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Strong solution for a mixed problem with nonlocal condition for certain pluriparabolic equations

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ABSTRACT. The present paper is devoted to a proof of the existence and uniqueness of a strong solution for a mixed problem with nonlocal condition for certain pluriparabolic equations. The proof is based on an a priori estimate and on the density of the range of the operator generated by the studied problem.

1. Statement of the problem

In the domain $Q = (0, b) \times (0, T_1) \times (0, T_2)$, with $b < \infty$, $T_1 < \infty$ and $T_2 < \infty$, we consider the one-dimensional pluriparabolic equation

$$\mathscr{L}v = \partial v/\partial t_1 + \partial v/\partial t_2 - \partial(a(x, t_1, t_2)\partial v/\partial x)/\partial x = f(x, t_1, t_2), \tag{1.1}$$

where $a(x, t_1, t_2)$ satisfy the following assumptions:

H1. $c_0 \le a(x, t_1, t_2) \le c_1$, $\partial a(x, t_1, t_2)/\partial x \le c_2$, $\partial a(x, t_1, t_2)/\partial t_p \le c_3$, $p = 1, 2$, $(x, t_1, t_2) \in \bar{Q}$.

H2. $\partial^2 a(x, t_1, t_2)/\partial t_p^2 \le c_4$, $\partial^2 a(x, t_1, t_2)/\partial x^2 \le c_5$, $\partial^2 a(x, t_1, t_2)/\partial t_p \partial x \le c_6$, $p = 1, 2$, $(x, t_1, t_2) \in \bar{Q}$.

We pose the following problem for equation (1.1): to determine its solution v in Q satisfying the initial conditions

$$\ell_1 v = v(x, 0, t_2) = \Phi_1(x, t_2), \qquad (x, t_2) \in Q_2 = (0, b) \times (0, T_2), \tag{1.2}$$

$$\ell_2 v = v(x, t_1, 0) = \Phi_2(x, t_1), \qquad (x, t_1) \in Q_1 = (0, b) \times (0, T_1), \tag{1.3}$$

the Neumann condition

$$\partial v(0, t_1, t_2)/\partial x = \mu(t_1, t_2), \qquad (t_1, t_2) \in (0, T_1) \times (0, T_2), \tag{1.4}$$

and the integral condition

$$\int_0^b v(x, t_1, t_2)dx = E(t_1, t_2), \qquad (t_1, t_2) \in (0, T_1) \times (0, T_2). \tag{1.5}$$

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Where $\Phi_1(x, t_2)$, $\Phi_2(x, t_1)$, $\mu(t_1, t_2)$, $E(t_1, t_2)$, $a(x, t_1, t_2)$ and $f(x, t_1, t_2)$ are known functions.

The data satisfies the following compatibility conditions:

$$\partial\Phi_1(0, t_2)/\partial x = \mu(0, t_2), \qquad \int_0^b \Phi_1(x, t_2)dx = E(0, t_2),$$

$$\partial\Phi_2(0, t_1)/\partial x = \mu(t_1, 0), \qquad \int_0^b \Phi_2(x, t_1)dx = E(t_1, 0),$$

and

$$\Phi_1(x, 0) = \Phi_2(x, 0).$$

This type of problems is propounded in the mathematical modelling of technologic process of external elimination of gas, practises in the refining of impurities of Silicon laminae. In this case, $v(x, t_1, t_2)$ is the distribution of impurities in the lamina $\{0 \leq x \leq b\}$ at the time t_1 and at the temperature t_2, $\Phi_1(x, t_2)$ is the distribution of impurities at the initial time and at the temperature t_2, $\Phi_2(x, t_1)$ is the distribution of impurities at the time t_1 and at the initial temperature. The condition (1.4) means that the flow of diffusion throughout the left boundary is equal of $\mu(t_1, t_2)$, and the condition (1.5) is the total mass of impurities in the lamina $\{0 \leq x \leq b\}$.

The first investigation of mixed problems with integral conditions goes back to Cannon [8] in 1963. The author proved, with the aid of integral equation, the existence and uniqueness of the solution for a mixed problem which combine Dirichlet and integral conditions for the homogeneous heat equation. Kamynin [14] extended the result of [8] to the general linear second order parabolic equation in 1964, by using a system of integral equations.

Along a different line, mixed problems for second order parabolic equations which combine local and integral conditions were considered by Ionkin [13], Cannon–van der Hoek [9], [10], Cannon–Esteva–van der Hoek [11], Lin [16], Kartynnik [15], Benouar–Yurchuk [1], Shi [17] and Yurchuk [18]. Recently, mixed problems with only integral conditions for parabolic and hyperbolic equations have been treated in Bouziani [3] and Bouziani–Benouar [5], [6].

In this paper, the existence and uniqueness of a strong solution of problem (1.1)–(1.5) is proved. The method in the present paper is further elaboration of that in Bouziani [2], [4] and Bouziani–Benouar [7].

To achieve the purpose, we reduce the non homogeneous boundary conditions (1.4), (1.5) to homogeneous conditions, by introducing a new unknown function u defined as follows:

$$u(x, t_1, t_2) = v(x, t_1, t_2) - \mathscr{U}(x, t_1, t_2),$$

where

$$\mathscr{U}(x, t_1, t_2) = \mu(t_1, t_2)x + 3x^2/b^3 \cdot \left(E(t_1, t_2) - \frac{b^2}{2}\mu(t_1, t_2) \right),$$

Then, the problem can be formulated in this way:

$$\mathscr{L}u = \not{f} - \mathscr{L}\mathscr{U} = f, \tag{1.6}$$

$$\ell_1 u = u(x, 0, t_2) = \Phi_1(x, t_2) - \ell_1 \mathscr{U} = \varphi_1(x, t_2), \tag{1.7}$$

$$\ell_2 u = u(x, t_1, 0) = \Phi_2(x, t_1) - \ell_2 \mathscr{U} = \varphi_2(x, t_1), \tag{1.8}$$

$$\partial u(0, t_1, t_2)/\partial x = 0, \tag{1.9}$$

$$\int_0^b u(x, t_1, t_2)dx = 0. \tag{1.10}$$

Here we assume that the functions φ_p, $p = 1, 2$, satisfies conditions of the form (1.9), (1.10), i.e., $\partial\varphi_p(0, \cdot)/\partial x = 0$, $\int_0^b \varphi_p(x, 0)dx = 0$, and such that $\varphi_1(x, 0) = \varphi_2(x, 0)$.

Instead of searching for the function v, we search for the function u. So, the strong solution of problem (1.1)–(1.5) will be given by: $v(x, t_1, t_2) = u(x, t_1, t_2) + \mathscr{U}(x, t_1, t_2)$.

