by } \tilde{\Theta} = \{s_\alpha | \alpha \in \Theta\},$$

and

$$P_\Theta = BW_\Theta B.$$

Then $(W_\Theta, \tilde{\Theta})$ is a Coxeter system with the root system $\Delta \cap \langle \Theta \rangle$ where $\langle \Theta \rangle =$ Z-span of Θ in $\mathfrak{h}^*$ and P_Θ is a subgroup of G containing B, which is called a (standard) parabolic subgroup. We know that the map $\Theta \mapsto P_\Theta$ is a lattice isomorphism between the lattice 2^Σ of all subsets of Σ and that of subgroups of G which contains B, e.g. $P_\varnothing = B$, $P_\Sigma = G$. The coset spaces G/P_Θ are (partial) flag manifolds, which contains Grassmannians in case of $G = GL_n$. We also define a subset W^Θ of W as follows,

$$\begin{aligned} W^\Theta &= \{w \in W | \ell(ws_\alpha) = \ell(w) + 1 \text{ for all } \alpha \in \Theta\} \\ &= \{w \in W | w\Theta \subset \Delta^+\} \qquad \text{(by Lemma 1.2(3))}. \end{aligned}$$

Then W^Θ is called a minimal coset representative of W/W_Θ since

1.3 LEMMA (cf. [8, 1.10] or [3, Ch. IV, §1, Exer. 3]). *We have*

$$W = W^\Theta \times W_\Theta, \qquad \textit{and hence} \quad W^\Theta \simeq W/W_\Theta \quad \textit{by} \quad u \mapsto uW_\Theta.$$

Given $w \in W$, there is a unique $(u, v) \in W^\Theta \times W_\Theta$ such that $w = uv$. Their lengths satisfy $\ell(w) = \ell(u) + \ell(v)$. Each $u \in W^\Theta$ is the unique element of smallest length in the coset $wW_\Theta = uW_\Theta$.

1.4 (*Bruhat decomposition for a partial flag manifold*). *We have the double coset decomposition* $B\backslash G/P_\Theta$.

$$G = \bigcup_{w \in W^\Theta} BwP_\Theta \quad (\textit{disjoint union}), \qquad \textit{and so}$$

$$G/P_\Theta = \bigcup_{w \in W^\Theta} Bw \cdot P_\Theta \quad (\textit{disjoint union})$$

is a cellular decomposition of the partial flag manifold G/P_Θ *into cells* $Bw \cdot P_\Theta$ *of dimension* $\ell(w)$.

SKETCH OF PROOF (cf. [14, 1.2.4.9]). Since $(G, B, N_G(H))$ is a Tits system, for subsets $X, Y \subset \Theta$ there is a bijection ([3, Ch. IV, §2, no. 5, Remarque 2]),

$$W_X \backslash W/W_Y \xrightarrow{\sim} P_X \backslash G/P_Y \qquad \text{by } W_X w W_Y \mapsto P_X w P_Y.$$

Put $X = \varnothing$ and $Y = \Theta$. Then we have from the above decomposition

$$W^\Theta \simeq W/W_\Theta \xrightarrow{\sim} B\backslash G/P_\Theta \qquad \text{by } w \mapsto BwP_\Theta.$$

As 1.1 we know that $\mathfrak{n}_w \xrightarrow{\sim} N_w \xrightarrow{\sim} Bw \cdot P_\Theta \subset G/P_\Theta$ are onto analytic diffeomorphisms and so $Bw \cdot P_\Theta$ $(w \in W^\Theta)$ is a cell in the space G/P_Θ of dimension $\dim_C \mathfrak{n}_w = \ell(w)$. □

We recall cohomology structure of flag manifolds G/P and results of B.G.G. [1]. Let X_w be the closure of a cell $Bw \cdot B$ in G/B, $[X_w] \in H_*(X_w, \boldsymbol{Z})$ be the fundamental cycle of the complex variety X_w of complex dimension $\ell(w)$ and $D_w \in H_*(G/B, \boldsymbol{Z})$ be the image of $[X_w]$ under the map induced by the embedding $X_w \subset G/B$. In this article we treat homology and cohomology of even dimensional only, so we write H_p and H^p instead of H_{2p} and H^{2p}. Then we can write $D_w \in H_{\ell(w)}(G/B, \boldsymbol{Z})$ (i.e. $D_w \in H_{2\ell(w)}$ in fact). In the same manner for each $w \in W^\Theta$, we define $D_w(\Theta) \in H_*(G/P_\Theta, \boldsymbol{Z})$ for a homology class determined by the cell $Bw \cdot P_\Theta$ in G/P_Θ. Then we have

1.5. (1) $\{D_w | w \in W\}$ *forms a free basis of the homology module* $H_*(G/B, \boldsymbol{Z})$, *i.e.* $H_*(G/B, \boldsymbol{Z}) = \bigoplus_{w \in W} \boldsymbol{Z} D_w$.

(2) *The natural map* $p: G/B \to G/P_\Theta$ *induces a epimorphism* $p_*: H_*(G/B, \boldsymbol{Z}) \to H_*(G/P_\Theta, \boldsymbol{Z})$ *such that* $p_* D_w = 0$ *if* $w \notin W^\Theta$ *and* $p_* D_w = D_w(\Theta)$ *if* $w \in W^\Theta$. *And* $H_*(G/P_\Theta, \boldsymbol{Z}) = \bigoplus_{w \in W^\Theta} \boldsymbol{Z} D_w(\Theta)$.

By (2) we will write simply $D_w \in H_*(G/P_\Theta, \boldsymbol{Z})$ instead of $D_w(\Theta)$ if there is no fear of confusion.

We introduce in $\mathfrak{h}$ the coroot system $\{H_\alpha | \alpha \in \Delta\}$ of Δ, i.e.

$$H_\alpha = 2h_\alpha/(\alpha, \alpha) \in \mathfrak{h} \qquad \text{where } h_\alpha \in \mathfrak{h} \text{ is given by}$$

$$(h_\alpha, H) = \alpha(H) \qquad \text{for all } H \in \mathfrak{h}.$$

Then $s_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha$ for all $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{h}_{\boldsymbol{Z}} = \{H \in \mathfrak{h} | \exp_G(2\pi i H) = 1\}$ be the unit lattice of G and $\mathfrak{h}_{\boldsymbol{Q}} = \mathfrak{h}_{\boldsymbol{Z}} \otimes \boldsymbol{Q}$. Then $\mathfrak{h}_{\boldsymbol{Z}}$ contains the coroot lattice $\mathfrak{h}_\Delta = \boldsymbol{Z}$-span $\{H_\alpha | \alpha \in \Delta\}$, $\mathfrak{h}_{\boldsymbol{Z}} \supset \mathfrak{h}_\Delta$. Let $\mathfrak{h}^*_{\boldsymbol{Z}} = \{\chi \in \mathfrak{h}^* | \chi(\mathfrak{h}_{\boldsymbol{Z}}) \subset \boldsymbol{Z}\}$, $\mathfrak{h}^*_{\boldsymbol{Q}} = \mathfrak{h}^*_{\boldsymbol{Z}} \otimes \boldsymbol{Q}$ and $\mathfrak{h}^*_\Delta = \{\chi \in \mathfrak{h}^* | \chi(\mathfrak{h}^*_\Delta) \subset \boldsymbol{Z}\}$. Then $\mathfrak{h}^*_\Delta$ is the weight lattice and $\mathfrak{h}^*_{\boldsymbol{Z}} \subset \mathfrak{h}^*_\Delta$.

