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Bifurcation theory for semilinear elliptic boundary value problems

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ABSTRACT. This expository paper is devoted to static bifurcation theory for a class of *degenerate* boundary value problems for semilinear second-order elliptic differential operators stimulated by a problem of chemical kinetics. Our approach is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of partial differential equations.

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0. Introduction and results

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$ with smooth boundary ∂D; its closure $\overline{D} = D \cup \partial D$ is an N-dimensional, compact smooth manifold

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with boundary. We let

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} a^{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x)$$

be a second-order, *elliptic* differential operator with real smooth coefficients on $\bar{D}$ such that:

(1) $a^{ij}(x) = a^{ji}(x)$, $1 \le i, j \le N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \ge a_0|\xi|^2, \qquad x \in \bar{D}, \xi \in \mathbf{R}^N.$$

(2) $c(x) \ge 0$ on $\bar{D}$.

First we consider the following linear boundary value problem: For given functions g and φ defined in D and on ∂D, respectively, find a function u in D such that

$$\begin{cases} Au = g & \text{in } D, \\ Bu := a\dfrac{\partial u}{\partial \nu} + bu = \varphi & \text{on } \partial D. \end{cases} \tag{0.1}$$

Here:

(1) $a \in C^\infty(\partial D)$.

(2) $b \in C^\infty(\partial D)$.

(3) $\partial/\partial\nu$ is the conormal derivative associated with the operator A:

$$\frac{\partial}{\partial \nu} = \sum_{i=1}^{N} a^{ij} n_j \frac{\partial}{\partial x_i},$$

where $\mathbf{n} = (n_1, n_2, \dots, n_N)$ is the unit exterior normal to ∂D (see Figure 1 below).

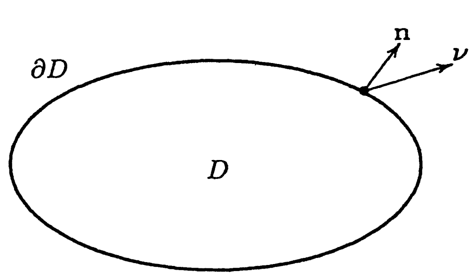

Figure 1

It is easy to see that problem (0.1) is nondegenerate (or coercive) if and only if either $a \neq 0$ on ∂D or $a \equiv 0$ and $b \neq 0$ on ∂D. In particular, if $a \equiv 1$ and $b \equiv 0$ on ∂D (resp. $a \equiv 0$ and $b \equiv 1$ on ∂D), then the boundary condition B is the so-called Neumann (resp. Dirichlet) condition.

In this paper we shall study problem (0.1) under the condition that $a \geq 0$ on ∂D. More precisely we make the following assumptions on the functions a, b and c:

(H.1) $a(x') \geq 0$ on ∂D.

(H.2) $b(x') \geq 0$ on ∂D and $b(x') > 0$ on $M = \{x' \in \partial D : a(x') = 0\}$.

(H.3) $c(x) > 0$ in D.

The probabilistic meaning of condition (H.2) is that a Markovian particle is definitely absorbed at the set M where no reflection phenomenon occurs; more precisely a Markovian particle does not stay on the boundary ∂D for any period of time until it "dies" at the time when it reaches the set M where the particle is definitely absorbed (cf. [23]). Condition (H.3) makes it possible to develop our machinery with a minimum of bother and the principal ideas can be presented concretely and explicitly.

We associate with problem (0.1) an unbounded linear operator $\mathfrak{A}$ from the Hilbert space $L^2(D)$ into itself as follows:

(a) The domain of definition $D(\mathfrak{A})$ is the space

$$D(\mathfrak{A}) = \{u \in W^{2,2}(D) : Bu = 0\}. \tag{0.2}$$

(b) $\mathfrak{A}u = Au,\ u \in D(\mathfrak{A})$.

The first purpose of this paper is to prove that the first eigenvalue of $\mathfrak{A}$ is simple with positive eigenfunction:

THEOREM 0. *If conditions* (H.1), (H.2) *and* (H.3) *are satisfied, then the operator* $\mathfrak{A}$ *is a nonnegative, selfadjoint operator. Moreover its first eigenvalue* λ_1 *is positive and* simple *and its corresponding eigenfunction* ψ_1 *is* positive *everywhere in* D:

$$\begin{cases} \mathfrak{A}\psi_1 = \lambda_1\psi_1 & \text{in } L^2(D), \\ \psi_1 > 0 & \text{in } D. \end{cases}$$

Secondly, as an application of Theorem 0, we study *local* static bifurcation problems for the following semilinear elliptic boundary value problem:

$$\begin{cases} Au - \lambda u + G(\lambda, u) = 0 & \text{in } D, \\ Bu = a\dfrac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D. \end{cases} \tag{0.3}$$

Here $G(\lambda, u)$ is a nonlinear operator, depending on a real parameter λ, which operates on the unknown function u. The word "bifurcation" means a "splitting", and in the context of nonlinear boundary value problems it connotes a situation wherein at some critical value of λ the number of solutions of the equation changes.

The second purpose of this paper is to discuss those aspects of local static bifurcation theory for problem (0.3) in the framework of Hölder spaces. A survey paper Amann [1] is a good reference to static bifurcation theory for *nondegenerate* boundary value problems for nonlinear second-order elliptic differential operators.

We introduce a closed subspace $C_B^{2+\theta}(\bar{D})$ of $C^{2+\theta}(\bar{D})$ defined by the formula

$$C_B^{2+\theta}(\bar{D}) = \{u \in C^{2+\theta}(\bar{D}) : Bu = 0 \text{ on } \partial D\},$$

and associate with problem (0.3) a nonlinear mapping $F(\lambda, u)$ of $\mathbf{R} \times C_B^{2+\theta}(\bar{D})$ into $C^{\theta}(\bar{D})$, $0 < \theta < 1$, as follows:

$$\begin{aligned} F : \mathbf{R} \times C_B^{2+\theta}(\bar{D}) &\longrightarrow C^{\theta}(\bar{D}) \\ (\lambda, u) &\longmapsto Au - \lambda u + G(\lambda, u). \end{aligned}$$

Suppose that there exists a curve Γ in the space $\mathbf{R} \times C_B^{2+\theta}(\bar{D})$ given by $\Gamma = \{w(t) : t \in I\}$, where I is an interval, such that $F(w) = 0$ for all $w \in \Gamma$. If there exists a number $\tau_0 \in I$ such that every neighborhood of $w(\tau_0)$ contains zeros of F not lying on Γ, then the point $w(\tau_0)$ is called a *bifurcation point* for the equation $F(w) = 0$ with respect to the curve Γ. In many situations the curve Γ is of the form $\{(\lambda, 0) : \lambda \in \mathbf{R}, 0 \in C_B^{2+\theta}(\bar{D})\}$. The basic problem of bifurcation theory is that of finding the bifurcation points for the equation $F(w) = 0$ with respect to Γ and studying the structure of the zeros of F near such points.

