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Local attractor for n-D Navier-Stokes system

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ABSTRACT. The n-D Navier-Stokes system ($n \geq 3$) is studied as an abstract equation with sectorial operator in a relevant Banach space X_r consisting of divergence-free functions. Existence of the local semiflow $\{T(t)\}$ on a 'sufficiently smooth' fractional power space X_r^α is then known in advance. This makes it possible to consider a subset $V \subset X_r^\alpha$ for which an *a priori* estimate asymptotically independent of initial data for originated in V solutions may be derived. The task of the present paper is to apply authors' previous result [4] to the Navier-Stokes system proving existence of a global attractor $\mathscr{A}_{\alpha,r}$ for the semigroup $\{T(t)\}$ restricted to V. Simultaneously $\mathscr{A}_{\alpha,r}$ is shown to be a local attractor in a neighborhood of zero.

1. Introduction

Since the publication in 1934 of Leray's famous paper, progress in understanding the dynamics of the Navier-Stokes system has been steady but slow. Difficulties encountered in dealing with this system became particularly intensive when 3-D flows were studied. A new trend, permitting simpler treatment of this problem, was the semigroup L^p-spaces approach appearing e.g. in [2], [8], [9], [11], [12], [17], [21]. This approach has been followed in our previous papers [4], [3], where the dynamics of semilinear parabolic equations was studied within the *dissipative systems* theory [11]. In the present paper the authors' previous result [4] is applied to the Navier-Stokes system and the existence of a global attractor $\mathscr{A}_{\alpha,r}$ for the semigroup $\{T(t)\}$ restricted to V is proved. Simultaneously $\mathscr{A}_{\alpha,r}$ is shown to be a local attractor in a neighbourhood of zero.

1.1. Overview. In the following two subsections the Navier-Stokes system, viewed as a sectorial equation in the relevant Banach space X_r, is discussed to generate local semiflow $\{T(t)\}$ on the fractional phase space X_r^α. Applying *introductory estimates* concept of [4] (Sections 2.1, 2.2) we choose suitable

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metric space $V \subset X_r^\alpha$ on which $\{T(t)\}$ becomes a dissipative, compact semigroup of global solutions. As a consequence the existence of a global attractor $\mathscr{A}$ for $\{T(t)\}$ restricted to V (Section 2.3) will be shown to follow from the one simple introductory estimate of solutions in $[W_0^{1,r}(\Omega)]^n$. Section 2.4 is devoted to the construction of a *local attractor* for $\{T(t)\}$ in a neighborhood of zero.

1.2. Formulation of the problem. Notation. We deal with the n-D Navier-Stokes equation:

$$(1) \qquad u_t = \nu\Delta u - \nabla p - (u, \nabla)u + f, \quad div\, u = 0, \qquad \text{for } t > 0, x \in \Omega,$$

where $n \geq 3, \nu > 0$ is a constant viscosity, $u = (u_1(t,x), \dots, u_n(t,x))$ denotes velocity, $p = p(t,x)$ pressure and $f = (f_1(x), \dots, f_n(x))$ external force. Here Ω is a bounded subdomain of R^n with the boundary $\partial\Omega$ of class C^2, whereas $(\cdot,\cdot)$ stands for the standard inner product in R^n.

Equation (1) is studied with a boundary condition of Dirichlet type

$$(2) \qquad u = 0, \quad t > 0, \quad x \in \partial\Omega,$$

and subject to an initial condition

$$(3) \qquad u(0,x) = u_0(x), \qquad \text{for } x \in \Omega.$$

For simplicity of further notation let us introduce the following list of function spaces:

$$\mathscr{L}^r(\Omega) := [L^r(\Omega)]^n,$$
$$\mathscr{W}^{2,r}(\Omega) := [W^{2,r}(\Omega)]^n,$$
$$X_r := cl_{\mathscr{L}^r(\Omega)}\{\phi \in [C_0^\infty(\Omega)]^n;\ div\ \phi = 0\},$$

and define in X_r an unbounded operator A_r by the formula

$$A_r = -\nu P_r \begin{bmatrix} \Delta & 0 & 0 & \dots & 0 \\ 0 & \Delta & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & \Delta \end{bmatrix}_{n\times n},$$

where P_r is the continuous projection from $\mathscr{L}^r(\Omega)$ to X_r which is given by the decomposition of $\mathscr{L}^r(\Omega)$ (cf. [9, p. 268]) onto the spaces of divergence-free vector fields and scalar-function gradients. Generically, P_r is thus an extension of the orthogonal projection in $\mathscr{L}^2(\Omega)$ (cf. [13]). It is known (cf. [9, Lem. 1.1]) that:

PROPOSITION 1. *Operator* $-A_r$ *considered on the domain*

$$D(A_r) := X_r \cap \{\phi \in \mathscr{W}^{2,r}(\Omega); \phi|_{\partial\Omega} = 0\}, \tag{4}$$

generates an analytic semigroup $\{e^{-tA_r}\}$ *in* X_r $(1 < r < \infty)$.

Considering the resolvent equation for A_2 in X_2 it is easy to see that $Re(\sigma(A_2)) \geq \nu\lambda_1$, where λ_1 is the first positive eigenvalue of $-\Delta$ in $L^2(\Omega)$ under homogeneous Dirichlet boundary conditions. It follows from the elliptic regularity theory that $Re(\sigma(A_r)) \geq \nu\lambda_1$ for each $r > 1$ (cf. [20, Th. 5.5.1]) and therefore, fractional powers A_r^α $(\alpha \in [0,1])$ of A_r may be defined on the domains $X_r^\alpha := D(A_r^\alpha)$ (see [12, Chapt. I]) and for each $r \in (1,\infty), \alpha \in (0,1]$:

$$\|A_r^\alpha e^{-tA_r}\|_{\mathscr{L}(X_r^\alpha, X_r)} \leq C_{\alpha,r} t^{-\alpha} e^{-\nu\lambda_1 t}. \tag{5}$$

Moreover, since as a result of [9, Lem. 3.1] the resolvent of A_r is compact, the embeddings $X_r^\beta \subset X_r^\alpha$ $(0 < \alpha < \beta, 1 < r < \infty)$ are compact (see [12, Th. 1.4.8]).

1.3. Local semiflow of fractional solutions. For $f \in \mathscr{L}^r(\Omega)$ the system (1)–(3) may be thus studied as an abstract Cauchy problem in X_r:

$$u_t + A_r u = F_r u + P_r f, \qquad t > 0, \quad u(0) = u_0, \tag{6}$$

where A_r considered with the domain (4) is sectorial in X_r and $F_r u = -P_r(u,\nabla)u$. Moreover, for $\alpha \in [\frac{1}{2},1), r > n$ and $f \in \mathscr{L}^r(\Omega)$ the nonlinear term F_r, acting from X_r^α into X_r, is Lipschitz continuous on bounded sets. Indeed, the estimate [9, Lem. 3.3, (iii)] reads:

$$\|P_r(w,\nabla)v\|_{\mathscr{L}^r(\Omega)} \leq c_r \|w\|_{\mathscr{W}^{1,r}(\Omega)} \|v\|_{\mathscr{W}^{1,r}}(\Omega), \quad w, v \in \mathscr{W}^{1,r}(\Omega), \quad r > n, \tag{7}$$

hence, when $\phi, \psi \in \mathscr{U}$ where $\mathscr{U} \subset X_r^\alpha$ is bounded, we have:

$$\begin{aligned} \|F_r\phi - F_r\psi\|_{X_r} &\leq c_r \|\phi\|_{\mathscr{W}^{1,r}(\Omega)} \|\phi - \psi\|_{\mathscr{W}^{1,r}(\Omega)} + c_r \|\phi - \psi\|_{\mathscr{W}^{1,r}(\Omega)} \|\psi\|_{\mathscr{W}^{1,r}(\Omega)} \\ &\leq c_r \max\{\|\phi\|_{\mathscr{W}^{1,r}(\Omega)}, \|\psi\|_{\mathscr{W}^{1,r}(\Omega)}\} \|\phi - \psi\|_{\mathscr{W}^{1,r}(\Omega)} \leq c_{r,\mathscr{U}} \|\phi - \psi\|_{\mathscr{W}^{1,r}(\Omega)}, \end{aligned}$$

whereas $X_r^\alpha \subset X_r^{1/2}$ for $\alpha \in [\frac{1}{2},1)$ and also $X_r^{1/2}$ is continuously embedded in $X_r \cap \mathscr{W}^{1,r}(\Omega)$ (cf. [9, Prop. 1.4]). Following [11, p. 72], [12, Chapt. 3] we recall the notion of the *fractional solution* of (6).

