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Uniqueness of nodal rapidly-decaying radial solutions to a linear elliptic equation on $\mathbf{R}^n$

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ABSTRACT. We consider a linear elliptic differential equation in the whole space and show the existence and uniqueness of nodal rapidly-decaying solutions with prescribed zeros. By using the Prüfer transformation, we give a comprehensive view to the problem. We also prove the existence and the uniqueness of solutions to the equation on the unit ball and the exterior of it with various boundary conditions.

1. Introduction

In this paper we consider the existence and uniqueness of nodal rapidly-decaying radial solutions to

$$\Delta u + \xi K(|x|)u = 0 \qquad \text{in } \boldsymbol{R}^n, \tag{1.1}$$

where $n > 2$ and $\xi > 0$ is a parameter. Concerning $K(r)$, we impose

(K) $\qquad K(r) > 0 \quad \text{on } (0, \infty), \qquad K(r) \in C^1(0, \infty), \qquad rK(r) \in L^1(0, \infty).$

Since we are interested in radial solutions, we consider the ordinary differential equation

$$\begin{cases} (r^{n-1}u_r)_r + r^{n-1}\xi K(r)u = 0, & r > 0, \\ u(0) = 1, \end{cases} \tag{1.2}$$

As for (1.2), it is unnecessary to restrict n to integer values. We do not require $u_r(0; \xi) = 0$, however, we can deduce $\lim_{r \downarrow 0} r^{n-1}u_r(r; \xi) = 0$ from $rK(r) \in L^1(0, 1)$ and (1.2) can be solved with only initial value $u(0) = 1$. Note that under (K), (1.2) has a unique global solution in the class $C[0, \infty) \cap C^2(0, \infty)$ for any $\xi > 0$ (see, e.g., Ni-Yotsutani [8]) and that any solution of (1.2) has

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only finite number of zeros on $(0, \infty)$ (see, e.g., Naito [6]). We denote the unique solution by $u(r; \xi)$.

From the O.D.E. theoretic point of view, there are many results on the existence and non-existence of oscillatory solutions and their behaviors. See for instance, Chapters 1 and 2 of Kiguradze and Chanturia [5], in which they treat equations of the form $u'' = p(t)u$, and the references therein. We should note that (1.2) is reduced to $u'' = p(t)u$ by $u(r) = v(t)/t$ and $t := r^{n-2}$ with a possible singularity of $p(t)$ at the origin. However, such an "eigenvalue problem" as in this paper has not been investigated with full of attention.

Moreover, from the Hilbert space approach (partial differential equations) point of view, it is worth seeking a solution which decays at the rate r^{2-n} as $r \to \infty$. We call a solution of this type a *rapidly-decaying solution.*

Once we know that $u(r; \xi)$ is not oscillatory, it is natural to ask whether there exists a rapidly-decaying solution with j zeros on $(0, \infty)$ for any $j = 0, 1, 2, \dots$.

Concerning positive solutions, M. Naito [6] proved the existence of a positive rapidly-decaying solution to (1.2) under (essentially the same condition as) (K). Later Edelson and Rumbos [2] showed the uniqueness of positive rapidly-decaying solutions. They treated (1.1) with nonradial $K(x)$ and generalized M. Naito's results to partial differential equations.

As for solutions with zeros, M. Naito [6] also proved the existence of nodal rapidly-decaying solutions under the additional condition $r^{n-1}K(r) \in L^1(0, \infty)$.

Here we efficiently use the Prüfer transformation to give a comprehensive view to prescribed zeros problems.

Let us describe the matter more precisely. From the Prüfer transformation point of view [9] (see also Hartman [3], p. 332), the condition $r^{n-1}K(r) \in L^1(0, \infty)$ ensures the continuity of the argument θ with respect to ξ, where we put $u = \rho \cos \theta$ and $-r^{n-1}u_r = \rho \sin \theta$. Then we have

$$\theta_r = r^{-(n-1)} \sin^2 \theta + \xi r^{n-1} K(r) \cos^2 \theta > 0$$

similar to (3.5) in Section 3. Since $r^{-(n-1)} \sin^2 \theta = r^{n-1} u_r^2/\rho^2$ and since $\rho(0; \xi) = 1$, the first term of the right-hand side is always integrable on $[0, \infty)$. As for the second term, on the other hand, if

$$\int_0^\infty r^{n-1} K(r) dr < \infty,$$

then we can show that $\theta(\infty; \xi)$ is defined and continuous with respect to ξ.

