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Oscillation and nonoscillation theorems for a class of second order quasilinear functional differential equations

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ABSTRACT. The equation to be studied in this paper is

$$(\mathrm{A}) \qquad (|y'(t)|^{\alpha-1}y'(t))' + f(t, y(g(t))) = 0, \qquad \alpha > 0.$$

Under certain assumptions on f and g, classification of nonosillatory solutions of (A) is given according to their asymptotic behavior as $t \to \infty$. Criteria are obtained for the existence and nonexistence of nonoscillatory solutions of (A). As a result one can indicate a class of equations of the form (A) for which the situation for oscillations of all solutions can be completely characterized.

0. Introduction

The purpose of this paper is to study the oscillatory and nonoscillatory behavior of quasilinear functional differential equations of the type

$$(\mathrm{A}) \qquad (|y'(t)|^{\alpha-1}y'(t))' + f(t, y(g(t))) = 0$$

for which the following conditions, collectively referred to as (H), are assumed to hold:

(a) α is a positive constant;
(b) $g(t)$ is a positive continuous function on $[a, \infty)$, $a \geq 0$, such that $\lim_{t\to\infty} g(t) = \infty$;
(c) $f(t, y)$ is a continuous function on $[a, \infty) \times \mathbf{R}$ which is nondecreasing in y and satisfies $yf(t, y) > 0$, $y \neq 0$, for each fixed $t \geq a$.

By a solution of (A) we mean a function $y \in C^1[T_y, \infty), T_y \geq a$, which has the property that $|y'|^{\alpha-1}y' \in C^1[T_y, \infty)$ and satisfies the equation at all sufficiently large t in $[T_y, \infty)$. Our attention will be restricted to those solutions $y(t)$ of (A) which are nontrivial in the sense that $\sup\{|y(t)| : t \geq T\} > 0$ for any $T \geq T_y$. A solution is said to be oscillatory if it has an infinite sequence of zeros clustering at $t = \infty$; otherwise a solution is said to be nonoscillatory.

It can be shown that, as regards the asymptotic behavior of a nonoscillatory solution $y(t)$ of (A), the following three cases are possible:

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(I) $\lim_{t\to\infty} \frac{y(t)}{t} = const \neq 0;$

(II) $\lim_{t\to\infty} \frac{y(t)}{t} = 0, \ \lim_{t\to\infty} y(t) = +\infty \text{ or } -\infty;$

(III) $\lim_{t\to\infty} y(t) = const \neq 0.$

Solutions of the types (I), (II) and (III) are called, respectively, *dominant*, *intermediate* and *subdominant* solutions. Our first task (Section 1) is to investigate the existence of these three types of solutions for (A), showing that necessary and sufficient conditions can be established for the existence of dominant and subdominant solutions.

Our next task (Section 2) is to find criteria for oscillation of all solution of (A). Additional restricitions on the nonlinearity of (A) are needed for this purpose. It will be shown that there exists a class of equations of the form (A) for which the oscillation situation can be completely characterized. An example of such equations is

$$(\mathrm{B}) \qquad (|y'(t)|^{\alpha-1}y'(t))' + q(t)|y(g(t))|^{\beta-1}y(g(t)) = 0,$$

where β is a positive constant and $q(t)$ is a positive continuous function on $[a, \infty)$.

In Section 3 we show that the results of Sections 1 and 2 can be extended to equations of the form

$$(\mathrm{C}) \qquad (p(t)|y'|^{\alpha-1}y')' + f(t, y(g(t))) = 0$$

provided $p(t)$ is a positive continuous function on $[a, \infty)$ satisfying

$$\int_a^{\infty} \frac{dt}{(p(t))^{1/\alpha}} = \infty.$$

Oscillation theory for the quasilinear ordinary differential equation

$$(|y'(t)|^{\alpha-1}y'(t))' + f(t, y(t)) = 0$$

has been developed by several authors including Elbert and Kusano [1]. To the best of the author's knowledge there is no previous work other than [3] concerning the oscillation of quasilinear functional differential equations with deviating arguments of the form (A) or (C).

1. Existence of nonoscillatory solutions

We begin with the existence of dominant solutions of (A).

