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Bipotential elliptic differential operators

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ABSTRACT. The classification of the second order elliptic differential operators with locally Lipschitz coefficients in a domain in $\boldsymbol{R}^n$ is considered. Using potential-theoretic techniques, it is modelled after the biharmonic classification of Riemannian manifolds.

1. Introduction

Let Ω be a domain in $\boldsymbol{R}^n$, $n \geq 2$, and $\mathscr{L}(\Omega)$ the family of all second order elliptic differential operators with locally Lipschitz coefficients in Ω. In this article, we put the elements in $\mathscr{L}(\Omega)$ into different classes depending on the existence of certain special solutions of the operators.

The classification is modelled after (and more general than) that of the Riemannian manifolds $\mathscr{R}$ based on the existence of Green functions, biharmonic functions, biharmonic Green functions etc.

The similarity between these two classifications of $\mathscr{L}(\Omega)$ and $\mathscr{R}$ arises from the fact that the C^2-solutions of $Lu = 0$ for any $L \in \mathscr{L}(\Omega)$ and the harmonic functions defined by $\Delta u = 0$ in $R \in \mathscr{R}$ (where Δ is the Laplace-Beltrami operator on R) both satisfy locally the basic assumptions in the axiomatic potential theory of M. Brelot [4].

2. Preliminaries

Let $\mathscr{L}(\Omega)$ denote the family of all second order elliptic differential operators with locally Lipschitz coefficients defined on a domain Ω in $\boldsymbol{R}^n, n \geq 2$. Assume that the last coefficient is 0 in each $L \in \mathscr{L}(\Omega)$. Precisely, $Lu(x) = \sum_{i,j} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u(x)}{\partial x_i}$, where the a_{ij}'s are in $C^{2,\lambda}$ and the b_i's are in $C^{1,\lambda}$; $a_{ij} = a_{ji}$; and the quadratic form $\sum_{i,j} a_{ij}\xi_i\xi_j$ is positive definite for every $x \in \Omega$.

Then the C^2-functions u in an open set $w \subset \Omega$ for which $Lu = 0$ are called the L-harmonic functions in w. Such solutions satisfy the basic assumptions of

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the Brelot axiomatic potential theory (Mme. Hervé [6], pp. 560–568), leading to the definitions of L-superharmonic functions, L-potentials associated with L in the open sets of Ω.

We also have in this context a Malgrange Approximation Lemma (A. De la Pradelle [5], p. 399); using this as in Theorem 4.2 [3], we can prove the following

LEMMA 2.1: *Given a Radon measure $\mu \geq 0$ on an open set $w \subset \Omega$, and any $L \in \mathscr{L}(\Omega)$ there exists an L-superharmonic function u in w such that in a local Riesz representation μ is the measure associated with u.*

NOTATIONS: 1) Given a Radon measure $\mu \geq 0$ on an open set $w \subset \Omega$, for $L \in \mathscr{L}(\Omega)$, we write $Lu = -\mu$ to denote that u is an L-superharmonic function in w generated by μ as in Lemma 2.1. (we ignore here any possible interpretation in the sense of distributions of the equation $Lu = -\mu$).

We remark that (i) u is not unique for a given μ, and (ii) if μ is a signed measure, one can find a $\delta - L$-superharmonic function u in w such that $Lu = -\mu$ in w.

2) Suppose $f(x)$ is a locally (Lebesgue) integrable function in w. Let μ be the signed measure defined by f as $d\mu(x) = f(x)\,dx$, dx the Lebesgue measure. If u is a $\delta - L$-superharmonic function generated by μ, we write $Lu = -f$ instead of $Lu = -\mu$.

3) On the other hand, instead of starting with a measure $\mu \geq 0$ in Ω as above, suppose we start with an L-superharmonic function v in Ω. Then locally v has a Riesz representation and an associated Radon measure $\mu \geq 0$ in Ω. Suppose now u is an L-superharmonic function in Ω generated by the measure μ as in Lemma 2.1. Then $v = u + h$ where h is an L-harmonic function in Ω. Consequently, we can just as well write $Lv = -\mu$ in Ω.