2. A priori estimate and its consequences

The problem (1.6)–(1.10) is equivalent to the operator equation

$$Lu = \mathscr{F},$$

where $Lu = (\mathscr{L}u, \ell_1 u, \ell_2 u)$, $\mathscr{F} = (f, \varphi_1, \varphi_2)$. The operator L acts from B to F, where B is the Banach space of functions $u \in L^2(Q)$, satisfying (1.9) and (1.10), with the finite norm

$$\|\partial u/\partial x\|_{0,Q}^2 + \sup_{0 \le \tau_1 \le T_1} \|u(x, \tau_1, t_2)\|_{0,Q_2}^2 + \sup_{0 \le \tau_2 \le T_2} \|u(x, t_1, \tau_2)\|_{0,Q_1}^2$$

and F is the Hilbert space of vector-valued functions $\mathscr{F} = (f, \varphi_1, \varphi_2)$, obtained by completing the space $L^2(Q) \times L^2(Q_2) \times L^2(Q_1)$ with respect to the norm

$$\|\mathscr{F}\|_F^2 = \|f\|_{0,Q}^2 + \|\varphi_1\|_{0,Q_2}^2 + \|\varphi_2\|_{0,Q_1}^2.$$

Let $D(L)$ be the set of all functions $u \in L^2(Q)$ for which $\partial u/\partial t_1$, $\partial u/\partial t_2$, $\partial u/\partial x$, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial t_1$, $\partial^2 u/\partial x \partial t_2 \in L^2(Q)$ and satisfying conditions (1.9)–(1.10).

THEOREM 1. *If the assumptions* H1 *are satisfied, then for any function* $u \in D(L)$, *we have*

$$\|u\|_B \leq c\|Lu\|_F \tag{2.1}$$

where $c > 0$ *is a constant independent of* u.

PROOF. Taking the scalar product in $L^2(Q^\tau)$ of equation (1.6) and the operator

$$Mu = 2(b-x)[\Im_x(\partial u/\partial t_1 + \partial u/\partial t_2) - a(x, t_1, t_2)\partial u/\partial x],$$

where $Q^\tau = (0, b) \times (0, \tau_1) \times (0, \tau_2)$ and $\Im_x g = \int_0^x g(\xi, t_1, t_2)d\xi$, we obtain

$$\begin{aligned}(\mathscr{L}u, Mu)_{0,Q} = {} & 2\int_{Q^\tau} (b-x)\partial u/\partial t_1 \cdot \Im_x(\partial u/\partial t_1)dxdt_1dt_2 \\ & + 2\int_{Q^\tau} (b-x)\partial u/\partial t_2 \cdot \Im_x(\partial u/\partial t_2)dxdt_1dt_2 \\ & + 2\int_{Q^\tau} (b-x)\partial u/\partial t_1 \cdot \Im_x(\partial u/\partial t_2)dxdt_1dt_2 \\ & + 2\int_{Q^\tau} (b-x)\partial u/\partial t_2 \cdot \Im_x(\partial u/\partial t_1)dxdt_1dt_2 \\ & - 2\int_{Q^\tau} (b-x)\partial(a(x, t_1, t_2)\partial u/\partial x)/\partial x \cdot \Im_x(\partial u/\partial t_2)dxdt_1dt_2 \\ & - 2\int_{Q^\tau} (b-x)a(x, t_1, t_2)\partial u/\partial x \cdot \partial u/\partial t_1 dxdt_1dt_2 \\ & - 2\int_{Q^\tau} (b-x)a(x, t_1, t_2)\partial u/\partial x \cdot \partial u/\partial t_2 dxdt_1dt_2 \\ & + 2\int_{Q^\tau} (b-x)\partial(a(x, t_1, t_2)\partial u/\partial x)/\partial x \\ & \cdot a(x, t_1, t_2)\partial u/\partial x dxdt_1dt_2.\end{aligned} \tag{2.2}$$

The successive integration by parts of integrals on the right-hand side of (2.2) are straightforward but somewhat tedious. We only give their results

$$\begin{aligned} & 2\int_{Q^\tau} (b-x)\partial u/\partial t_p \cdot \Im_x(\partial u/\partial t_p)dxdt_1dt_2 \\ & = \int_{Q^\tau} (\Im_x(\partial u/\partial t_p))^2 dxdt_1dt_2, \qquad p = 1,\ 2,\end{aligned} \tag{2.3}$$

$$(2.4)\qquad 2\int_{Q^\tau}(b-x)\partial u/\partial t_2\cdot\Im_x(\partial u/\partial t_1)dxdt_1dt_2$$
$$=2\int_{Q^\tau}\Im_x(\partial u/\partial t_2)\cdot\Im_x(\partial u/\partial t_1)dxdt_1dt_2$$
$$-2\int_{Q^\tau}(b-x)\Im_x(\partial u/\partial t_2)\cdot\partial u/\partial t_1dxdt_1dt_2,$$

$$(2.5)\qquad -2\int_{Q^\tau}(b-x)\partial(a(x,t_1,t_2)\partial u/\partial x)/\partial x\cdot\Im_x\partial u/\partial t_1dxdt_1dt_2$$
$$=\int_{Q_2^{\tau_2}}a(x,\tau_1,t_2)\cdot(u(x,\tau_1,t_2))^2dxdt_2-\int_{Q_2^{\tau_2}}a(x,0,t_2)\cdot(\varphi_1(x,t_2))^2dxdt_2$$
$$-\int_{Q^\tau}\partial a(x,t_1,t_2)/\partial t_1\cdot u^2dxdt_1dt_2$$
$$-2\int_{Q^\tau}\partial a(x,t_1,t_2)/\partial x\cdot u\cdot\Im_x(\partial u/\partial t_1)dxdt_1dt_2$$
$$+2\int_{Q^\tau}(b-x)a(x,t_1,t_2)\partial u/\partial x\cdot\partial u/\partial t_1dxdt_1dt_2,$$

$$(2.6)\qquad -2\int_{Q^\tau}(b-x)\partial(a(x,t_1,t_2)\partial u/\partial x)/\partial x\cdot\Im_x(\partial u/\partial t_2)dxdt_1dt_2$$
$$=\int_{Q_1^{\tau_1}}a(x,t_1,\tau_2)\cdot(u(x,t_1,\tau_2))^2dxdt_1-\int_{Q_1^{\tau_1}}a(x,t_1,0)\cdot(\varphi_2(x,t_1))^2dxdt_1$$
$$-\int_{Q^\tau}\partial a(x,t_1,t_2)/\partial t_2\cdot u^2dxdt_1dt_2$$
$$-2\int_{Q^\tau}\partial a(x,t_1,t_2)/\partial x\cdot u\cdot\Im_x(\partial u/\partial t_2)dxdt_1dt_2$$
$$+2\int_{Q^\tau}(b-x)a(x,t_1,t_2)\partial u/\partial x\cdot\partial u/\partial t_2dxdt_1dt_2,$$

$$(2.7)\qquad 2\int_{Q^\tau}(b-x)\partial(a(x,t_1,t_2)\partial u/\partial x)/\partial x\cdot a(x,t_1,t_2)\cdot\partial u/\partial xdxdt_1dt_2$$
$$=\int_{Q^\tau}(a(x,t_1,t_2))^2(\partial u/\partial x)^2dxdt_1dt_2.$$