THEOREM 1.1. *The equation* (A) *possesses a dominant solution if and only if there exists a constant* $c \neq 0$ *such that*

$$\int_a^\infty |f(t, cg(t))| dt < \infty. \tag{1.1}$$

PROOF. (The "only if" part) Let $y(t)$ be a dominant solution of (A). We may assume that $y(t)$ is eventually positive. Then, $y(t) > 0$ and $y'(t) > 0$ for $t \geq t_0$, $t_0 \geq a$ being sufficiently large, and there exist positive constants c, c' and $T \geq t_0$ such that

$$cg(t) \leq y(g(t)) \leq c'g(t) \qquad \text{for } t \geq T. \tag{1.2}$$

By (A), we get

$$\int_T^t f(s, y(g(s))) ds = (y'(T))^\alpha - (y'(t))^\alpha < (y'(T))^\alpha, \qquad t \leq T,$$

which implies that

$$0 < \int_T^\infty f(s, y(g(s))) ds < \infty. \tag{1.3}$$

Using (1.2) in (1.3), we obtain

$$\int_T^\infty f(s, cg(s)) ds < \infty.$$

(The "if" part) Assume that (1.1) holds. Without loss of generality the constant c therein may be supposed to be positive. Let $\omega > 0$ be a constant such that $2\omega \leq c$. Choose $T > a$ large enough so that $T_* = \min\{T, \inf_{t \geq T} g(t)\} \geq a$ and that

$$\int_T^\infty f(t, cg(t)) dt \leq (2^\alpha - 1)\omega^\alpha.$$

Define

$$Y = \{y \in C[T_*, \infty) : \omega(t - T)_+ \leq y(t) \leq 2\omega(t - T)_+, t \geq T_*\},$$

where

$$(t - T)_+ = \begin{cases} t - T, & t \geq T, \\ 0, & t < T. \end{cases}$$

Let $\mathscr{F}$ denote the mapping from Y to $C[T_*, \infty)$ denoted by

$$(\mathscr{F}y)(t) = \begin{cases} \displaystyle\int_T^t \left(\omega^\alpha + \int_s^\infty f(r, y(g(r))) dr\right)^{1/\alpha} ds, & t \geq T, \\ 0, & T_* \leq t \leq T. \end{cases}$$

It is a matter of routine calculations to verify that (i) $\mathscr{F}$ maps Y into itself; (ii) $\mathscr{F}$ is a continuous mapping; and (iii) $\mathscr{F}(Y)$ is relatively compact in $C[T_*, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $y \in Y$ such that $y = \mathscr{F}y$, which implies that the function $y = y(t)$ satisfies

$$y(t) = \int_T^t \left(\omega^\alpha + \int_s^\infty f(r, y(g(r)))dr\right)^{1/\alpha} ds, \qquad t \geq T.$$

Differentiating this integral equation, we see that $y(t)$ is a positive solution of (A) with the property that $\lim_{t\to\infty}[y(t)/t] = \omega$, that is, $y(t)$ is a desired dominant solution of (A). This completes the proof of Theorem 1.1.

A characterization for the existence of subdominant solutions of (A) is given in the following theorem.

THEOREM 1.2. *The equation* (A) *possesses a subdominant solution if and only if there exists a constant* $c \neq 0$ *such that*

$$\int_a^\infty \left(\int_t^\infty |f(s, c)|ds\right)^{1/\alpha} dt < \infty. \tag{1.4}$$

PROOF. (The "only if" part) Assume the existence of a subdominant solution $y(t)$ of (A) which is eventually positive. Then, $y(t) > 0$ and $y'(t) > 0$ for sufficiently large t, say $t \geq t_0 \geq a$, and there exist positive constants c, c' and $T \geq t_0$ such that

$$c \leq y(g(t)) \leq c' \qquad \text{for } t \geq T. \tag{1.5}$$

We now integrate $f(s, y(g(s)))$ from $t \geq T$ to ∞. Noting that $y'(t) \to 0$ as $t \to \infty$, we have

$$(y'(t))^\alpha = \int_t^\infty f(s, y(g(s)))ds, \qquad t \geq T,$$

or

$$y'(t) = \left(\int_t^\infty f(s, y(g(s)))ds\right)^{1/\alpha}, \qquad t \geq T. \tag{1.6}$$

Integrating the both sides of (1.6) from T to ∞, we have

$$0 < \int_T^\infty \left(\int_t^\infty f(s, y(g(s)))ds\right)^{1/\alpha} dt < \infty,$$

which, combined with (1.5), yields

$$0 < \int_T^\infty \left(\int_t^\infty f(s,c)ds \right)^{1/\alpha} dt < \infty.$$

The case where $y(t)$ is eventually negative can be treated similarly.

(The "if" part) Suppose that (1.4) holds for some $c > 0$. Take $T > a$ so large that $T_* = \min\{T, \inf_{t \geq T} g(t)\} \geq a$ and that

$$\int_T^\infty \left(\int_t^\infty f(s,c)ds \right)^{1/\alpha} dt \leq \frac{c}{2}.$$

If we define

$$Z = \left\{ z \in C[T_*, \infty) : \frac{c}{2} \leq z(t) \leq c, t \geq T_* \right\}$$

and

$$(\mathscr{G}z)(t) = \begin{cases} c - \displaystyle\int_t^\infty \left(\int_s^\infty f(r, z(g(r)))dr \right)^{1/\alpha} ds, & t \geq T, \\ (\mathscr{G}z)(T), & T_* \leq t \leq T, \end{cases}$$

then it can be shown that $\mathscr{G}$ is a continuous mapping which sends Z into a relatively compact subset of Z. Consequently, $\mathscr{G}$ has a fixed element z in $Z : z = \mathscr{G}z$, which clearly satisfies the integral equation

$$z(t) = c - \int_t^\infty \left(\int_s^\infty f(r, z(g(r)))dr \right)^{1/\alpha} ds, \qquad t \geq T.$$