Let $R = S(\mathfrak{h}^*_{\boldsymbol{Q}}) = \boldsymbol{Q}[\mathfrak{h}_{\boldsymbol{Q}}]$ be the ring of polynomial functions on $\mathfrak{h}_{\boldsymbol{Q}}$ with rational coefficients. The Weyl group acts naturally on R. Let $I = R^W$ be the subring of W-invariants in R, $I^+ = \{f \in I | f(0) = 0\}$ and $J = I^+R$ be an ideal of R generated by I^+.

We construct a homomorphism $\beta: R \to H^*(G/B, \boldsymbol{Q})$ as follows. First let $\chi \in \mathfrak{h}^*_{\boldsymbol{Z}}$. Then χ lifts to a character $\theta: H \to \boldsymbol{C}^*$ by $\theta(\exp X) = \exp \chi(X)$, $X \in \mathfrak{h}$. We extend θ to a character of B by $\theta(hn) = \theta(h)$, $h \in H$, $n \in N$. Since $G \to G/B$ is a principal fiber bundle with structure group B, θ defines a line bundle E_χ on G/B. We let $\beta(\chi) = c_1(E_\chi) \in H^1(G/B, \boldsymbol{Z})$ be the 1-st Chern class of E_χ. Then β is a homomorphism $\mathfrak{h}^*_{\boldsymbol{Z}} \to H^1(G/B, \boldsymbol{Z})$, which extends naturally to a ring-homomorphism $\beta: R \to H^*(G/B, \boldsymbol{Q})$.

1.6 (A. Borel [2]). (1) *The homomorphism β commutes with the actions of W on R and $H^*(G/B)$.*

(2) Ker $\beta = J$ *and the induced map $\bar{\beta}: \bar{R} = R/J \to H^*(G/B, \boldsymbol{Q})$ is an onto ring-isomorphism. $\bar{R} = R/J$ is a truncated polynomial ring of finite dimension $|W|$ over $\boldsymbol{Q}$.*

(3) *The natural map $p: G/B \to G/P_\Theta$ induces a ring-monomorphism p^*: $H^*(G/P_\Theta) \to H^*(G/B)$. The cohomology ring $H^*(G/P_\Theta, \boldsymbol{Q})$ is isomorphic to the subring $H^*(G/B, \boldsymbol{Q})^{W_\Theta} = (R/J)^{W_\Theta}$* of *$W_\Theta$-invariants* by p^*.

B.G.G. [1] established a connection between homology and cohomology of G/P. They introduced polynomials $\{P_w \in R | w \in W\}$ in such a way that the induced set $\{\bar{P}_w = \beta(P_w) \in \bar{R} | w \in W\}$ forms a basis of $\bar{R} = H^*(G/B)$ dual to the basis $\{D_w | w \in W\}$ of $H_*(G/B)$ by the natural pairing $\langle\ ,\ \rangle$ of homology and cohomology: $\langle D_w, \bar{P}_u \rangle = \delta_{wu}$, and determine P_w.

The polynomial P_w is constructed by the following divided difference operator Δ_w. For each root $\alpha \in \Delta$, we define the operator $\Delta_\alpha: R \to R$ of degree -1 by

$$\Delta_\alpha f = (f - s_\alpha f)/\alpha, \qquad \text{i.e.}$$

$$(\Delta_\alpha f)(H) = (f(H) - f(s_\alpha H))/\alpha(H), \qquad H \in \mathfrak{h}_{\boldsymbol{Q}}.$$

The Δ-operators have the following properties.

1.7 (cf. [1], [5] and [7]). (1) *Let $w = s_1 s_2 \cdots s_l \in W$, $s_i = s_{\alpha_i} \in \tilde{\Sigma}$. If $\ell(w) < l$ then $\Delta_{\alpha_1}\Delta_{\alpha_2}\cdots\Delta_{\alpha_l} = 0$. If $\ell(w) = l$, i.e. this expression of w is reduced then the operator $\Delta_{\alpha_1}\Delta_{\alpha_2}\cdots\Delta_{\alpha_l}$ depends only on w and does not depend on the* *reduced expression of* w. *We thus put* $\Delta_w = \Delta_{\alpha_1}\Delta_{\alpha_2}\cdots\Delta_{\alpha_l}$ *for the reduced expression* $w = s_1 s_2 \cdots s_l$, $s_i = s_{\alpha_i} \in \tilde{\Sigma}$.

(2) $\Delta_w \cdot \Delta_u = \begin{cases} \Delta_{wu} & \text{if } \ell(wu) = \ell(w) + \ell(u) \\ 0 & \text{if } \ell(wu) < \ell(w) + \ell(u) \end{cases}, \qquad w, u \in W.$

(3) $\Delta_{-\alpha} = -\Delta_\alpha$, $\Delta_\alpha^2 = 0$, $w\Delta_\alpha w^{-1} = \Delta_{w\alpha}$.

(4) $s_\alpha \Delta_\alpha = -\Delta_\alpha s_\alpha = \Delta_\alpha$, $s_\alpha = 1 - \alpha\Delta_\alpha$.

(5) $\Delta_\alpha(fg) = f(\Delta_\alpha g) + (\Delta_\alpha f)s_\alpha g$, $f, g \in R$.

(6) $\Delta_\alpha f = 0$ *iff* $s_\alpha f = f$.

(7) $\Delta_\alpha J \subset J$.

From (5) and (6) $\Delta_\alpha\colon R \to R$ is a R^W-endomorphism and by (7) it induces an endomorphism Δ_α (we use the same letter) of $\bar{R} = R/J$. The homology basis $D_w \in H_*(G/B)$ viewed as a functional on the cohomology $H^*(G/B) = \bar{R}$ is described by Δ_w as

1.8.

$$\langle D_w, \beta(f)\rangle = (\Delta_w f)(0), \qquad f \in R,\ w \in W.$$

The polynomials $\{P_w\}$ which induce the dual basis of $\{D_w\}$ are determined mod J and given as follows:

1.9 ([1]). (1) *Let* $P_0 = P_{s_0} \in H^r(G/B, \mathbf{Q})$ *be the fundamental cohomology class of top order* $r = \ell(s_0) = \dim_{\mathbf{C}} G/B$. *Then*

$$P_0 = |W|^{-1} \textstyle\prod_{\alpha\in\Delta^+} \alpha = \rho^r/r! \qquad (\text{mod } J),$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ *is half the sum of the positive roots.*

(2) $$\bar{P}_w = \Delta_{w^{-1}s_0}\bar{P}_0 \qquad \text{for } w \in W.$$

(3) *By the natural ring-monomorphism* $p^*\colon H^*(G/P_\Theta) \to H^*(G/B)\{p^{*-1}\bar{P}_w | w \in W^\Theta\}$ *is the basis of* $H^*(G/P_\Theta\}$ *dual to the basis* $\{D_w | w \in W^\Theta\}$ *of* $H_*(G/P_\Theta)$.

As to (1) we put $D_0 = D_{s_0} \in H_r(G/B)$ for the fundamental homology class of top order. As to (2) note that $\deg(\Delta_{w^{-1}s_0}P_0) = \ell(s_0) - \ell(w^{-1}s_0) = \ell(s_0) - (\ell(s_0) - \ell(w^{-1})) = \ell(w) = \deg P_w$. From 1.5(2) and 1.9(3), we will write simply $P_w \in H^*(G/P_\Theta)$ instead of $p^{*-1}\bar{P}_w$. In other words we identify $H^*(G/P_\Theta)$ with the subring $H^*(G/B)^{W_\Theta}$ of $H^*(G/B)$ by the natural map p^*. These polynomials P_w have the following properties:

1.10 ([1]). (1) *Let* $w \in W$, $\alpha \in \Sigma$. *Then*

$$\Delta_\alpha P_w = \begin{cases} 0 & \text{if } \ell(ws_\alpha) = \ell(w) + 1 \\ P_{ws_\alpha} & \text{if } \ell(ws_\alpha) = \ell(w) - 1. \end{cases}$$

(2) *Let* w, $u \in W$, $\ell(w) + \ell(u) = r$. *Then*

$$P_w P_u = \begin{cases} P_0 & \text{if } u = s_0 w \\ 0 & \text{if } u \neq s_0 w. \end{cases}$$

(3) (*The Poincaré duality of* G/B) *Let* $\mathscr{P} = D_0 \cap : H^*(G/B, \boldsymbol{Q}) \xrightarrow{\sim} H_*(G/B, \boldsymbol{Q})$ *be the Poincaré duality of the full flag manifold* G/B. *Then we have*

$$\mathscr{P} P_w = D_{s_0 w}.$$

We give a proof of the following, for this is a key fact in §3.

