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Based modules and good filtrations in algebraic groups

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Abstract. Let $\mathfrak{G}_{\mathfrak{k}}$ be a simply connected semisimple algebraic group over an algebraically closed field $\mathfrak{k}$ of positive characteristic with simple system of roots Π. After the initial efforts by Wang J.-P. and S. Donkin, O. Mathieu proved, using the Frobenius splitting of the flag variety, Donkin's conjectures that (i) if Π' is a subset of Π and if $\mathfrak{G}'_{\mathfrak{k}}$ is the semisimple subgroup of $\mathfrak{G}_{\mathfrak{k}}$ generated by the root subgroups associated to Π', then any Weyl module of $\mathfrak{G}_{\mathfrak{k}}$ admits a filtration by $\mathfrak{G}'_{\mathfrak{k}}$-modules all of whose subquotients are Weyl modules for $\mathfrak{G}'_{\mathfrak{k}}$; (ii) the tensor product of any two Weyl modules of $\mathfrak{G}_{\mathfrak{k}}$ admits a filtration by $\mathfrak{G}_{\mathfrak{k}}$-modules all of whose subquotients are Weyl modules of $\mathfrak{G}_{\mathfrak{k}}$. In this note we explain that the conjectures can also be obtained as immediate consequences of Lusztig's results on based modules.

Introduction

Let $\mathfrak{G}_{\mathfrak{k}}$ be a simply connected semisimple algebraic group over an algebraically closed field $\mathfrak{k}$ of positive characteristic with simple system of roots Π. After the initial efforts by Wang J.-P. and S. Donkin, using the Frobenius splitting of the flag variety O. Mathieu [M] proved Donkin's conjectures that (i) if Π' is a subset of Π and if $\mathfrak{G}'_{\mathfrak{k}}$ is the semisimple subgroup of $\mathfrak{G}_{\mathfrak{k}}$ generated by the root subgroups associated to Π', then any Weyl module of $\mathfrak{G}_{\mathfrak{k}}$ admits a filtration by $\mathfrak{G}'_{\mathfrak{k}}$-modules all of whose subquotients are Weyl modules for $\mathfrak{G}'_{\mathfrak{k}}$; (ii) the tensor product of any two Weyl modules of $\mathfrak{G}_{\mathfrak{k}}$ admits a filtration by $\mathfrak{G}_{\mathfrak{k}}$-modules all of whose subquotients are Weyl modules of $\mathfrak{G}_{\mathfrak{k}}$. Since then J. Paradowski [P] has given another proof using Lusztig's canonical basis. There is yet a third proof using Kashiwara's crystal base; Donkin's conjectures are immediate consequences of Lusztig's results on based modules [L], which may be worth pointing out after the appearence of a friendly account [J] of crystal bases. The third proof works naturally over $\boldsymbol{Z}$, hence over any commutative ring, and is free of Donkin's cohomological criterion for the existence of good filtrations [JG, II.4.16].

In §1 of the present note we will restate Lusztig's results in the framework of [J], and show Donkin's conjectures. We see that the proof is logically

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Key words and phrases, Based modules, Crystal Bases, Good filtrations independent of the construction of canonical bases using perverse sheaves, and hence elementary (but by no means easy). In §2 we will append a note on the quasi-$\mathscr{R}$-matrix that plays an important role in the construction of tensor products of based modules.

We will adopt the notations of [J] unless otherwise specified. Thus k will denote the rational function field $\boldsymbol{Q}(q)$ over $\boldsymbol{Q}$ in indeterminate q and U the quantized enveloping algebra over k associated to the simple system of roots Π. All tensor products without subscripts are taken over k. Let Λ^+ be the subset of dominant weights in the weight group Λ and $\boldsymbol{N}\Pi = \sum_{\alpha \in \Pi} \boldsymbol{N}\alpha$. Let $A = \boldsymbol{Q}[q]_{(q)}$ the localization of the polynomial algebra $\boldsymbol{Q}[q]$ at the maximal ideal (q) and $\mathscr{A} = \boldsymbol{Z}[q, q^{-1}]$. Accordingly, $U_{\mathscr{A}}$ instead of $U_{\boldsymbol{Z}}$ will denote Lusztig's $\boldsymbol{Z}[q, q^{-1}]$-form of U [J, 11.1] and $L_{\mathscr{A}}(\lambda)$ instead of $L_{\boldsymbol{Z}}(\lambda)$ will denote the $\boldsymbol{Z}[q, q^{-1}]$-form of simple module $L(\lambda)$ over U of highest weight $\lambda \in \Lambda^+$. If $\mathscr{C}$ is a category, $\mathscr{C}(A, B)$ will denote the set of morphisms from object A to object B in $\mathscr{C}$.