It follows that $z(t)$ is a positive solution of (A) with the property that $\lim_{t\to\infty} z(t) = c$, that is, $z(t)$ is a subdominant solution of (A). Similarly, if (1.4) holds for some $c < 0$, then one can prove in a similar manner the existence of a subdominant solution which is eventually negative. This completes the proof of Theorem 1.2.

A sufficient condition for the existence of an intermediate solution of (A) is given in the next theorem.

THEOREM 1.3. *Suppose that* (1.1) *holds for some* $c \neq 0$. *Suppose in addition that*

$$\int_a^\infty \left(\int_t^\infty |f(s,d)|ds \right)^{1/\alpha} dt = \infty \tag{1.7}$$

for all $d \neq 0$ *with* $cd > 0$. *Then* (A) *has a nonoscillatory solution* $y(t)$ *such that* $\lim_{t\to\infty}[y(t)/t] = 0$ *and* $\lim_{t\to\infty}|y(t)| = \infty$.

PROOF. We may suppose that $c > 0$. Choose k such that $0 < k < c$. Let T be so large that $T_* = \min\{T, \inf_{t \geq T} g(t)\} \geq a$ and that

$$\int_T^\infty f(t, k(g(t) + 1))dt \leq k.$$

Consider the set $W \subset C[T_*, \infty)$ and the mapping $\mathscr{H} : W \to C[T_*, \infty)$ defined by

$$W = \{w \in C[T_*, \infty) : k \leq w(t) \leq k(t+1), t \geq T_*\}$$

and

$$(\mathscr{H}w)(t) = \begin{cases} k + \int_T^t \left(\int_s^\infty f(r, w(g(r)))dr \right)^{1/\alpha} ds, & t \geq T, \\ k, & T_* \leq t \leq T. \end{cases}$$

Then, the Schauder-Tychonoff fixed point theorem is applicable, and there exists an element $w \in W$ such that $w = \mathscr{H}w$. This function $w = w(t)$ satisfies

$$w(t) = k + \int_T^t \left(\int_s^\infty f(r, w(g(r)))dr \right)^{1/\alpha} ds, \qquad t \geq T, \tag{1.8}$$

which implies that $w(t)$ is a positive solution of (A). From (1.8) we see that

$$\lim_{t \to \infty} \frac{w(t)}{t} = \lim_{t \to \infty} w'(t) = \lim_{t \to \infty} \left(\int_t^\infty f(s, w(g(s)))ds \right)^{1/\alpha} = 0$$

and

$$\lim_{t \to \infty} w(t) \geq \lim_{t \to \infty} \left[k + \int_T^t \left(\int_s^\infty f(r, k)dr \right)^{1/\alpha} ds \right] = \infty,$$

which is a consequence of (1.7). This finishes the proof.

REMARK 1.1. It seems to be a very difficult question to find a necessary and sufficient condition for the existence of an intermediate solution of (A) even in the special case where $g(t) = t$.

EXAMPLE 1.1. An important special case of (A) is

$$(|y'(t)|^{\alpha-1}y'(t))' + q(t)|y(g(t))|^{\beta-1}y(g(t)) = 0, \tag{B}$$

where α and β are positive constants and $q(t)$ is a positive continuous function on $[a, \infty)$. Clearly, the hypothesis (H) is satisfied for this equation. It is easy to see that the conditions (1.1) and (1.4) reduce, respectively, to

$$\int_a^\infty (g(t))^\beta q(t)dt < \infty \tag{1.9}$$

and

$$\int_a^\infty \left(\int_t^\infty q(s)ds\right)^{1/\alpha} dt < \infty. \tag{1.10}$$

It follows that

(i) (B) has a dominant solution if and only if (1.9) holds;
(ii) (B) has a subdominant solution if and only if (1.10) holds;
(iii) (B) has an intermediate solution if

$$\int_a^\infty (g(t))^\beta q(t)dt < \infty \quad \text{and} \quad \int_a^\infty \left(\int_t^\infty q(s)ds\right)^{1/\alpha} dt = \infty. \tag{1.11}$$

Suppose in particular that $q(t) = t^{-\delta}$, where $\delta > 0$ is a constant. As is easily verified,