Suppose that $u(r; \xi) \to c > 0$ as $r \to \infty$ (as we will see in Lemma 2.4 that $\lim_{r\to\infty} u(r; \xi) \neq 0$ exists unless $\lim_{r\to\infty} r^{n-2} u(r; \xi)$ is finite). If $rK(r) \in L^1(1, \infty)$

and if $r^{n-1}K(r) \notin L^1(1, \infty)$, then we get

$$\rho \sin \theta = -r^{n-1}u_r = \xi \int_0^r s^{n-1}K(s)u ds \to \infty$$

as $r \to \infty$, which implies $\rho \to \infty$ as $r \to \infty$. However, as we will see in Lemma 2.2, $\theta(\infty; \xi) < \infty$ for any ξ (u has only a finite number of zeros). Hence $\theta(\infty; \xi)$ must satisfy $\cos \theta(\infty; \xi) = 0$. Otherwise, u cannot converge to c. This indicates that $\theta(\infty; \xi)$ is not continuous with respect to ξ. So we employ the method used in Kabeya, Yanagida and Yotsutani [4], Y. Naito [7] and Yanagida and Yotsutani [10]. That is, we must connect the solution $u(r; \xi)$ at $r = 1$ with the solution $\tilde{u}(r; \xi)$ to

$$\begin{cases} (r^{n-1}\tilde{u}_r)_r + r^{n-1}\xi K(r)\tilde{u} = 0, & 1 < r, \\ \lim_{r\to\infty} r^{n-2}\tilde{u}(r) = \beta, \end{cases} \tag{1.3}$$

with suitable β. By the connection, we can get the desired asymptotic behavior.

THEOREM 1.1. *Suppose that* (K) *holds. Then there exists a unique increasing positive sequence* $\{\xi_j\}_{j=1}^{\infty}$ *with* $\xi_j \to \infty$ *as* $j \to \infty$ *such that* $u(r; \xi_j)$ *has exactly* $(j-1)$ *zeros and* $\lim_{r\to\infty} r^{n-2}|u(r; \xi_j)| \in (0, \infty)$.

As a by-product of the proof of Theorem 1.1, we show the existence and uniqueness of solutions with prescribed zeros to

$$\begin{cases} (r^{n-1}\hat{u}_r)_r + r^{n-1}\xi K(r)\hat{u} = 0, & 0 < r < 1, \\ \hat{u}(0) = 1, \\ \hat{u}_r(1) \sin \varphi + \hat{u}(1) \cos \varphi = 0, \end{cases} \tag{1.4}$$

where $\varphi \in (-\pi/2, \pi/2]$.

THEOREM 1.2. *Suppose that* (K) *holds.*

(i) *If* $0 < \varphi \leq \pi/2$, *then there exists a unique positive increasing sequence* $\{\hat{\xi}_j\}$ *such that the unique solution* $\hat{u}(r; \hat{\xi}_j)$ *to* (1.4) *has exactly* j *zeros on* $(0, 1)$ *for* $j = 0, 1, 2, \ldots$.
(ii) *If* $-\pi/2 < \varphi \leq 0$, *then there exists a unique positive increasing sequence* $\{\hat{\xi}_j\}$ *such that the unique solution* $\hat{u}(r; \hat{\xi}_j)$ *to* (1.4) *has exactly* j *zeros on* $(0, 1)$ *for* $j = 1, 2, 3, \ldots$.

Similarly, we consider a singular boundary value problem

$$(1.5)\qquad \begin{cases} (r^{n-1}\tilde{u}_r)_r + r^{n-1}\xi K(r)\tilde{u} = 0, & 1 < r, \\ \lim_{r\to\infty} r^{n-2}\tilde{u}(r) = \beta, \\ \tilde{u}_r(1)\sin\varphi + \tilde{u}(1)\cos\varphi = 0, \end{cases}$$

where $\varphi \in [-\pi/2, \pi/2)$ and $\beta > 0$. It is sufficient to put $\beta = 1$ only for (1.5), however, we need to vary β to show Theorem 1.1.

THEOREM 1.3. *Suppose that* (K) *holds.*

(i) *If* $-\pi/2 \le \varphi < \tan^{-1}(n-2)$, *then there exists a unique positive increasing sequence* $\{\hat{\xi}_j\}$ *such that the unique solution* $\tilde{u}(r; \hat{\xi}_j\}$ *to* (1.5) *has exactly* j *zeros on* $(1, \infty)$ *for* $j = 0, 1, 2, \ldots$.

(ii) *If* $\tan^{-1}(n-2) \le \varphi < \pi/2$, *then there exists a unique positive increasing sequence* $\{\hat{\xi}_j\}$ *such that the unique solution* $\tilde{u}(r; \hat{\xi}_j)$ *to* (1.5) *has exactly* j *zeros on* $(1, \infty)$ *for* $j = 1, 2, 3, \ldots$.

It is proved that the condition $rK(r) \in L^1(0, \infty)$ does not admit any oscillatory solutions (see e.g., Lemma 2.2 or M. Naito [6]). However, we will show that the number of zeros of $u(r; \xi)$ increases as $\xi \to \infty$ (Proposition 2.1).

To make sure that there is a solution with prescribed zeros, we use the Prüfer transformation as used in Y. Naito [7] and Yanagida and Yotsutani [10]. By this transformation, it becomes easy to show the existence of solutions in Theorems 1.2 and 1.3 having prescribed zeros and satisfying the boundary condition.

Concerning Theorem 1.1, we must choose a suitable β to match $\hat{u}(r; \xi)$ with $\tilde{u}(r; \xi)$ at $r = 1$.

