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# An approximation of Hausdorff dimensions of generalized cookie-cutter Cantor sets

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ABSTRACT. We give a method of approximation of Hausdorff dimensions of generalized cookie-cutter Cantor sets using the thermodynamic formalism.

## 1. Introduction

Cookie-cutter Cantor sets in the line are studied as simple examples of fractals which are invariant sets of dynamical systems. This enables us to develop applications of some ideas from dynamical systems theory to fractal geometry (see [1], [5]). In this paper, we study an approximation of Hausdorff dimensions of 'generalized' cookie-cutter Cantor sets using the thermodynamic formalism. The reader should refer to the thermodynamic formalism, for example, in [2], [8], [9].

The original cookie-cutter Cantor set, whose prototype is the standard middle-third Cantor set $\{\sum_{n=1}^{\infty} x_n 3^{-n} : x_n = 0, 2\}$, was defined and studied minutely by Bedford [1]. We define a generalized cookie-cutter map and a generalized cookie-cutter Cantor set in the line as follows.

Write $I = [0, 1]$, and take $0 = x_0 < x_1 < \cdots < x_{2p-1} = 1$. Put $J_0 = [x_0, x_1]$, $J_i = (x_{2i}, x_{2i+1}]$ for $i = 1, \ldots, p-1$ and $D = \bigcup_{i=0}^{p-1} J_i$. A *generalized cookie-cutter map* $f$ is a mapping $f : D \to I$ with the properties that

(i) $f|_{\mathring{J}_i}$ is a one-to-one map onto $(0, 1)$,

(ii) $f$ is continuous on $J_i$ and in $C^{1+\gamma}(\gamma > 0)$, i.e., $|f'(x) - f'(y)| \leq L|x - y|^{\gamma}$ for any $x, y \in J_i$ with some $L > 0$ (independ of $x, y$), moreover $\inf\{|f'(x)| : x \in J_i\} > 1$ for $i = 1, \ldots, p-1$, where $\mathring{J}$ denotes the interior of $J$.

A *generalized cookie-cutter set* $C(f)$ associated with $f$ is the set:

$$C(f) = \overline{\{x \in D : f^n(x) \in D \text{ for any } n \geq 1\}}.$$

Put

$$C_n = \overline{f^{-n}(I)}, \qquad \text{for any } n \in \mathbf{N}.$$

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Then $C_n$ is equal to the union of $p^n$ disjoint closed intervals $I_1^{(n)}, \dots, I_N^{(n)}$, $(N = p^n)$, such that $\max I_i^{(n)} < \min I_j^{(n)}$ if $i < j$ for any $n \in \mathbf{N}$. Since we have $C_n \supset C_{n+1}$ for any $n \in \mathbf{N}$, we see that $C(f) = \bigcap_{n=1}^{\infty} C_n = \lim_{n\to\infty} C_n$. By definition, the generalized cookie-cutter map $f$ has a repelling fixed point, so that $C(f) \neq \emptyset$. If we take $x_1 = 1/3$, $x_2 = 2/3$, $f(x) = 3x \pmod 1$ and skip $(x_1, x_2]$, then $C(f)$ is the middle-third Cantor set.

For a generalized cookie-cutter map $f$, set

$$A^{(n)} = (a_{ij}^{(n)})_{i,j=1}^{N}, \qquad \text{where } a_{ij}^{(n)} = \begin{cases} 1 & \text{if } f^{-1}(I_i^{(n)}) \cap I_j^{(n)} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_s^{(n)} = (b_{ij}^{(n)})_{i,j=1}^{N}, \qquad \text{where } b_{ij}^{(n)} = \begin{cases} s_i & \text{if } a_{ij}^{(n)} = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $s = (s_1, \dots, s_N)$. We call $A^{(n)}$ and $B_s^{(n)}$ the *structure matrix* and the *'weighted' structure matrix* with weight $s$ of $n$-th step, respectively. Set

$$r_{(n)}^{\delta} = ((r_1^{(n)})^{\delta}, (r_2^{(n)})^{\delta}, \dots, (r_N^{(n)})^{\delta}),$$
$$(\text{resp. } R_{(n)}^{\Delta} = ((R_1^{(n)})^{\Delta}, (R_2^{(n)})^{\Delta}, \dots, (R_N^{(n)})^{\Delta})),$$

where

$$r_i^{(n)} = \inf_{x \in \mathring{I}_i^{(n)}} 1/|f'(x)|, \qquad R_i^{(n)} = \sup_{x \in \mathring{I}_i^{(n)}} 1/|f'(x)| \qquad \text{for } i = 1, \dots, N.$$