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1. Based modules and good filtrations

(1.1) Let $\mathscr{F}$ be the category of finite dimensional U-modules of type 1. Recall $\boldsymbol{Q}$-algebra involution ψ on U such that $E_\alpha \mapsto E_\alpha$, $F_\alpha \mapsto F_\alpha$, $K_\alpha \mapsto K_\alpha^{-1}$ for each $\alpha \in \Pi$, and that $q \mapsto q^{-1}$. A based module of U is a triple (M, B, ψ_M) of $M \in \mathscr{F}$, B a k-basis of M, and a $\boldsymbol{Q}$-linear in volution ψ_M on M such that

(BM1) $B = \sqcup_{\nu \in \Lambda} B_\nu$ with $B_\nu = B \cap M_\nu$,

(BM2) If $M_{\mathscr{A}} = \sum_{b \in B} \mathscr{A} b$, $M_{\mathscr{A}}$ is $U_{\mathscr{A}}$-stable,

(BM3) ψ_M fixes each element of B and $\psi_M(um) = \psi(u)\psi_M(m)$ for each $u \in U$ and $m \in M$,

(BM4) If $\mathscr{L}(M) = \coprod_{b \in B} Ab$ and if $\bar{B}$ is the image of B in $\mathscr{L}(M)/q\mathscr{L}(M)$, $(\mathscr{L}(M), \bar{B})$ forms a crystal base of M.

A morphism from a based module (M, B, ψ_M) to another based module $(M', B', \psi_{M'})$ is a U-linear map $f : M \to M'$ such that $f(B) \subseteq B' \sqcup \{0\}$. Thus $\psi_{M'} \circ f = f \circ \psi_M$ and $B \cap \ker f$ automatically forms a k-basis of $\ker f$. We will denote by $\mathscr{BM}$ the category of based modules. Note that $\mathscr{BM}$ is not an additive category.

(1.2) Let $(\mathscr{L}(\infty), \mathscr{B}(\infty))$ be the crystal base of U^- and $B(\infty) = \{G(b) | b \in \mathscr{B}(\infty)\}$ the global crystal base of U^-. If $\lambda \in \Lambda^+$, let $B(\lambda) =$

$\{G(b)v_\lambda | b \in \mathscr{B}(\infty), G(b)v_\lambda \neq 0\}$ be a global crystal base of $L(\lambda)$ [J, 11.10], where $v_\lambda \in L(\lambda)_\lambda \backslash 0$. Then with the $\boldsymbol{Q}$-linear involution ψ_λ on $L(\lambda)$ as in [J, 11.9]

(1) $(L(\lambda), B(\lambda), \psi_\lambda)$ forms a simple object of $\mathscr{BM}$ with $L(\lambda)_{\mathscr{A}} = L_{\mathscr{A}}(\lambda)$.

(1.3) It is easy to see

Lemma *Let* (M, B, ψ_M), $(M', B', \psi_{M'}) \in \mathscr{BM}$ *and* $f \in \mathscr{BM}((M, B, \psi_M), (M', B', \psi_{M'}))$. *Let* (M_i, B_i, ψ_{M_i}) *be a subobject of* (M, B, ψ_M) *in* $\mathscr{BM}, i = 1, 2$.

(i) [L, 27.1.2] $(M \oplus M', B \sqcup B', \psi_M \oplus \psi_{M'})$ *is the coproduct of* (M, B, ψ_M) *and* $(M', B', \psi_{M'})$ *in* $\mathscr{BM}$.

(ii) $(M_1 \cap M_2, B_1 \cap B_2, \psi_M|_{M_1 \cap M_2}) \in \mathscr{BM}$.