REMARK. Dealing with the Laplacian operator $\Delta \in \mathscr{L}(\Omega)$, instead of writing Δ-superharmonic, Δ-potential etc., we suppress Δ and simply refer to superharmonic, potential etc.. Thus, a superharmonic function in Ω is locally Lebesgue integrable.

3. Bipotential operators

For an $L \in \mathscr{L}(\Omega)$, the domain Ω is called L-hyperbolic or L-parabolic depending on whether there exists or not an L-potential in Ω. If $L = \Delta$, we simply refer to Ω as hyperbolic or parabolic. We assume in the sequel that Ω is always hyperbolic, that is there are Δ-potentials in Ω.

DENITION 3.1. An operator $L \in \mathscr{L}(\Omega)$ is called a *bipotential operator* in Ω if there exist an L-potential q and a potential $p > 0$ (that is, p is a Δ-potential) in Ω such that $Lq = -p$.

EXAMPLES:

1) Δ is a bipotential operator in $\boldsymbol{R}^n$, $n \geq 5$. For, $p = 2(n-4)r^{2-n}$ and $q = r^{4-n}$ are potentials in $\boldsymbol{R}^n$ such that $\Delta q = -p$.

2) Since $\boldsymbol{R}^2$ is parabolic, Δ cannot be a bipotential operator in $\boldsymbol{R}^2$.

3) Though $\boldsymbol{R}^3$ and $\boldsymbol{R}^4$ are hyperbolic, yet Δ is not a bipotential operator there. For $\Delta q = -p$ would imply [7] that $\int_1^\infty \int_{\partial B} \frac{\Delta q}{r^{n-2}} r^{n-1}\, dr\, d\sigma$ is finite, where ∂B is the unit sphere and $d\sigma$ is its surface area; this in turn would imply (since a potential majorizes mr^{2-n}, for some $m > 0$, near infinity) that $\int_1^\infty \frac{1}{r^{n-2}} \frac{1}{r^{n-2}} r^{n-1}\, dr$ is finite which is false when $n = 3, 4$.

PROPOSITION 3.2. *Let Ω be a relatively compact domain in $\boldsymbol{R}^n$, $n \geq 2$. Then, any $L \in \mathscr{L}(\Omega_1)$, Ω_1 a domain $\supset \overline{\Omega}$, is a bipotential operator in Ω.*

PROOF. Let Ω_0 be a relatively compact domain such that $\Omega_1 \supset \Omega_0 \supset \overline{\Omega}$. Then, by Lemma 2.1, there exists an L-superhamonic function q_0 in Ω_0 such that $Lq_0 = -1$ in Ω_0.

Since $\overline{\Omega} \subset \Omega_0$, q_0 has an L-harmonic minorant in Ω; let h_0 be the greatest L-harmonic minorant of q_0 in Ω. Then $q_1 = q_0 - h_0$ is an L-potential in Ω such that $Lq_1 = -1$ in Ω.

Now, for a nonpolar compact set K in Ω, $p = (\hat{R}_1^K)_\Omega$ is a potential in Ω. Let s be an L-superharmonic function in Ω such that $Ls = -p$; and let t be an L-superharmonic function in Ω such that $Lt = -(1-p)$.

Then there exists an L-harmonic function h_1 in Ω such that $s + t = q_1 + h_1$; since $q_1 > 0$, this implies that s has an L-subharmonic minorant in Ω. Let h be the greatest L-harmonic minorant of s in Ω and write $q = s - h$.

Then q is an L-potential in Ω such that $Lq = -p$. Hence L is a bipotential operator in Ω.

DEFINITION 3.3. An operator $L \in \mathscr{L}(\Omega)$ is called a *biharmonic potential operator* in Ω if there exists an L-potential q and a harmonic function $h > 0$ in Ω such that $Lq = -h$.