We denote the maximal eigenvalue of $B_{r_{(n)}^{\delta}}^{(n)}$ (resp. $B_{R_{(n)}^{\Delta}}^{(n)}$) by $\lambda_{\max}^{-}(\delta, n)$ (resp. $\lambda_{\max}^{+}(\Delta, n)$). We can show that there exists a unique solution $\delta_n$ (resp. $\Delta_n$) of the following equation

$$\lambda_{\max}^{-}(\delta, n) = 1 \qquad (\text{resp. } \lambda_{\max}^{+}(\Delta, n) = 1). \tag{1.1}$$

Our main result is the following. The reader should refer to examples in §4 for motivation.

THEOREM. *The Hausdorff dimension* H-dim$(C(f))$ *of the generalized cookie-cutter set* $C(f)$ *satisfies*

$$\delta_n \leq \text{H-dim}(C(f)) \leq \Delta_n \qquad \textit{for any } n \in \mathbf{N}. \tag{1.2}$$

*Moreover we have*

$$\delta_n \leq \delta_{n+1}, \qquad \Delta_n \geq \Delta_{n+1}, \qquad \textit{for any } n \in \mathbf{N}, \tag{1.3}$$

*and*

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \Delta_n = \text{H-dim}(C(f)). \tag{1.4}$$

*The above limits converge exponentially, i.e., there exists $C > 0$ and $0 < \xi < 1$ such that*

$$0 \le \text{H-dim}(C(f)) - \delta_n \le C\xi^n, \quad 0 \le \Delta_n - \text{H-dim}(C(f)) \le C\xi^n \quad \textit{for any } n \in \mathbf{N}. \tag{1.5}$$

REMARK 1.1

(i) For the right end $x_{2i+1}$ of $J_i$, $f'(x_{2i+1})$ denotes the left derivative of $f$ at $x_{2i+1}$ and $f'(0)$ denotes the right derivative of $f$ at 0.

(ii) To simplify the notation and the statement, we defined the domain $D$ as above, but we can also prove the theorem even if $D$ is the union which skips at least one interval in $[x_0, x_1]$ and $(x_i, x_{i+1}]$, $i = 1, \dots, 2p-2$.

## 2. Proofs

We first prepare some tools and results on the thermodynamic formalism. Bowen [2] and Walters [9] used the pressure of some real analytic functions (potentials) to study the equilibrium state. We study pressures for some function corresponding to the generalized cookie-cutter map $f$. Put $\varphi(x) = -\log|f'(x)|$ for $x \in C(f)$ and

$$\varphi_n^-(x) = \log r_k^{(n)}, \qquad \varphi_n^+(x) = \log R_k^{(n)} \qquad x \in I_k^{(n)}, k = 1, \dots, N$$

for each $n \in \mathbf{N}$. Clearly we see that

$$\varphi_n^-(x) \le \varphi(x) \le \varphi_n^+(x) \qquad \text{for any } x \in C(f) \text{ and } n \in \mathbf{N}. \tag{2.1}$$

The pressures for potentials $\varphi_n^-(x)$ and $\varphi_n^+(x)$ are given as follows:

LEMMA 2.1

$$P(\alpha\varphi_n^-) = \log \lambda_{\max}^-(\alpha, n), \qquad P(\alpha\varphi_n^+) = \log \lambda_{\max}^+(\alpha, n) \qquad \textit{for any } \alpha > 0, \tag{2.2}$$

*where $P(\phi)$ denotes the pressure for $\phi$.*

PROOF. We show only $P(\alpha\varphi_n^-) = \log \lambda_{\max}^-(\alpha, n)$. By elementary calculus, we see that

$$\sum_{x \in \mathrm{Fix} f^m} \exp \sum_{k=0}^{m-1} \alpha\varphi_n^-(f^k(x)) = \mathrm{Trace}(B_{r_{(n)}^\alpha}^{(n)})^m \qquad \text{for any } m \in \mathbf{N}.$$