(iii) [L, 27.1.4] *If* $\pi_1 : M \to M/M_1$ *is the U-linear quotient, then* ψ_M *induces a* $\boldsymbol{Q}$*-linear involution* ψ_{M/M_1} *and* $(M/M_1, \pi_1(B \backslash B_1), \psi_{M/M_1})$ *gives the quotient of* (M, B, ψ_M) *by* $(M', B', \psi_{M'})$ *in* $\mathscr{BM}$.

(iv) $(f^{-1}(0), B \cap f^{-1}(0), \psi_M|_{f^{-1}(0)})$ *is the kernel of* f *in* $\mathscr{BM}$.

(v) *If* $\Pi' \subseteq \Pi$ *and if* U' *is the subalgebra of* U *generated by* E_α, F_α, $K_\alpha^{\pm 1}$, $\alpha \in \Pi'$, *then* (M, B, ψ_M) *is naturally a based module for* U'.

(1.4) Proposition [L, 27.1.7] *Let* $(M, B, \psi_M) \in \mathscr{BM}$. *Assume* $M \neq 0$ *and let* λ *be a maximal weight of* M. *If* $M[\lambda]$ *is the* $L(\lambda)$*-isotypic component of* M, *then in* $\mathscr{BM}$

$$(M[\lambda], B \cap M[\lambda], \psi_M|_{M[\lambda]}) \simeq (L(\lambda), B(\lambda), \psi_\lambda)^{\oplus |B_\lambda|}.$$

(1.5) From (1.3.iii, iv) and (1.4) one obtains

Corollary (*cf.* [X, 3.3(v)]) *The simple objects of* $\mathscr{BM}$ *are parametrized by* Λ^+, *i.e., every simple object of* $\mathscr{BM}$ *is isomorphic to* $(L(\lambda), B(\lambda), \psi_\lambda)$ *for a unique* $\lambda \in \Lambda^+$. *Hence any nonzero based module admits a filtration in* $\mathscr{BM}$ *with all the subquotients isomorphic to some* $(L(\lambda), B(\lambda), \psi_\lambda)$, $\lambda \in \Lambda^+$.

(1.6) Let (M, B, ψ_M), $(M', B', \psi_{M'}) \in \mathscr{BM}$. If we regard $M \otimes M' \in \mathscr{F}$ via the twisted comultiplication $\Delta' : U \to U \otimes U$ such that for each $\alpha \in \Pi$

$$E_\alpha \mapsto E_\alpha \otimes K_\alpha^{-1} + 1 \otimes E_\alpha, \qquad F_\alpha \mapsto F_\alpha \otimes 1 + K_\alpha \otimes F_\alpha, \qquad K_\alpha \mapsto K_\alpha \otimes K_\alpha,$$

then [J, 9.17]

(1) $(\mathscr{L}(M) \otimes_A \mathscr{L}(M'), \bar{B} \otimes_{\boldsymbol{Q}} \bar{B}')$ forms a crystal base of $M \otimes M'$.

The $\boldsymbol{Q}$-linear involution $\psi_M \otimes \psi_{M'}$, however, does not satisfy the condition

(BM3). To remedy the situation, recall from [J, 6.12] a unique k-bilinear pairing $(\ ,\ ) : U^{\leq 0} \times U^{\geq 0} \to k$ such that for each $x, x' \in U^{\geq 0}$; $y, y' \in U^{\leq 0}$ and $\alpha, \beta \in \Pi$

$$(y, xx') = (\Delta(y), x' \otimes x), \qquad (yy', x) = (y \otimes y', \Delta(x)), \qquad (K_\alpha, K_\beta) = q^{-(\alpha,\beta)},$$
$$(F_\alpha, E_\beta) = \delta_{\alpha,\beta}(q_\alpha - q_\alpha^{-1})^{-1}, \qquad (K_\alpha, E_\beta) = 0 = (F_\alpha, K_\beta).$$