NOTE: The argument in the proof of Proposition 3.2 shows that a biharmonic potential operator L in Ω is a bipotential operator. But Δ which is a bipotential operator in $\boldsymbol{R}^n$, $n \geq 5$, is not a biharmonic potential operator there.

For, any positive harmonic function in $\boldsymbol{R}^n$ being a constant, assume that there exists a potential q in $\boldsymbol{R}^n$ such that $\Delta q = -1$. Then $\int_1^\infty \int_{\partial B} \frac{\Delta q}{r^{n-2}} r^{n-1}\, dr\, d\sigma$ should be finite, that is $\int_1^\infty r\, dr$ should be finite, a contradiction.

However, if Ω is a bounded domain in $\boldsymbol{R}^n$, $n \geq 2$, any operator $L \in \mathscr{L}(\Omega_1)$, Ω_1 a domain $\supset \overline{\Omega}$, is a biharmonic potential operator in Ω. For, as shown in the proof of Proposition 3.2, there exists an L-potential q_1 in Ω such that $Lq_1 = -1$.

4. Weak bipotential operators

In this section we will obtain some necessary and sufficient conditions for an operator $L \in \mathscr{L}(\Omega)$ to be a bipotential operator.

Given the local nature of the solutions of the operator L, we can apply these results in the context of Ω being a Riemannian manifold and L_0 the Laplace-Beltrami operator defined on Ω, to show that $-L_0$ is a bipotential operator if and only if the biharmonic Green potential is defined on the Riemannian manifold (Chapter VIII, Sario-Nakai-Wang-Chung [9]). Consequently, Theorem 4.4 below provides some additional characteristics of Riemannian manifolds with biharmonic Green potentials not explicitly mentioned in [9].

Actually, in this section we want to work in a more inclusive context; the definition of a bipotential operator $L \in \mathscr{L}(\Omega)$ presupposes that Ω is an L-hyperbolic domain. With a view to remove this restriction on Ω, we introduce the notion of a weak bipotential operator.

We call an L-superharmonic function u in a domain $\Omega \subset \boldsymbol{R}^n$ an *admissible* L-superharmonic function (p. 145 [1]) if u has an L-harmonic minorant outside a compact set in Ω.

DEFINITION 4.1. An operator $L \in \mathscr{L}(\Omega)$ is called a *weak bipotential operator* in Ω if there exists an admissible L-superharmonic function u and a potential $p > 0$ in Ω such that $Lu = -p$ in Ω.

REMARKS: 1) A bipotential operator $L \in \mathscr{L}(\Omega)$ is a weak bipotential operator. On the other hand, if Ω is an L-hyperbolic domain and if L is a weak bipotential operator in Ω, then L is actually a bipotential operator.

For, in this case suppose u is an L-superharmonic function in Ω having an L-harmonic minorant h outside a compact set. Then there exists an L-harmonic function H in Ω such that $|H - h|$ is bounded outside a compact set (Extension Theorem 1.20 [1], originally proved by M. Nakai in [8]); consequently, u is the unique sum of an L-potential q and an L-harmonic function in Ω. Hence $Lu = -p$ implies that $Lq = -p$ and L is a bipotential operator in Ω.

2) From the above remark it follows that Δ cannot be a weak bipotential operator in $\boldsymbol{R}^n n = 3, 4$. For, $\boldsymbol{R}^3$ and $\boldsymbol{R}^4$ are hyperbolic domains where Δ is not a bipotential operator.

LEMMA 4.2. *Let $L \in \mathscr{L}(\Omega)$. For a measure $\mu \geq 0$ on an open set $w \subset \Omega$, suppose there exists an admissible L-superharmonic function u in w such that $Lu = -\mu$. Then, for any measure $\lambda, 0 \leq \lambda \leq \mu$ in w, there exists an admissible L-superharmonic function v in w such that $Lv = -\lambda$.*

PROOF. Let v and v_1 be L-superharmonic functions in w (Lemma 2.1) such that $Lv = -\lambda$ and $Lv_1 = -(\mu - \lambda)$.