DEFINITION 1. By a fractional solution of (6) we understand a continuous function $u : [0,\tau_{u_0}) \longrightarrow X_r^\alpha$ satisfying (6), such that $u_t : (0,\tau_{u_0}) \longrightarrow X_r$ and $F_r(u(\cdot)) : (0,\tau_{u_0}) \longrightarrow X_r$ are continuous and $u(t)$ belongs to $D(A_r)$ for each $t \in (0,\tau_{u_0})$.

Since we have shown the appropriate Lipschitz continuity of F_r, it follows immediately from the general results of [11, Sec. 2.2] (cf. also [12, Chapt. 3]) that:

PROPOSITION 2. *For* $r \in (n, \infty), \alpha \in [\frac{1}{2}, 1)$ *and* $f \in \mathscr{L}^r(\Omega)$ *a local semiflow* $\{T(t)\}$ (*where* $T(t)u_0 = u(t, u_0)$ *for* $t \in [0, \tau_{max}(u_0))$) *of maximal fractional solutions of* (6) *is defined on* X_r^α.

In the following section a subset $V_r^\alpha \subset X_r^\alpha$ ($r \in (n, \infty), \alpha \in [\frac{1}{2}, 1)$) will be distinguished such that fractional solutions $T(t)u_0$ of (6) with $u_0 \in V_r^\alpha$ are defined globally in time. In addition, the existence of a *restricted global attractor* for $\{T(t)\}$, see Definition 2 below, will be shown.

DEFINITION 2. Let $r \in (n, \infty), \alpha \in [\frac{1}{2}, 1)$ and $\{T(t)\}$ be a local semiflow on X_r^α defined in Proposition 2. We say that $S \subset X_r^\alpha$ is a restricted global attractor for $\{T(t)\}$ in X_r^α if for some closed, nonempty subset V of $X_r^\alpha, T(t) : V \longrightarrow V$ $(t \geq 0)$ is a global semiflow on V such that S is a global attractor for $\{T(t)\}$ restricted to V as stated in [11, Sec. 3.4] (that is (i) $T(t)S = S$ for $t \geq 0$, (ii) S is compact, (iii) S attracts trajectories of bounded subsets of V).

2. Global solutions of the problem (6)

2.1. Background. As shown in our previous papers [4] and [3], to obtain the relevant results concerning global existence of solutions of a sectorial equation sufficiently smooth global a priori estimate of solutions is needed. In addition, the nonlinear term taken on solutions of this equation should also be subordinated to some θ-power ($\theta \in [0, 1)$) of the 'main part operator'. In particular these properties may not be known to hold for all solutions but only for those originating in some proper subset of the phase space on which the equation is studied. Hence in further considerations the concept of [3, Th. 5] (cf. also earlier results of [4, Th. 1.2]) will generally be followed as described in Corollary 1 below.

COROLLARY 1. *Let* $r \in (n, \infty), \alpha \in [\frac{1}{2}, 1)$ *and* $\{T(t)\}$ *denotes, generated by* (6), *local semiflow on* X_r^α *defined in Proposition* 2. *Let us also recall that the resolvent of* A_r *is known to be compact. Then in order to prove the existence of a* ***restricted global attractor*** *for* $\{T(t)\}$ *in* X_r^α *it suffices to show that there is a Banach space* $Y \supset D(A_r)$ *and a nondecreasing function* $g : [0, \infty) \longrightarrow [0, \infty)$ *for which the conjunction of conditions* (8) *and* (9) *stated below holds with some closed and positively invariant nonempty subset* V *of* X_r^α; *where*

$$\exists_{c>0} \forall_{u_0 \in V} \forall_{t \in (0, \tau_{max}(u_0))} \|T(t)u_0\|_Y \leq c \tag{8}$$

and

$$\begin{aligned} &\exists_{\theta \in [0,1)} \forall_{u_0 \in V} \forall_{t \in (0, \tau_{max}(u_0))} \|F(T(t)u_0)\|_{\mathscr{L}^r(\Omega)} \\ &\quad \leq g(\|T(t)u_0\|_Y)(1 + \|T(t)u_0\|_{X_r^\alpha}^\theta). \end{aligned} \tag{9}$$