Substituting (2.3)–(2.7) into (2.2), we obtain

(2.8)

$$\int_{Q^\tau}(\Im_x(\partial u/\partial t_1)+\Im_x(\partial u/\partial t_2))^2dxdt_1dt_2+\int_{Q_2^{\tau_2}}a(x,\tau_1,t_2)\cdot(u(x,\tau_1,t_2))^2dxdt_2$$
$$+\int_{Q_1^{\tau_1}}a(x,t_1,\tau_2)(u(x,t_1,\tau_2))^2dxdt_1+\int_{Q^\tau}(a(x,t_1,t_2))^2(\partial u/\partial x)^2dxdt_1dt_2$$
$$=(\mathscr{L}u,Mu)_{0,Q^\tau}-2\int_{Q^\tau}\partial a(x,t_1,t_2)/\partial x\cdot u\cdot(\Im_x(\partial u/\partial t_1)+\Im_x(\partial u/\partial t_2))dxdt_1dt_2$$
$$+\int_{Q_2^{\tau_2}}a(x,0,t_2)\cdot(\varphi_1(x,t_2))^2dxdt_2+\int_{Q_1^{\tau_1}}a(x,t_1,0)\cdot(\varphi_2(x,t_1))^2dxdt_1$$
$$+\int_{Q^\tau}(\partial a(x,t_1,t_2)/\partial t_1+\partial a(x,t_1,t_2)/\partial t_2)u^2dxdt_1dt_2.$$

We estimate the first term on the right-hand side of (2.8) by applying the Cauchy-Schwarz inequality and the Cauchy inequality

$$(2.9)\quad (\mathscr{L}u,Mu)_{0,Q^\tau}\le 2b^2\int_{Q^\tau}f^2dxdt_1dt_2+2b^2/c_0\cdot\int_{Q^\tau}(a(x,t_1,t_2))^2f^2dxdt_1dt_2$$
$$+c_0/2\int_{Q^\tau}(\partial u/\partial x)^2dxdt_1dt_2$$
$$+1/2\int_{Q^\tau}(\Im_x(\partial u/\partial t_1)+\Im_x(\partial u/\partial t_2))^2dxdt_1dt_2.$$

The remaining integral throughout Q^τ on the same side of (2.8) can be estimated as follows

$$(2.10)\qquad -2\int_{Q^\tau}\partial a(x,t_1,t_2)/\partial x\cdot u\cdot(\Im_x(\partial u/\partial t_1)+\Im_x(\partial u/\partial t_2))dxdt_1dt_2$$
$$\le 2\int_{Q^\tau}(\partial a(x,t_1,t_2)/\partial x)^2u^2dxdt_1dt_2$$
$$+1/2\int_{Q^\tau}(\Im_x(\partial u/\partial t_1)+\Im_x(\partial u/\partial t_2))^2dxdt_1dt_2.$$

By virtue of (2.9) and (2.10) and the conditions H1, we can transform (2.8) into (2.11)

$$(2.11)\qquad c_0/2\|\partial u/\partial x\|_{0,Q^\tau}^2+c_0\|u(x,\tau_1,t_2)\|_{0,Q_2^{\tau_2}}^2+c_0\|u(x,t_1,\tau_2)\|_{0,Q_1^{\tau_1}}^2$$
$$\le 2b^2(1+c_1^2/c_0)\|f\|_{0,Q}^2+c_1\|\varphi_1\|_{0,Q_2}^2+c_1\|\varphi_2\|_{0,Q_1}^2$$
$$+2(c_2^2+c_3)\|u\|_{0,Q^\tau}^2.$$

We eliminate the last term on the right-hand side of (2.11). To do that we use the following Lemma:

LEMMA 1. *If* $f_1(\tau_1, \tau_2)$, $f_2(\tau_1, \tau_2)$ *and* $f_3(\tau_1, \tau_2)$ *are nonnegative functions on the rectangle* $(0, T_1) \times (0, T_2)$, $f_1(\tau_1, \tau_2)$ *and* $f_2(\tau_1, \tau_2)$ *are integrable, and* $f_3(\tau_1, \tau_2)$ *is nondecreasing in each of its variables separately, then it follows from*

$$(2.12)\qquad \int_0^{\tau_1}\int_0^{\tau_2} f_1(t_1, t_2)dt_1dt_2 + f_2(\tau_1, \tau_2) \le c\left(\int_0^{\tau_1} f_2(t_1, \tau_2)dt_1 + \int_0^{\tau_2} f_2(\tau_1, t_2)dt_2\right) + f_3(\tau_1, \tau_2)$$

that

$$(2.13)\qquad \int_0^{\tau_1}\int_0^{\tau_2} f_1(t_1, t_2)dt_1dt_2 + f_2(\tau_1, \tau_2) \le \exp\left(2c(\tau_1 + \tau_2)\right)\cdot f_3(\tau_1, \tau_2).$$

PROOF OF LEMMA 1. We write (2.12) in the form

$$(2.14)\qquad Tf_1 + f_2 \le Kf_2 + f_3,$$

where

$$Tf_1 = \int_0^{\tau_1}\int_0^{\tau_2} f_1(t_1, t_2)dt_1dt_2$$

and

$$Kf_2 = \int_0^{\tau_1} f_2(t_1, \tau_2)dt_1 + \int_0^{\tau_2} f_2(\tau_1, t_2)dt_2.$$