(i) if $g(t) = \sigma t + \tau$, $\sigma > 0$, $\tau \in \mathbf{R}$, then
$(1.9) \Longleftrightarrow \delta > 1 + \beta$; $(1.10) \Longleftrightarrow \delta > 1 + \alpha$;
$(1.11) \Longleftrightarrow \alpha > \beta$ and $1 + \beta < \delta \leq 1 + \alpha$;
(ii) if $g(t) = t^\theta$, $\theta > 0$, then
$(1.9) \Longleftrightarrow \delta > 1 + \beta\theta$; $(1.10) \Longleftrightarrow \delta > 1 + \alpha$;
$(1.11) \Longleftrightarrow \alpha > \beta\theta$ and $1 + \beta\theta < \delta \leq 1 + \alpha$;
(iii) if $g(t) = \log t$, then
$(1.9) \Longleftrightarrow \delta > 1$; $(1.10) \Longleftrightarrow \delta > 1 + \alpha$;
$(1.11) \Longleftrightarrow 1 < \delta \leq \alpha + 1$.

2. Oscillation criteria

We are interested in obtaining conditions under which all solutions of (A) are oscillatory.

LEMMA 2.1. *Let $y \in C^1[T, \infty)$ be a function such that $y(t) > 0$, $y'(t) > 0$ and $y'(t)$ is nonincreasing for $t \geq T$. Let $g \in C[T, \infty)$ be a function such that $g(t) \geq 0$ and $\lim_{t\to\infty} g(t) = \infty$. Then, for every $k \in (0, 1)$ there exists $T_k \geq T$ such that*

$$y(g(t)) \geq k\frac{g_*(t)}{t} y(t), \qquad t \geq T_k, \tag{2.1}$$

where

$$g_*(t) = \min\{t, g(t)\}. \tag{2.2}$$

PROOF. This lemma is essentially due to Erbe [2]. Since $y(t)$ is increasing and $y'(t)$ is nonincreasing, we have

$$y(g(t)) \geq y(g_*(t)), \qquad t \geq T$$

and

$$y(t) - y(g_*(t)) = \int_{g_*(t)}^{t} y'(s)ds \leq y'(g_*(t))(t - g_*(t)), \qquad t \geq g_*(t) \geq T,$$

from which it follows that

$$\frac{y(t)}{y(g(t))} \leq 1 + \frac{y'(g_*(t))}{y(g_*(t))}(t - g_*(t)), \qquad t \geq g_*(t) \geq T. \tag{2.3}$$

On the other hand,

$$y(g_*(t)) - y(T) = \int_{T}^{g_*(t)} y'(s)ds \geq y'(g_*(t))(g_*(t) - T), \qquad g_*(t) \geq T,$$

which implies that for each $k \in (0,1)$ there exists $T_k \geq T$ such that

$$\frac{y(g_*(t))}{y'(g_*(t))} \geq kg_*(t), \qquad t \geq T_k. \tag{2.4}$$

Combining (2.3) with (2.4), we find

$$\frac{y(t)}{y(g(t))} \leq 1 + \frac{t - g_*(t)}{kg_*(t)} = \frac{t + (k-1)g_*(t)}{kg_*(t)} \leq \frac{t}{kg_*(t)}, \qquad t \geq T_k,$$

which shows that (2.1) is true.

First we present a criterion for oscillation of all bounded solutions of (A).

THEOREM 2.1. *All bounded solutions of* (A) *are oscillatory if*

$$\int_a^\infty \left(\int_t^\infty \left| f\left(s, c\,\frac{g_*(s)}{s}\right) \right| ds \right)^{1/\alpha} dt = \infty \tag{2.5}$$

for all $c \neq 0$.

PROOF. Suppose to the contrary that (A) has a bounded nonoscillatory solution $y(t)$. With no loss of generality we may assume that $y(t)$ is eventually positive. Then, applying Lemma 2.1, we see that for every $k \in (0,1)$ $y(t)$ satisfies (2.1) provided T_k is sufficiently large. Since $f(t,y)$ is nondecreasing in y, we obtain the following inequality from (A):

$$(|y(t)|^{\alpha-1}y'(t))' + f\left(t, k\,\frac{g_*(t)}{t}\,y(t)\right) \leq 0, \qquad t \geq T_k.$$

From Lemma 1 of [3] it follows that there exists a function $z(t)$ such that $0 < z(t) \le y(t)$ and that

$$(2.6) \qquad (|z'(t)|^{\alpha-1}z'(t))' + f\left(t, k\,\frac{g_*(t)}{t}\,z(t)\right) = 0, \qquad t \ge T_k.$$

Thus $z(t)$ is a bounded nonoscillatory solution (i.e. a subdominant solution) of (2.6), and so application of Theorem 1.2 shows that

$$\int_{T_k}^{\infty}\left(\int_t^{\infty} f\left(s, k\,\frac{g_*(s)}{s}\right)ds\right)^{1/\alpha} dt < \infty.$$

This contradicts (2.5) and the proof is complete.