Observation 1. Fixing $r \in (n, \infty), \alpha \in [\frac{1}{2}, 1)$ and using (7) we can estimate the right side of (6) as follows:

$$(10) \qquad \|F_r u + P_r f\|_{\mathscr{L}^r(\Omega)} \le \|P_r(u, \nabla)u\|_{\mathscr{L}^r(\Omega)} + C_r\|f\|_{\mathscr{L}^r(\Omega)} \le c_r\|u\|^2_{\mathscr{W}^{1,r}(\Omega)} + C_r\|f\|_{\mathscr{L}^r(\Omega)}.$$

Since

$$\|u\|_{\mathscr{W}^{1,r}(\Omega)} = \|u\|^{1/4}_{\mathscr{W}^{1,r}(\Omega)}\|u\|^{3/4}_{\mathscr{W}^{1,r}(\Omega)} \le c_{r,1/2}\|u\|^{1/4}_{X_r^{1/2}}\|u\|^{3/4}_{X_r^{1/2}} \le c_{r,\alpha}\|u\|^{1/4}_{X_r^\alpha}\|u\|^{3/4}_{X_r^{1/2}},$$

then the condition (10) leads to the inequality:

$$(11) \qquad \|F_r u + P_r f\|_{\mathscr{L}^r(\Omega)} \le c_r c^2_{r,\alpha}\|u\|^{1/2}_{X_r^\alpha}\|u\|^{3/2}_{X_r^{1/2}} + C_r\|f\|_{\mathscr{L}^r(\Omega)} \le (c_r c^2_{r,\alpha}\|u\|^{3/2}_{X_r^{1/2}} + C_r\|f\|_{\mathscr{L}^r(\Omega)})(1 + \|u\|^{1/2}_{X_r^\alpha}).$$

For $Y := X_r^{1/2}$ the estimate (11) becomes the required counterpart of (9). It is thus seen from Corollary 1 that $X_r^{1/2}$-*a priori* estimate of solutions is all we need to obtain a restricted global attractor for $\{T(t)\}$ in X_r^α.

Remark 1. It should be noted here that up to now the global existence of regular solutions of the n-D Navier-Stokes system is generally unknown unless sufficiently small initial data or large viscosity ν is considered. In this paper we decide to fix the viscosity coefficient ν. Hence, in our following studies we shall get, for small $\|f\|_{\mathscr{L}^r(\Omega)}$, the semiflow $\{T(t)\}$ globally defined merely in the vicinity of zero and on such a 'small' set V, validity of the estimate (8) will be shown.

2.2. Estimate of the $X_r^{1/2}$-norm ($r \in (n, \infty)$) of fractional solutions.

Lemma 1. *Let $r \in (n, \infty)$ and $\{T(t)\}$ be a local semiflow on $X_r^{1/2}$ defined in Proposition* 2. *If the norm $\|f\|_{\mathscr{L}^r(\Omega)}$ fulfills the 'smallness' restriction* (18), *then there are $R > 0$ and $\eta > 0$ (η and R defined, respectively, in* (19), (20) *below) such that*

$$(12) \qquad \|T(t)u_0\|_{X_r^{1/2}} \le R \qquad \text{for each } u_0 \in \mathscr{B}_{X_r^{1/2}}(0, \eta),$$

where $\mathscr{B}_{X_r^{1/2}}(0, \eta)$ denotes an open ball in $X_r^{1/2}$ centered at 0 *with radius η.*

Proof. Let $r > n$. Since (6) is equivalent to the integral equation

$$(13) \qquad u(t) = e^{-tA_r}u_0 + \int_0^t e^{-(t-s)A_r}(F_r u(s) + P_r f)\,ds,$$

then taking the $X_r^{1/2}$-norm of both sides in (13) and applying (5) we obtain:

$$(14)\quad \|u(t)\|_{X_r^{1/2}} \le e^{-\nu\lambda_1 t}\|u_0\|_{X_r^{1/2}} + \int_0^t C_{1/2,r}(t-s)^{-1/2}e^{-\nu\lambda_1(t-s)}(\|F_r u(s)\|_{\mathscr{L}^r(\Omega)} + \|P_r f\|_{\mathscr{L}^r(\Omega)})\,ds.$$