Since f_1 is nonnegative function, (2.12) gives rise to

$$(2.15)\qquad f_2 \le cKf_2 + f_3.$$

Obviously the operator K preserves the inequality. If we apply it to (2.15) and multiply the result by c, we obtain

$$cKf_2 \le c^2K^2f_2 + cKf_3.$$

Hence

$$Tf_1 + f_2 \le c^2K^2f_2 + cKf_3 + f_3.$$

Continuing this process, we obtain

$$Tf_1 + f_2 \le c^{n+1}K^{n+1}f_2 + \textstyle\sum_{m=0}^{n} c^mK^mf_3.$$

It is easy to see that

$$c^{n+1}K^{n+1}f_2 \leq c^{n+1}2^{n+1}/(n+1)!\cdot(\tau_1+\tau_2)^{n+1}\cdot\sup f_2,$$

which implies that the first term tends to zero as $n\to\infty$, while the second term on the right-hand side is majored by the function $\exp(2c(\tau_1+\tau_2))\cdot f_3(\tau_1,\tau_2)$. The proof of Lemma 1 is complete. □

Returning to the proof of Theorem, we denote the first term on the left-hand side of (2.11) by $f_1(\tau_1,\tau_2)$, the sum of the three first terms on the right-hand side of (2.11) by $f_3(\tau_1,\tau_2)$, and the last term on the same side of (2.11) by Kf_2, by Lemma 1 we obtain

$$\|\partial u/\partial x\|^2_{0,Q^\tau} + \|u(x,\tau_1,t_2)\|^2_{0,Q_2^{\tau_2}} + \|u(x,t_1,\tau_2)\|^2_{0,Q_1^{\tau_1}} \leq c_7\cdot(\|f\|^2_{0,Q} + \|\varphi_1\|^2_{0,Q_2} + \|\varphi_2\|^2_{0,Q_1}),$$

where

$$c_7 = 2/c_0 \max(2b^2(1+c_1^2/c_0), c_1)\exp(2(c_2^2+c_3)(T_1+T_2)).$$

The right-hand side here is independent of (τ_1,τ_2), hence replacing the left-hand side by its upper bound with respect to τ_p from 0 to T_p, $p=1, 2$, thus obtaining (2.1), where $c=c_7^{1/2}$. □

PROPOSITION. *The operator L from B into F is closable.*

PROOF. Suppose that $u_n \in D(L)$ is a sequence such that

(2.16) $$u_n \xrightarrow[n\to\infty]{} 0 \quad \text{in } B$$

and

(2.17) $$Lu_n \xrightarrow[n\to\infty]{} \mathcal{F} = (f,\varphi_1,\varphi_2) \quad \text{in } F,$$

we must prove that $f\equiv 0$, $\varphi_1\equiv 0$, and $\varphi_2\equiv 0$.

Since $u_n \xrightarrow[n\to\infty]{} 0$ in B, then

(2.18) $$u_n \xrightarrow[n\to\infty]{} 0 \quad \text{in } \mathcal{D}'(Q).$$

By virtue of the continuity of derivation of $\mathcal{D}'(Q)$ in $\mathcal{D}'(Q)$, (2.18) implies

(2.19) $$\mathcal{L}u_n \xrightarrow[n\to\infty]{} 0 \quad \text{in } \mathcal{D}'(Q).$$

But, since $\mathcal{L}u_n \xrightarrow[n\to\infty]{} f$ in $L^2(Q)$, then

(2.20) $$\mathcal{L}u_n \xrightarrow[n\to\infty]{} f \quad \text{in } \mathcal{D}'(Q).$$

By virtue of the uniqueness of the limit in $\mathcal{D}'(Q)$, we conclude that $f\equiv 0$.

Moreover, by the fact that

(2.21) $$\ell_1 u_n \xrightarrow[n\to\infty]{} \varphi_1 \quad \text{in } L^2(Q_2)$$

and the canonical injection from $L^2(Q_2)$ into $\mathscr{D}'(Q_2)$ is continuous, (2.21) implies

(2.22) $$\ell_1 u_n \xrightarrow[n\to\infty]{} \varphi_1 \qquad \text{in } \mathscr{D}'(Q_2).$$

Moreover, since

$$u_n \xrightarrow[n\to\infty]{} 0 \qquad \text{in } B$$

and

$$\|\ell_1 u_n\|^2_{0,Q_2} \leq \|u_n\|_B, \qquad \forall n$$

then, we have

(2.23) $$\ell_1 u_n \xrightarrow[n\to\infty]{} 0 \qquad \text{in } L^2(Q_2),$$

consequently

(2.24) $$\ell_1 u_n \xrightarrow[n\to\infty]{} 0 \qquad \text{in } \mathscr{D}'(Q_2).$$

By virtue of the uniqueness of the limit in $\mathscr{D}'(Q_2)$, (2.23) and (2.24) imply that $\varphi_1 \equiv 0$. The reasoning is similar for proving that $\varphi_2 \equiv 0$. □

Let $\bar{L}$ be the closure of the operator L with domain of definition $D(\bar{L})$.

Definition. A solution of the operator equation

$$\bar{L}u = \mathscr{F}$$

is called *a strong solution* of the problem (1.6)–(1.10).

By passing to limit, inequality (2.1) extends to strong solutions, i.e., we have the inequality

(2.24) $$\|u\|_B \leq c\|\bar{L}u\|_F, \qquad \forall u \in D(\bar{L})$$

Inequality (2.24) leads to the following results:

Corollary 1. *If a strong solution of* (1.6)–(1.10) *exists, it is unique and depends continuously on* $\mathscr{F} = (f, \varphi_1, \varphi_2) \in F$.

Corollary 2. *The range* $R(\bar{L})$ *of the operator* $\bar{L}$ *is closed and equals to* $\overline{R(L)}$.

Thus, to prove the existence of a strong solution of the problem (1.6)–(1.10) for any $\mathscr{F} \in F$, it remains to prove that the range $R(L)$ of the operator L is dense in F.

3. Solvability of the problem

Theorem 2. *Suppose the conditions of Theorem 1 are satisfied. Assume that $a(x, t_1, t_2)$ satisfies the conditions H2. If, for some function $\omega \in L^2(Q)$ and for all $u \in D_0(L) = \{u/u \in D(L): \ell_1 u = 0, \ell_2 u = 0\}$, we have*

$$(\mathscr{L}u, \omega)_{0,Q} = 0 \tag{3.1}$$

then ω vanishes almost everywhere in Q.

Proof. Relation (3.1) holds for any function u of $D_0(L)$, using this fact we can express it in a special form. First define g_p by the relation:

$$g_p = \Im_t^* \omega_p = \int_{t_p}^{T_p} \omega_p d\tau_p, \qquad p = 1,\ 2.$$

Let $\partial u/\partial t_p$ be a solution of the equation

$$-a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_p) = g_p, \qquad p = 1,\ 2, \tag{3.2}$$

where σ is a fixed number belonging to $[0, b]$ and $\Im_x^* g = \int_x^b g(\xi, t)d\xi$.