Corollary 2.1. *Suppose that*

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0.$$

Then all bounded solutions of (A) *are oscillatory if and only if*

$$(2.7) \qquad \int_a^{\infty}\left(\int_t^{\infty} |f(s,c)|\,ds\right)^{1/\alpha} dt = \infty$$

for all $c \ne 0$.

In order to establish a criterion for oscillation of all solutions of (A) a further restriction on its nonlinearity is required.

Theorem 2.2. *Suppose that there exists a continuous function* $\phi(u)$ *on* $\mathbf{R}$ *which is nondecreasing and satisfies* $u\phi(u) > 0$, $u \ne 0$,

$$(2.8) \qquad \int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty,$$

and

$$(2.9) \qquad \liminf_{|u|\to\infty} \frac{|f(t,uv)|}{|\phi(u)|^{\alpha}} \ge k|f(t,v)|, \qquad t \ge a,$$

for some constant $k > 0$ *and all* v *with* $|v| \le 1$. *If* (2.5) *holds, then all solutions of* (A) *are oscillatory.*

Proof. Assume to the contrary that (A) has a nonoscillatory solution $y(t)$ which is eventually positive. Let $\delta, 0 < \delta < 1$, be fixed. Take T_δ large enough so that

$$y(g(t)) \ge \delta\,\frac{g_*(t)}{t}\,y(t) \qquad \text{for } t \ge T_\delta.$$

Then, we have

$$(|y'(t)|^{\alpha-1}y'(t))' + f\left(t, \delta\,\frac{g_*(t)}{t}\,y(t)\right) \le 0, \qquad t \ge T_\delta,$$

so that, by Lemma 1 of [3], there exists a positive solution $z(t)$ of the equation

$$(|z'(t)|^{\alpha-1}z'(t))' + f\left(t, \delta\,\frac{g_*(t)}{t}\,z(t)\right) = 0, \qquad t \ge T_\delta. \tag{2.10}$$

From Theorem 2.1 applied to (2.10) it follows that $z(t)$ can not be bounded for $t \ge T_\delta$, i.e., $\lim_{t\to\infty} z(t) = \infty$. By (2.10), we obtain

$$z'(t) \ge \left(\int_t^\infty f\left(s, \delta\,\frac{g_*(s)}{s}\,z(s)\right)ds\right)^{1/\alpha}, \qquad t \ge T_\delta,$$

and

$$\begin{aligned} \frac{z'(t)}{\phi(\delta z(t))} &\ge \frac{1}{\phi(\delta z(t))}\left(\int_t^\infty f\left(s, \delta\,\frac{g_*(s)}{s}\,z(s)\right)ds\right)^{1/\alpha} \\ &\ge \left(\int_t^\infty \frac{f\left(s, \delta\,\frac{g_*(s)}{s}\,z(s)\right)}{(\phi(\delta z(s)))^\alpha}\,ds\right)^{1/\alpha}, \qquad t \ge T_\delta. \end{aligned} \tag{2.11}$$

Because of (2.9) we have the following inequality for the integrand of the last integral in (2.11);

$$\frac{f(t, \delta\,\frac{g_*(t)}{t}\,z(t))}{(\phi(\delta z(t)))^\alpha} \ge \frac{k}{2} f\left(t, \frac{g_*(t)}{t}\right), \qquad t \ge T, \tag{2.12}$$

where $T \ge T_\delta$ is taken sufficiently large. From (2.11) and (2.12) we get

$$\frac{z'(t)}{\phi(\delta z(t))} \ge \left(\frac{k}{2}\right)^{1/\alpha}\left(\int_t^\infty f\left(s, \frac{g_*(s)}{s}\right)ds\right)^{1/\alpha}, \qquad t \ge T,$$

which, after integration over $[T, t]$, gives

$$\frac{1}{\delta}\int_{\delta z(T)}^{\delta z(t)} \frac{du}{\phi(u)} \ge \left(\frac{k}{2}\right)^{1/\alpha}\int_T^t \left(\int_s^\infty f\left(r, \frac{g_*(r)}{r}\right)dr\right)^{1/\alpha} ds, \qquad t \ge T.$$

Letting $t \to \infty$ and using (2.8), we conclude that

$$\int_T^\infty \left(\int_s^\infty f\left(r, \frac{g_*(r)}{r}\right)dr\right)^{1/\alpha} ds < \infty,$$

which contradicts (2.5). This completes the proof.