From (7) and [9, Prop. 1.4] we have:

$$(15)\qquad \|F_r u(s)\|_{\mathscr{L}^r(\Omega)} \le \frac{M_{1/2,r}}{\nu}\|u(s)\|^2_{X_r^{1/2}},$$

whereas, since P_r is a projector from $\mathscr{L}^r(\Omega)$ onto X_r, then

$$(16)\qquad \|P_r f\|_{\mathscr{L}^r(\Omega)} \le C_r\|f\|_{\mathscr{L}^r(\Omega)}.$$

Estimates (13)–(16) lead to the condition

$$\|u(t)\|_{X_r^{1/2}} \le e^{-\nu\lambda_1 t}\|u_0\|_{X_r^{1/2}} + \left(\frac{M_{1/2,r}}{\nu}\sup_{s\in[0,t]}\|u(s)\|^2_{X_r^{1/2}} + C_r\|f\|_{\mathscr{L}^r(\Omega)}\right)\int_0^\infty C_{1/2,r}y^{-1/2}e^{-\nu\lambda_1 y}\,dy,$$

which for $r(t) := \sup_{s\in[0,t]}\|u(s)\|_{X_r^{1/2}}$ gives

$$(17)\qquad r(t) \le \frac{M_{1/2,r}C_{1/2,r}\Gamma(\frac{1}{2})}{\nu^{3/2}\lambda_1^{1/2}}r^2(t) + \|u_0\|_{X_r^{1/2}} + \frac{C_r C_{1/2,r}\Gamma(\frac{1}{2})}{(\nu\lambda_1)^{1/2}}\|f\|_{\mathscr{L}^r(\Omega)} \equiv Ar^2(t) + \|u_0\|_{X_r^{1/2}} + B.$$

If further the $\|f\|_{\mathscr{L}^r(\Omega)}$-norm is required to satisfy:

$$(18)\qquad \|f\|_{\mathscr{L}^r(\Omega)} < \frac{\nu^2\lambda_1}{4C_r M_{1/2,r}C^2_{1/2,r}\Gamma^2(\frac{1}{2})},$$

the determinant $1 - 4A(\|u_0\|_{X_r^{1/2}} + B)$ of the quadratic inequality (17) will be positive provided that

$$(19)\quad \|u_0\|_{X_r^{1/2}} < \frac{1}{4A} - B = \left(\frac{\nu^{3/2}(\lambda_1)^{1/2}}{4M_{1/2,r}C_{1/2,r}\Gamma(\frac{1}{2})} - \frac{C_r C_{1/2,r}\Gamma(\frac{1}{2})}{(\nu\lambda_1)^{1/2}}\|f\|_{\mathscr{L}^r(\Omega)}\right) =: \eta.$$

It is thus seen that if u_0 is taken from the ball $\mathscr{B}_{X_r^{1/2}}(0,\eta)$ then, according to (17) and continuity of $T(t)u_0$ in $X_r^{1/2}$, the norm $\|T(t)u_0\|_{X_r^{1/2}}$ will never exceed the smaller root of the equation $Az^2 - z + \|u_0\|_{X_r^{1/2}} + B = 0$. Therefore for $u_0 \in \mathscr{B}_{X_r^{1/2}}(0,\eta)$, $T(t)u_0$ is a global fractional solution of (6) and $\|T(t)u_0\|_{X_r^{1/2}}$

satisfies, for all $t \geq 0$, the estimate:

(20)
$$\|T(t)u_0\|_{X_r^{1/2}} \leq \frac{1-\sqrt{1-4A(\|u_0\|_{X_r^{1/2}}+B)}}{2A} < \frac{1}{2A} = \frac{\nu^{3/2}(\lambda_1)^{1/2}}{2M_{1/2,r}C_{1/2,r}\Gamma(\frac{1}{2})} =: R.$$

The proof of Lemma 1 is completed. □

On account of the continuity of $T(t)$ (cf. [12, Th. 3.4.1]) Lemma 1 may be extended to:

LEMMA 2. *Let $r \in (n, \infty)$ and $\{T(t)\}$ be a local semiflow on $X_r^{1/2}$ defined in Proposition 2. If the norm $\|f\|_{\mathscr{L}^r(\Omega)}$ fulfills the 'smallness' restriction* (18), *then*

(21) $$\|T(t)u_0\|_{X_r^{1/2}} \leq R \qquad \textit{for each } u_0 \in cl_{X_r^{1/2}}\gamma^+(\mathscr{B}_{X_r^{1/2}}(0,\eta)),$$

where $\gamma^+(\mathscr{B}_{X_r^{1/2}}(0,\eta))$ denotes a positive orbit of $\mathscr{B}_{X_r^{1/2}}(0,\eta)$, i.e. $\gamma^+(\mathscr{B}_{X_r^{1/2}}(0,\eta)) := \bigcup_{t\geq 0} T(t)\mathscr{B}_{X_r^{1/2}}(0,\eta)$.

2.3. Restricted global attractors in $X_r^\alpha (r \in (n,\infty), \alpha \in [\frac{1}{2}, 1))$.

THEOREM 1. *Let $r \in (n,\infty), \alpha \in [\frac{1}{2}, 1)$ and let $\{T(t)\}$ denote a local semiflow on X_r^α defined in Proposition 2. Let the $\|f\|_{\mathscr{L}^r(\Omega)}$-norm also be restricted by* (18). *Then for*

$$V_r^\alpha := cl_{X_r^\alpha}(\gamma^+(\mathscr{B}_{X_r^{1/2}}(0,\eta)) \cap X_r^\alpha),$$

all fractional solutions $T(t)u_0$ with $u_0 \in V_r^\alpha$ are globally defined, $T(t)(V_r^\alpha) \subset V_r^\alpha$ $(t \geq 0)$ and the semigroup $\{T(t)\}$ restricted to V_r^α has a global attractor $\mathscr{A}_{\alpha,r}$.

PROOF. Let $r \in (n,\infty)$. Choosing $Y := X_r^{1/2}$ it is easy to see that the estimates (11) and (21) are the required counterparts of (8), (9). Thus Theorem 1 is a direct consequence of Corollary 1. □

OBSERVATION 2. Let $\omega_{X_r^{1/2}}(G)$ denote the *ω-limit set* of G in $X_r^{1/2}$, i.e. $\omega_{X_r^{1/2}}(G) := \bigcap_{s\geq 0} cl_{X_r^{1/2}} \bigcup_{t\geq s} T(t)G$. When $r \in (n,\infty)$ and $\alpha \in (\frac{1}{2}, 1)$ then V_r^α is an unbounded, complete (although not linear) metric subspace of X_r^α. For $\alpha = \frac{1}{2}$, the set $V_r^{1/2} = cl_{X_r^{1/2}}\gamma^+(\mathscr{B}_{X_r^{1/2}}(0,\eta))$ is bounded in $X_r^{1/2}$ and in the presence of [11, Lem. 3.2.1] the restricted attractor $\mathscr{A}_{1/2,r}$ from Theorem 1 is then equal to $\omega_{X_r^{1/2}}(V_r^{1/2})$. Since, as may easily be seen, $\omega_{X_r^{1/2}}(V_r^{1/2})$ coincides with $\omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta))$, then also

$$\mathscr{A}_{1/2,r} = \omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta)), \qquad r \in (n,\infty).$$

We shall prove below that for fixed $r \in (n,\infty)$ all attractors $\mathscr{A}_{\alpha,r}$ with $\alpha \in [\frac{1}{2}, 1)$ coincide.

Theorem 2. *Under the assumptions of Theorem* 1:

$$\mathscr{A}_{\alpha,r} = \omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta)), \quad r \in (n,\infty), \quad \alpha \in [\tfrac{1}{2},1). \tag{22}$$

Proof. Indeed, since for $r \in (n,\infty), \beta \in [\alpha,1)$ and G bounded in V_r^α the image $T(t)G$ is bounded in V_r^β (cf. [12, Th. 3.3.6]), then we obtain by invariance that $\omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta)) \subset \mathscr{A}_{\alpha,r}$. To get the converse inclusion, it suffices to use compactness of the embedding $X_r^\beta \subset X_r^\alpha$. The proof of Theorem 2 is completed. □

Having obtained $\mathscr{A}_{1/2,r}$ as a *restricted global attractor* for $\{T(t)\}$ in V_r^α, our additional task will be then to prove that $\mathscr{A}_{1/2,r}$ is also a *local attractor* for $\{T(t)\}$, i.e. $\mathscr{A}_{1/2,r}$ attracts some its open neighborhood in X_r^α. In particular, the results of [4] will be thus extended to cover the case of local attractors connected with systems for which global semiflow is known to exist merely on some neighborhood of zero.