These results are closely related to the limiting behavior of radial solutions to $\Delta u + K(|x|)|u|^{p-1}u = 0$ in $\mathbf{R}^n$ as $p \to 1 + 0$. For positive solutions, the limiting behavior of solutions to this equation was investigated by Yanagida and Yotsutani [11]. It is known that in $rK(r) \in L^1(0, \infty)$ case the behavior has the most various features (see [11]). These Theorems enable us to generalize the results of [11] to nodal solutions. We will discuss this in the forthcoming paper.

2. Preliminaries

We collect the fundamental properties of solutions to (1.2) and (1.3), and the asymptotic behavior of them as $\xi \to \infty$. While many of the properties are seen in M. Naito [6] and Ni-Yotsutani [8], we give proofs for the self-containedness. As for the asymptotic behavior of zeros of a solution, we use the Prüfer transformation.

LEMMA 2.1. *Let u be a solution to* (1.2). *Then u satisfies*

(i) $r^{n-1}u_r \to 0$ *as* $r \to +0$.
(ii) $r^{n-2}|u|$ *is nondecreasing near* $r = \infty$ *provided* $|u| > 0$ *near* $r = \infty$.

PROOF. From (1.2), $r^{n-1}u_r$ is monotone decreasing near $r = 0$. So $\lim_{r\to+0} r^{n-1}u_r$ exists. If $\lim_{r\to+0} r^{n-1}u_r < 0$, then $\lim_{r\to+0} u = \infty$. This contradicts $u(0) = 1$. Similarly, if $\lim_{r\to+0} r^{n-1}u_r > 0$, then we also get a contradiction. (i) is proved.

As for (ii), the equation

$$(r^{n-1}u_r)_r + \xi r^{n-1}K(r)u = 0$$

is written as

$$\left\{\frac{1}{r^{n-3}}(r^{n-2}u)_r\right\}_r + \xi rK(r)u = 0. \tag{2.1}$$

We may suppose that $u > 0$ near $r = \infty$. From (2.1), $(r^{n-2}u)_r/r^{n-3}$ is montone decreasing near $r = \infty$. If $\lim_{r\to\infty}(r^{n-2}u)_r/r^{n-3} \geq 0$, then the assertion is proved. If $\lim_{r\to\infty}(r^{n-2}u)_r/r^{n-3} < 0$, then there exist positive constants $c > 0$ and $r_0 > 0$ such that $(r^{n-2}u)_r \leq -cr^{n-3}$ for $r \geq r_0$. Then we have

$$-r_0^{n-2}u(r_0) \leq r^{n-2}u(r) - r_0^{n-2}u(r_0) \leq -\frac{c}{n-2}(r^{n-2} - r_0^{n-2}). \tag{2.2}$$

Letting $r \to \infty$ in (2.2), we get a contradiction since $n > 2$. The proof is complete. ∎

LEMMA 2.2. *Suppose that* (K) *holds. Then the solution $u(r; \xi)$ has at most finite number of zeros on* $(0, \infty)$.

PROOF. To the contrary, suppose that $u(r; \xi)$ has infinitely many zeros on $(0, \infty)$. Let $a > 0$ be sufficiently large such that $u(a; \xi) > 0$ and $u_r(a; \xi) = 0$. Moreover, let $z > a$ be the smallest zero of $u(r; \xi)$ on $[a, \infty)$, that is, $u(r; \xi) > 0$ on $[a, z)$. Then, from (1.2) we have $u_r(r; \xi) < 0$ on (a, z). Using this property, we obtain

$$\begin{aligned} 0 = u(z; \xi) &= u(a; \xi) - \frac{\xi}{n-2}\int_0^z \left(1 - \left(\frac{s}{r}\right)^{n-2}\right) sK(s)uds \\ &\geq \left\{1 - \frac{\xi}{n-2}\int_a^z sK(s)ds\right\} u(a; \xi) \\ &\geq \left\{1 - \frac{\xi}{n-2}\int_a^\infty sK(s)ds\right\} u(a; \xi) > 0 \end{aligned}$$

by $sK(s) \in L^1(0, \infty)$ provided $a > 0$ is sufficiently large. This is a contradiction. ■

Using the argument similar to that of the proof of Lemma 2.2, we get the asymptotic behavior of $u(r; \xi)$ as $r \to \infty$ for sufficiently small $\xi > 0$.

LEMMA 2.3. *Suppose that* (K) *holds. Then for the solution* $u(r; \xi)$, *there hold* $u(r; \xi) > 0$ *on* $[0, \infty)$ *and*

$$\lim_{r \to \infty} u(r; \xi) > 0$$

for any sufficiently small $\xi > 0$.