COROLLARY 2.2. *Suppose that*

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0.$$

Suppose moreover the existence of a function $\phi(u)$ with the properties as stated in Theorem 2.2. Then, all solutions of (A) *are oscillatory if and only if* (2.7) *holds.*

THEOREM 2.3. *Suppose that there exists a continuous function $\psi(u)$ on* $\mathbf{R}$ *which is nondecreasing and satisfies $u\psi(u) > 0$, $u \neq 0$,*

$$\int_{\pm 0}^{\pm M} \frac{du}{\psi(u^{*/\alpha})} < \infty, \qquad u^{*/\alpha} = |u|^{1/\alpha - 1}u, \tag{2.13}$$

and

$$\liminf_{u\to\infty} \frac{|f(t,uv)|}{|\psi(u)|} \geq k|f(t,v)|, \qquad t \geq a, \tag{2.14}$$

for some $k > 0$ and all v with $|v| \geq 1$. If

$$\int_a^\infty |f(t, cg_*(t))| dt = \infty \tag{2.15}$$

for all $c \neq 0$, then all solutions of (A) *are oscillatory.*

PROOF. Let $y(t)$ be a nonoscillatory solution of (A). We may suppose that $y(t)$ is eventually positive. Since (2.15) implies

$$\int_a^\infty |f(t, cg(t))| dt = \infty$$

for all $c \neq 0$, (A) cannot possess dominant solutions by Theorem 1.1, so that $y(t)$ is either subdominant or intermediate. This means that $\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} [y(t)/t] = 0$. Since $y'(t) > 0$ is decreasing, we have

$$y(t) - y(t_0) = \int_{t_0}^t y'(s) ds \geq y'(t)(t - t_0), \qquad t \geq t_0,$$

which implies that, for a fixed δ, $0 < \delta < 1$,

$$y(t) \geq \delta t y'(t), \qquad y(g_*(t)) \geq \delta g_*(t) y'(g_*(t)), \qquad t \geq T,$$

provided $T \geq t_0$ is taken sufficiently large. We then have

$$y(g(t)) \geq y(g_*(t)) \geq \delta g_*(t) y'(g_*(t)) \geq \delta g_*(t) y'(t), \qquad t \geq T,$$

and hence

$$\frac{y(g(t))}{y'(t)} \geq \delta g_*(t), \qquad t \geq T. \tag{2.16}$$

Let $M > 0$ be a constant such that $0 < y'(t) \leq M$ for $t \geq T$ and define

$$\Psi(u) = \int_u^M \frac{dv}{\psi(v^{1/\alpha})}, \qquad 0 < u \leq M.$$

We observe that

$$\begin{aligned}\frac{d}{dt}\Psi((y'(t))^\alpha) &= \frac{f(t, y(g(t)))}{\psi(y'(t))} \\ &= \frac{f\left(t, \frac{y(g(t))}{y'(t)} y'(t)\right)}{\psi(y'(t))} \geq \frac{f(t, \delta g_*(t) y'(t))}{\psi(y'(t))}, \qquad t \geq T,\end{aligned} \tag{2.17}$$

where (2.16) has been used. In view of (2.14) $T_1 > T$ can be chosen so large that $\delta g_*(t) \geq 1$ for $t \geq T_1$ and that

$$\frac{f(t, \delta g_*(t) y'(t))}{\psi(y'(t))} \geq \frac{k}{2} f(t, \delta g_*(t)), \qquad t \geq T_1,$$

which, combined with (2.17), yields

$$\frac{d}{dt}\Psi((y'(t))^\alpha) \geq \frac{k}{2} f(t, \delta g_*(t)), \qquad t \geq T_1. \tag{2.18}$$

This implies that

$$\int_T^\infty f(t, \delta g_*(t)) dt < \infty,$$

since it follows from (2.18) and (2.13) that for $t \geq T_1$

$$\begin{aligned}\frac{k}{2}\int_{T_1}^t f(s, \delta g_*(s)) ds &\leq \Psi((y'(t))^\alpha) - \Psi((y'(T_1))^\alpha) \\ &< \int_{(y'(t))^\alpha}^M \frac{dv}{\psi(v^{1/\alpha})} < \int_0^M \frac{dv}{\psi(v^{1/\alpha})}.\end{aligned}$$

This is a contradiction and completes the proof.

COROLLARY 2.3. *Suppose the existence of a function $\psi(u)$ with the properties as stated in Theorem 2.3.*

(i) *Suppose that* $g(t) \le t$ *for* $t \ge a$. *Then, all solutions of* (A) *are oscillatory if and only if*

$$\int^{\infty} |f(t, cg(t))|dt = \infty \tag{2.19}$$

for all $c \neq 0$

(ii) *Suppose that*

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{g(t)}{t} < \infty.$$

Then, all solutions of (A) *are oscillatory if and only if*

$$\int^{\infty} |f(t, ct)|dt = \infty$$

for all $c \neq 0$.

REMARK 2.1. (i) The equation (A) is said to be *super-half-linear* if there exists a constant $\beta > \alpha$ such that $|f(t,y)|/|y|^{\beta}$ is nondecreasing in $|y|$ for each fixed $t \ge a$. In this case, for any u, v with $|u| \ge 1$, $|v| \le 1$, we have

$$\frac{|f(t,uv)|}{|uv|^{\beta}} \ge \frac{|f(t,v)|}{|v|^{\beta}} \quad \text{or} \quad \frac{|f(t,uv)|}{|u|^{\beta}} \ge |f(t,v)|, \qquad t \ge a, \tag{2.20}$$

so that (2.9) holds with the choice $\phi(u) = |u|^{\beta/\alpha - 1}u$ which is increasing and satisfies $u\phi(u) > 0$, $u \neq 0$, and (2.8).