1.11 PROPOSITION ([5, Lemma 4], [7, 2.5]). *The operator* $\Delta_{s_0}: R \to R$ *is given by*

$$\Delta_{s_0} f = \sum_{w \in W} \varepsilon(w) w f / \prod_{\alpha \in \Delta^+} \alpha, \qquad f \in R,$$

where $\varepsilon(w) = (-1)^{\ell(w)} = \pm 1$ *is the sign of* $w \in W$.

PROOF. First we have $s_\alpha \Delta_{s_0} = \Delta_{s_0}$ for $\alpha \in \Delta^+$, hence $w\Delta_{s_0} = \Delta_{s_0}$ for $w \in W$. In fact we have $\Delta_\alpha \Delta_{s_0} = 0$ by $\ell(s_\alpha s_0) = \ell(s_0) - 1$ and 1.7(2), then use 1.7(4). Fix a reduced expression $s_0 = s_1 s_2 \cdots s_r$, $s_i = s_{\alpha_i}$, $\alpha_i \in \Sigma$. Then we see that $\Delta_{s_0} = \Delta_{\alpha_1} \circ \Delta_{\alpha_2} \circ \cdots \circ \Delta_{\alpha_r} = \alpha_1^{-1}(1 - s_1) \circ \alpha_2^{-1}(1 - s_2) \circ \cdots \circ \alpha_r^{-1}(1 - s_r)$. Expanding out we have

$$\Delta_{s_0} = \sum_{w \in W} q_w w$$

where $q_w \in \boldsymbol{Q}(\mathfrak{h}_{\boldsymbol{Q}})$ is a rational function. The comparison of coefficients in $w\Delta_{s_0} = \Delta_{s_0}$ then implies that $wq_u = q_{wu}$, w, $u \in W$. Here we use the fact that w's are linearly independent over $\boldsymbol{Q}(\mathfrak{h}_{\boldsymbol{Q}})$, which follows from the Dedekind theorem of the Galois extension $\boldsymbol{Q}(\mathfrak{h}_{\boldsymbol{Q}})/\boldsymbol{Q}(\mathfrak{h}_{\boldsymbol{Q}})^W$. We know that $q_{s_0} s_0 = (-1)^r \alpha_1^{-1} s_1 \circ \alpha_2^{-1} s_2 \circ \cdots \circ \alpha_r^{-1} s_r = (-1)^r \{\alpha_1 (s_1 \alpha_2) \cdots (s_1 \cdots s_{r-1} \alpha_r)\}^{-1} s_0 = (-1)^r (\prod_{\alpha \in \Delta^+} \alpha)^{-1} s_0$, hence $q_{s_0} = \varepsilon(s_0)(\prod_{\alpha \in \Delta^+} \alpha)^{-1}$ by 1.2(1). Note that $w(\prod_{\alpha \in \Delta^+} \alpha) = (-1)^{\ell(w)} \prod_{\alpha \in \Delta^+} \alpha = \varepsilon(w) \prod_{\alpha \in \Delta^+} \alpha$ for $w \in W$ by 1.2(2). We thus obtain that $q_w = ws_0 \cdot q_{s_0} = \varepsilon(w)/\prod_{\alpha \in \Delta^+} \alpha$. □

2. The Poincaré duality and the Gysin homomorphism for partial flag manifolds

In this section we shall describe the Poincaré duality and the Gysin homomorphism for partial flag manifolds G/P in terms of the Weyl group W. For each subset $\Theta \subset \Sigma$ of simple roots we obtain a parabolic subgroup $P_\Theta = BW_\Theta B$, the partial flag manifold G/P_Θ and the cellular decomposition $G/P_\Theta = \bigcup_{w \in W^\Theta} Bw \cdot P_\Theta$. The homology and cohomology of G/P_Θ is given by

$$H_*(G/P_\Theta, \boldsymbol{Q}) = \bigoplus_{w \in W^\Theta} \boldsymbol{Q} D_w, \qquad H^*(G/P_\Theta, \boldsymbol{Q}) = \bar{R}^{W_\Theta} = \bigoplus_{w \in W^\Theta} \boldsymbol{Q} P_w.$$

According to the left coset decomposition $W = W^{\Theta} \times W_{\Theta}$, we put

$$s_0 = s^{\Theta} s_{\Theta}, \qquad s^{\Theta} \in W^{\Theta}, \qquad s_{\Theta} \in W_{\Theta}.$$

Then s_{Θ} is the unique element of maximal length in W_{Θ}. In fact, if there is an element $t \in W_{\Theta}$ such that $\ell(t) \geq \ell(s_{\Theta})$ then $\ell(s^{\Theta} t) = \ell(s^{\Theta}) + \ell(t) \geq \ell(s^{\Theta}) + \ell(s_{\Theta}) = \ell(s_0)$ by 1.3, the uniqueness of s_0 in W implies that $s_0 = s^{\Theta} s_{\Theta} = s^{\Theta} t$, and hence $s_{\Theta} = t$. Since $(W_{\Theta}, \tilde{\Theta})$ is a Weyl group of the root system $\Delta \cap \langle \Theta \rangle$, s_{Θ} has the same properties as s_0 for W_{Θ}. We have $\ell(s_{\Theta}) = |\Delta^+ \cap \langle \Theta \rangle|$, $s_{\Theta}(\Theta) = -\Theta$ and $s_{\Theta}^2 = 1$ for example. Similarly we know that s^{Θ} is the unique element of maximal length in W^{Θ} and that $\ell(s^{\Theta}) = \ell(s_0) - \ell(s_{\Theta}) = |\Delta^+ \backslash \langle \Theta \rangle| = \dim_C G/P_{\Theta}$ by $s^{\Theta} = s_0 s_{\Theta}$. We put $D_{\Theta} = D_{s^{\Theta}} \in H_*(G/P_{\Theta})$ and $P_{\Theta} = P_{s^{\Theta}} \in H^*(G/P_{\Theta})$ for these top order elements of homology and cohomology. (There will be no confusion between the notation P_{Θ} of cohomology class and that of parabolic subgroup.)