PROOF. Let $\xi > 0$ satisfy

$$1 - \frac{\xi}{n-2} \int_0^\infty sK(s)ds > 0. \tag{2.3}$$

Now for such $\xi > 0$, put

$$z(\xi) = \sup\{r > 0 | u(r; \xi) > 0 \text{ on } [0, r)\}.$$

Suppose that there exists $\xi > 0$ satisfying (2.3) such that $z(\xi) < \infty$. Then we have

$$\begin{aligned} u(z(\xi); \xi) &= 1 - \frac{\xi}{n-2} \int_0^{z(\xi)} \left(1 - \left(\frac{s}{z(\xi)}\right)^{n-2}\right) sK(s)u ds \\ &\geq 1 - \frac{\xi}{n-2} \int_0^{z(\xi)} sK(s)ds \\ &\geq 1 - \frac{\xi}{n-2} \int_0^\infty sK(s)ds > 0 \end{aligned}$$

since $1 > u(r; \xi) > 0$ on $(0, z(\xi))$ and since (2.3). This is a contradiction. Thus we have $z(\xi) = \infty$ for $\xi > 0$ sufficiently small. In this case, we can show $u_r < 0$ for any $r > 0$ and we have

$$u(r; \xi) > 1 - \frac{\xi}{n-2} \int_0^\infty sK(s)ds > 0.$$

Since $u(r; \xi)$ is monotone decreasing, we obtain

$$\lim_{r \to \infty} u(r; \xi) > 1 - \frac{\xi}{n-2} \int_0^\infty sK(s)ds > 0$$

by (2.3). The proof is complete. ■

By Lemma 2.3, $u(r;\xi)$ does not change its sign near $r=\infty$. Concerning the limiting behavior of the solution $u(r;\xi)$, we can classify it into two classes by (ii) of Lemma 2.1: $\lim_{r\to\infty} r^{n-2}|u|$ is finite or not. We call a solution of the latter class a *slowly-decaying solution.* As we have seen in the previous lemma, we have specified the limiting behavior of $u(r;\xi)$ if $\xi>0$ is sufficiently small. The next lemma implies that a slowly-decaying solution always has a non-zero limit.

LEMMA 2.4. *Suppose* (K) *holds. Then the solution* $u(r;\xi)$ *satisfies either*

$$\lim_{r\to\infty} |u(r;\xi)| \in (0,\infty)$$

or

$$\lim_{r\to\infty} r^{n-2}|u(r;\xi)| \in (0,\infty).$$

PROOF. The assertion was proved in Lemma 1 of M. Naito [6], however, we give a proof for the sake of self-containedness.

Let $v(r)$ be a solution of (1.2) with $\lim_{r\to\infty} r^{n-2}v=\beta>0$. This implies that there exists $r_0>0$ such that $u>0$ on $[r_0,\infty)$. Put

$$w(r) := v(r)\int_{r_0}^{r} \frac{ds}{s^{n-1}v(s)^2}.$$

Then we can see that $w(r)$ is a solution of

$$(r^{n-1}w_r)_r + \xi r^{n-1}K(r)w = 0.$$

From l'Hospital's rule, we have

$$\lim_{r\to\infty} w(r) = \lim_{r\to\infty} \frac{\int_{r_0}^{r} s^{1-n}v^{-2}ds}{v(r)^{-1}} = -\lim_{r\to\infty} \frac{1}{r^{n-1}v_r(r)} = \frac{1}{(n-2)\beta}$$

since

$$\beta = \lim_{r\to\infty} \frac{v}{r^{2-n}} = -\frac{1}{n-2}\lim_{r\to\infty} r^{n-1}v_r$$

and since $\lim_{r\to\infty} r^{n-1}v_r$ exists. Thus we find that v and w are linearly independent. Since the equation is linear, any solution of (1.2) can be written as a linear combination of v and w, that is, $u=c_1v+c_2w$ with constants c_1 and c_2. The conclusion comes from whether $c_2=0$ or not. ∎

By Lemma 2.2, the number of zeros of $u(r;\xi)$ is finite for any fixed ξ. Now we show the number of zeros tends to infinity as $\xi\to\infty$.

PROPOSITION 2.1. *For any $N \in \mathbf{N} \cup \{0\}$, there exists $\hat{\xi}_N > 0$ such that the solution $u(r; \hat{\xi}_N)$ to* (1.2) *has at least N zeros on* $(0, 1)$.

PROOF. Let $0 < r_0 < r_1 < 1$ be fixed. Set $K_0 = \min_{[r_0, r_1]} K(r) > 0$. Now we consider the ordinary differential equation

$$(r_1^{n-1} u_r)_r + \xi r_0^{n-1} K_0 u = 0. \tag{2.4}$$

Any solution to (2.4) can be written explicitly as

$$u(r) = \alpha(\xi) \sin\left(\sqrt{\frac{\xi r_0^{n-1} K_0}{r_1^{n-1}}}\, r\right) + \beta(\xi) \cos\left(\sqrt{\frac{\xi r_0^{n-1} K_0}{r_1^{n-1}}}\, r\right), \tag{2.5}$$

where $\alpha(\xi)$ and $\beta(\xi)$ are constants dependent on ξ but independent of r.

The number of zeros of (2.5) on $[r_0, r_1]$ tends to infinity as $\xi \to \infty$. Since $r^{n-1} \le r_1^{n-1}$ and $r_0^{n-1} K_0 \le r^{n-1} K(r)$ on $[r_0, r_1]$, the number of zeros of $u(r; \xi)$ on $[r_0, r_1]$ also tends to infinity as $\xi \to \infty$ by Sturm's comparison theorem (see e.g., Hartman [3] Theorem 3.1, p. 334). The proof is complete. ■

Similar to Proposition 2.1, for the problem (1.3) we have a dual version of Proposition 2.1.