Therefore in this particular case we have

$$P(\alpha\varphi_n^-) = \lim_{m\to\infty}\frac{1}{m}\log \operatorname{Trace}(B^{(n)}_{r^\alpha_{(n)}})^m = \lim_{m\to\infty}\frac{1}{m}\log\sum_{i=1}^{N}\lambda_i^-(\alpha,n)^m$$
$$= \log\lambda^-_{\max}(\alpha,n)$$

by the definition of the pressure. ■

REMARK 2.2 We can calculate the Ruelle's zeta function [7, Proposition 1] for $\delta\varphi_n^-$ and $\Delta\varphi_n^+$ as follows:

$$\zeta_{\delta\varphi_n^-}(u) = \prod_{i=1}^{N}\frac{1}{1-u\lambda_i^-(\delta,n)}, \qquad \left(\text{resp. } \zeta_{\Delta\varphi_n^+}(u) = \prod_{i=1}^{N}\frac{1}{1-u\lambda_i^+(\Delta,n)},\right) \tag{2.3}$$

where $\lambda_i^-(\delta,n)$ (resp. $\lambda_i^+(\Delta,n)$), $i=1,\dots,N$, are the eigenvalues of $B^{(n)}_{r^\delta_{(n)}}$ (resp. $B^{(n)}_{R^\Delta_{(n)}}$). Mori [6, Theorem B] calculated the zeta function using a renewal equation.

Now since $f$ is in $C^{1+\gamma}$ and expanding on $C(f)$, the matrices $A^{(n)}$ and $B^{(n)}_{r^\delta_{(n)}}$ are non-negative and irreducible by [2, Lemma 1.3]. The following well-known lemma is applicable to the matrices.

LEMMA 2.3 *Let $P=(p_{ij})_{i,j=1}^n$ and $Q=(q_{ij})_{i,j=1}^n$ be non-negative irreducible matrices. If $p_{ij}\le q_{ij}$ and there exist $1\le i,j\le n$ such that $p_{ij}<q_{ij}$, then $\lambda_P<\lambda_Q$, where $\lambda_P$ and $\lambda_Q$ are the maximal eigenvalues of $P$ and $Q$, respectively.*

In the present setting, we assume that $|f'(x)|>1$ for any $x\in C(f)$. Hence we see that $(b_{ij}^{(n)})^\alpha > (b_{ij}^{(n)})^\beta$ for $0<\alpha<\beta$ and $1\le i,j\le N$ for any $n\in\mathbf{N}$. Therefore we can apply Lemma 2.3 to $B^{(n)}_{r^\alpha_{(n)}}$ and $B^{(n)}_{r^\beta_{(n)}}$ and we obtain that

$$\lambda^-_{\max}(\alpha,n) > \lambda^-_{\max}(\beta,n), \qquad \lambda^+_{\max}(\alpha,n) > \lambda^+_{\max}(\beta,n) \qquad \text{for } 0<\alpha<\beta \tag{2.4}$$

for any $n\in\mathbf{N}$. Let us remark the following fact which is called the *Bowen-Ruelle formula*. The Hausdorff dimension of the mixing repeller is a unique solution $t$ of the equation $P(-t\log|f'|)=0$ (see [7, Proposition 4]).

Using these arguments, we prove the theorem.

PROOF OF (1.2) IN THEOREM. By (2.1) and [9, Theorem 9.7 (ii)], we have $P(\alpha\varphi_n^-)\le P(\alpha\varphi)\le P(\alpha\varphi_n^+)$. Using (2.2) and the Bowen-Ruelle formula we obtain the following inequalities.

$$\lambda^-_{\max}(t,n)\le 1\le\lambda^+_{\max}(t,n) \qquad \text{for any } n\in\mathbf{N}, \tag{2.5}$$

where $t=\text{H-dim}(C(f))$. By the definition of $\delta_n$, we have $\lambda^-_{\max}(t,n)\le 1 = \lambda^-_{\max}(\delta_n,n)$, so we have $\delta_n\le t$ by (2.4). Similarly we see that $t\le\Delta_n$.