The next theorem asserts that the point $(\lambda_1, 0)$ is a bifurcation point for the equation $F(\lambda, u) = 0$:

THEOREM 1. *Let λ_1 be the first eigenvalue of $\mathfrak{A}$ with positive eigenfunction ψ_1, and let $G(\lambda, u)$ be a C^k map, $k \geq 3$, of a neighborhood of $(\lambda_1, 0)$ in $\mathbf{R} \times C_B^{2+\theta}(\bar{D})$ into $C^{\theta}(\bar{D})$. Assume that the following four conditions are satisfied*:

(i) $G(\lambda_1, 0) = 0$, $G_\lambda(\lambda_1, 0) = 0$.

(ii) $G_u(\lambda_1, 0) = 0$.

(iii) $\int_D G_{\lambda\lambda}(\lambda_1, 0) \cdot \psi_1 \, dx = 0$.

(iv) $\int_D (G_{\lambda u}(\lambda_1, 0)\psi_1 - \psi_1) \cdot \psi_1 \, dx \neq 0$.

Then the point $(\lambda_1, 0)$ is a bifurcation point for the equation $F(\lambda, u) = 0$. In fact, the set of solutions of $F(\lambda, u) = 0$ near $(\lambda_1, 0)$ consists of two C^{k-2} curves Γ_1 and Γ_2 intersecting only at the point $(\lambda_1, 0)$. Furthermore the curve Γ_1 is *tangent to the λ-axis at $(\lambda_1, 0)$ and may be parametrized by λ as*

$$\Gamma_1 = \{(\lambda, u_1(\lambda)) : |\lambda - \lambda_1| < \varepsilon\},$$

while the curve Γ_2 may be parametrized by a variable s as

$$\Gamma_2 = \{(\lambda_2(s), s\psi_1 + u_2(s)) : |s| < \varepsilon\}.$$

Here the functions $\lambda_2(s)$ and $u_2(s)$ satisfy the conditions

$$\lambda_2(0) = \lambda_1, \quad u_2(0) = \frac{du_2}{ds}(0) = 0.$$

We give two simple examples for Theorem 1 which deal with bifurcation theory under conditions on the *quadratic* term and on the *cubic* term, respectively:

Example 1. We let

$$F(\lambda, u) = Au - \lambda u + u^2.$$

Then the set of solutions of $F(\lambda, u) = 0$ near $(\lambda_1, 0)$ consists of two smooth curves Γ_1 and Γ_2 which may be parametrized respectively by λ and s as follows (see Figure 2 below):

$$\Gamma_1 = \{(\lambda, 0) : |\lambda - \lambda_1| < \varepsilon\},$$
$$\Gamma_2 = \{(\lambda_2(s), s\psi_1 + u_2(s)) : |s| < \varepsilon\}.$$

Here the function $\lambda_2(s)$ satisfies the conditions

$$\lambda_2(0) = \lambda_1, \quad \frac{d\lambda_2}{ds}(0) > 0.$$

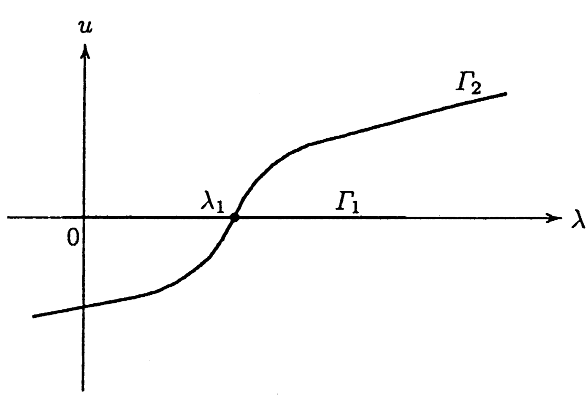

Figure 2

Example 2. We let

$$F(\lambda, u) = Au - \lambda u + u^3.$$

Then the set of solutions of $F(\lambda, u) = 0$ near $(\lambda_1, 0)$ consists of a *pitchfork*. More precisely the two smooth curves Γ_1 and Γ_2 may be parametrized respectively by λ and s as follows (see Figure 3 below):

$$\Gamma_1 = \{(\lambda, 0) : |\lambda - \lambda_1| < \varepsilon\},$$
$$\Gamma_2 = \{(\lambda_2(s), s\psi_1 + u_2(s)) : |s| < \varepsilon\}.$$

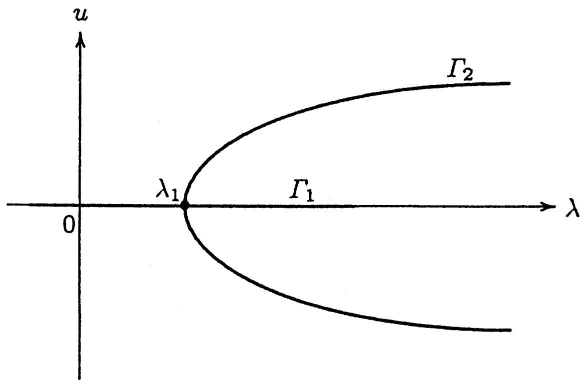

Figure 3

Here the function $\lambda_2(s)$ satisfies the conditions

$$\lambda_2(0) = \lambda_1, \quad \frac{d\lambda_2}{ds}(0) = 0, \quad \frac{d^2\lambda_2}{ds^2}(0) > 0.$$

Thirdly we consider the following general nonlinear elliptic boundary value problem: For given function $f(x, \xi)$ defined on $\bar{D} \times [0, \infty)$, find a nonnegative function $u(x)$ in D such that

$$\begin{cases} Au = f(x, u) & \text{in } D, \\ Bu = a\dfrac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D. \end{cases} \tag{0.4}$$

Problem (0.4) is the prototype of a class of nonlinear second-order elliptic boundary value problems which arise in numerous application in physical problems and in problems of Riemannian geometry.