PROPOSITION 2.2. *For any $N \in \mathbf{N} \cup \{0\}$, there exists $\hat{\xi}_N > 0$ such that the solution $\tilde{u}(r; \hat{\xi}_N)$ to* (1.3) *has at least N zeros on* $(1, \infty)$.

PROOF. Using the Kelvin transformation $v(t) = r^{n-2} \tilde{u}(r)$, $t = 1/r$, we reduce the problem (1.3) to (1.2) and get the conclusion by Proposition 2.1. ■

3. The Prüfer transformation

To prove the existence of a solution with prescribed zeros, we efficiently use the Prüfer transformation.

Let

$$u(r; \xi) = \rho \cos \theta \tag{3.1}$$

and

$$-r^{n-1} u_r(r; \xi) = \rho \sin \theta \tag{3.2}$$

where $\rho(r; \xi) = (u^2 + (r^{n-1} u_r)^2)^{1/2}$ and $\theta = \theta(r; \xi)$ are continuous functions of r satisfying $\rho(0; \xi) = 1$ and $\theta(0; \xi) = 0$, respectively. Moreover they are also continuously dependent on parameter ξ (see e.g., Theorem 7.4, p. 9 of Coddington and Levinson [1]). We begin with enumerating fundamental lemmas.

LEMMA 3.1. *Under* (K), *there holds $\theta_r(r; \xi) > 0$ for $r > 0$.*

PROOF. Since $u_r(r; \xi) = r^{-(n-1)}(r^{n-1}u_r)$ and $-(r^{n-1}u_r)_r = \xi r^{n-1}K(r)u$, we get

$$\rho_r \cos\theta - (\rho\sin\theta)\theta_r = -r^{-(n-1)}\rho\sin\theta \tag{3.3}$$

and

$$\rho_r \sin\theta + (\rho\cos\theta)\theta_r = \xi r^{n-1}K(r)\rho\cos\theta. \tag{3.4}$$

Multiplying (3.3) and (3.4) by $-\sin\theta$ and by $\cos\theta$, respectively, we obtain

$$\rho\theta_r = r^{-(n-1)}\rho\sin^2\theta + \xi r^{n-1}K(r)\rho\cos^2\theta > 0. \tag{3.5}$$

Since $\rho > 0$ and $K(r) > 0$, we get the conclusion. ■

LEMMA 3.2. *Under* (K), *there holds* $\limsup_{\xi\to\infty}\theta(1; \xi) = \infty$.

PROOF. This comes from Proposition 2.1. ■

LEMMA 3.3. *Under* (K), *there hold* $\lim_{\xi\to+0}\rho(1; \xi) = 1$ *and* $\lim_{\xi\to+0}\theta(1; \xi) = 0$.

PROOF. Let us put

$$r_0(\xi) = \sup\{r > 0 | u(r; \xi) > 0 \text{ on } (0, r)\}.$$

Then for $0 < r < r_0(\xi)$, we have

$$0 \geq u_r(r; \xi) = -\frac{\xi}{r^{n-1}}\int_0^r s^{n-1}K(s)u\,ds \geq -\frac{\xi}{r^{n-1}}\int_0^r s^{n-1}K(s)ds$$

and

$$1 \geq u(r; \xi) \geq 1 - \frac{\xi}{n-2}\int_0^r sK(s)ds.$$

Hence $r_0(\xi) \to \infty$ as $\xi \to +0$, and we have

$$\begin{cases} \lim\limits_{\xi\to 0} u(1; \xi) = 1, \\ \lim\limits_{\xi\to 0} u_r(1; \xi) = 0. \end{cases}$$

This establishes the conclusion. ■

In turn, we put

$$\tilde{u}(r; \xi) = \tilde{\rho}_\beta \cos\tilde{\theta}_\beta, \tag{3.6}$$

$$-r^{n-1}\tilde{u}_r(r; \xi) = \tilde{\rho}_\beta \sin\tilde{\theta}_\beta, \tag{3.7}$$

and $\lim_{r\to\infty}\tilde{\theta}_\beta(r; \xi) = \pi/2$. By l'Hospital's rule, we get

$$\beta = \lim_{r\to\infty} \frac{\tilde{u}}{r^{2-n}} = -\frac{1}{n-2} \lim_{r\to\infty} r^{n-1}\tilde{u}_r.$$

Hence we have

$$\lim_{r\to\infty} \tilde{\rho}_\beta(r; \xi) = (n-2)\beta.$$

Hereafter, until Section 5, we omit the suffix β. The following are dual versions of Lemma 3.1, 3.2, and 3.3, respectively. Lemma 3.4 is nothing but Lemma 3.1.

LEMMA 3.4. *Under* (K), *there holds* $\tilde{\theta}_r(r; \xi) > 0$ *for* $r > 0$.