(ii) The equation (A) is said to be *sub-half-linear* if there exists a positive constant $\beta < \alpha$ such that $|f(t,y)|/|y|^{\beta}$ is nonincreasing in $|y|$ for each fixed $t \ge a$. In this case, we see that (2.20) holds for any u, v with $|u| \le 1$, $|v| \ge 1$, so that (2.14) holds with the choice $\psi(u) = |u|^{\beta - 1}u$ which is increasing and satisfies $u\psi(u) > 0$, $u \neq 0$, and (2.13).

EXAMPLE 2.1. Consider the equation (B) in which β is a positive constant and $q(t)$ and $g(t)$ are positive continuous functions on $[a, \infty)$ with $\lim_{t\to\infty} g(t) = \infty$. According to the above definitions, (B) is *super-half-linear* if $\beta > \alpha$ and *sub-half-linear* if $\beta < \alpha$.

(I) Let $\beta > \alpha$. All solutions of (B) are oscillatory if

$$\int_a^{\infty} \left(\int_t^{\infty} q(s) \left(\frac{g_*(s)}{s} \right)^{\beta} ds \right)^{1/\alpha} dt = \infty. \tag{2.21}$$

Suppose in addition that $g(t)$ satisfies

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0.$$

Then, all solutions of (B) are oscillatory if and only if

$$\int_a^\infty \left(\int_t^\infty q(s)ds\right)^{1/\alpha} dt = \infty. \tag{2.22}$$

(II) Let $\beta < \alpha$. All solutions of (B) are oscillatory if

$$\int_a^\infty q(t)(g_*(t))^\beta dt = \infty. \tag{2.23}$$

Suppose in addition that $g(t) \leq t$ for $t \geq a$. Then, all solutions of (B) are oscillatory if and only if

$$\int_a^\infty q(t)(g(t))^\beta dt = \infty. \tag{2.24}$$

On the other hand, in case $g(t)$ satisfies

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0 \quad \text{and} \quad \liminf_{t\to\infty} \frac{g(t)}{t} < \infty,$$

a necessary and sufficient condition for oscillation of all solutions of (B) is that

$$\int_a^\infty t^\beta q(t)dt = \infty. \tag{2.25}$$

REMARK 2.2. Let $\alpha < \beta$ and consider the equation (B) in which $g(t)$ is continuously differentiable and satisfies $g'(t) > 0$ and $g(t) < t$ for $t \geq a$. As mentioned above, all solutions of this equation are oscillatory if (2.21) is satisfied, that is,

$$\int_a^\infty \left(\int_t^\infty q(s)\left(\frac{g(s)}{s}\right)^\beta ds\right)^{1/\alpha} dt = \infty. \tag{2.26}$$

On the other hand, a comparison theorem given in [3] (Corollary to Theorem 6) shows that the same conclusion holds for (B) if all solutions of the ordinary differential equation

$$(|y'(t)|^{\alpha-1}y'(t))' + \frac{q(g^{-1}(t))}{g'(g^{-1}(t))}|y(t)|^{\beta-1}y(t) = 0 \tag{2.27}$$

are oscillatory, where $g^{-1}(t)$ denotes the inverse function of $g(t)$. It is known [1] that such a situation holds for (2.27) if and only if

$$\int_a^\infty \left(\int_t^\infty \frac{q(g^{-1}(s))}{g'(g^{-1}(s))} ds\right)^{1/\alpha} dt = \infty,$$

which is equivalent to

$$\int_a^\infty \left(\int_{g^{-1}(t)}^\infty q(s)ds \right)^{1/\alpha} dt = \infty. \tag{2.28}$$

Suppose, for example, that $q(t) = t^{-2}$ and $g(t) = t^\gamma$, $0 < \gamma < 1$. In this case (2.26) holds if $\gamma \geq (\beta - \alpha + 1)/\beta$, while (2.28) holds if $\gamma \geq 1/\alpha$. This shows that α must be larger than 1 and that (2.28) is better than (2.26).