And let

$$u = \begin{cases} 0 & 0 \le t_p \le s_p \\ \displaystyle\int_{s_1}^{t_1}\int_{s_2}^{t_2} \partial^2 u/\partial\tau_1\partial\tau_2 d\tau_1 d\tau_2 & s_p \le t_p \le T_p \end{cases}, \qquad p = 1,\ 2. \tag{3.3}$$

We now have

$$\omega = \sum_{p=1}^2 \Im_t^{*^{-1}} g_p = \sum_{p=1}^2 \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_p))/\partial t_p. \tag{3.4}$$

Lemma 2. *The function ω defined by the relation (3.4) is in $L^2(Q)$.*

Proof of Lemma 2. Let the inequality

$$\int_0^b (\Im_x^*((\xi - x)\partial u/\partial t_p))^2 dx \le b^4/12 \cdot \int_0^b (\partial u/\partial t_p)^2 dx. \tag{3.5}$$

Indeed, the Cauchy-Schwarz inequality gives

$$(\Im_x^*((\xi - x)\partial u/\partial t_p))^2 = \left(\int_x^b (\xi - x)\partial u/\partial t_p d\xi\right)^2 \le \left(\int_x^b (\xi - x)^2 d\xi\right)\int_0^b (\partial u/\partial t_p)^2 dx$$
$$\le (b - x)^3/3 \cdot \int_0^b (\partial u/\partial t_p)^2 dx.$$

Therefore, we have

$$\int_0^b (\Im_x^*((\xi - x)\partial u/\partial t_p))^2 dx \le 1/3 \cdot \int_0^b (\partial u/\partial t_p)^2 dx \cdot \left(\int_0^b (\xi - x)^3 d\xi \right)$$

$$= b^4/12 \cdot \int_0^b (\partial u/\partial t_p)^2 dx.$$

By virtue (3.5) and by the fact that the conditions H1 are satisfied, we deduce that $\partial a(\sigma, t_1, t_2)/\partial t_p \cdot \Im_x^*((\xi - x)\partial u/\partial t_p)$ is in $L^2(Q)$.

It remains to prove that $a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial^2 u/\partial t_p^2)$ belongs to $L^2(Q)$. For this, we use t-averaging operators ρ_ε of the form

$$(\rho_\varepsilon g)(x, t) = 1/\varepsilon \cdot \int_{-\infty}^{+\infty} \omega(s - t/\varepsilon) g(x, s) ds,$$

where $\omega \in C_0^\infty(0, T)$, $\omega(t) \ge 0$, $\int_{-\infty}^{+\infty} \omega(t)dt = 1$.

Applying the operators ρ_ε and $\partial/\partial t_p$ to equation (3.2), we obtain

$$(3.6)\qquad \begin{aligned} &a(\sigma, t_1, t_2)\partial(\rho_\varepsilon \Im_x^*((\xi - x)\partial u/\partial t_p))/\partial t_p \\ &= -\partial a(\sigma, t_1, t_2)/\partial t_p \cdot \rho_\varepsilon \Im_x^*((\xi - x)\partial u/\partial t_p) - \partial(\rho_\varepsilon g_p)/\partial t_p \\ &\quad + \partial(a(\sigma, t_1, t_2)\rho_\varepsilon \Im_x^*((\xi - x)\partial u/\partial t_p) \\ &\quad - \rho_\varepsilon a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_p))/\partial t_p. \end{aligned}$$

It follows from (3.6) that

$$\begin{aligned} &\|a(\sigma, t_1, t_2)\partial(\rho_\varepsilon \Im_x^*((\xi - x)\partial u/\partial t_p))/\partial t_p\|_{0,Q}^2 \\ &\le 3c_3^2 \|\rho_\varepsilon \Im_x^*((\xi - x)\partial u/\partial t_p)\|_{0,Q}^2 + 3\|\partial(\rho_\varepsilon g_p)/\partial t_p\|_{0,Q}^2 \\ &\quad + 3\|\partial(a(\sigma, t_1, t_2)\rho_\varepsilon \Im_x^*((\xi - x)\partial u/\partial t_p) \\ &\quad - \rho_\varepsilon a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_p))/\partial t_p\|_{0,Q}^2. \end{aligned}$$

Using properties of ρ_ε introduced in [12], yields

$$\|a(\sigma, t_1, t_2)\partial(\rho_\varepsilon \Im_x^*((\xi - x)\partial u/\partial t_p))/\partial t_p\|_{0,Q}^2 \le c_8(\|\partial u/\partial t_p\|_{0,Q}^2 + \|\partial g_p/\partial t_p\|_{0,Q}^2).$$

where

$$c_8 = \max (c_3^2 b^4/4, 3).$$

Since $\rho_\varepsilon g \xrightarrow[\varepsilon \to 0]{} g$ in $L^2(Q)$, and the norms of $a(\sigma, t_1, t_2)\partial(\rho_\varepsilon \Im_x^*((\xi - x)\partial u/\partial t_p))/\partial t_p$ in $L^2(Q)$ are bounded, we conclude $a(\sigma, t_1, t_2)\partial(\Im_x^*((\xi - x)\partial u/\partial t_p))/\partial t_p \in L^2(Q)$. The proof of Lemma 2 is complete. □

Returning to the proof of Theorem 2, replacing ω in (3.1) by its representation (3.4), we have

$$(3.7)\quad (\partial u/\partial t_1, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_1))/\partial t_1)_{0,Q}$$
$$+ (\partial u/\partial t_1, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_2))/\partial t_2)_{0,Q}$$
$$+ (\partial u/\partial t_2, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_1))/\partial t_1)_{0,Q}$$
$$+ (\partial u/\partial t_2, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_2))/\partial t_2)_{0,Q}$$
$$- (\partial(a(x, t_1, t_2)\partial u/\partial x)/\partial x, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_1))/\partial t_1)_{0,Q}$$
$$- (\partial(a(x, t_1, t_2)\partial u/\partial x)/\partial x, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_2))/\partial t_2)_{0,Q} = 0.$$