Proof of (1.3). Clearly we have $\varphi_n^-(x) \le \varphi_{n+1}^-(x)$ for any $x \in C(f)$ and $n \in \mathbf{N}$. Therefore we have $P(\alpha\varphi_n^-) \le P(\alpha\varphi_{n+1}^-)$ for any $\alpha > 0$ and $n \in \mathbf{N}$. By (2.2), we have $\log \lambda_{\max}^-(\alpha, n) \le \log \lambda_{\max}^-(\alpha, n+1)$, so $\lambda_{\max}^-(\alpha, n) \le \lambda_{\max}^-(\alpha, n+1)$. Similarly we obtain that $\lambda_{\max}^+(\alpha, n) \ge \lambda_{\max}^+(\alpha, n+1)$ for $\alpha > 0$. Therefore by definition, $\delta_n$ and $\delta_{n+1}$ satisfy the following relations

$$1 = \lambda_{\max}^-(\delta_{n+1}, n+1) = \lambda_{\max}^-(\delta_n, n) \le \lambda_{\max}^-(\delta_n, n+1).$$

By (2.4), we obtain $\delta_n \le \delta_{n+1}$. Similarly we obtain $\varDelta_n \ge \varDelta_{n+1}$.

Proof of (1.4). Since we assume that $f$ is in $C^{1+\gamma}$, there exists $C_1 > 0$ such that $\|\varphi - \varphi_n^-\| \le C_1\xi^n$, where $\|\phi\| = \sup_{x \in C(f)}|\phi(x)|$ and $0 < \xi = \max_{x \in C(f)} 1/|f'(x)| < 1$ (see [3, Lemma 4]). Since we have $\varphi \le \varphi_n^- + \|\varphi - \varphi_n^-\| \le \varphi_n^- + C_1\xi^n$, we see that

$$P(\alpha\varphi) \le P(\alpha\varphi_n^- + C_2\xi^n) = P(\alpha\varphi_n^-) + C_2\xi^n,$$

where $C_2 = \alpha C_1$ (see [9, Theorem 9.7 (iv)]). This implies that

$$0 \le P(\alpha\varphi) - P(\alpha\varphi_n^-) \le C_2\xi^n \to 0 \qquad \text{as } n \to \infty. \tag{2.6}$$

By the Bowen-Ruelle formula and (2.2), we have

$$0 = P(t\varphi) = \lim_{n\to\infty} P(t\varphi_n^-) = \lim_{n\to\infty} \log \lambda_{\max}^-(t, n) \qquad \text{for } t = \text{H-dim}(C(f)). \tag{2.7}$$

By (1.2) and (1.3), $\delta_n$ converges to a limit $s$. Lemma 2.1 and (2.6) guarantee that $1 = \lim_{n\to\infty} \lambda_{\max}^-(\delta_n, n) = \lim_{n\to\infty} \lambda_{\max}^-(s, n) = e^{P(s\varphi)}$. By the Bowen-Ruelle formula, we obtain that $t = s$. Similarly we obtain that $\lim_{n\to\infty} \varDelta_n = t$.

Proof of (1.5). By (2.6), the definition of $\delta_n$ and Lemma 2.1, we have

$$(C_3)^{\xi^n} = (C_3)^{\xi^n} \lambda_{\max}^-(\delta_n, n) \le \lambda_{\max}^-(t, n) \qquad \text{for any } n \in \mathbf{N},$$

where $0 < C_3 = e^{-C_2} < 1$. We can choose $C > 0$ such that

$$\max_{1 \le i \le N} (r_i^{(n)})^C \le C_3.$$

Put $\delta_n' = \delta_n + C\xi^n$. Then we can apply Lemma 2.3 to both $B^{(n)}_{r_{(n)}^{\delta_n'}}$ and $(C_3)^{\xi^n} B^{(n)}_{r_{(n)}^{\delta_n}}$. Therefore we obtain that

$$\lambda_{\max}^-(\delta_n', n) \le (C_3)^{\xi^n} \lambda_{\max}^-(\delta_n, n) \le \lambda_{\max}^-(t, n). \tag{2.8}$$

By (2.4), we have $\delta_n \le t \le \delta_n' = \delta_n + C\xi^n$. So we deduce that $0 \le t - \delta_n \le C\xi^n$. Similarly we obtain that $0 \le \varDelta_n - t \le C\xi^n$. ■

We can obtain (1.2) without using the theory of the thermodynamic formalism. Yin [10] proved (1.2) for a fixed $n \in \mathbf{N}$ in a setting of iterated function systems. But in this paper, by only simple calculus, we show it easily using the theory of the thermodynamic formalism.