Then $v + v_1 = u+$ (an L-harmonic function) in w.

Since u has an L-harmonic minorant outside a compact set K in w, v has an L-subharmonic minorant outside K. This implies that v is an admissible L-superharmonic function in w.

THEOREM 4.3. *For $L \in \mathscr{L}(\Omega)$, the following are equivalent:*

1) For some (or any) compact nonpolar set K in Ω (with respect to Δ, the notion of polarity being the same with respect to any L, Théorème 36.1 [6]), there exists an L-superharmonic function u in $\Omega \backslash K$, having an L-harmonic minorant in a neighbourhood of the Alexandrov point of Ω, such that $Lu = -\hat{R}_1^K$ in $\Omega \backslash K$.

2) L is a weak bipotential operator in Ω.

3) There exist an admissible L-superharmonic function u that is not L-harmonic and a superharmonic function $s > 0$ in Ω such that $Lu = -s$.

4) For some (or any) $y \in \Omega$, if p_y denotes a potential with harmonic point support $\{y\}$, there exists an admissible L-superharmonic function q_y in Ω such that $Lq_y = -p_y$.

PROOF.

1) $\Rightarrow$ 2): Let $Lu = -\hat{R}_1^K$ in $\Omega \backslash K$.

Let A be an outerregular compact set and w a regular domain such that $K \subset \mathring{A} \subset A \subset w$. *(Recall the identity between the L-regular and the Δ-regular boundary points for an open set $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$).*

We can assume that u is L-harmonic in a neighbourhood of $\overline{w \backslash A}$.

Then as in [2], we can find L-harmonic functions u_1 in $\Omega \backslash A$ and u_2 in w such that $u = u_1 - u_2$ in $w \backslash A$.

Then the function v equal to $u - u_1$ in $\Omega \backslash A$ and to $-u_2$ in w is an L-superharmonic function in Ω; it is admissible also since by assumption u has an L-harmonic minorant in a neighbourhood of the Alexandrov point of Ω; moreover, in $\Omega \backslash A$, $Lv = Lu = -\hat{R}_1^K$.

Let v_1 be an L-superharmonic function in Ω (Lemma 2.1) such that $Lv_1 = -\chi_A$ the characteristic fuction of A. Since v_1 has compact (harmonic) support, it is an admissible L-superharmonic function in Ω.

Thus $v+v_1$ is an admissible L-superharmonic function in Ω with the associated measure μ such that $\mu \geq \lambda$ where λ is the measure defined by $d\lambda(x) = \hat{R}_1^K(x)\,dx$.

Hence, by Lemma 4.2, there exists an admissible L-superharmonic function q in Ω such that $Lq = -\hat{R}_1^K$ in Ω. That is L is a weak bipotential operator in Ω.

2) $\Rightarrow$ 3): Suppose there exists an admissible L-superharmonic function q and a potential $p > 0$ in Ω such that $Lq = -p$. Then clearly q is not L-harmonic in Ω.

3) $\Rightarrow$ 4): Suppose $Lu = -s$ where u is a non L-harmonic admissible L-superharmonic function and $s > 0$ is superharmonic in Ω.

Now, given $y \in \Omega$, choose an outerregular compact set k in Ω such that $y \in \mathring{k}$ and consider a potential p_y with point harmonic support $\{y\}$. Then $p_y = Bp_y$ in $\Omega \backslash k$ where Bp_y is the Dirichlet solution in $\Omega \backslash k$ with boundary values p_y on ∂k and 0 at the Alexandrov point of Ω.

Note that $Bp_y \leq ms$ in $\Omega \backslash k$ for some $m > 0$. Consequently, we can construct as in the proof of 1) $\Rightarrow$ 2), an admissible L-superharmonic function v_1 in Ω such that $Lv_1 = -Bp_y = -p_y$ in $\Omega \backslash k$.

Let v_2 be an admissible L-superharmonic function in Ω for which $Lv_2 = -p_y \chi_k$ in Ω.