Integrating each term of (3.7) by parts with respect to t, we obtain

$$(3.8)\quad (\partial u/\partial t_1, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_1))/\partial t_1)_{0,Q}$$
$$= 1/2 \int_{Q_{2s_2}} a(\sigma, s_1, t_2)(\Im_x^*(\partial u(x, s_1, t_2)/\partial t_1))^2 dxdt_2$$
$$- 1/2 \int_{Q_s} \partial a(\sigma, t_1, t_2)/\partial t_1 \cdot (\Im_x^*(\partial u/\partial t_1))^2 dxdt_1dt_2,$$

$$(3.9)\quad (\partial u/\partial t_1, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_2))/\partial t_2)_{0,Q}$$
$$= 1/2 \int_{Q_{2s_2}} a(\sigma, T_1, t_2)(\Im_x^*(\partial u(x, T_1, t_2)/\partial t_2))^2 dxdt_2$$
$$- 1/2 \int_{Q_s} \partial a(\sigma, t_1, t_2)/\partial t_1 \cdot (\Im_x^*(\partial u/\partial t_2))^2 dxdt_1dt_2,$$

$$(3.10)\quad (\partial u/\partial t_2, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_1))/\partial t_1)_{0,Q}$$
$$= 1/2 \int_{Q_{1s_1}} a(\sigma, t_1, T_2)(\Im_x^*(\partial u(x, t_1, T_2)/\partial t_1))^2 dxdt_1$$
$$- 1/2 \int_{Q_s} \partial a(\sigma, t_1, t_2)/\partial t_2 \cdot (\Im_x^*(\partial u/\partial t_1))^2 dxdt_1dt_2,$$

$$(3.11)\quad (\partial u/\partial t_2, \partial(a(\sigma, t_1, t_2)\Im_x^*((\xi - x)\partial u/\partial t_2))/\partial t_2)_{0,Q}$$
$$= 1/2 \int_{Q_{1s_1}} a(\sigma, t_1, s_2)(\Im_x^*(\partial u(x, t_1, s_2)/\partial t_2))^2 dxdt_1$$
$$- 1/2 \int_{Q_s} \partial a(\sigma, t_1, t_2)/\partial t_2 \cdot (\Im_x^*(\partial u/\partial t_2))^2 dxdt_1dt_2.$$

$$(3.12)\quad -(\partial(a(x,t_1,t_2)\partial u/\partial x)/\partial x, \partial(a(\sigma,t_1,t_2)\Im_x^*((\xi-x)\partial u/\partial t_1))/\partial t_1)_{0,Q}$$

$$= \int_{Q_s} a(x,t_1,t_2)a(\sigma,t_1,t_2)(\partial u/\partial t_1)^2 dxdt_1dt_2$$

$$+ 1/2\int_{Q_{2s_2}} \partial a(x,T_1,t_2)/\partial t_1\cdot a(\sigma,T_1,t_2)(u(x,T_1,t_2))^2 dxdt_2$$

$$- 1/2\int_{Q_s} (\partial^2 a(x,t_1,t_2)/\partial t_1^2\cdot a(\sigma,t_1,t_2) + \partial a(x,t_1,t_2)/\partial t_1$$

$$\cdot\partial a(\sigma,t_1,t_2)/\partial t_1)u^2 dxdt_1dt_2$$

$$- \int_{Q_s} \partial^2 a(x,t_1,t_2)/\partial x\partial t_1\cdot a(\sigma,t_1,t_2)u\Im_x^*(\partial u/\partial t_1)dxdt_1dt_2$$

$$- 1/2\int_{Q_s} \partial^2 a(x,t_1,t_2)/\partial x^2\cdot a(\sigma,t_1,t_2)(\Im_x^*(\partial u/\partial t_1))^2 dxdt_1dt_2.$$

$$(3.13)\quad -(\partial(a(x,t_1,t_2)\partial u/\partial x)/\partial x, \partial(a(\sigma,t_1,t_2)\Im_x^*((\xi-x)\partial u/\partial t_2))/\partial t_2)_{0,Q}$$

$$= \int_{Q_s} a(x,t_1,t_2)a(\sigma,t_1,t_2)(\partial u/\partial t_2)^2 dxdt_1dt_2$$

$$+ 1/2\int_{Q_{1s_1}} \partial a(x,T_1,t_2)/\partial t_2\cdot a(\sigma,t_1,T_2)(u(x,t_1,T_2))^2 dxdt_1$$

$$- 1/2\int_{Q_s} (\partial^2 a(x,t_1,t_2)/\partial t_2^2\cdot a(\sigma,t_1,t_2) + \partial a(x,t_1,t_2)/\partial t_2$$

$$\cdot\partial a(\sigma,t_1,t_2)/\partial t_2)u^2 dxdt_1dt_2$$

$$- \int_{Q_s} \partial^2 a(x,t_1,t_2)/\partial x\partial t_2\cdot a(\sigma,t_1,t_2)u\Im_x^*(\partial u/\partial t_2)dxdt_1dt_2$$

$$- 1/2\int_{Q_s} \partial^2 a(x,t_1,t_2)/\partial x^2\cdot a(\sigma,t_1,t_2)(\Im_x^*(\partial u/\partial t_2))^2 dxdt_1dt_2.$$

By virtue the conditions of Theorem 2, we obtain

$$(3.14)\quad c_0/2\cdot\int_{Q_{2s_2}} (\Im_x^*(\partial u(x,s_1,t_2)/\partial t_1))^2 dxdt_2$$

$$\le c_3/2\cdot\int_{Q_s} (\Im_x^*(\partial u/\partial t_1))^2 dxdt_1dt_2$$

$$+ (\partial u/\partial t_1, \partial(a(\sigma,t_1,t_2)\Im_x^*((\xi-x)\partial u/\partial t_1))/\partial t_1)_{0,Q},$$

$$(3.15)\qquad c_0/2\cdot\int_{Q_{2s_2}}(\Im_x^*(\partial u(x,T_1,t_2)/\partial t_2))^2dxdt_2$$

$$\le c_3/2\cdot\int_{Q_s}(\Im_x^*(\partial u/\partial t_2))^2dxdt_1dt_2$$

$$+(\partial u/\partial t_1,\partial(a(\sigma,t_1,t_2)\Im_x^*((\xi-x)\partial u/\partial t_2))/\partial t_2)_{0,Q},$$

$$(3.16)\qquad c_0/2\cdot\int_{Q_{1s_1}}(\Im_x^*(\partial u(x,t_1,T_2)/\partial t_1))^2dxdt_1$$

$$\le c_3/2\cdot\int_{Q_s}(\Im_x^*(\partial u/\partial t_1))^2dxdt_1dt_2$$

$$+(\partial u/\partial t_2,\partial(a(\sigma,t_1,t_2)\Im_x^*((\xi-x)\partial u/\partial t_1))/\partial t_1)_{0,Q},$$

$$(3.17)\qquad c_0/2\cdot\int_{Q_{2s_2}}(\Im_x^*(\partial u(x,t_1,s_2)/\partial t_2))^2dxdt_1$$