REMARK 2.4 Recall that $\delta_n$ (resp. $\Delta_n$) was defined by the unique solution of (1.1). Now we can see that

$$\delta_n = \max\{\delta \in [0,1] : \det(B^{(n)}_{r^{\delta}_{(n)}} - E_N) = 0\}, \tag{2.9}$$

$$\Delta_n = \max\{\Delta \in [0,1] : \det(B^{(n)}_{R^{\Delta}_{(n)}} - E_N) = 0\}, \tag{2.10}$$

where $E_N$ is an $N$-dimensional identity matrix. We had better use (2.9) and (2.10) to calculate $\delta_n$ and $\Delta_n$ with computer simulations.

## 3. Examples

To help understanding our theorem, we apply our method to the approximation of the Hausdorff dimension of a cookie-cutter set $C(f)$ given by a map $f$ satisfying following properties:
(i) the domain of $f$ is $[0,x_1]\cup(x_2,1]$,
(ii) $f$ is unimodal and $f(x)>1$ for some $x\in(0,1)$,
(iii) $f''(x)<0$ for $x\in[0,x_1]$, $(x_1<1/2)$ and
(iv) $f$ is symmetrical with respect to 1/2-axis.

A prototype of above maps is the logistic map: $f_a(x)=f(x)=ax(1-x)$ for $a>2+\sqrt{5}$.

REMARK 3.1 As to the above case, if we assume that $a>4$, we can apply our theorem. In fact, if $a>4$, $f_a$ is expanding on $C(f_a)$, i.e., there exists $m_0\in\mathbf{N}, \lambda>1$ and const. $>0$ such that

$$|(f_a^m)'(x)| \geq \text{const.}\,\lambda^m \qquad \text{for } m\geq m_0, \quad x\in C(f_a).$$

Hence for sufficiently large $m$, we can apply our theorem to $f_a^m$.

THE FIRST STEP.
Set $I_1^{(1)}=[0,\alpha]$ and $I_2^{(1)}=[\alpha_+,1]$, where $\alpha=\min f^{-1}(1), \alpha_+=\max f^{-1}(1)$. Then the matrices in the first step are

$$A^{(1)}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \qquad B_s^{(1)}=\begin{pmatrix}s_1 & s_1\\ s_2 & s_2\end{pmatrix}.$$

The maximal eigenvalue of $B_s^{(1)}$ is $s_1+s_2$. We see that

$$r^{\delta}_{(1)}=(f'(0)^{-\delta},f'(0)^{-\delta}), \qquad R^{\Delta}_{(1)}=(f'(\alpha)^{-\Delta},f'(\alpha)^{-\Delta}).$$

Since $\delta_1$ and $\Delta_1$ are the unique solutions of the following equations respectively:

$$2f'(0)^{-\delta} = 1, \qquad 2f'(\alpha)^{-\Delta} = 1,$$

they are expressed as

$$\begin{aligned} \delta_1 &= \frac{\log 2}{\log f'(0)} = \frac{\log 2}{\log \max_{x \in C(f)} |f'(x)|}, \\ \Delta_1 &= \frac{\log 2}{\log f'(\alpha)} = \frac{\log 2}{\log \min_{x \in C(f)} |f'(x)|}. \end{aligned} \tag{3.1}$$

If $f(x) = f_a(x) = ax(1-x)$ for $a > 2 + \sqrt{5}$ then we have

$$\delta_1 = \frac{\log 2}{\log a} = \frac{\log 2}{\log \max_{x \in J(f_a)} |f'(x)|},$$

$$\Delta_1 = \frac{\log 2}{\log \sqrt{a^2 - 4a}} = \frac{\log 2}{\log \min_{x \in J(f_a)} |f'(x)|}.$$

The estimation $\delta_1$ is equivalent to Brolin's one [4].