A solution $u \in C^2(\bar{D})$ of problem (0.4) is said to be *nontrivial* if it does not identically equal zero on $\bar{D}$. A nontrivial solution u of problem (0.4) is said to be *positive* if $u(x) \geq 0$ on $\bar{D}$.

In order to state our existence theorem of positive solutions of problem (0.4), we introduce a fundamental condition (*slope condition*) on the nonlinear term $f(x, \xi)$:

For a positive number σ, there exists a constant $\omega = \omega(\sigma) > 0$, independent of $x \in \bar{D}$, such that

$$f(x, \xi) - f(x, \eta) > -\omega \cdot (\xi - \eta), \quad x \in \bar{D}, 0 \leq \eta < \xi \leq \sigma. \tag{R$_\sigma$}$$

Geometrically, this condition means that the slope of the function $f(x, \cdot)$ is bounded below, uniformly with respect to $x \in \bar{D}$.

A nonnegative function $\psi \in C^2(\bar{D})$ is called a *supersolution* of problem (0.4) if it satisfies the conditions

$$\begin{cases} A\psi - f(x, \psi) \geq 0 & \text{in } D, \\ B\psi \geq 0 & \text{on } \partial D. \end{cases}$$

Similarly a nonnegative function $\phi \in C^2(\bar{D})$ is called a *subsolution* of problem (0.4) if it satisfies the conditions

$$\begin{cases} A\phi - f(x, \phi) \leq 0 & \text{in } D, \\ B\phi \leq 0 & \text{on } \partial D. \end{cases}$$

The next theorem, which is a generalization of [1, Theorem 9.4] to the degenerate case, asserts that the existence of an ordered pair of sub- and supersolutions implies the existence of a solution of problem (0.4):

THEOREM 2. *Assume that $f(x, \xi)$ belongs to $C^\theta(\bar{D} \times [0, \sigma])$, $0 < \theta < 1$, and satisfies condition $(\mathrm{R})_\sigma$ for some $\sigma > 0$. If ψ and ϕ are respectively super- and subsolutions of problem (0.4) satisfying $0 \leq \phi(x) \leq \psi(x) \leq \sigma$ on $\bar{D}$, then there exists a solution $u \in C^{2+\theta}(\bar{D})$ of problem (0.4) such that $\phi(x) \leq u(x) \leq \psi(x)$ on $\bar{D}$.*

In order to formulate our uniqueness theorem of positive solutions of problem (0.4), we introduce another fundamental condition (*sublinearity*) on the nonlinear term $f(x, \xi)$:

We have for all $0 < \tau < 1$

$$f(x, \tau\xi) \geq \tau f(x, \xi), \quad x \in \bar{D}, \xi > 0, \tag{S1}$$

and

$$f(x, 0) \geq 0, \quad x \in \bar{D}. \tag{S2}$$

Then our uniqueness theorem for problem (0.4) is stated as follows:

THEOREM 3. *Assume that* $f(x,\xi)$ *belongs to* $C^\theta(\bar{D}\times[0,\sigma])$, $0<\theta<1$, *for every* $\sigma>0$, *and satisfies condition* $(\mathrm{R})_\sigma$ *for every* $\sigma>0$ *and also condition* (S). *Then problem* (0.4) *has at most one positive solution.*

As an application of Theorem 2, we can prove the following existence theorem for positive solutions of problem (0.4) (cf. [7, Theorem 2]):

THEOREM 4. *Assume that* $f(x,\xi)$ *belongs to* $C^\theta(\bar{D}\times[0,\sigma])$, $0<\theta<1$, *for every* $\sigma>0$, *and satisfies condition* $(\mathrm{R})_\sigma$ *for every* $\sigma>0$. *If in addition the two limits*

$$l_0(x)=\lim_{\xi\downarrow 0}\frac{f(x,\xi)}{\xi},\quad l_\infty(x)=\lim_{\xi\to\infty}\frac{f(x,\xi)}{\xi}$$

exist uniformly for all $x\in\bar{D}$ *and if we have*

$$l_\infty(x)<\lambda_1<l_0(x),\qquad x\in\bar{D},\tag{0.5}$$

then problem (0.4) *has a positive solution* $u\in C^{2+\theta}(\bar{D})$.

If the nonlinear term $f(x,\xi)$ is *independent of* x, then we can prove that condition (0.5) is necessary and sufficient for the existence of positive solutions of problem (0.4); more precisely we have the following generalization of [7, Theorem 1] to the degenerate case:

THEOREM 5. *Assume that* $f(x,\xi)=f(\xi)$ *belongs to* $C^\theta([0,\sigma])$, $0<\theta<1$, *for every* $\sigma>0$, *and satisfies condition* $(\mathrm{R})_\sigma$ *for every* $\sigma>0$, *and further that the function* $f(\xi)/\xi$ *is strictly decreasing for* $0<\xi<\infty$. *We let*

$$l_0(x)=\lim_{\xi\downarrow 0}\frac{f(\xi)}{\xi},\quad l_\infty(x)=\lim_{\xi\to\infty}\frac{f(\xi)}{\xi}.$$

Then problem (0.4) *has a positive solution* $u\in C^{2+\theta}(\bar{D})$ *if and only if we have*

$$l_\infty<\lambda_1<l_0.\tag{0.6}$$

Furthermore the solution u *is unique in the space* $C^2(\bar{D})$.

Now, as an application of Theorem 5, we study *global* static bifurcation problems for semilinear elliptic boundary value problems. We shall only restrict ourselves to some aspects which have been discussed in our papers with K. Umezu [26], [27] and [28].

We consider the following semilinear elliptic boundary value problem:

$$\begin{cases} Au-\lambda u+h(u)=0 & \text{in } D,\\ Bu=a\dfrac{\partial u}{\partial\nu}+bu=0 & \text{on } \partial D,\end{cases}\tag{0.7}$$

where the nonlinear term $h(\xi)$ is a function independent of x.