The Poincaré duality $\mathscr{P}_{\Theta}$ of the partial flag G/P_{Θ} is defined as follows. Since D_{Θ} is the fundamental homology class of G/P_{Θ},

$$\mathscr{P}_{\Theta} = D_{\Theta} \cap \colon H^p(G/P_{\Theta}, \boldsymbol{Q}) \xrightarrow{\sim} H_{\ell(s^{\Theta})-p}(G/P_{\Theta}, \boldsymbol{Q}), \qquad 0 \leq p \leq \ell(s^{\Theta}),$$

$$\langle \mathscr{P}_{\Theta} f, g \rangle = \langle D_{\Theta} \cap f, g \rangle = \langle D_{\Theta}, fg \rangle, \qquad f, g \in \bar{R}^{W_{\Theta}} = H^*(G/P_{\Theta}).$$

The Poincaré duality of G/P_{Θ} is given by the following:

2.1 THEOREM.

$$\mathscr{P}_{\Theta} P_w = D_{s_0 w s_{\Theta}}, \qquad w \in W^{\Theta}.$$

We first check two points that if $w \in W^{\Theta}$ then also $s_0 w s_{\Theta} \in W^{\Theta}$ and $\ell(s_0 w s_{\Theta}) = \ell(s^{\Theta}) - \ell(w)$. These guarantee that if $P_w \in H^p(G/P_{\Theta})$ then $D_{s_0 w s_{\Theta}} \in H_{\ell(s^{\Theta})-p}(G/P_{\Theta})$. Indeed if $w \in W^{\Theta}$, then $s_0 w s_{\Theta}(\Theta) = -s_0 w(\Theta) \subset -s_0 \Delta^+ = -\Delta^- = \Delta^+$ so $s_0 w s_{\Theta} \in W^{\Theta}$ by definition of W^{Θ}, and $\ell(s_0 w s_{\Theta}) = \ell(s_0) - \ell(w s_{\Theta}) = \ell(s_0) - (\ell(w) + \ell(s_{\Theta})) = (\ell(s_0) - \ell(s_{\Theta})) - \ell(w) = \ell(s^{\Theta}) - \ell(w)$ by 1.3.

Next we extend 1.10(2).

2.2 LEMMA. *Let $w, u \in W^{\Theta}$, $\ell(w) + \ell(u) = \ell(s^{\Theta})$. Then*

$$P_w P_u = \begin{cases} P_{\Theta} & \text{if } u = s_0 w s_{\Theta}, \\ 0 & \text{if } u \neq s_0 w s_{\Theta}. \end{cases}$$

PROOF. The fundamental cohomology class P_{Θ} of G/P_{Θ} is given by 1.9(2) as

$$P_{\Theta} = P_{s^{\Theta}} = \Delta_{(s^{\Theta})^{-1} s_0} P_0 = \Delta_{s_{\Theta}} P_0.$$

First let $u = s_0 w s_{\Theta}$. Then $P_u = \Delta_{u^{-1} s_0} P_0 = \Delta_{s_{\Theta} w^{-1}} P_0$. We know that $P_0 =$

$P_w P_{s_0 w}$ by 1.10(2). Applying Δ_{s_Θ} to this both sides we get

$$P_\Theta = \Delta_{s_\Theta} P_0 = \Delta_{s_\Theta}(P_w P_{s_0 w}).$$

The reduced expression of $s_\Theta \in W_\Theta$ is of the form $s_\Theta = s_1 s_2 \cdots s_n$ where $n = \ell(s_\Theta)$, $s_i = s_{\alpha_i}$ with $\alpha_i \in \Theta$. And so $\Delta_{s_\Theta} = \Delta_{\alpha_1}\Delta_{\alpha_2}\cdots\Delta_{\alpha_n}$. For any $\alpha \in \Theta$, $\ell(ws_\alpha) = \ell(w) + 1$ since $w \in W^\Theta$, hence $\Delta_\alpha P_w = 0$ by 1.10(1). We thus obtain 1.7(5),

$$\Delta_\alpha(P_w P_{s_0 w}) = P_w(\Delta_\alpha P_{s_0 w}) + (\Delta_\alpha P_w)(s_\alpha P_{s_0 w}) = P_w(\Delta_\alpha P_{s_0 w}),$$

we iterate this and get

$$P_\Theta = \Delta_{s_\Theta}(P_w P_{s_0 w}) = P_w(\Delta_{s_\Theta} P_{s_0 w}) = P_w(\Delta_{s_\Theta}\Delta_{w^{-1}} P_0) = P_w \Delta_{s_\Theta w^{-1}} P_0 = P_w P_u,$$

since $\ell(s_\Theta w^{-1}) = \ell(ws_\Theta) = \ell(w^{-1}) + \ell(s_\Theta)$ and 1.7(2). Next let $u \neq s_0 w s_\Theta$. Then $us_\Theta \neq s_0 w$, and by 1.10(2), $P_w P_{us_\Theta} = 0$. Applying Δ_{s_Θ} to both sides and calculating as above, we obtain

$$0 = \Delta_{s_\Theta}(P_w P_{us_\Theta}) = P_w \Delta_{s_\Theta} P_{us_\Theta} = P_w P_u.$$

Here we again use 1.7(2) and compute

$$\Delta_{s_\Theta} P_{us_\Theta} = \Delta_{s_\Theta}\Delta_{s_\Theta u^{-1} s_0} P_0 = \Delta_{u^{-1}s_0} P_0 = P_u,$$

since $\ell(s_\Theta) + \ell(s_\Theta u^{-1} s_0) = \ell(s_\Theta) + (\ell(s_0) - \ell(s_\Theta u^{-1})) = \ell(s_\Theta) + \ell(s_0) - \ell(s_\Theta) - \ell(u^{-1}) = \ell(s_0) - \ell(u^{-1}) = \ell(u^{-1}s_0)$. □

Proof of Theorem 2.1. For $w \in W^\Theta$ we have by the choice of basis

$$\mathscr{P}_\Theta P_w = \textstyle\sum_{u \in W^\Theta} \langle \mathscr{P}_\Theta P_w, P_u \rangle D_u = \sum_{\ell(u)+\ell(w)=\ell(s^\Theta)} \langle D_\Theta, P_w P_u \rangle D_u.$$

By Lemma 2.2, the only one term $u = s_0 w s_\Theta$ remains and we get $\mathscr{P}_\Theta P_w = D_{s_0 w s_\Theta}$ as required. □

We shall describe the Gysin homomorphism between partial flag manifolds. For two subsets $\Theta \subset \Phi$ of simple roots Σ we have Weyl groups $W_\Theta \subset W_\Phi$, $W^\Theta \supset W^\Phi$, parabolic subgroups $P_\Theta \subset P_\Phi$ and the natural map π between the partial flag manifolds:

$$\pi\colon G/P_\Theta \to G/P_\Phi,$$

which induces naturally the homomorphisms $\pi_*\colon H_*(G/P_\Theta) \to H_*(G/P_\Phi)$ of homology and $\pi^*\colon H^*(G/P_\Phi) \to H^*(G/P_\Theta)$ of cohomology.

The Gysin homomorphism for π is defined by $\pi_! = \mathscr{P}_\Phi^{-1} \circ \pi_* \circ \mathscr{P}_\Theta$: $H^*(G/P_\Theta) \to H^*(G/P_\Phi)$, i.e. given by the following commutative diagram:

$$\begin{array}{ccc} H^p(G/P_\Theta) & \xrightarrow{\pi_!} & H^{p-(\ell(s^\Theta)-\ell(s^\Phi))}(G/P_\Phi) \\ \mathscr{P}_\Theta \wr \downarrow & & \downarrow \wr \mathscr{P}_\Phi \\ H_{\ell(s^\Theta)-p}(G/P_\Theta) & \xrightarrow{\pi_*} & H_{\ell(s^\Theta)-p}(G/P_\Phi). \end{array}$$

Note that the Gysin homomorphism $\pi_!$ decreases the dimension of cohomology by $\ell(s^\Theta) - \ell(s^\Phi) = \dim_C G/P_\Theta - \dim_C G/P_\Phi \geq 0$. Under the above notation we can calculate $\pi_!$ as follows.