THE SECOND STEP.
Set $I_1^{(2)} = [0, \gamma], I_2^{(2)} = [\beta, \alpha] I_3^{(2)} = [\alpha_+, \gamma_+]$ and $I_4^{(2)} = [\beta_+, 1]$, where

$$\beta = \min f^{-1}(\alpha_+), \gamma = \min f^{-1}(\alpha), \beta_+ = \max f^{-1}(\alpha_+), \gamma_+ = \max f^{-1}(\alpha).$$

Then the matrices in the second step are

$$A^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad B_s^{(2)} = \begin{pmatrix} s_1 & 0 & 0 & s_1 \\ s_2 & 0 & 0 & s_2 \\ 0 & s_3 & s_3 & 0 \\ 0 & s_4 & s_4 & 0 \end{pmatrix}.$$

The maximal eigenvalue of $B_s^{(2)}$ is $\frac{1}{2}\left\{s_1 + s_3 + \sqrt{(s_1 - s_3)^2 + 4s_2s_4}\right\}$. We see that

$$r_{(2)}^{\delta} = (f'(0)^{-\delta}, f'(\beta)^{-\delta}, f'(\beta)^{-\delta}, f'(0)^{-\delta}),$$
$$R_{(2)}^{\Delta} = (f'(\gamma)^{-\Delta}, f'(\alpha)^{-\Delta}, f'(\alpha)^{-\Delta}, f'(\gamma)^{-\Delta}),$$

Now $\delta_2$ and $\Delta_2$ are the unique solutions of the following equations respectively:

$$f'(0)^{-\delta} + f'(\beta)^{-\delta} = 1, \qquad f'(\gamma)^{-\Delta} + f'(\alpha)^{-\Delta} = 1.$$

Clearly we see that $\delta_1 < \delta_2 < \text{H-dim}(C(f_a)) < \Delta_2 < \Delta_1$. Figure 1 and Figure 2 show the relation between approximations of the Hausdorff dimensions of $C(f_a)$ and the parameter $a$, where $f_a(x) = ax(1-x)$ for steps $1 \sim 4$.

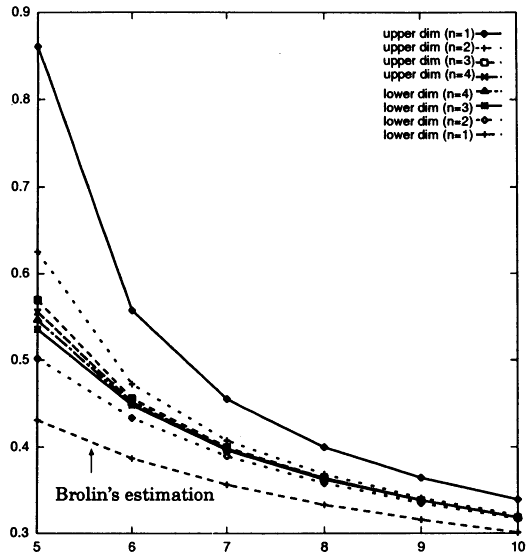


Figure 1: Upper and lower estimations of Hausdorff dimensions of $C(f_a)$, $f_a(x) = ax(1-x)$ horizontal axis: $a$, vetical axis: Hausdorffdimension

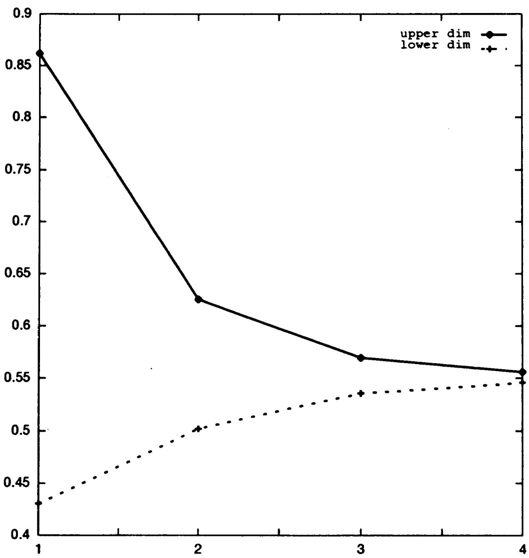


Figure 2: Upper and lower estimations of Hausdorff dimensions of $C(f_5)$, horizontal axis: approximation step, vetical axis: Hausdorff dimension

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