2.4. Local attractor for $\{T(t)\}$ in $X_r^{1/2}$ ($r \in (n,\infty)$). It has been proved so far that for each $r \in (n,\infty), \omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta))$ is compact invariant and attracts $\mathscr{B}_{X_r^{1/2}}(0,\eta)$. Strengthening slightly the assumption (18) we shall prove in Theorem 3 that $\omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta)) \subset \mathscr{B}_{X_r^{1/2}}(0,\eta)$, i.e. $\omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta))$ is a local attractor for $\{T(t)\}$ in $X_r^{1/2}$. Recall that ([11]) a compact invariant set is a *local attractor* if it attracts some bounded neighborhood of itself.

Lemma 3. *Let $r \in (n,\infty)$ and $\{T(t)\}$ denotes a global semiflow on $V_r^{1/2}$ defined in Theorem* 1. *If $\|f\|_{\mathscr{L}^r(\Omega)}$-norm is restricted by* (18) *then, using notation of Lemma* 1,

$$\limsup_{t\to+\infty} \sup_{u_0 \in \mathscr{B}_{X_r^{1/2}}(0,\eta)} \|T(t)u_0\|_{X_r^{1/2}} \le \frac{1-\sqrt{1-4AB}}{2A}. \tag{23}$$

Proof. Starting as in the proof of Lemma 1 (see formulae (13)–(16)), for each $u_0 \in \mathscr{B}_{X_r^{1/2}}(0,\eta)$ we get:

(24)

$$\begin{aligned}
&\|u(t)\|_{X_r^{1/2}} \le e^{-\nu\lambda_1 t}\|u_0\|_{X_r^{1/2}} \\
&\quad + \left(\int_0^\tau + \int_\tau^t\right) C_{1/2,r}\frac{e^{-\nu\lambda_1(t-s)}}{(t-s)^{1/2}}\left(\frac{M_{1/2,r}}{\nu}\|u(s)\|^2_{X_r^{1/2}} + C_r\|f\|_{\mathscr{L}^r(\Omega)}\right) ds \\
&\le e^{-\nu\lambda_1 t}\|u_0\|_{X_r^{1/2}} + \left(\frac{M_{1/2,r}}{\nu}\sup_{s\in[0,\tau]}\|u(s)\|^2_{X_r^{1/2}} + C_r\|f\|_{\mathscr{L}^r(\Omega)}\right)\int_{t-\tau}^t C_{1/2,r}\frac{e^{-\nu\lambda_1 y}}{y^{1/2}}\,dy \\
&\quad + \left(\frac{M_{1/2,r}}{\nu}\left[\sup_{s\in[\tau,t]}\sup_{u_0\in\mathscr{B}_{X_r^{1/2}}(0,\eta)}\|u(s)\|_{X_r^{1/2}}\right]^2 + C_r\|f\|_{\mathscr{L}^r(\Omega)}\right)\int_0^{t-\tau} C_{1/2,r}\frac{e^{-\nu\lambda_1 y}}{y^{1/2}}\,dy.
\end{aligned}$$

Fixing arbitrarily $\varepsilon > 0$ it is possible to choose $\tau = \tau_\varepsilon$ such that:

(25) $$\sup_{s \in [\tau_\varepsilon, t]} \sup_{u_0 \in \mathscr{B}_{X_r^{1/2}}(0,\eta)} \|u(s)\|_{X_r^{1/2}} \leq \limsup_{t \to +\infty} \sup_{u_0 \in \mathscr{B}_{X_r^{1/2}}(0,\eta)} \|T(t)u_0\|_{X_r^{1/2}} + \varepsilon.$$

Inserting (25) into the right side of (24), using Lemma 1 and taking the supremum we obtain that