The next corollary is an immediate consequence of Theorem 5:

COROLLARY 1. *Assume that* $h(\xi) \in C^{\theta}([0,\sigma])$, $0 < \theta < 1$, *for every* $\sigma > 0$ *and that the function* $-h(\xi)$ *satisfies condition* $(\mathrm{R})_{\sigma}$ *for every* $\sigma > 0$, *and further that the function* $h(\xi)/\xi$ *is strictly increasing for* $0 < \xi < \infty$. *We let*

$$\alpha_0 = \lim_{\xi \downarrow 0} \frac{h(\xi)}{\xi}, \quad \alpha_\infty = \lim_{\xi \to \infty} \frac{h(\xi)}{\xi}.$$

Then problem (0.7) *has a unique positive solution* $u \in C^{2+\theta}(\bar{D})$ *if and only if* $\lambda_1 + \alpha_0 < \lambda < \lambda_1 + \alpha_\infty$.

It is worth while pointing out that the bifurcation solution curve (λ, u) of problem (0.7) is "formally" given by the formula

$$\lambda = \lambda_1 + \frac{h(u)}{u}. \tag{0.8}$$

Indeed, Theorem 0 tells us that the first eigenvalue λ_1 is the *unique* eigenvalue corresponding to a positive eigenfunction of the operator $\mathfrak{A}$. Hence, if we write problem (0.7) as

$$\begin{cases} \mathfrak{A}u = \lambda u - h(u) = \left(\lambda - \dfrac{h(u)}{u}\right)u, \\ u > 0 \qquad \text{in } D, \end{cases}$$

then we have $\lambda_1 = \lambda - h(u)/u$. This proves formula (0.8).

For Corollary 1, we give four simple examples of the function $h(\xi)$:

EXAMPLE 3 (The asymptotic linear case). If k is a positive number, we define a function $h(\xi)$ of class C^1 by the formula

$$h(\xi) = \begin{cases} \dfrac{k}{6}\xi^3 & \text{for } 0 \le \xi \le 1, \\ k\left(\xi + \dfrac{1}{2\xi} - \dfrac{4}{3}\right) & \text{for } \xi > 1. \end{cases}$$

Then we have $\alpha_0 = 0$, $\alpha_\infty = k$ and so $\lambda_1 < \lambda < \lambda_1 + k$. The situation may be represented schematically by Figure 4.

EXAMPLE 4 (The asymptotic nonlinear case). $h(\xi) = \xi^p$, $p > 1$. In this case we have $\alpha_0 = 0$, $\alpha_\infty = \infty$ and so $\lambda_1 < \lambda < \infty$. The situation may be represented schematically by Figure 5.

EXAMPLE 5. $h(\xi) = -\sqrt{\xi}$. In this case we have $\alpha_0 = -\infty$, $\alpha_\infty = 0$ and so $-\infty < \lambda < \lambda_1$. More precisely the point (λ_1, ∞) is a *bifurcation point from*

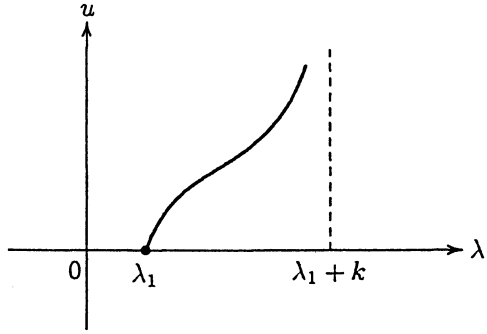

Figure 4

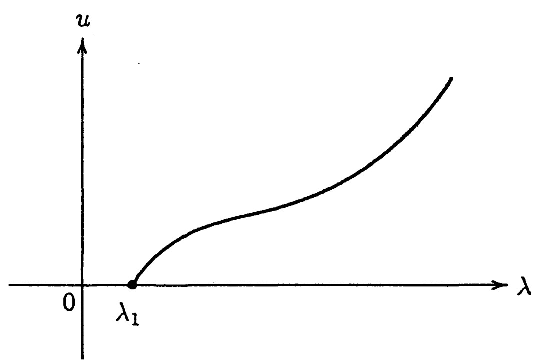

Figure 5

infinity for problem (0.7) in the sense of Amann [1, Section 19]. The situation may be represented schematically by Figure 6 below.

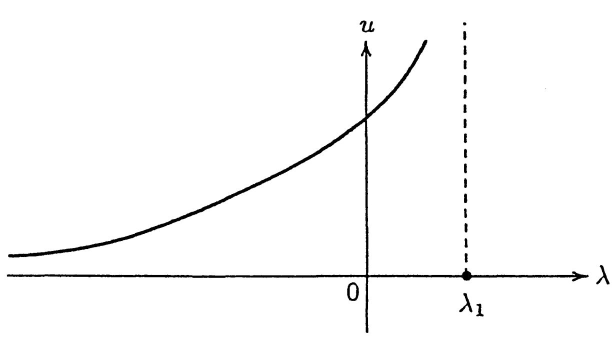

Figure 6

EXAMPLE 6. $h(\xi) = -\exp[-\xi]$. In this case we have $\alpha_0 = -\infty$, $\alpha_\infty = 0$ and so $-\infty < \lambda < \lambda_1$. The point (λ_1, ∞) is a bifurcation point from infinity for problem (0.7) (see Figure 6).

Finally we consider the following semilinear elliptic eigenvalue problem:

$$\begin{cases} Au - \lambda g(u) = 0 & \text{in } D, \\ Bu = a\dfrac{\partial u}{\partial \nu} + bu = 0 & \text{on } \partial D, \end{cases} \tag{0.9}$$

where the nonlinear term $g(\xi)$ is a function independent of x.

The next corollary, which is also an immediate consequence of Theorem 5, generalizes [20, Theorem 2.1] and [30, Theorem 2.6] to the degenerate case:

COROLLARY 2. *Assume that $g(\xi) \in C^\theta([0,\sigma])$, $0 < \theta < 1$, for every $\sigma > 0$ and that the function $g(\xi)$ satisfies condition $(\mathrm{R})_\sigma$ for every $\sigma > 0$, and further that the function $g(\xi)/\xi$ is strictly decreasing for $0 < \xi < \infty$. We let*

$$\beta_0 = \lim_{\xi \downarrow 0} \frac{g(\xi)}{\xi}, \quad \beta_\infty = \lim_{\xi \to \infty} \frac{g(\xi)}{\xi}.$$

Then problem (0.9) has a unique positive solution $u \in C^{2+\theta}(\bar{D})$ if and only if $\lambda_1/\beta_0 < \lambda < \lambda_1/\beta_\infty$.