$$\le c_3/2\cdot\int_{Q_s}(\Im_x^*(\partial u/\partial t_2))^2dxdt_1dt_2$$

$$+(\partial u/\partial t_2,\partial(a(\sigma,t_1,t_2)\Im_x^*((\xi-x)\partial u/\partial t_2))/\partial t_2)_{0,Q},$$

$$(3.18)\qquad c_0^2\int_{Q_s}(\partial u/\partial t_1)^2dxdt_1dt_2+c_0c_2/2\cdot\int_{Q_{2s_2}}(u(x,T_1,t_2))^2dxdt_2$$

$$\le(3c_1^2/4+c_3^2/2+c_4^2/4)\int_{Q_s}u^2dxdt_1dt_2$$

$$+(c_1^2/4+c_5^2/4+c_6^2/2)\int_{Q_s}(\Im_x^*(\partial u/\partial t_1))^2dxdt_1dt_2$$

$$-(\partial(a(x,t_1,t_2)\partial u/\partial x)/\partial x,\partial(a(\sigma,t_1,t_2)\Im_x^*((\xi-x)\partial u/\partial t_1))/\partial t_1)_{0,Q}.$$

$$(3.19)\qquad c_0^2\int_{Q_s}(\partial u/\partial t_2)^2dxdt_1dt_2+c_0c_2/2\cdot\int_{Q_{2s_2}}(u(x,t_1,T_2))^2dxdt_1$$

$$\le(3c_1^2/4+c_3^2/2+c_4^2/4)\int_{Q_s}u^2dxdt_1dt_2$$

$$+(c_1^2/4+c_5^2/4+c_6^2/2)\int_{Q_s}(\Im_x^*(\partial u/\partial t_2))^2dxdt_1dt_2$$

$$-(\partial(a(x,t_1,t_2)\partial u/\partial x)/\partial x,\partial(a(\sigma,t_1,t_2)\Im_x^*((\xi-x)\partial u/\partial t_2))/\partial t_2)_{0,Q}.$$

Combining the relations (3.14)–(3.19) and using (3.7), this yields

$$(3.20)\quad \|\partial u/\partial t_1\|_{0,Q_s}^2 + \|\partial u/\partial t_2\|_{0,Q_s}^2 + \|\Im_x^*(\partial u(x,t_1,T_2)/\partial t_1)\|_{0,Q_{1s_1}}^2 + \|\Im_x^*(\partial u(x,s_1,t_2)/\partial t_1)\|_{0,Q_{2s_2}}^2 + \|\Im_x^*(\partial u(x,t_1,s_2)/\partial t_2)\|_{0,Q_{1s_1}}^2 + \|\Im_x^*(\partial u(x,T_1,t_2)/\partial t_2)\|_{0,Q_{2s_2}}^2 + \|u(x,t_1,T_2)\|_{0,Q_{1s_1}}^2 + \|u(x,T_1,t_2)\|_{0,Q_{2s_2}}^2 \le c_9(\|\Im_x^*(\partial u/\partial t_1)\|_{0,Q_s}^2 + \|\Im_x^*(\partial u/\partial t_2)\|_{0,Q_s}^2 + \|u\|_{0,Q_s}^2),$$

where

$$c_9 = \max(c_3/2,\, 3c_1^2/4 + c_3^2/2 + c_4^2/4,\, (c_1^2 + c_5^2 + c_6^2)/4)/\min(c_0^2,\, c_0/2,\, c_0c_2/2).$$

Inequality (3.20) is basic in our proof. In order to use it, we introduce the new function

$$\theta(x,t_1,t_2) = \int_{t_1}^{T_1} u_{\tau_1} d\tau_1 + \int_{t_2}^{T_2} u_{\tau_2} d\tau_2$$

Then

$$u(x,T_1,t_2) = \theta(x,s_1,t_2), \qquad u(x,t_1,T_2) = \theta(x,t_1,s_2),$$

$$\partial u(x,t_1,T_2)/\partial t_1 = \partial\theta(x,t_1,s_2)/\partial t_1, \qquad \partial u(x,T_1,t_2)/\partial t_2 = \partial\theta(x,s_1,t_2)/\partial t_2,$$

$$\partial u(x,s_1,t_2)/\partial t_1 = -1/2\cdot\partial\theta(x,s_1,t_2)/\partial t_1,$$

$$\partial u(x,t_1,s_2)/\partial t_2 = -1/2\partial\theta(x,t_1,s_2)/\partial t_2.$$

Then (3.20) becomes

$$(3.21)\quad \|\partial u/\partial t_1\|_{0,Q_s}^2 + \|\partial u/\partial t_2\|_{0,Q_s}^2 + (1 - 3c_9(T_1 - s_1)/4)\|\Im_x^*(\partial\theta(x,s_1,t_2)/\partial t_1)\|_{0,Q_{1s_1}}^2 + \|\Im_x^*(\partial\theta(x,t_1,s_2)/\partial t_1)\|_{0,Q_{2s_2}}^2 + (1 - 3c_9(T_1 - s_1)/4)\|\Im_x^*(\partial\theta(x,s_1,t_2)/\partial t_2)\|_{0,Q_{2s_2}}^2 + \|\Im_x^*(\partial\theta(x,t_1,s_2)/\partial t_2)\|_{0,Q_{1s_1}}^2 + (1 - 3c_9(T_2 - s_2)/4)\|\theta(x,t_1,s_2)\|_{0,Q_{1s_1}}^2 + (1 - 3c_9(T_1 - s_1)/4)\|\theta(x,s_1,t_2)\|_{0,Q_{2s_2}}^2 \le \frac{3c_9}{4}(\|\Im_x^*(\partial\theta/\partial t_1)\|_{0,Q_s}^2 + \|\Im_x^*(\partial\theta/\partial t_2)\|_{0,Q_s}^2 + \|\theta\|_{0,Q_s}^2).$$

Hence if $s_{p_0} > 0$ satisfies $1 - 3c_9(T_p - s_{p_0})/4 = 1/2$, $p = 1, 2$, (3.21) implies

$$
\begin{aligned}
(3.22)\quad & \|\partial u/\partial t_1\|^2_{0,Q_s} + \|\partial u/\partial t_2\|^2_{0,Q_s} + \|\Im^*_x(\partial\theta(x, s_1, t_2)/\partial t_1)\|^2_{0,Q_{1s_1}} \\
& + \|\Im^*_x(\partial\theta(x, t_1, s_2)/\partial t_1)\|^2_{0,Q_{2s_2}} + \|\Im^*_x(\partial\theta(x, s_1, t_2)/\partial t_2)\|^2_{0,Q_{2s_2}} \\
& + \|\Im^*_x(\partial\theta(x, t_1, s_2)/\partial t_2)\|^2_{0,Q_{1s_1}} + \|\theta(x, t_1, s_2)\|^2_{0,Q_{1s_1}} \\
& + \|\theta(x, s_1, t_2)\|^2_{0,Q_{2s_2}} \\
& \le 3c_9/2\cdot(\|\Im^*_x(\partial\theta/\partial t_1)\|^2_{0,Q_s} + \|\Im^*_x(\partial\theta/\partial t_2)\|^2_{0,Q_s} + \|\theta\|^2_{0,Q_s})
\end{aligned}
$$

for all $(s_1, s_2) \in [T_1 - s_{1_0}, T_1] \times [T_2 - s_{2_0}, T_2]$.