One has [J, 6.13.2, 6.18] that for each μ and $\nu \in N\Pi$

(2) $$(\ ,\ )|_{U^-_{-\mu} \times U^+_\nu} = 0 \qquad \text{unless}\ \mu = \nu$$

while

(3) $$(\ ,\ )|_{U^-_{-\mu} \times U^+_\mu} = \text{is nondegenerate.}$$

Then under the identification of $U^-_{-\mu} \otimes U^+_\mu$ with $\mathbf{Mod}_k(U^+_\mu, U^+_\mu)$ via $(\ ,\ )|_{U^-_{-\mu} \times U^+_\mu}$ let $\Theta_\mu \in U^-_{-\mu} \otimes U^+_\mu$ correspond to $id_{U^+_\mu}$, and $\Theta_{M,M'}$ the k-linear endomorphism of $M \otimes M'$ defined by the action of $\sum_{\mu \in N\Pi} \Theta_\mu$. If $P : U \otimes U \to U \otimes U$ is the transposition $y \otimes x \mapsto x \otimes y$, let $\Theta^P_{M,M'}$ be the k-linear endomorphism of $M \otimes M'$ defined by the action of $\sum_{\mu \in N\Pi} P(\Theta_\mu)$. If ${}^\psi\Delta' = (\psi \otimes \psi) \circ \Delta' \circ \psi$, one finds from [J, 7.2.4]

(4) $${}^\psi\Delta'(u)\Theta^P_{M,M'} = \Theta^P_{M,M'}\Delta'(u) \qquad \text{for each}\ u \in U\ \text{on}\ M \otimes M'.$$

Set

(5) $$\Psi_{M,M'} = (\psi_M \otimes \psi_{M'}) \circ \Theta^P_{M,M'} : M \otimes M' \to M \otimes M'.$$

Then by (2) for each $u \in U$ and $x \in M \otimes M'$

(6) $$\Psi_{M,M'}(ux) = \Psi_{M,M'}(\Delta'(u)x) = \Delta'(\psi(u))\Psi_{M,M'}(x) = \psi(u)\Psi_{M,M'}(x).$$

Also if $\Theta^\psi_{M,M'}$ is the $\boldsymbol{Q}$-linear endomorphism of $M \otimes M'$ defined by the action of $\sum_{\mu \in N\Pi} (\psi \otimes \psi)(\Theta_\mu)$, then (cf. (2.2))

(7) $$\Theta^\psi_{M,M'} = (\Theta_{M,M'})^{-1},$$

from which one obtains

(8) $$\Psi^2_{M,M'} = id_{M \otimes M'}.$$

Define a partial order on $B \times B'$ such that $(b_1, b_1') \leq (b_2, b_2')$ iff

(9) $$(b_1, b_1') \in M_{\lambda_1} \otimes M'_{\lambda_1'} \text{ and } (b_2, b_2') \in M_{\lambda_2} \otimes M'_{\lambda_2'} \text{ with}$$
$$\lambda_1 \leq \lambda_2, \lambda_1' \geq \lambda_2' \text{ and } \lambda_1 + \lambda_1' = \lambda_2 + \lambda_2'.$$

(1.7) Theorem (cf. [L, 27.3.2]) *Let* (M, B, ψ_M), $(M', B', \psi_{M'}) \in \mathscr{BM}$, *and let* $\mathscr{L} = \coprod_{(b,b') \in B \times B'} \boldsymbol{Z}[q](b \otimes b')$.

(i) *For each* $(b, b') \in B \times B'$ *there is unique* $b \diamondsuit b' \in \mathscr{L}$ *such that* $\Psi_{M,M'}(b \diamondsuit b') = b \diamondsuit b'$ *and that* $b \diamondsuit b' - b \otimes b' \in q\mathscr{L}$.

(ii) *For each* $(b, b') \in B \times B'$

$$b \diamondsuit b' - b \otimes b' \in \sum_{\substack{(b_1, b_1') \in B \times B' \\ (b_1, b_1') > (b, b')}} q\boldsymbol{Z}[q](b_1 \otimes b_1').$$

Hence $(M \otimes M', (b \diamondsuit b' | b \in B, b' \in B'), \Psi_{M,M'}) \in \mathscr{BM}$ *with* $(M \otimes M')_{\mathscr{A}} = \coprod_{(b,b') \in B \times B'} \mathscr{A}(b \diamondsuit b') = M_{\mathscr{A}} \otimes_{\mathscr{A}} M'_{\mathscr{A}}$, *where the U-action on* $M \otimes M'$ *is given by* Δ'.

(1.8) Remark Precisely, Lusztig uses his canonical bases to define based modules and in the formulation of his theorem [27.3.2]. In its restatement above, due to Kashiwara's twisted action Δ' of U on the tensor products in dealing with their crystal bases, one has to adjust Lusztig's Ψ [L, 27.3.1] by transposition P, that causes reversing of the order in the sum in the expression of $b \diamondsuit b'$ in (ii).