2.3 Theorem. (1) *$\pi_!$ operates on the basis $\{P_w | w \in W^\Theta\}$ of $H^*(G/P_\Theta)$ as*

$$\pi_! P_w = \begin{cases} P_{ws_\Theta s_\Phi} & \text{if } s_0 w s_\Theta \in W^\Phi \\ 0 & \text{if } s_0 w s_\Theta \notin W^\Phi, \end{cases} \qquad w \in W^\Theta.$$

(2) *$\pi_!$ is written by the Δ-operator as*

$$\pi_! = \Delta_{s_\Phi s_\Theta}\colon \bar{R}^{W_\Theta} = H^*(G/P_\Theta) \to \bar{R}^{W_\Phi} = H^*(G/P_\Phi).$$

Proof. (1) By 1.5(2) the induced homology map $\pi_*\colon H_*(G/P_\Theta) \to H_*(G/P_\Phi)$ is given by, for $w \in W^\Theta$,

$$\pi_* D_w(\Theta) = \begin{cases} D_w(\Phi) & \text{if } w \in W^\Phi \\ 0 & \text{if } w \notin W^\Phi. \end{cases}$$

Hence for $P_w \in H^*(G/P_\Theta)$, $w \in W^\Theta$, we have

$$\begin{aligned} \pi_! P_w &= \mathscr{P}_\Phi^{-1} \circ \pi_* \circ \mathscr{P}_\Theta(P_w) = \mathscr{P}_\Phi^{-1} \circ \pi_*(D_{s_0 w s_\Theta}) \\ &= \begin{cases} \mathscr{P}_\Phi^{-1} D_{s_0 w s_\Theta} = P_{w s_\Theta s_\Phi} & \text{if } s_0 w s_\Theta \in W^\Phi \\ 0 & \text{if } s_0 w s_\Theta \notin W^\Phi. \end{cases} \end{aligned}$$

(2) First let $w \in W^\Theta$ with $s_0 w s_\Theta \in W^\Phi$. Then by 1.9(2) we have

$$\pi_! P_w = P_{w s_\Theta s_\Phi} = \Delta_{s_\Phi s_\Theta w^{-1} s_0} P_0.$$

Since $s_\Theta \in W^\Theta \subset W^\Phi$, we have $\ell(s_\Phi s_\Theta) = \ell(s_\Phi) - \ell(s_\Theta)$. We thus get a length relation: $\ell(s_\Phi s_\Theta w^{-1} s_0) = \ell(s_0 w s_\Theta s_\Phi) = \ell(s_0 w s_\Theta) + \ell(s_\Phi)$ (by $s_0 w s_\Theta \in W^\Phi$ and 1.3) $= \ell(s_0) - \ell(w s_\Theta) + \ell(s_\Phi) = \ell(s_0) - \ell(w) - \ell(s_\Theta) + \ell(s_\Phi) = (\ell(s_\Phi) - \ell(s_\Theta)) + (\ell(s_0) - \ell(w)) = \ell(s_\Phi s_\Theta) + \ell(w^{-1} s_0)$. Hence by 1.7(2) we obtain

$$\pi_! P_w = \Delta_{s_\Phi s_\Theta w^{-1} s_0} P_0 = \Delta_{s_\Phi s_\Theta} \Delta_{w^{-1} s_0} P_0 = \Delta_{s_\Phi s_\Theta} P_w.$$

Since $H^*(G/P_\Theta) = \text{span}\,\{P_w | w \in W^\Theta\}$ it suffices to show that if $w \in W^\Theta$ and $s_0 w s_\Theta \notin W^\Phi$ then $\Delta_{s_\Phi s_\Theta} P_w = 0$. Again in this case we have

$$\Delta_{s_\Phi s_\Theta} P_w = \Delta_{s_\Phi s_\Theta} \Delta_{w^{-1} s_0} P_0.$$

Since $s_0 w s_\Theta \notin W^\Phi$ there is some $\alpha \in \Phi$ such that $\ell(s_0 w s_\Theta s_\alpha) = \ell(s_0 w s_\Theta) - 1$. We thus have $\ell(s_0 w s_\Theta s_\Phi) < \ell(s_0 w s_\Theta) + \ell(s_\Phi)$ since $s_\Phi \in W_\Phi$ has the reduced expression of the form $s_\Phi = s_1 s_2 \cdots s_m$ with $s_1 = s_\alpha$, $s_i \in \tilde{\Phi}$ and $m = \ell(s_\Phi)$. So we get $\ell(s_\Phi s_\Theta w^{-1} s_0) = \ell(s_0 w s_\Theta s_\Phi) < \ell(s_0 w s_\Theta) + \ell(s_\Phi) = \ell(s_0) - \ell(w) - \ell(s_\Theta) + \ell(s_\Phi) = \ell(s_\Phi s_\Theta) + \ell(w^{-1} s_0)$. By 1.7(2) again we finally get $\Delta_{s_\Phi s_\Theta} \Delta_{w^{-1} s_0} = 0$, which implies that $\Delta_{s_\Phi s_\Theta} P_w = 0$. □

3. Complex flag manifolds and the Schubert cell decomposition

In this section we investigate the classical case that $G = GL_n(\boldsymbol{C})$ and $B =$ the large upper triangular matrix subgroup of G. In this case the coset space G/B is identified with the set $Fl_n(\boldsymbol{C})$ of full flags in $\boldsymbol{C}^n$. We shall first look at the cell structure of the Bruhat decomposition. Let H be the diagonal matrix subgroup of G, $\simeq (\boldsymbol{C}^\times)^n$ and N be the upper triangular matrices with all the diagonal entry 1. Let E_{ij} denote a square matrix with (i, j)-entry 1, all other entries being 0, $E_i = E_{ii}$ and $D(a_1, a_2, \ldots, a_n) = \sum_{i=1}^n a_i E_i$ denote a diagonal matrix. Then Lie algebras of H and N are $\mathfrak{h} = \sum_{i=1}^n \boldsymbol{C} E_i$ and $\mathfrak{n} = \sum_{i<j} \boldsymbol{C} E_{ij}$ respectively. The root system is $\Delta = \{\alpha_{ij} = x_i - x_j,\ 1 \le i \ne j \le n\}$ where $x_i \in \mathfrak{h}^*$ with $x_i(D(a_1, \ldots, a_n)) = a_i$. The α_{ij}-root space is $\mathfrak{g}_{ij} = \boldsymbol{C} E_{ij}$. $\Delta^+ = \{\alpha_{ij}, i < j\} \supset \Sigma = \{\alpha_i = \alpha_{i,i+1} = x_i - x_{i+1},\ 1 \le i < n\}$ are the set of positive roots and the set of simple roots respectively. Let $M = N_G(H)$. Then the Weyl group is $W = M/H$ and M is the subgroup of G comprised of all monomial matrices which have only one non-zero entry in each row and in each column. We see that M is isomorphic to a semidirect product $M \simeq \mathfrak{S}_n \ltimes H$ of the symmetric group $\mathfrak{S}_n$ on n letters and the diagonal subgroup H. The isomorphism is given by $\mathfrak{S}_n \ltimes H \xrightarrow{\sim} M$, $(w, h) \to m_w h$ where $w = \begin{pmatrix} 1, 2, \ldots, n \\ w_1, w_2, \ldots, w_n \end{pmatrix} \in \mathfrak{S}_n$ and m_w is a permutation matrix with (w_j, j)-entry 1 for each $j = 1, 2, \ldots, n$, all other entries being 0. We see that $m_w E_i m_w^{-1} = E_{w(i)}$ and if $h = D(a_1, a_2, \ldots, a_n) \in H$ then $m_w h$ is a matrix with (w_j, j)-entry a_j for $1 \le j \le n$, all other entries being 0. Hence the Weyl group W is isomorphic to $\mathfrak{S}_n$ and its action on $\mathfrak{h}$ and $\mathfrak{h}^*$ is a permutation of the coordinate axes: $W = M/H = \mathfrak{S}_n$ and $w \cdot E_i = m_w E_i m_w^{-1} = E_{w(i)}$, $w \cdot x_i = x_{w(i)}$ for $w \in W = \mathfrak{S}_n$. Since $(G = GL_n, B, M)$ is a Tits system (cf. [3, Ch. IV, §2, no. 2]) we have the Bruhat decomposition:

$$GL_n(\boldsymbol{C})/B = \bigcup_{w \in W} Bw \cdot B \quad \text{(disjoint union)}.$$