LEMMA 3.5. *Under* (K), *there holds* $\liminf_{\xi\to\infty} \tilde{\theta}(1; \xi) = -\infty$.

These are proved in the same way as the proofs of Lemmas 3.2 and 3.3.

LEMMA 3.6. *Under* (K), *there hold* $\lim_{\xi\downarrow 0} \tilde{\rho}(1; \xi) = \beta\sqrt{(n-2)^2+1}$ *and* $\lim_{\xi\downarrow 0} \tilde{\theta}(1; \xi) = \tan^{-1}(n-2)$.

PROOF. Let $\hat{R}(\xi) = \inf\{r > 0 | \tilde{u}(r; \xi) > 0$ and $(r^{n-2}\tilde{u})_r \geq 0$ on $[r, \infty)\}$. $\hat{R}(\xi)$ is well-defined by Lemma 2.1. Then for $r \in (\hat{R}(\xi), \infty)$, we have

$$r^{n-1}\tilde{u}_r(r; \xi) = \xi \int_r^\infty sK(s)\tilde{U}ds - (n-2)\beta$$

and

$$\begin{aligned}\tilde{U}(r) &= r^{n-2}\int_r^\infty s^{-(n-1)}\left\{-\xi\int_s^\infty tK(t)\tilde{U}dt + (n-2)\beta\right\}ds \\ &= \beta - \frac{\xi}{n-2}\int_r^\infty \left(1 - \left(\frac{r}{t}\right)^{n-2}\right)tK(t)\tilde{U}dt\end{aligned}$$

where $\tilde{U} = r^{n-2}u$. Then we have

$$-(n-2)\beta \leq r^{n-1}u_r(r) \leq \xi\beta\int_r^\infty sK(s)ds - (n-2)\beta,$$

$$0 \leq \xi\int_r^\infty \left(\frac{r}{s}\right)^{n-3} K(s)\tilde{U}ds = \tilde{U}_r(r),$$

and

$$\beta\left(1 - \frac{\xi}{n-2}\int_r^\infty tK(t)dt\right) \leq \tilde{U}(r) \leq \beta,$$

for $r \in (\hat{R}(\xi), \infty)$. Letting $\xi \to 0$, we get $\hat{R}(\xi) \to 0$, $\lim_{\xi\downarrow 0} \tilde{u}_r(1; \xi) = -(n-2)\beta$

and $\lim_{\xi\to 0}\tilde{u}(1;\xi)=\beta$. This implies that

$$\lim_{\xi\downarrow 0}\tilde{\theta}(1;\xi)=\tan^{-1}(n-2)$$

and

$$\lim_{\xi\to 0}\tilde{\rho}(1;\xi)=\beta\sqrt{(n-2)^2+1}. \qquad \blacksquare$$

4. Proof of the Uniqueness

Before showing the existence of a solution, we prove the uniqueness. The comparison lemma is very useful.

LEMMA 4.1. *Suppose that* $\xi'>\xi>0$. *Then* $u(r;\xi')$ *oscillates faster than* $u(r;\xi)$; *more precisely, if* $\xi'>\xi>0$, *then* $\theta(r;\xi')>\theta(r;\xi)$ *for* $r\in(0,\infty)$ *where* θ *is defined in* (3.1)–(3.2).

PROOF. As we have seen in (3.5), θ satisfies

$$\rho\theta_r=r^{-(n-1)}\rho\sin^2\theta+\xi r^{n-1}K(r)\rho\cos^2\theta.$$

In the interval $[r_0,\infty)$ for any $r_0>0$ there is no singularity in the right-hand side, so we can use Prüfer's comparison theorem to get the conclusion. Thus we have only to show the statement near the origin. First we choose $r_0>0$ so that $u(r;\xi')$, $u(r;\xi)>0$ on $[0,r_0]$. Such $r_0>0$ exists because $u(0;\xi)=u(0;\xi')=1$. By Green's formula, for any $0<\varepsilon<r<r_0$, we have

$$[s^{n-1}(u_s(s;\xi')u(s;\xi)-u(s;\xi')u_s(s;\xi))]_\varepsilon^r=-(\xi'-\xi)\int_\varepsilon^r s^{n-1}K(s)u(s;\xi)u(s;\xi')ds.$$

From (i) of Lemma 2.1, we get $\varepsilon^{n-1}u_r(\varepsilon;\xi)\to 0$ and $\varepsilon^{n-1}u_r(\varepsilon;\xi')\to 0$ as $\varepsilon\to 0$. By $s^{n-1}K(s)u(s;\xi)u(s;\xi')\in L^1(0,1)$, we obtain

$$r^{n-1}(u_r(r;\xi')u(r;\xi)-u(r;\xi')u_r(r;\xi))=-(\xi'-\xi)\int_0^r s^{n-1}K(s)u(s;\xi)u(s;\xi')ds.$$

This implies

$$\rho(r;\xi)\rho(r;\xi')(\sin\theta(r;\xi')\cos\theta(r;\xi)-\sin\theta(r;\xi)\cos\theta(r;\xi'))>0,$$

that is,

$$\sin(\theta(r;\xi')-\theta(r;\xi))>0.$$

Thus $\theta(r;\xi)<\theta(r;\xi')$ for $0<r<r_0$ by $\theta(0;\xi)=0$ and the continuity of θ with respect to ξ. Recalling that we can apply Prüfer's comparison theorem on $[r_0,\infty)$, we obtain

$$\theta(r;\xi)<\theta(r;\xi') \qquad \text{for } r>0. \qquad \blacksquare$$

Proposition 4.1. *There exists at most one solution to* (1.4).