In the application to $\mathfrak{G}$-modules in (1.10), however, the twisted action Δ' will not cause any difference to the standard diagonal G-action on the tensor products as $K_\alpha = 1$ in Dist($\mathfrak{G}$).

(1.9) Putting together (1.3.vi), (1.5) and (1.7) yields Donkin's conjectures for the quantum algebra.

Corollary *Let* $(M, B, \psi_M) \in \mathscr{BM}$.

(i) *If* $\prod' \subseteq \prod$ *and if* U' *is the subalgebra of* U *associated with* $\prod'$, *then* $M_{\mathscr{A}}$ *admits a filtration of* $U'_{\mathscr{A}}$*-modules with each subquotient isomorphic to some* $L'_{\mathscr{A}}(\lambda')$, *where* $L'(\lambda')$ *is a simple* U'*-module of highest weight* λ'.

(ii) [X, 3.3(vi)] *If* $(M', B', \psi_{M'}) \in \mathscr{BM}$, *then* $M_{\mathscr{A}} \otimes_{\mathscr{A}} M'_{\mathscr{A}}$ *admits a filtration of* $U_{\mathscr{A}}$*-modules with each subquotient isomorphic to some* $L_{\mathscr{A}}(\lambda)$, $\lambda \in \Lambda^+$.

(1.10) Regarding $\boldsymbol{Z}$ as an $\mathscr{A}$-algebra by specializing q to 1, on has a ring isomorphism [LF, 6.7.b], [LR, 8.16]

$$\{U_{\mathscr{A}}/(K_\alpha - 1)_{\alpha \in \Pi}\} \otimes_{\mathscr{A}} \boldsymbol{Z} \simeq \mathrm{Dist}(\mathfrak{G}), \tag{1}$$

where $\mathfrak{G}$ is the Chevalley $\boldsymbol{Z}$-form of the algebraic group $\mathfrak{G}_{\mathfrak{k}}$. As the Dist($\mathfrak{G}$)-modules that are free of finite rank over $\boldsymbol{Z}$ are naturally $\mathfrak{G}$-modules such that each $L_{\mathscr{A}}(\lambda) \otimes_{\mathscr{A}} \boldsymbol{Z}$, $\lambda \in \Lambda^+$, is the Weyl module for $\mathfrak{G}$ of highest weight λ, one obtains from (1.9) Donkin's conjectures for algebraic group $\mathfrak{G}$. Let $\mathscr{F}_{\mathfrak{G}}(\Delta)$ be the full subcategory of $\mathfrak{G}$-modules admitting a filtration all of whose subquotients are Weyl modules.

COROLLARY *Let* $M \in \mathscr{F}_{\mathfrak{G}}(\Delta)$.

(i) *If* $\Pi' \subseteq \Pi$ *and if* $\mathfrak{G}'$ *is the semisimple subgroup of* $\mathfrak{G}$ *associated to* Π', *then* M *admits a filtration of* $\mathfrak{G}'$*-modules all of whose subquotients are Weyl modules for* $\mathfrak{G}'$.

(ii) *If* $M' \in \mathscr{F}_{\mathfrak{G}}(\Delta)$, *then* $M \otimes M' \in \mathscr{F}_{\mathfrak{G}}(\Delta)$.

2. The quasi-$\mathscr{R}$-matrix

In this section we will complement the proof of (1.7).

(2.1) We begin with a lemma that dispenses us with the introduction of $\Theta = \sum_\mu \Theta_\mu$ in a completion of $U \otimes U$. Let ${}^{\psi}\Delta = (\psi \otimes \psi) \circ \Delta \circ \psi \in k\,\mathbf{Alg}(U, U \otimes U)$.