Then, if $v = v_1 + v_2$, v is an admissible L-superharmonic function in Ω with the associated measure μ such that $\mu \geq \lambda$ where λ is the measure defined by $d\lambda(x) = p_y(x)\,dx$.

By Lemma 4.2, there exists an admissible L-superharmonic function q_y in Ω such that $Lq_y = -p_y$.

4) $\Rightarrow$ 1): For some y, let $Lq_y = -p_y$ in Ω where q_y is an admissible superharmonic function in Ω; that is q_y has an L-harmonic minorant outside some compact set A in Ω.

For a compact nonpolar set K, $\hat{R}_1^K \leq \left(\inf_K p_y\right)^{-1} p_y$ in $\Omega \backslash K$. Consequently, as in Lemma 4.2, there exists an L-superharmonic function u in $\Omega \backslash K$ such that $Lu = -\hat{R}_1^K$; moreover, u has an L-harmonic minorant outside the compact set $K \cup A$.

This completes the proof of the theorem.

We have remarked earlier that if Ω is an L-hyperbolic domain, then any admissible L-superharmonic function u in Ω is the unique sum of an L-potential and an L-harmonic function and also if L is a weak bipotential operator on Ω it is actually a bipotential operator. Consequently, it is easy to deduce from Theorem 4.3 the following

THEOREM 4.4. *For $L \in \mathscr{L}(\Omega)$, let Ω be an L-hyperbolic domain. Then the following are equivalent*:

1) For some (or any) compact nonpolar set K in Ω, there exists an L-superharmonic function $u \geq 0$ in $\Omega \backslash K$ such that $Lu = -\hat{R}_1^K$ in $\Omega \backslash K$.

2) L is a bipotential operator in Ω.

3) There exist an L-superharmonic function $u > 0$ and a superharmonic function $s > 0$ in Ω such that $Lu = -s$.

4) For some (or any) $y \in \Omega$, if p_y denotes a potential with harmonic support $\{y\}$, there exists a unique L-potential q_y in Ω such that $Lq_y = -p_y$.

Corollary 4.5. *Let L be a bipotential operator in Ω. For $y \in \Omega$ fixed, let Ω_n be a sequence of relatively compact domains, $y \in \Omega_n$, $\overline{\Omega}_n \subset \Omega_{n+1}$, and $\Omega = \cup \Omega_n$. Let $p_y^{\Omega_n}$ be the potential in Ω_n with harmonic point support $\{y\}$ such that $p_y^{\Omega_n} = p_y + a$ harmonic function in Ω_n; let $q_y^{\Omega_n}$ be the unique L-potential in Ω_n such that $Lq_y^{\Omega_n} = -p_y^{\Omega_n}$ in Ω_n. Then $\sup q_y^{\Omega_n}$ is an L-potential in Ω.*

Proof. Since L is a bipotential operator in Ω, there exists a unique L-potential q_y in Ω such that $Lq_y = -p_y$.

Since $p_y \geq p_y^{\Omega_n}$ in Ω_n, $q_y^{\Omega_n} = q_y + s_n$ in Ω_n where s_n is an L-subharmonic function in Ω_n; since $q_y^{\Omega_n}$ is an L-potential in Ω_n and $s_n \leq q_y^{\Omega_n}$, we have $s_n \leq 0$ in Ω_n.

Consequently, $q_y^{\Omega_n} \leq q_y$ in Ω_n for every n.

A similar argument also shows that $q_y^{\Omega_n}$ is an increasing sequence and hence $\sup q_y^{\Omega_n}$ is an L-potential in Ω.

Remarks. 1) The proof of the above corollary follows the construction of the biharmonic Green function on a hyperbolic Riemannian manifold (p. 300 [9]).

2) When Ω is a hyperbolic Riemannian manifold with the harmonic functions defined locally as the solutions of the Laplace-Beltrami operator, the statement (1) in Theorem 4.4 expresses the condition that the biharmonic measure of the ideal boundary of Ω is finite (p.310 [9]).

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