Proof. This is immediate from Lemma 4.1. ■

Proposition 4.2. *There exists at most one solution to* (1.5).

Proof. The conclusion comes from Lemma 4.1 and the Kelvin transformation. ■

5. Proof of Theorems

Before proving Theorem 1.1, we prove Theorems 1.2 and 1.3. Let

$$\Gamma = \{(\theta(1; \xi), \rho(1; \xi)) | \xi \in (0, \infty)\}$$

and

$$\tilde{\Gamma}_j(\beta) = \{(\tilde{\theta}_1(1; \xi) + j\pi, \beta\tilde{\rho}_1(1; \xi)) | \xi \in (0, \infty)\}$$

where $j = 0, 1, 2, \ldots$. Then Γ is a curve in $\mathbf{R}^2$ starting from $(0, 1)$ and $\tilde{\theta}_1(1; \xi) \to \infty$ as $\beta \to \infty$. Similarly, $\tilde{\Gamma}_j(\beta)$ is a curve starting from $(\tan^{-1}(n-2) + j\pi, \beta\sqrt{(n-2)^2 + 1})$ and $\tilde{\theta}(1: \xi) \to -\infty$ as $\xi \to \infty$.

Proof of Theorem 1.2. We note that

$$\lim_{\xi \to +0} \theta(1; \xi) = 0 \qquad \text{and} \qquad \limsup_{\xi \to \infty} \theta(1; \xi) = \infty$$

by Lemmas 3.2 and 3.3. Let ξ_j be the smallest number such that

$$\theta(1; \xi_j) = \begin{cases} \varphi + j\pi & (j = 0, 1, 2, \ldots) \qquad 0 < \varphi \le \pi/2, \\ \varphi + j\pi & (j = 1, 2, 3, \ldots) \qquad -\pi/2 < \varphi \le 0, \end{cases}$$

which implies that

$$u(1; \xi_j) \sin \varphi + u_r(1; \xi_j) \cos \varphi = 0.$$

In view of Lemma 3.1, $\theta(r; \xi_j)$ is a strictly increasing function of $r \in (0, 1)$, so $\theta(r; \xi_j)$ varies from 0 to $\theta(1; \xi_j)$ as r does from 0 to 1. Hence $u(r; \xi_j)$ has exactly j zeros on $(0, 1)$. Moreover, by Proposition 4.1, such ξ_j is unique. Finally the monotonicity of $\{\xi_j\}$ follows from Lemma 4.1 and the divergence of ξ_j comes from Proposition 2.1. The proof is complete. ■

Proof of Theorem 1.3. We note that

$$\lim_{\xi \to +0} \tilde{\theta}(1; \xi) = \tan^{-1}(n - 2) \qquad \text{and} \qquad \lim_{\xi \to \infty} \tilde{\theta}(1; \xi) = -\infty$$

by Lemmas 3.5 and 3.6. Let $\tilde{\xi}_j$ be the smallest number such that

$$\tilde{\theta}(1;\tilde{\xi}_j)=\begin{cases}\varphi-j\pi & (j=0,1,2,\ldots) \quad -\pi/2\le\varphi<\tan^{-1}(n-2),\\ \varphi-j\pi & (j=1,2,3,\ldots) \quad \tan^{-1}(n-2)\le\varphi<\pi/2,\end{cases}$$

which implies that

$$\tilde{u}(1;\tilde{\xi}_j)\sin\varphi+\tilde{u}_r(1;\tilde{\xi}_j)\cos\varphi=0.$$

In view of Lemma 3.1, $\tilde{\theta}(r;\tilde{\xi}_j)$ is a strictly increasing function of $r\in(1,\infty)$, which varies from $\tilde{\theta}(1;\tilde{\xi}_j)$ to $\pi/2$. Hence $\tilde{u}(1;\tilde{\xi}_j)$ has exactly j zeros on $(1,\infty)$. Moreover, by Proposition 4.2, such $\tilde{\xi}_j$ is unique. This shows the conclusion. Finally the monotonicity of $\{\tilde{\xi}_j\}$ follows from Lemma 4.1 and the divergence of $\tilde{\xi}_j$ comes from Proposition 2.2. The proof is complete. ∎

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. First we note that by the linearity of the equation,

$$\tan\tilde{\theta}_1(1;\xi)=-\frac{\tilde{u}_r(1;\xi)}{\tilde{u}(1;\xi)}=-\frac{\beta\tilde{u}_r(1;\xi)}{\beta\tilde{u}(1;\xi)}=\tan\tilde{\theta}_\beta(1;\xi).$$

This implies that $P\tilde{\Gamma}_j(\beta)=P\tilde{\Gamma}_j(1)$, where P is the orthogonal projection from $\mathbf{R}^2$ to $\mathbf{R}_\theta$. Hence from $\tilde{\rho}(1;\xi)>0$ for $\xi>0$, $\tilde{\Gamma}_j(\beta)$ lies entirely beneath Γ if $\beta>0$ is sufficiently small, and $\tilde{\Gamma}_j(\beta)$ lies entirely above Γ if β is sufficiently large. We claim that in the interval $[0,\tan^{-1}(n-2)+j\pi]$, there exists $\xi_j>0$ such that $\theta(1;\xi_j)=\tilde{\theta}(1;\xi_j)+j\pi$.