LEMMA *Given* $\theta = (\theta_\mu)_{\mu \in N\Pi}$ *with* $\theta_\mu \in U^-_{-\mu} \otimes U^+_\mu$. *For each* $M, M' \in \mathscr{F}$ *let* $\theta_{M,M'}$ *be the* k*-linear endomorphism of* $M \otimes M'$ *defined by the action of* $\sum_\mu \theta_\mu$. *The following are equivalent*:

(i) *For each* $u \in U$ *and* $M, M' \in \mathscr{F}$

$$\Delta(u)\theta_{M,M'} = \theta^{\psi}_{M,M'}\Delta(u) \qquad \text{on } M \otimes M'.$$

(ii) *For each* $\mu \in N\Pi$ *and* $\alpha \in \Pi$

$$(1) \qquad (E_\alpha \otimes 1)\theta_\mu + (K_\alpha \otimes E_\alpha)\theta_{\mu-\alpha} = \theta_\mu(E_\alpha \otimes 1) + \theta_{\mu-\alpha}(K_\alpha^{-1} \otimes E_\alpha)$$

$$(2) \qquad (1 \otimes F_\alpha)\theta_\mu + (F_\alpha \otimes K_\alpha^{-1})\theta_{\mu-\alpha} = \theta_\mu(1 \otimes F_\alpha) + \theta_{\mu-\alpha}(F_\alpha \otimes K_\alpha).$$

PROOF. As Δ and ${}^{\psi}\Delta$ are both k-algebra homomorphisms, to see that (ii) implies (i), it is enough to check the equality on generators $E_\alpha, K_\alpha, F_\alpha, \alpha \in \Pi$, of U. In $\Delta(E_\alpha)\theta_{M,M'}$ (resp. $\theta_{M,M'}{}^{\psi}\Delta(E_\alpha)$) the only contributions to the shift of the weight spaces by $(-(\mu-\alpha), \mu)$ come from the left hand side (resp. the right hand side) of (1). Likewise for F_α while the corresponding equality for K_α is automatic. Hence (i) follows from (ii).

Conversely, if (i) holds, we have with $u = E_\alpha$

$$(3) \qquad (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha)\theta_{M,M'} = \theta_{M,M'}(E_\alpha \otimes 1 + K_\alpha^{-1} \otimes E_\alpha).$$

Let $\mu \in N\Pi$. From [J, 5.18] there is $\lambda \in \Lambda^+$ such that

(4) the k-linear map $U^-_{-\mu} \to L(\lambda)_{\lambda-\mu}$ via $u \mapsto uv_\lambda$ is bijective.

Recall from [J, 4.6] the k-algebra involution ω on U such that $E_\alpha \mapsto F_\alpha$ and $K_\alpha \mapsto K_\alpha^{-1}$ $\forall \alpha \in \Pi$. Take $M' = {}^{\omega}L(\lambda)$ the k-linear space $L(\lambda)$ with U acting through ω. Write

$$\{(E_\alpha \otimes 1)\theta_\mu + (K_\alpha \otimes E_\alpha)\theta_{\mu-\alpha}\} - \{\theta_\mu(E_\alpha \otimes 1) + \theta_{\mu-\alpha}(K_\alpha^{-1} \otimes E_\alpha)\} = \sum_i v_i \otimes u_i$$

with $v_i \in U^-_{-\mu+\alpha}$ and $u_i \in U^+_\mu$ such that $(u_i)_i$ is k-linearly independent. Then by (3) for each $m \in M$

$$0 = \left(\sum_i v_i \otimes u_i\right)(m \otimes v_\lambda) = \sum_i v_i m \otimes \omega(u_i)v_\lambda. \tag{5}$$

The set $(\omega(u_i)v_\lambda)_i$ remains k-linearly independent by (4), hence $v_i m = 0$ for each $m \in M$. Then by [J, 5.11] all $v_i = 0$, hence (1). Likewise (2).

(2.2) One can then show as in [L, 4.1.2, 3]

Theorem *There is unique* $\theta = (\theta_\mu)_{\mu \in N\Pi}$ *with* $\theta_\mu \in U^-_{-\mu} \otimes U^+_\mu$ *and* $\theta_0 = 1 \otimes 1$ *satisfying* (2.1.ii). *The element* $\Theta = (\Theta_\mu)_\mu$ *of* (1.6) *is such, and hence for each* $M, M' \in \mathscr{F}$

$$\Theta_{M,M'}\Theta^\psi_{M,M'} = id_{M \otimes M'} = \Theta^\psi_{M,M'}\Theta_{M,M'}.$$

(2.3) **Theorem** [L, 24.1.6] *If we write for each* $\mu \in N\Pi$

$$\Theta_\mu = \sum_{b,b' \in B(\infty)_{-\mu}} \gamma_{b,b'} b \otimes \omega(b'), \qquad \gamma_{b,b'} \in k,$$

then $\gamma_{b,b'} \in \mathscr{A}$ *for all* b, b'.