Roughly speaking, the bifurcation solution curve (λ, u) of problem (0.9) is given by the formula

$$\lambda = \frac{u}{g(u)}\lambda_1.$$

This can be shown by an argument similar to that for formula (0.8).

For Corollary 2, we give two important examples of the function $g(\xi)$ stimulated by a problem of chemical kinetics (cf. [2], [5]):

EXAMPLE 7 (The simple Arrhenius rate law). $g(\xi) = \exp[\xi/(1 + \varepsilon\xi)]$ with $\varepsilon > 0$. The function $g(\xi)$ describes the temperature dependence of reaction rate for exothermic reactions obeying the simple Arrhenius rate law in circumstances in which heat flow is purely conductive. In this context ε is a dimensionless ambient temperature and λ is a dimensionless heat evolution rate. The equation $Au - \lambda g(u) = 0$ represents heat balance with reactant consumption ignored, where u is a dimensionless temperature excess, and the boundary condition $Bu = 0$ represents the exchange of heat at the surface of the reactant by Newtonian cooling. It is easy to verify that the function $g(\xi)/\xi$ is strictly decreasing for all $\xi > 0$ if $\varepsilon > 1/4 = 0.25$. If this is the case, we have $\beta_0 = \infty$, $\beta_\infty = 0$ and so $0 < \lambda < \infty$. The situation may be represented schematically by Figure 7 below. Roughly speaking, if the ambient temperature is so high (or the activation energy is so low) that the parameter ε exceeds the value 0.25, then only a smooth progression of reaction rate with imposed ambient temperature can occur; such a reaction may be very rapid but it is only accelerating and lacks the discontinuous change associated with criticality and ignition (cf. [5, Table 1]). This result is a generalization of [21, Example] to the degenerate case (cf. [8], [10]).

EXAMPLE 8 (The bimolecular rate law). $g(\xi) = \sqrt{1+\varepsilon\xi}\exp[\xi/(1+\varepsilon\xi)]$ with $\varepsilon > 0$. The function $g(\xi)$ describes the temperature dependence of reaction rate obeying the bimolecular rate law. It is easy to verify that the function $g(\xi)/\xi$ is strictly decreasing for all $\xi > 0$ if $6-4\sqrt{2} < \varepsilon < 6+4\sqrt{2}$. If this is the case, we have $\beta_0 = \infty$, $\beta_\infty = 0$ and so $0 < \lambda < \infty$ (see Figure 7 below). Rephrased, if the ambient temperature parameter ε exceeds the value $6-4\sqrt{2} \doteq 0.343146$, then only a smooth progression of reaction rate with imposed ambient temperature can occur (cf. [5, Table 1]).

By Examples 7 and 8, we find that ignition phenomena are normally shown by strongly exothermic reactions in which heat is transferred only by conduction but such criticality is lost in systems of high ambient temperature or low activation energy. The hydrogen-atom torch and the sodium-halogen flame exemplify such systems.

The rest of this paper is organized as follows.

In the first section, Section 1, we present a brief description of the theory of positive mappings in ordered Banach spaces ([1], [15]) and local static bifurcation theory from a simple eigenvalue ([9], [11], [16]) which form a function analytic background in the proof of main results. The material in this section is given for completeness, to minimize the necessity of consulting many references.

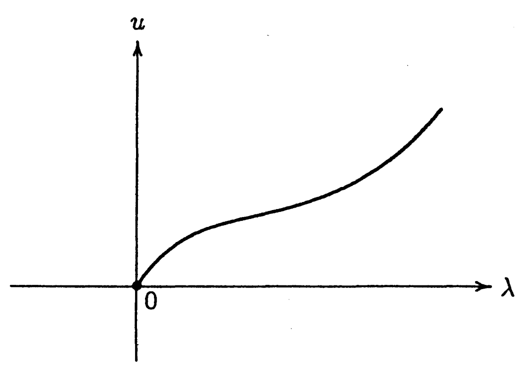

Figure 7

A general class of semilinear second-order elliptic boundary value problems satisfies the maximum principle. Roughly speaking, this additional information means that the operators associated with the boundary value problems are compatible with the natural ordering of the underlying function spaces. Consequently we are led to the study of nonlinear equations in the framework of ordered Banach spaces. In particular we give a sharper version of the famous Kreĭn-Rutman theorem for strongly positive, compact linear operators (Theorem 1.1).

Section 2 is devoted to the proof of Theorem 0. There is a standard method of reducing problem (0.1) to an equivalent integral equation on the boundary in an appropriate function space. More precisely, by using the Green and Poisson operators for problem (0.1) we transform problem (0.1) to the study of a pseudo-differential operator T on the boundary (Proposition 2.2), which may be considered as a generalization of the classical potential approach. The main difficulty in this approach lies in the fact that we have to establish *a priori* estimates for problem (0.1). To do so, we use the L^p theory of pseudo-differential operators to prove that conditions (H.1) and (H.2) are sufficient for the existence of a parametrix for the operator T (Lemma 2.3). Next the maximum principle, which stems from a second-order equation, gives us various *a priori* information about the possible solutions of problem (0.1). In this way we can prove an existence and uniqueness theorem for problem (0.1) in the framework of Hölder spaces (Theorem 2.1).

Furthermore the maximum principle tells us that the resolvent K of problem (0.1) is a positive operator in the ordered Banach space $C(\bar{D})$ (Proposition 2.8). In order to obtain an abstract formulation of this fact, we introduce an ordered Banach subspace $C_e(\bar{D})$ of $C(\bar{D})$ which combines the good properties of the resolvent K with the good properties of the natural ordering of $C(\bar{D})$. Here the function e is the unique solution of the linear boundary value problem

$$\begin{cases} Ae = 1 & \text{in } D, \\ Be = 0 & \text{on } \partial D, \end{cases}$$

and the ordered Banach space $C_e(\bar{D})$ is defined by the formula

$$C_e(\bar{D}) = \{u \in C(\bar{D}) : \text{there exists a constant } c > 0 \text{ such that } -ce \leq u \leq ce\},$$

with norm

$$\|u\|_e = \inf\{c > 0 : -ce \leq u \leq ce\}.$$

This setting has the advantages that it takes into consideration in an optimal way the *a priori* information given by the maximum principle and that it is amenable to the methods of abstract functional analysis. We recall that Taira

[25] proved Theorem 0 by using the theory of Feller semigroups in functional analysis. In this paper we give a direct proof of Theorem 0 by making use of the theory of positive mappings in ordered Banach spaces (Theorem 2.9).