We also note that Γ is continuously parametrized by ξ and so is $\tilde{\Gamma}_j(\beta)$.

Now, for any $\phi\ge0$, let $\xi(\phi)\ge0$ and $\tilde{\xi}(\phi)\ge0$ satisfy $\theta(1;\xi(\phi))=\phi$ and $\theta(1;\tilde{\xi}(\phi))+j\pi=\phi$, respectively. That is, $\xi(\phi)$ and $\tilde{\xi}(\phi)$ are the inverse function of $\theta(1;\cdot)$ and $\tilde{\theta}(1;\cdot)$, respectively. We show that they are continuous and monotone.

We define $\Phi(\phi)=\xi(\phi)-\tilde{\xi}(\phi)$. Taking Lemma 4.1, Propositions 4.1, and 4.2 into account, we find that $\theta(1;\xi)$ and $\tilde{\theta}(1;\xi)$ are one-to-one onto mappings from $\mathbf{R}_+=[0,\infty)$ to $\mathbf{R}_+$. Moreover, by Lemma 4.1, $\theta(1;\xi)$ is an increasing function of ξ and $\tilde{\theta}(1;\tilde{\xi})$ is a decreasing one. In addition, by the continuity of $\theta(1;\xi)$ and $\tilde{\theta}(1;\xi)$, there hold

$$\overline{\theta(1;M)}=\theta(1;\overline{M})$$

and

$$\overline{\tilde{\theta}(1;M)}=\tilde{\theta}(1;\overline{M})$$

for any subset $M\subset\mathbf{R}_+$. Thus $\theta(1;\xi)$ and $\tilde{\theta}(1;\xi)$ are homeomorphisms from $\mathbf{R}_+$ onto itself. So the inverse mappings $\xi(\phi)$ and $\tilde{\xi}(\phi)$ are continuous.

We note that $\Phi(0) = -\tilde{\xi}(0) < 0$ and $\Phi(\tan^{-1}(n-2) + j\pi) = \xi(\tan^{-1}(n-2) + j\pi) > 0$. Then by the intermediate value theorem, there exists ϕ_j such that $\Phi(\phi_j) = 0$. That is, $\xi(\phi_j) = \tilde{\xi}(\phi_j)$ and we set $\xi(\phi_j) =: \xi_j$. Since $\xi(\phi)$ is increasing and since $\tilde{\xi}(\phi)$ is decreasing, ϕ_j is unique. This shows that uniqueness of the desired solution.

Now we choose β_j so that

$$(\tilde{\theta}_1(1; \xi(\phi_j)) + j\pi, \beta_j\tilde{\rho}_1(1; \xi(\phi_j)) = (\phi_j + j\pi, \beta_j\rho_1(1; \xi_j)) \in \tilde{\Gamma}_j(\beta_j)$$

is on Γ. This implies that

$$\theta(1; \xi_j) = \tilde{\theta}(1; \xi_j) + j\pi, \qquad \rho(1; \xi_j) = \tilde{\rho}_{\beta_j}(1; \tilde{\xi}_j),$$

that is,

$$u(1; \xi_j) = (-1)^j\beta_j\tilde{u}(1; \xi_j), \qquad u_r(1; \xi_j) = (-1)^j\beta_j\tilde{u}_r(1; \xi_j).$$

Hence

$$u(r; \xi_j) \equiv (-1)^j\beta_j\tilde{u}(r; \xi_j) \qquad \text{on } (0, \infty)$$

and

$$\lim_{r\to\infty} r^{n-2}u(r; \xi_j) = (-1)^j\beta_j$$

by the uniqueness of solutions to (1.1). Moreover, since

$$\theta(0; \xi_j) = 0$$

and

$$\lim_{r\to\infty} \tilde{\theta}(r; \xi_j) = \frac{\pi}{2},$$

the total variation of $\theta(r; \xi_j)$ as r varies from 0 to ∞ is $(j + 1/2)\pi$. This implies that $u(r; \xi_j)$ has exactly j zeros on $[0, \infty)$ and $\lim_{r\to\infty} r^{n-2}|u(r; \xi_j)| \in (0, \infty)$. The monotonicity and the divergence of ξ is ensured by Lemmas 4.1, the continuity of $\theta(r; \theta)$ with respect to ξ for any fixed $r > 0$, Proposition 2.1 and Proposition 2.2. The proof is complete. ■

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