(26)
$$\sup_{u_0 \in \mathscr{B}_{X_r^{1/2}}(0,\eta)} \|u(t)\|_{X_r^{1/2}} \leq e^{-\nu\lambda_1 t}\eta + \left(\frac{M_{1/2,r}}{\nu} R^2 + C_r\|f\|_{\mathscr{L}^r(\omega)}\right) C_{1/2,r} \int_{t-\tau}^{t} \frac{e^{-\nu\lambda_1 y}}{y^{1/2}}\, dy$$
$$+ \left(\frac{M_{1/2,r}}{\nu}\left[\limsup_{t \to +\infty} \sup_{u_0 \in \mathscr{B}_{X_r^{1/2}}(0,\eta)} \|T(t)u_0\|_{X_r^{1/2}} + \varepsilon\right]^2 + C_r\|f\|_{\mathscr{L}^r(\Omega)}\right)$$
$$\times C_{1/2,r} \int_0^{\infty} \frac{e^{-\nu\lambda_1 y}}{y^{1/2}}\, dy.$$

In consequence, the quantity $r_\infty := \limsup_{t \to +\infty} \|u(t)\|_{X_r^{1/2}}$ needs to satisfy the quadratic inequality

(27) $$r_\infty \leq A[r_\infty + \varepsilon]^2 + B$$

and, since $\varepsilon > 0$ could be arbitrarily small, (23) follows. Lemma 3 is proved. □

From Lemma 3 our final conclusion follows.

THEOREM 3. *Let* $r \in (n, \infty)$ *and* $\{T(t)\}$ *denotes a global semiflow on* $V_r^{1/2}$ *defined in Theorem* 1. *If* $\|f\|_{\mathscr{L}^r(\Omega)}$*-norm satisfies stronger then* (18) *restriction*:

(28) $$\|f\|_{\mathscr{L}^r(\Omega)} < (2\sqrt{3} - 3)\frac{\nu^2\lambda_1}{4C_r M_{1/2,r} C_{1/2,r}^2 \Gamma^2(\frac{1}{2})},$$

then $\{T(t)\}$ *has a local attractor which coincides with the set* $\mathscr{A}_{1/2,r} = \omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta))$ *introduced in Theorem* 2.

PROOF. When (28) holds, then $\dfrac{1 - \sqrt{1 - 4AB}}{2A} < \dfrac{1}{4A} - B = \eta$ so that from Lemma 3

(29) $$\mathscr{A}_{1/2,r} = \omega_{X_r^{1/2}}(\mathscr{B}_{X_r^{1/2}}(0,\eta)) \subset cl_{X_r^{1/2}}\mathscr{B}_{X_r^{1/2}}\left(0, \frac{1 - \sqrt{1 - 4AB}}{2A}\right) \subset \mathscr{B}_{X_r^{1/2}}(0,\eta).$$

In Theorems 1, 2, the set $\mathscr{A}_{1/2,r}$ was shown to be a restricted global attractor for $\{T(t)\}$ in V_r^α. Therefore, (29) ensures that $\mathscr{A}_{1/2,r}$ attracts its open neighborhood and is also a local attractor for $\{T(t)\}$. The proof is completed. □

REMARK 2. We have shown in Theorem 1 that for sufficiently small $\|f\|_{\mathscr{L}^r(\Omega)}$ the Navier-Stokes system (1) generates a C^0-semigroup in the neighborhood of zero, and consequently, on its minimal, positively invariant, closed superset V_r^α. This semigroup has a global attractor $\mathscr{A}_{\alpha,r}$ which is independent of $\alpha \in [\frac{1}{2}, 1)$. Moreover, under the additional restriction (28), the n-D Navier-Stokes system has local attractor in a neighborhood of zero as shown in Theorem 3.

REMARK 3. It should be noted finally, that similar considerations remain true for sectorial problems in a Banach space X having the norm of nonlinear term $F(u)$ bounded by $\|u\|_{X^\alpha}^q$ with $q > 1$. In that case global in time solvability of the problem

$$u_t + Au = F(u), \quad t > 0, \quad u(0) = u_0,$$

for arbitrary initial data u_0 is usually excluded. Nevertheless, the global semigroup may be obtained in some neighborhood of zero and, moreover, it possesses strong stability properties which follow from the existence of the attractor.

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