3. Extensions

We will show that, by means of a simple change of variables, the results for (A) can be carried over to more general equations of the form

$$(p(t)|y'(t)|^{\alpha-1}y'(t))' + f(t, y(g(t))) = 0, \tag{C}$$

where α and $f(t,y)$ are as in (A) and $p(t)$ is a positive continuous function on $[a, \infty)$ satisfying

$$\int_a^\infty \frac{dt}{(p(t))^{1/\alpha}} = \infty. \tag{3.1}$$

In fact, define the function $P(t)$ by

$$P(t) = \int_a^t \frac{ds}{(p(s))^{1/\alpha}}, \qquad t \geq a, \tag{3.2}$$

and introduce the change of variables $(t, y) \to (\tau, Y)$ given by

$$\tau = P(t), \qquad Y(\tau) = y(t). \tag{3.3}$$

Then the equation (C) is transformed into

$$(|\dot{Y}(\tau)|^{\alpha-1}\dot{Y}(\tau))^{\cdot} + F(\tau, Y(G(\tau))) = 0, \tag{C$'$}$$

where a dot denotes differentiation with respect to τ,

$$F(\tau, Y) = (p(t))^{1/\alpha} f(t, y), \tag{3.4}$$

and

$$G(\tau) = P \circ g \circ P^{-1}(\tau), \tag{3.5}$$

P^{-1} designating the inverse function of P.

Since (C′) is of the same type as (A), all the theorems of §§1–2 can be applied to (C′), and the results thus obtained give rise to the corresponding oscillation and nonoscillation theorems for (C), some of which will be stated below.

A nonoscillatory solution $y(t)$ of (C) is called *dominant* if $\lim_{t\to\infty}[y(t)/P(t)] = const \neq 0$ and *subdominant* if $\lim_{t\to\infty} y(t) = const \neq 0$.

THEOREM 3.1. *Let $f(t,y)$ be as in Theorems 1.1 and 1.2.*

(i) (C) *has a dominant solution if and only if*

$$\int_a^\infty |f(t, cP(g(t)))|dt < \infty \tag{3.6}$$

for some $c \neq 0$.

(ii) (C) *has a subdominant solution if and only if*

$$\int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty |f(s,c)|ds\right)^{1/\alpha} dt < \infty \tag{3.7}$$

for some $c \neq 0$.

This nonoscillation theorem is an immediate consequence of Theorems 1.1 and 1.2.

Oscillation criteria for (C) follow from Theorems 2.2 and 2.3 by noting that, for the function $G = P \circ g \circ P^{-1}$,

$$G_*(\tau) = \min\{\tau, G(\tau)\} = P \circ g_* \circ P^{-1}(\tau),$$

where g_* is defined by (2.2), and that

$$\int_0^\infty \left(\int_\tau^\infty \left|F\left(\sigma, c\,\frac{G_*(\sigma)}{\sigma}\right)\right| d\sigma\right)^{1/\alpha} d\tau = \int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty \left|f\left(s, c\frac{P(g_*(s))}{P(s)}\right)\right| ds\right)^{1/\alpha} dt,$$

$$\int_0^\infty |F(\tau, cG_*(\tau))|d\tau = \int_a^\infty |f(t, cP(g_*(t)))|dt.$$

THEOREM 3.2. *Let $f(t,y)$ be as in Theorem 2.2. All solutions of* (C) *are oscillatory if*

$$\int_a^\infty \left(\frac{1}{p(t)}\int_t^\infty \left|f\left(s, c\frac{P(g_*(s))}{P(s)}\right)\right| ds\right)^{1/\alpha} dt = \infty \tag{3.8}$$

for all $c \neq 0$.

THEOREM 3.3. *Let $f(t, y)$ be as in Theorem* 2.3. *All solutions of* (C) *are oscillatory if*

$$\int_a^\infty |f(t, cP(g_*(t)))| dt = \infty \tag{3.9}$$

for all $c \neq 0$.

We conclude with necessary and sufficient conditions for oscillation of all solutions of the equation

$$(p(t)|y'(t)|^{\alpha-1}y'(t))' + q(t)|y(g(t))|^{\beta-1}y(g(t)) = 0, \tag{D}$$

where $\alpha, \beta, q(t)$ and $g(t)$ are as in (B), and $p(t)$ is a positive continuous function on $[a, \infty)$ satisfying (3.1).

(I) Let $\beta > \alpha$. Suppose that

$$\liminf_{t\to\infty} \frac{P(g(t))}{P(t)} > 0.$$

Then, all solutions of (D) are oscillatory if and only if

$$\int_a^\infty \left(\frac{1}{p(t)} \int_t^\infty q(s) ds \right)^{1/\alpha} dt = \infty.$$

(II) Let $\beta < \alpha$. Suppose that $g(t) \leq t$ for $t \geq a$. Then, all solutions of (D) are oscillatory if and only if

$$\int_a^\infty q(t)(P(g(t)))^\beta dt = \infty.$$

On the other hand, in case $g(t)$ satisfies

$$\liminf_{t\to\infty} \frac{P(g(t))}{P(t)} > 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{P(g(t))}{P(t)} < \infty,$$

all solutions of (D) are oscillatory if and only if

$$\int_a^\infty q(t)(P(t))^\beta dt = \infty.$$

References

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