On the other hand the identification $GL_n(\boldsymbol{C})/B \xrightarrow{\sim} Fl_n(\boldsymbol{C})$ is given as follows. The set $Fl_n(\boldsymbol{C})$ of full flags in $\boldsymbol{C}^n$ is by definition, $Fl_n(\boldsymbol{C}) = \{(V_1, V_2, \ldots, V_n)$, sequences of linear subspaces of $\boldsymbol{C}^n | 0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \boldsymbol{C}^n$, $\dim_{\boldsymbol{C}} V_j = j\}$. Let $\{e_i\}$ be the standard basis of $\boldsymbol{C}^n$ and fix a base point $o = (\boldsymbol{C}^1, \boldsymbol{C}^2, \ldots, \boldsymbol{C}^n) \in$

$Fl_n(C^n)$ with $C^j = \text{span}\{e_1, e_2, \dots, e_j\}$. Then we have a bijection which identifies the two spaces,

$$GL_n(C)/B \cong Fl_n(C), \qquad gB \mapsto go = (gC^1, gC^2, \dots, gC^n).$$

If a matrix $g \in GL_n(C)$ is written by $g = (g_1, g_2, \dots, g_n)$ where $g_j = ge_j$ = the j-th column vector of g then $gC^j = \text{span}\{g_1, g_2, \dots, g_j\}$ = linear subspace of C^n spanned by the first j column vectors of the matrix g. By this identification a subset $Bw \cdot B = Bm_w \cdot B = Nm_w \cdot B \subset GL_n(C)/B$ corresponds to

$$Bw \cdot B = Nm_w \cdot B = \begin{bmatrix} 1 & & & & \\ & 1 & & * & \\ & & 1 & & \\ & O & & 1 & \\ & & & & 1 \end{bmatrix} \begin{array}{r} \\ \\ \\ w_j) \\ \\ \end{array} \begin{bmatrix} & \overset{j}{|} & \\ & | & \\ \text{———} & 1 & \\ O & & \\ & & \end{bmatrix} \cdot B$$

$$= \begin{array}{r} \\ \\ \\ \\ \\ w_j) \\ \\ \\ \end{array} \begin{bmatrix} & \overset{j}{*} \\ & * \\ & * \\ & * \\ & * \\ \text{———} & 1 \\ O & 0 \\ & 0 \end{bmatrix} \cdot \begin{bmatrix} * & & & & & & & \\ & * & & & * & & & \\ & & * & & & & & \\ & & & * & & & & \\ & & & & * & & & \\ & O & & & & * & & \\ & & & & & & * & \\ & & & & & & & * \end{bmatrix}$$

$$= \begin{array}{r} \\ \\ \\ w_j) \\ \\ \\ \end{array} \begin{bmatrix} & \overset{j}{*} & & \\ & * & & \\ & * & & \\ \text{———} & 1 & 0 & 0 \\ O & 0 & & \\ & 0 & & \end{bmatrix} \cdot B \cong \begin{array}{r} \\ \\ \\ w_j) \\ \\ \\ \end{array} \begin{bmatrix} & \overset{j}{*} & & \\ & * & & \\ & * & & \\ \text{———} & 1 & 0 & 0 \\ O & 0 & & \\ & 0 & & \end{bmatrix} o \subset Fl_n(C)$$

Hence $Bw \cdot B$ corresponds to the set of all matrices $b = (b_{ij})$ with $b_{w_j, j} = 1$, $b_{ij} = b_{w_j, k} = 0$ if $i > w_j$, $k > j$ for each $j = 1, 2, \dots, n$ and all other entries are free $b_{ij} = *$, or the flags which are made by column vectors of those matrices. Since $go \neq g'o$ in $Fl_n(C)$ if $gB \neq g'B$ in G/B, and two different matrices of the above form cannot transform each other by the right B-action, two matrices which have different free parameter $*$ correspond to different flags in C^n. Since the topology of $Fl_n(C)$ is induced from matrix topology by the above bijection, $Bw \cdot B$ thus forms a cell in $Fl_n(C)$ which is expressed by matrices of the above form. $Bw \cdot B$ is sometimes called a Bruhat cell.

Next let $\Phi = \Sigma \backslash \{\alpha_p\}$ and $p + q = n$. Then

$$W_\Phi = \mathfrak{S}_p \times \mathfrak{S}_q = \left(\begin{array}{c|c} \mathfrak{S}_p & O \\ \hline O & \mathfrak{S}_q \end{array}\right), \qquad P_\Phi = BW_\Phi B = \left(\begin{array}{c|c} GL_p & * \\ \hline O & GL_q \end{array}\right),$$

and $G/P_\Phi = Gr_{p,q}(C) =$ the Grassmann manifold of p-subspaces in C^n. The identification is given by a bijection:

$$GL_n(C)/P_\Phi \simeq Gr_{p,q}(C), \qquad gP_\Phi \mapsto gC^p = \text{span}\,\{g_1, g_2, \dots, g_p\}.$$

We look at the cell structure of the Bruhat decomposition:

$$GL_n(C)/P_\Phi = \textstyle\bigcup_{w \in W^\Phi} Bw \cdot P_\Phi \quad \text{(disjoint union)}.$$

First from $w\alpha_i = w(x_i - x_{i+1}) = x_{w(i)} - x_{w(i+1)}$ it is easy to see that $W^\Phi = \{w \in W \mid w\Phi \subset \Delta^+\} = \left\{ w = \begin{pmatrix} 1, 2, \dots, n \\ w_1, w_2, \dots, w_n \end{pmatrix} \in \mathfrak{S}_n = W \mid w_1 < w_2 < \cdots < w_p,\ w_{p+1} < \cdots < w_n \right\}$. Therefore the Bruhat cell $Bw \cdot P_\Phi = Nm_w \cdot P_\Phi$ corresponds to

$Bw \cdot P_\Phi = Nm_w \cdot P_\Phi$

$$= \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & * & & \\ & & & & & \\ & O & & & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{array}{r} \\ \\ w_1) \\ w_2) \\ \\ w_p) \\ \\ \end{array} \left[\begin{array}{ccc|cccc} & & O & 1 & & & \\ & & & & 1 & & \\ 1 & & & & & & \\ -1 & & & & & & \\ & O & & & & 1 & \\ \hline & & -1 & & & & \\ & & & & & & 1 \end{array}\right] \cdot P_\Phi$$

$$= \left[\begin{array}{ccccc|cccc} * & * & * & & * & 1 & & & \\ * & * & * & & * & & 1 & * & \\ 1 & * & * & * & * & & & & \\ 0 & * & * & * & * & & & & \\ 0 & 1 & * & & * & & & & \\ & & & & * & & & & \\ & & & & * & & 1 & & \\ & & O & & 1 & & & & \\ & & & & & & & & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} * & * \\ \hline O & * \end{array}\right]$$

$$= \left[\begin{array}{ccccc|ccc} * & * & * & & * & 1 & & \\ * & * & * & * & * & & 1 & \\ 1 & 0 & 0 & & 0 & & & O \\ 0 & * & * & * & * & & & \\ 0 & 1 & 0 & & 0 & & & \\ & & & & * & & & \\ & & & & * & & & 1 \\ & & O & & 1 & & O & \\ & & & & & & & & 1 \end{array}\right] \cdot P_\Phi \simeq \text{span} \begin{array}{c} \\ w_1) \\ \\ w_2) \\ \\ w_p) \end{array} \left[\begin{array}{ccccc} * & * & * & & * \\ * & * & * & * & * \\ 1 & 0 & 0 & & 0 \\ 0 & * & * & * & * \\ 0 & 1 & 0 & & 0 \\ & & & & * \\ & & O & & * \\ \hline & & & & 1 \end{array}\right]$$

$$\subset Gr_{p,q}(C),$$

where the above indicates a p-subspace spanned by column vectors of the matrix. Thus the Bruhat cell $Bw \cdot P_\Phi$ corresponds exactly to the classical Schubert cell of the symbol $(w_1, w_2, \ldots, w_p)$ in $Gr_{p,q}(C)$: $e(w_1, w_2, \ldots, w_p) = \left\{ W \in Gr_{p,q}(C) | 0 \subset W \cap C^1 \subset W \cap C^2 \subset \cdots \subset W \cap C^n = W, \dim \frac{W \cap C^i}{W \cap C^{i-1}} = 0 \text{ or } 1, \dim (W \cap C^{w_i}) = i, \dim (W \cap C^{w_i - 1}) = i - 1 \right\}$ (cf., e.g. [11, p. 75]).