In Section 3 we prove Theorem 1. Theorem 1 follows by applying local static bifurcation theory from a simple eigenvalue (Theorem 1.2).

Section 4 is devoted to the proof of Theorems 2 and 3. We transpose the nonlinear problem (0.4) into an equivalent fixed point equation for the resolvent K in an appropriate ordered Banach space. More precisely, by applying the resolvent K for problem (0.1) we transform problem (0.4) into a nonlinear operator equation in the ordered Banach space $C(\bar{D})$

$$u = K(F(u)) = K(f(\cdot, u(\cdot))) \tag{0.10}$$

in such a way that as much information as possible is carried over to the abstract setting. By condition $(\mathrm{R})_\sigma$, it follows that the map H, defined by $H(u) = K(F(u))$, leaves invariant the ordering of the space $C(\bar{D})$ (Lemma 4.1). In the case of an increasing map it suffices to verify that H maps *two* points of a bounded, closed and convex set into itself in order to apply Schauder's fixed point theorem (Lemma 4.2). This is a much easier task than to verify the standard hypotheses for an application of the same theorem.

The fact that the resolvent K is strongly positive has important consequences. Namely, if $u \geq v$ and $u \not\equiv v$ on $\bar{D}$, then the function $H(u) - H(v)$ is an interior point of the positive cone P_e of the ordered Banach space $C_e(\bar{D})$. This implies that the map H is a strongly increasing self-map of $C_e(\bar{D})$ (Lemma 4.6). The proof of Theorem 3 is based on a uniqueness theorem of fixed points of strongly increasing and strongly sublinear mappings in ordered Banach spaces (Theorem 4.4).

Section 5 is devoted to the proof of Theorem 4. Theorem 4 follows from a straightforward application of Theorem 2 if we construct explicitly super- and subsolutions of problem (0.4). First, by using the positive eigenfunction ψ_1 of problem (0.1) we have a subsolution $\phi_\varepsilon = \varepsilon\psi_1$ for $\varepsilon > 0$ sufficiently small. On the other hand, in order to construct a supersolution of problem (0.4) we make good use of the positivity lemma (Lemma 5.1) and the existence and uniqueness theorem for problem (0.1) (Theorem 2.1).

In the final section, Section 6, by applying Green's formula we show that condition (0.6) is necessary for the existence of positive solutions of problem (0.4). This proves Theorem 5.

1. Functional analytic preliminaries

In this section we present a brief description of basic definitions and results about the theory of positive mappings in ordered Banach spaces ([1], [15]) and local static bifurcation theory from a simple eigenvalue ([9], [11], [16]) which form a functional analytic background in the sequel.

1.1. Theory of positive mappings in ordered Banach spaces. Let X be a nonempty set. An ordering $\leq$ in X is a relation in X which is reflexive, transitive and antisymmetric. A nonempty set together with an ordering is called an ordered set.

Let V be a real vector space. An ordering $\leq$ in V is said to be *linear* if the following two conditions are satisfied:

(i) If $x, y \in V$ and $x \leq y$, then we have $x + z \leq y + z$ for all $z \in V$.

(ii) If $x, y \in V$ and $x \leq y$, then we have $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

A real vector space together with a linear ordering is called an *ordered vector space*.

If $x, y \in V$ and $x \leq y$, then the set $[x, y] = \{z \in X : x \leq z \leq y\}$ is called an *order interval.*

If we let

$$Q = \{x \in V : x \geq 0\},$$

then it is easy to verify that the set Q has the following two conditions:

(iii) If $x, y \in Q$, then $\alpha x + \beta y \in Q$ for all $\alpha, \beta \geq 0$.

(iv) If $x \neq 0$, then at least one of x and $-x$ does not belong to Q.

The set Q is called the *positive cone* of the ordering $\leq$.

Let E be a Banach space E with a linear ordering $\leq$. The Banach space E is called an *ordered Banach space* if the positive cone Q is closed in E. We say that Q is *generating* if, for each $x \in E$ there exist vectors $u, v \in Q$ such that $x = u - v$. It is to be expected that the topology and the ordering of an ordered Banach space are closely related if the norm is *monotone*: If $0 \leq u \leq v$, then $\|u\| \leq \|v\|$.

A linear operator $L : E \to E$ is said to be *strongly positive* if Lx is an interior point of Q for every $x \in Q \backslash \{0\}$.

The next theorem, which is a sharper version of the famous Kreĭn-Rutman theorem for strongly positive compact linear operators, will play a fundamental role in the sequel (cf. [15, Chapter 2]):

THEOREM 1.1. *Let E be an ordered Banach space and $L : E \to E$ a linear operator. Assume that L is strongly positive and compact and that all the eigenvalues of L are positive. Then we have the following*:

(a) *The operator L has a* positive *eigenfunction $x_0 \in Q$ with eigenvalue $\lambda_0 > 0 : Lx_0 = \lambda_0 x_0$.*

(b) *An eigenvalue corresponding to a positive eigenfunction of L is* simple.

(c) *The operator L has a* unique *positive eigenfunction.*

(d) *An eigenvalue corresponding to a positive eigenfunction is* greater *than the remaining eigenvalues of L.*

1.2. Local bifurcation theory. This subsection is devoted to local static bifurcation theory for problem (0.3). By making use of bifurcation theory from a simple eigenvalue essentially due to Crandall-Rabinowitz [11], we discuss the changes that occur in the structure of the solutions of $F(\lambda, u) = 0$ as λ varies near the first eigenvalue λ_1 of the operator $\mathfrak{A}$. For detailed studies of bifurcation theory, the reader is referred to Chow-Hale [9] and Nirenberg [16].