We denote the sum of three terms on the right-hand side of (3.22) by $y(s_1, s_2)$. Hence, we obtain

$$\|\partial u/\partial t_1\|^2_{0,Q_s} + \|\partial u/\partial t_2\|^2_{0,Q_s} - (\partial/\partial s_1 + \partial/\partial s_2)y \le 3c_9 y/2.$$

Consequently,

$$-(\partial/\partial s_1 + \partial/\partial s_2)(y.\exp(3c_9(s_1 + s_2)/2)) \le 0.$$

Taking into account that $y(T_1, T_2) = 0$, we obtain

$$(3.23)\qquad (y.\exp(3c_9(s_1 + s_2)/2)) \le 0.$$

It follows that $\omega = 0$ almost everywhere in Q_{T-s_0}. Proceeding in this way step by step, we prove that $\omega = 0$ almost everywhere in Q. Therefore, the proof of Theorem 2 is complete. □

THEOREM 3. *The range $R(L)$ of L coincides with F.*

PROOF. Since F is a Hilbert space, we have $R(L) = F$ is equivalent to the orthogonality of vector $W = (\omega, \omega_1, \omega_2) \in F$ to the set $R(L)$, i.e., if and only if the relation

$$(3.24)\qquad (\mathscr{L}u, \omega)_{0,Q} + (\ell_1 u, \omega_1)_{0,Q_2} + (\ell_2 u, \omega_2)_{0,Q_1} = 0$$

where u runs over B and $W = (\omega, \omega_1, \omega_2) \in F$, implies that $W = 0$.

Putting $u \in D_0(L)$ in (3.24), we obtain

$$(\mathscr{L}u, \omega)_{0,Q} = 0$$

Hence Theorem 2 implies that $\omega = 0$. Thus, (3.24) takes the form

$$(\ell_1 u, \omega_1)_{0,Q_2} + (\ell_2 u, \omega_2)_{0,Q_1} = 0, \qquad u \in D(L).$$

Since the quantities $\ell_1 u$, $\ell_2 u$ can vanish independently and the range of the operators ℓ_1, ℓ_2 are dense in $L^2(Q_1)$ and $L^2(Q_2)$, respectively, the last equality above implies that $\omega_1 = \omega_2 = 0$. Hence $W = 0$. The proof of Theorem 3 is complete. □

REMARK. We can prove that our results remain in force for the case of multidimensional time:

$$\sum_{m=1}^{n} \partial u/\partial t_m - \partial(a(x, t_1, t_2, \ldots, t_n)\partial u/\partial x)/\partial x = f$$

with the appropriate initial conditions

$$\ell_m u = u|_{t_m=0} = \varphi_m(x, t_1, \ldots, t_{m-1}, t_{m+1}, \ldots, t_n), \qquad m = 1, \ldots, n. \quad \square$$

References

[1] N. E. Benouar–N. I. Yurchuk, Mixed problem with an integral condition for parabolic equations with the Bessel operator, Differ. Uravn., **27** (1991), 2094–2098.

[2] A. Bouziani, Problème aux limites pour certaines équations de type non classique du troisième ordre, Bull. Class. Sci., Académie Royale de Belgique, **6** (1995), 215–221.

[3] A. Bouziani, Mixed problem with integral conditions for a certain parabolic equation, J. Appl. Math. Stochastic Anal., **9** (1996), 323–330.

[4] A. Bouziani, Mixed problem for certain nonclassical equations with a small parameter, Bull. Class. Sci., Académie Royale de Belgique, **5** (1994), 389–400.

[5] A. Bouziani–N. E. Benouar, Problème mixte avec conditions intégrales pour une classe d'équations paraboliques, C.R.A. Sci., Paris, Série 1, **321** (1995), 1177–1182.

[6] A. Bouziani–N. E. Benouar, Problème mixte avec conditions intégrales pour une classe d'équations hyperboliques, Bull. Belg. Math. Soc. **3** (1996), 137–145.

[7] A. Bouziani–N. E. Benouar, Problème aux limites pour une classe d'équations de type non classique pour une structure pluri-dimensionnelle, Bull. Polish Acad. Sci. Math. **43** (1995), 317–328.

[8] J. R. Cannon, The solution of the heat equation subject to the specification of energy, Quart. Appl. Math. **21** (1963), 155–160.

[9] J. R. Cannon–J. van der Hoek, The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy, Boll. Uni. Math. Ital. Suppl. **1** (1981), 253–282.

[10] J. R. Cannon–J. van der Hoek, An implicit finite difference scheme for the diffusion of mass in a portion of the domain, Numerical Solutions of Partial Differential equations, (J. Noye, ed), North-Holland, Amsterdam (1982), 527–539.

[11] J. R. Cannon–S. P. Esteva–J. van der Hoek, A Galerkin procedure for the diffusion equation subject to the specification of mass, S.I.A.M. Numer. Anal. **24** (1987), 499–515.

[12] L. Gårding, Cauchy's problem for hyperbolic equations, University of Chicago, Lecture notes, 1957.

[13] N. I. Ionkin, Solution of boundary value problem in heat conduction theory with non local boundary conditions, Diff. Uravn. **13** (1977), 294–304.

[14] N. I. Kamynin, A boundary value problem in the theory of the heat conduction with non classical boundary condition, Th., Vychisl., Mat., Mat., Fiz. **43** (1964), 1006–1024.

[15] A. V. Kartynnik, Three point boundary value problem with an integral space variables conditions for second order parabolic equations, Diff. Uravn. **26** (1990), 1568–1575.

[16] Y. Lin, Parabolic partial differential equations subject to non-local boundary conditions, Ph.D. thesis, Washington state University, Pullman, WA, 1988.

[17] P. Shi, Weak solution to an evolution problem with a nonlocal contraint, SIAM. J. Math. Anal. **24** (1993), 46–58.
[18] N. I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations, Diff. Uravn. **22** (1986), 2217–2126.

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