Now let $\Theta = \{\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{n-1}\} = \Sigma \backslash \{\alpha_1, \alpha_2, \ldots, \alpha_p\} \subset \Phi$ and also let $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_{p-1}\} \subset \Phi$. Then

$$W_\Theta = \mathfrak{S}_q = \left(\begin{array}{c|c} I_p & O \\ \hline O & \mathfrak{S}_q \end{array}\right), \quad W_\Gamma = \mathfrak{S}_p = \left(\begin{array}{c|c} \mathfrak{S}_p & O \\ \hline O & I_q \end{array}\right), \quad \text{hence } W_\Phi = W_\Gamma \times W_\Theta,$$

$$\text{and} \qquad P_\Theta = \left[\begin{array}{cccc|c} * & & & & \\ & * & & * & \\ O & & & & * \\ & & * & & \\ \hline & & & * & \\ & O & & & * \end{array}\right] \subset P_\Phi.$$

We shall calculate the Gysin homomorphism $\pi_!$ associated to the natural map:

$$\pi: G/P_\Theta \to G/P_\Phi = Gr_{p,q}(C).$$

We first review the cohomology structure of these spaces (cf. [2], [13, Theorem 4.2]). For $G = GL_n(C)$ we have the unit lattice $\mathfrak{h}_Z = \bigoplus_{i=1}^n Ze_i$ and its dual lattice $\mathfrak{h}_Z^* = \bigoplus_{i=1}^n Zx_i$. So $R = Q[\mathfrak{h}_Q] = Q[x_1, x_2, \ldots, x_n]$ and $I = R^W = Q[e_1, e_2, \ldots, e_n] =$ the ring of symmetric polynomials where e_k is the k-th elementary symmetric function. Thus the cohomology ring of the full flag manifold $Fl_n(C) = G/B$ is

$$H^*(G/B) = \boldsymbol{Q}[x_1, \dots, x_n]/J$$

where $J = (e_1, e_2, \dots, e_n)$ = the ideal generated by symmetric polynomials without constant term. We know that $H^*(G/B)$ has an additive basis $\{x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} | 0 \le \alpha_i \le n-i,\ i = 1, 2, \dots, n\}$. We next see the cohomology ring of the Grassmannian $G/P_\Phi = Gr_{p,q}(\boldsymbol{C})$ as,

$$H^*(G/P_\Phi) = (R/J)^{W_\Phi} = R^{W_\Phi}/I^+ R^{W_\Phi},$$

$$R^{W_\Phi} = \boldsymbol{Q}[x]^{\mathfrak{S}_p \times \mathfrak{S}_q} = \boldsymbol{Q}[c_1, c_2, \dots, c_p, c'_1, c'_2, \dots, c'_q],$$

where $c_i = e_i(x_1, \dots, x_p)$ and $c'_j = e_j(x_{p+1}, \dots, x_n)$. $I^+ R^{W_\Phi} = (e_1, \dots, e_n)$ is an ideal generated by symmetric polynomials in this ring. By using the generating function for the elementary symmetric function: $E(t) = \sum_{r \ge 0} e_r t^r = \prod_{i=1}^n (1 + x_i t)$, we easily see a relation: $e_r = \sum_{i+j=r} c_i c'_j$. Hence we have

$$H^*(G/P_\Phi) = \bar{R}^{W_\Phi} = \boldsymbol{Q}[c_1, \dots, c_p, c'_1, \dots, c'_q]/(\textstyle\sum_{i+j=r} c_i c'_j = 0 | r \ge 1).$$

We note that c_i and c'_j are the canonical Chern classes of the tautological bundles on the Grassmannian $Gr_{p,q}(\boldsymbol{C}) = G/P_\Phi$ and that the above identity is also deduced directly by algebraic topology. The ring $H^*(G/P_\Phi)$ is generated by $c_1, c_2, \dots, c_p$ (as ring) and has an additive basis $\{c_{j_1} c_{j_2} \cdots c_{j_k} | 0 \le k \le q\}$. In the same way we have for the partial flag manifold G/P_Θ,

$$H^*(G/P_\Theta) = \bar{R}^{W_\Theta} = \boldsymbol{Q}[x_1, x_2, \dots, x_p, c'_1, c'_2, \dots, c'_q]/J \qquad \text{where}$$

$$J = (e_r = \textstyle\sum_{1 \le i_1 < i_2 < \cdots < i_k \le p, 0 \le k \le r} x_{i_1} x_{i_2} \cdots x_{i_k} c'_{r-k} | r \ge 1).$$

Moreover $H^*(G/P_\Theta)$ has the ring-generators $\{x_1, x_2, \dots, x_p\}$ and has an additive basis $\{x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_p^{\alpha_p} | 0 \le \alpha_i \le n-i,\ i = 1, 2, \dots, p\}$.

We recall several facts about symmetric polynomials (cf. [10] for example). For indeterminates $x_1, x_2, \dots, x_n$ let $A = \boldsymbol{Z}[x_1, x_2, \dots, x_n]$. Let $e_r = e_r(x_1, x_2, \dots, x_n) = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$ be the elementary symmetric function which has the generating function (that we have already used above):

$$E(t) = \textstyle\sum_{r \ge 0} e_r t^r = \prod_{i=1}^n (1 + x_i t).$$

Also let $h_r = h_r(x_1, x_2, \dots, x_n)$ be the complete symmetric function with generating function:

$$H(t) = \textstyle\sum_{r \ge 0} h_r t^r = \prod_{i=1}^n (1 - x_i t)^{-1} \qquad \text{in } A[[t]].$$

The identity $E(-t)H(t) = 1$ implies that

$$\textstyle\sum_{r=0}^l (-1)^r e_r h_{l-r} = h_l - e_1 h_{l-1} + e_2 h_{l-2} + \cdots + (-1)^l e_l = 0,$$

for all $l \ge 1$. For $\alpha \in \boldsymbol{N}^n$ ($\boldsymbol{N} = \{0, 1, 2, \dots\}$) let

$$a_\alpha = a_\alpha(x_1, x_2, \dots, x_n) = \textstyle\sum_{w \in \mathfrak{S}_n} \varepsilon(w) w \cdot x^\alpha = \det(x_i^{\alpha_j}) \in A,$$

where $w\cdot x^{\alpha} = x_{w(1)}^{\alpha_1} x_{w(2)}^{\alpha_2} \cdots x_{w(n)}^{\alpha_n} = x^{w^{-1}\cdot\alpha}$ for $w \in \mathfrak{S}_n$. Then a_α is skew-symmetric; $w\cdot a_\alpha = a_{w\cdot\alpha} = \varepsilon(w)a_\alpha$ for $w \in \mathfrak{S}_n$ and for $\delta = (n-1, n-2, \dots, 1, 0)$,

$$a_\delta = \det(x_i^{n-j}) = \prod_{i<j}(x_i - x_j) = \prod_{\alpha\in\Delta^+}\alpha$$

is the Vandermonde determinant. For $\lambda \in N^n$ the Schur function S_λ is defined by a homogeneous symmetric polynomial of degree $|\lambda|$:

$$S_\lambda = S_\lambda(x_1, x_2, \dots, x_n) = a_{\lambda+\delta}(x)/a_\delta(x) \in A.$$

If $\lambda \in N^n$ is a partition, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, let $d(\lambda) = \#\{i|\lambda_i \neq 0\}$ be the depth of λ and let λ' denote the conjugate partition of λ, i.e. $\lambda'_j = \#\{i|\lambda_i \geq j\}$. Then the Schur function S_λ is expressed by elementary symmetric functions e_r or by complete symmetric functions h_r as

3.1.

$$S_\lambda = \det(h_{\lambda_i - i+j})_{1\leq i,j\leq n} = \det(e_{\lambda'_i - i+j})_{1\leq i,j\leq m},$$

where $n \geq d(\lambda)$ and $m \geq d(\lambda')$.