1.2A. Differentiability. Let X, Y be Banach spaces, U an open set in X and $f : U \to Y$ a map. We say that the map f is (Fréchet) *differentiable at a point* $x \in U$ if there exist a continuous linear operator $A : X \to Y$ and a map ψ defined for all sufficiently small h in X, with values in Y, such that

$$\begin{cases} f(x+h) = f(x) + Ah + \|h\|\psi(h), \\ \lim_{h\to 0} \psi(h) = 0. \end{cases}$$

We remark that the continuous linear operator A is uniquely determined by f and x. The operator A is called the (Fréchet) *derivative* of f at x, and is denoted by $Df(x)$ or $f'(x)$. A map f is said to be (Fréchet) *differentiable on* U if it is (Fréchet) differentiable at every point of U. In this case the derivative Df is a map of U into the Banach space $B(X, Y)$ of all continuous (bounded) linear operators:

$$\begin{aligned} Df : U &\longrightarrow B(X, Y) \\ u &\longmapsto Df(u). \end{aligned}$$

If in addition Df is continuous from U into $B(X, Y)$, we say that f is *of class* C^1.

If the derivative Df is differentiable at a point $x \in U$ (resp. in U), we say that f is *twice differentiable* at x (resp. in U). The derivative of Df at x is called the *second derivative* of f at x, and is denoted by $D^2f(x)$. This is an element of the Banach space $B(X, B(X, Y))$ which can be naturally identified with the space $B_2(X, Y) = B(X, X; Y)$ of all continuous bilinear mappings of $X \times X$ into Y.

By induction on k, we define a k times differentiable mapping f of U into Y as a $(k-1)$ times differentiable mapping whose $(k-1)$-th derivative $D^{k-1}f$ is differentiable in U. The derivative $D^kf = D(D^{k-1}f)$ is called the *k-th derivative* of f. The derivative $D^kf(x)$ at a point $x \in U$ can be identified with an element of the space $B_k(X, Y)$ of all continuous k-linear mappings of $X \times X \times \cdots \times X$ into Y. A map $f : U \to Y$ is said to be *of class* C^r $(r \geq 2)$ in U if all the derivatives D^kf exist and are continuous in U for $1 \leq k \leq r$.

Here it is worth while pointing out that if $X = \mathbf{R}$, then the space $B(X, Y)$ can be identified with the space Y; so the space $B_k(\mathbf{R}, Y)$ can be identified with the space Y for general $k \geq 2$.

Now we assume that the Banach space X is the product space of two Banach spaces X_1 and X_2:

$$X = X_1 \times X_2.$$

For each point $x = (x_1, x_2) \in U \subset X$, one can consider the partial mappings

$$F_1 : u_1 \mapsto f(u_1, x_2),$$
$$F_2 : u_2 \mapsto f(x_1, u_2)$$

of open subsets of X_1 and X_2 respectively into Y. We say that f is *differentiable with respect to the first* (resp. *second*) *variable* if the mapping $F_1(u_1)$ (resp. $F_2(u_2)$) is differentiable at x_1 (resp. at x_2). The derivative $DF_1(x_1)$ (resp. $DF_2(x_2)$) is an element of the Banach space $B(X_1, Y)$ (resp. $B(X_2, Y)$), and is called the *partial* (*Fréchet*) *derivative* of f at (x_1, x_2) with respect to the first (resp. second) variable. We write

$$D_{x_1} f(x_1, x_2) = f_{x_1}(x_1, x_2) = DF_1(x_1),$$
$$D_{x_2} f(x_1, x_2) = f_{x_2}(x_1, x_2) = DF_2(x_2).$$

One can define inductively the partial (Fréchet) derivatives $D^j_{x_1} D^k_{x_2} f$ for general j and k.

1.2B. Bifurcation from a simple eigenvalue. In this subsection we study the equation of the form

$$F(t, x) = 0,$$

where $F(\cdot, t)$ depends on a real parameter t. In other words, $F(t, x)$ is a nonlinear operator, depending on the parameter t, which operates on the unknown vector x. One of the first questions to be answered is whether or not the equation $F(t, x) = 0$ has any solution x for a given value of t. If it does, the question of how many solutions it has arises, and then how this number varies with t. Of particular interest is the process of bifurcation whereby a given solution of $F(t, x) = 0$ splits into two or more solutions as t passes through some critical value.

Let $F(t, x)$ be a mapping of a Banach space $\mathbf{R} \times X$ into a Banach space Y. Suppose that there exists a curve Γ in the space $\mathbf{R} \times X$ given by $\Gamma = \{w(t) : t \in I\}$, where I is an interval, such that $F(w) = 0$ for all $w \in \Gamma$. If there exists a number $\tau_0 \in I$ such that every neighborhood of $w(\tau_0)$ contains zeros of F not lying on Γ, then the point $w(\tau_0)$ is called a *bifurcation point* for the equation $F(w) = 0$ with respect to the curve Γ.

The next theorem gives sufficient conditions in order that a point $(t_0, 0)$ be a bifurcation point for the equation $F(t, x) = 0$ (see [11, Theorem 1.7], [16, Theorem 3.2.2], [9, Chapter 6, Theorem 6.1]):

THEOREM 1.2 (The bifurcation theorem). *Let $F(t,x)$ be a C^k map, $k \geq 3$, of a neighborhood of $(t_0, 0)$ in a Banach space $\mathbf{R} \times X$ into a Banach space Y such that*

$$F(t_0, 0) = 0.$$

Assume that the following four conditions are satisfied:

(i) $F_t(t_0, 0) = 0$.

(ii) *The null space $N(F_x(t_0, 0))$ is one dimensional, spanned by a vector x_0.*

(iii) *The range $R(F_x(t_0, 0))$ has codimension one in the space Y.*

(iv) $F_{tt}(t_0, 0) \in R(F_x(t_0, 0))$ *and* $F_{tx}(t_0, 0)x_0 \notin R(F_x(t_0, 0))$.

Then the point $(t_0, 0)$ is a bifurcation point for the equation $F(t,x) = 0$. In fact, the set of solutions of $F(t,x) = 0$ near $(t_0, 0)$ consists of two C^{k-2} curves Γ_1 and Γ_2 intersecting only at the point $(t_0, 0)$. Furthermore the curve Γ_1 is tangent to the t-axis at $(t_0, 0)$ and may be parametrized by t as

$$\Gamma_1 = \{(t, x_1(t)) : |t - t_0| < \varepsilon\},$$

while the curve Γ_2 may be parametrized by a variable s as

$$\Gamma_2 = \{(t_2(s), sx_0 + x_2(s)) : |s| < \varepsilon\}.$$

Here the functions $t_2(s)$ and $x_2(s)$ satisfy the conditions

$$t_2(0) = t_0, \quad x_2(0) = \frac{dx_2}{ds}(0) = 0.$$

The conditions in Theorem 1.2 are based on the linear approximation, and are independent of the nonlinearities. The following two corollaries analyze in detail the nonlinear nature of the problem; it is essential to know some properties of the nonlinearities in x in the map $F(t,x)$.