Proof. Let b_0, $b'_0 \in B(\infty)_{-\mu}$, $\mu \in N\Pi$. Choose $\lambda, \lambda' \in \Lambda^+$ such that $b_0 v_\lambda \in B(\lambda)$, $b'_0 v'_\lambda \in B(\lambda')$ [J,5.18]. The introduction of based modules makes verification of [L, 21.1.2] trivial; if $w_0 \in W$ with $w_0\Pi = -\Pi$, ${}^\omega L(-w_0\lambda) \simeq L(\lambda)$ in $\mathscr{F}$, hence there is by (1.4) an isomorphism $\chi \in \mathscr{BM}(({}^\omega L(-w_0\lambda), B(-w_0\lambda), \psi_{-w_0\lambda}), (L(\lambda), B(\lambda), \psi_\lambda))$ and likewise an isomorphism $\chi' \in \mathscr{BM}((L(-w_0\lambda'), B(-w_0\lambda'), \psi_{-w_0\lambda'}), ({}^\omega L(\lambda'), B(\lambda'), \psi_{\lambda'}))$.

If $M(\nu) = U/\{\sum_{\alpha \in \Pi} UE_\alpha + \sum_{\alpha \in \Pi} U(K_\alpha - q^{(\nu,\alpha)})\}$, $\nu \in \Lambda$, and if $M_\mathscr{A}(\nu) = U_\mathscr{A}\bar{1}$ with $\bar{1}$ the image of $1 \in U$ in $M(\nu)$, one can show [L, 24.1.4]

$$\Theta_{{}^\omega M(-w_0\lambda) \otimes M(-w_0\lambda')} \text{ stabilizes } {}^\omega M_\mathscr{A}(-w_0\lambda) \otimes_\mathscr{A} M_\mathscr{A}(-w_0\lambda'). \tag{1}$$

If we write Θ for $\Theta_{{}^\omega L(-w_0\lambda) \otimes L(-w_0\lambda')}$, taking the quotient Θ stabilizes ${}^\omega L_\mathscr{A}(-w_0\lambda) \otimes_\mathscr{A} L_\mathscr{A}(-w_0\lambda')$, hence $(\chi \otimes \chi') \circ \Theta \circ (\chi^{-1} \otimes (\chi')^{-1})$ stabilizes $L_\mathscr{A}(\lambda) \otimes_\mathscr{A} {}^\omega L_\mathscr{A}(\lambda')$. In particular,

$$L_\mathscr{A}(\lambda) \otimes_\mathscr{A} {}^\omega L_\mathscr{A}(\lambda') \ni (\chi \otimes \chi') \circ \Theta \circ (\chi^{-1} \otimes (\chi')^{-1})(v_\lambda \otimes v_{\lambda'}) = \sum_\mu \sum_{b,b'} \gamma_{b,b'} b v_\lambda \otimes b' v_{\lambda'}.$$

As $bv_\lambda \in B(\lambda) \sqcup \{0\}$ and $b'v_{\lambda'} \in B(\lambda') \sqcup \{0\}$, we must have $\gamma_{b_0,b'_0} \in \mathscr{A}$.

(2.4) One can now argue as in [L, 27.3.2] to obtain (1.7).

References

[1] Jantzen, J. C., Representations of Algebraic Groups, PAM **131**, 1987 (Academic Press)
[2] Jantzen, J. C., Lectures on Quantum Groups, GSM **6**, 1995 (AMS)
[3] Lusztig, G., Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, J. Amer. Math. Soc. **3** (1990), 257–296
[4] Lusztig, G., Quantum groups at roots of 1, Geom. Ded. **35** (1990), 89–114
[5] Lusztig, G., Introduction to Quantum Groups, PM110, 1993 (Birkhäuser)
[6] Mathieu, O., Filtrations of G-modules, Ann. Sci. Ecol. Norm. Sup. **23** (1990), 625–644
[7] Paradowski, J., Filtrations of modules over the quantum algebra, Proc. Symp. Pure Math. **56–2**, 93–108 1994 (AMS)
[8] Xi, N., Irreducible modules of quantized enveloping algebras at roots of 1, Publ. RIMS **32–2** (1996), 235–276

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