We can now describe the Gysin homomorphisms $\pi_!$ for $\pi: G/P_\Theta \to G/P_\Phi = Gr_{p,q}(C)$ by the Schur function S_λ and by elementary symmetric functions c_1, c_2, …, c_p those are the Chern classes on $Gr_{p,q}(C)$. We thus regain results of J. Damon [4, Cor. 2 of Theorem 1] and T. Sugawara [13, Theorem 6.2 and Cor. 6.3] in our context.

3.2 THEOREM. *Keep the notation above. Let $s_\Phi = s_\Gamma s_\Theta$ be the decomposition of elements of maximal length according to $W_\Phi = W_\Gamma \times W_\Theta$. The Gysin homomorphism $\pi_!$: $H^*(G/P_\Theta) = \bar{R}^{W_\Theta} \to H^*(G/P_\Phi) = \bar{R}^{W_\Phi}$ for π is given as follows.*

(1) *For a polynomial $f \in \bar{R}^{W_\Phi}$*

$$\pi_! f = \Delta_{s_\Gamma} f = \sum_{w\in W_\Gamma} \varepsilon(w) w\cdot f / \prod_{1\leq i<j\leq p}(x_i - x_j),$$

where $w \in W_\Gamma = \mathfrak{S}_p$ acts on p variables x_1, x_2, …, x_p of the polynomial f. In particular $\pi_!(wf) = \varepsilon(w)\pi_! f$, $w \in W_\Gamma$.

(2) *For a monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_p^{\alpha_p} \in \bar{R}^{W_\Theta}$ ($\alpha \in N^p$),*

$$\pi_!(x^\alpha) = S_{\alpha-\delta}(x_1, \dots, x_p)$$

where $\delta = (p-1, p-2, \dots, 1, 0) \in N^p$. In particular if $\lambda \in N^p$ is a partition, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$ then

$$\pi_!(x^{\lambda+\delta}) = \det(c_{\lambda'_i - i+j}) = \det(\bar{c}_{\lambda_i - i+j})$$

where we put $\bar{c}_j = (-1)^j c'_j$.

PROOF. (1) By 2.3(2), $\pi_! = \Delta_{s_\Gamma}$ since $s_\Gamma = s_\Phi s_\Theta$. Note that s_Γ is the element of maximal length in $W_\Gamma = \mathfrak{S}_p$ and Δ_{s_Γ} acts on the first p variables

$x_1, \dots, x_p$ of a polynomial in R by definition of the Δ-operator. Then our assertion follows from 1.11.

(2) Since $a_\delta(x_1, \dots, x_p) = \prod_{1 \le i < j \le p}(x_i - x_j) = \prod_{\alpha \in \Delta^+ \cap \langle\Gamma\rangle} \alpha$ we know that for a monomial x^α ($\alpha \in N^p$).

$$\pi_!(x^\alpha) = a_\alpha(x_1, \dots, x_p)/a_\delta(x_1, \dots, x_p) = S_{\alpha-\delta}(x_1, \dots, x_p),$$

by the very definition of the Schur function. Note that $S_{\alpha-\delta}(x_1, \dots, x_p)$ is a symmetric polynomial of $x_1, \dots, x_p$ and so it belongs to $\bar{R}^{W_\Phi} = H^*(G/P_\Phi)$. We shall express it by the Chern classes c_i and c'_j. Now there is a partition $\lambda \in N^p$ and $w \in \mathfrak{S}_p$ such that $w \cdot \alpha = \lambda$. Note that $\pi_!(x^{w \cdot \alpha}) = \pi_!(w^{-1}x^\alpha) = \varepsilon(w)\pi_!(x^\alpha)$. We thus consider $\pi_!(x^\alpha)$ for a strict partition $\alpha = \lambda + \delta$. In view of 3.1 we know that for the last identity, it suffices to show that

$$h_j(x_1, \dots, x_p) = (-1)^j e_j(x_{p+1}, \dots, x_n)$$

in $\bar{R} = R/J = Q[x_1, \dots, x_n]/(e_1, \dots, e_n)$ since $c_i = e_i(x_1, \dots, x_p)$ and $c'_j = e_j(x_{p+1}, \dots, x_n)$. Let $E_1(t) = \prod_{i=1}^p (1 + x_i t)$ and $E_2(t) = \prod_{i=p+1}^n (1 + x_i t)$ be generating functions of $e_r(x_1, \dots, x_p)$ and $e_r(x_{p+1}, \dots, x_n)$. Then we have

$$E_1(t)E_2(t) = E(t) = \sum_{r \ge 0} e_r t^r = 1 \qquad \text{in } (R/J)[t].$$

Let $H_1(t) = \prod_{i=1}^p (1 - x_i t)^{-1}$ be the generating function of $h_r(x_1, \dots, x_p)$. Then $E_1(-t)H_1(t) = 1$. Hence we obtain that $H_1(t) = E_1(-t)^{-1} = E_2(-t)$ in $(R/J)[[t]]$, which implies our identity. □

References

[1] I. N. Bernstein, I. M. Gel'fand and S. I. Gel'fand, Schubert cells and cohomology of the spaces G/P, Russ. Math. Surveys **28**, No. 3 (1973) 1–26.

[2] A. Borel, Sur la cohomologie des espaces fibré principaux et des espaces homogènes des groupes de Lie compacts, Ann. of Math. (2) **57** (1953), 115–207.

[3] N. Bourbaki, Groupes et algèbres de Lie, Ch. IV, V, VI, Masson, Paris 1981.

[4] J. Damon, The Gysin homomorphism for flag bundles: Application, Amer. J. Math. **96** (1974), 248–260.

[5] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, Invent. Math. **21** (1973), 287–301.

[6] M. Demazure, Désingularisation des variétés de Schubert généralisées, Ann. Sci. Ec. Norm. Sup., **7** (1974), 53–88.

[7] H. Hiller, Schubert calculus of a Coxeter group, l'Enseign. Math. **27** (1981), 57–84.

[8] J. E. Humphreys, Reflection groups and Coxeter groups, Cambridge Studies in Advanced Math. 29, Cambridge Univ. Press, 1990.

[9] B. Kostant, Lie algebra cohomology and generalized Schubert cells, Ann. of Math. (2) **77** (1963), 72–144.

[10] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd edition, Oxford University Press, 1995.

[11] J. Milnor and J. Stasheff, Characteristic Classes, Annals of Math. Studies 76, Princeton University Press, 1974.
[12] W. Stoll, Invariant Forms on Grassmann Manifolds, Annals of Math. Studies 89, Princeton Univ. Press, 1977.
[13] T. Sugawara, The Gysin homomorphism for generalized flag bundles, Mem. Fac. Sci. Kyushu Univ., Ser. A, Vol. **42**, No. 2 (1988), 131–144.
[14] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups, I, Springer-Verlag, 1972.

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