The first corollary deals with local bifurcation theory under generic conditions on the *quadratic* term (see [9, Chapter 6, Corollary 6.2]):

COROLLARY 1.3. *Let $F(t,x)$ be a C^k map, $k \geq 3$, of a neighborhood of $(t_0, 0)$ in a Banach space $\mathbf{R} \times X$ into a Banach space Y. Assume that the following five conditions are satisfied:*

(i) $F(t, 0) = 0$ *for all* $|t - t_0|$ *sufficiently small.*

(ii) *The null space $N(F_x(t_0, 0))$ is one-dimensional, spanned by a vector x_0.*

(iii) *The range $R(F_x(t_0, 0))$ has codimension one in the space Y.*

(iv) $F_{tx}(t_0, 0)x_0 \notin R(F_x(t_0, 0))$.

(v) $F_{xx}(t_0, 0)(x_0, x_0) \notin R(F_x(t_0, 0))$.

Then the set of solutions of $F(t,x) = 0$ near $(t_0, 0)$ consists of two C^{k-2} curves Γ_1 and Γ_2 which may be parametrized respectively by t and s as *follows*:

$$\Gamma_1 = \{(t,0) : |t - t_0| < \varepsilon\},$$

$$\Gamma_2 = \{(t_2(s), sx_0 + x_2(s)) : |s| < \varepsilon\}.$$

Here the functions $t_2(s)$ *and* $x_2(s)$ *satisfy the conditions*

$$t_2(0) = t_0, \quad \frac{dt_2}{ds}(0) \neq 0,$$

$$x_2(0) = \frac{dx_2}{ds}(0) = 0.$$

The second corollary deals with local bifurcation theory under generic conditions on the *cubic* term (see [9, Chapter 6, Corollary 6.4]):

COROLLARY 1.4. *Let* $F(t,x)$ *be a* C^k *map*, $k \geq 3$, *of a neighborhood of* $(t_0, 0)$ *in a Banach space* $\mathbf{R} \times X$ *into a Banach space* Y. *Assume that the following six conditions are satisfied*:

(i) $F(t,0) = 0$ *for all* $|t - t_0|$ *sufficiently small.*
(ii) *The null space* $N(F_x(t_0,0))$ *is one-dimensional, spanned by a vector* x_0.
(iii) *The range* $R(F_x(t_0,0))$ *has codimension one in the space* Y.
(iv) $F_{tx}(t_0,0)x_0 \notin R(F_x(t_0,0))$.
(v) $F_{xx}(t_0,0)(x_0,x_0) \in R(F_x(t_0,0))$.
(vi) $F_{xxx}(t_0,0)(x_0,x_0,x_0) \notin R(F_x(t_0,0))$.

Then the set of solutions of $F(t,x) = 0$ *near* $(t_0, 0)$ *consists of a* pitchfork. *More precisely the two* C^{k-2} *curves* Γ_1 *and* Γ_2 *may be parametrized respectively by* t *and* s *as follows*:

$$\Gamma_1 = \{(t,0) : |t - t_0| < \varepsilon\},$$

$$\Gamma_2 = \{(t_2(s), sx_0 + x_2(s)) : |s| < \varepsilon\}.$$

Here the functions $t_2(s)$ *and* $x_2(s)$ *satisfy the conditions*

$$t_2(0) = t_0, \quad \frac{dt_2}{ds}(0) = 0, \quad \frac{d^2t_2}{ds^2}(0) \neq 0,$$

$$x_2(0) = \frac{dx_2}{ds}(0) = 0.$$

2. Proof of Theorem 0

In this section we give a simple and direct proof of Theorem 0 by making use of the theory of positive mappings in ordered Banach spaces (Theorem

2.9). The proof of Theorem 0 is divided into three subsections, Subsections 2.1, 2.2 and 2.3. In Subsection 2.1 we prove an existence and uniqueness theorem for problem (0.1) in the framework of Hölder spaces (Theorem 2.1). By using this theorem, we prove the selfadjointness of $\mathfrak{A}$ in Subsection 2.2 (Theorem 2.6) and the positivity of the resolvent associated with problem (0.1) in Subsection 2.3 (Proposition 2.8), respectively.

2.1. Existence and uniqueness theorem for problem (0.1). In this subsection we prove an existence and uniqueness theorem for problem (0.1) in the framework of Hölder spaces which will play an important role in the proof of Theorem 0.

First we introduce a subspace of the Hölder space $C^{1+\theta}(\partial D)$, $0 < \theta < 1$, which is associated with the boundary condition

$$Bu = a\frac{\partial u}{\partial \nu} + bu$$

in the following way: We let

$$C_*^{1+\theta}(\partial D = \{\varphi = a\varphi_1 + b\varphi_2 : \varphi_1 \in C^{1+\theta}(\partial D), \varphi_2 \in C^{2+\theta}(\partial D)\},$$

and define a norm

$$|\varphi|_{C_*^{1+\theta}(\partial D)} = \inf\{|\varphi_1|_{C^{1+\theta}(\partial D)} + |\varphi_2|_{C^{2+\theta}(\partial D)} : \varphi = a\varphi_1 + b\varphi_2\}.$$

Then it is easy to verify that the space $C_*^{1+\theta}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{C_*^{1+\theta}(\partial D)}$. We remark that the space $C_*^{1+\theta}(\partial D)$ is an "interpolation space" between the spaces $C^{2+\theta}(\partial D)$ and $C^{1+\theta}(\partial D)$.

The purpose of this subsection is to prove the following:

THEOREM 2.1. *If conditions* (H.1), (H.2) *and* (H.3) *are satisfied, then the mapping*

$$(A, B) : C^{2+\theta}(\bar{D}) \longrightarrow C^{\theta}(\bar{D}) \oplus C_*^{1+\theta}(\partial D)$$

is an algebraic and topological isomorphism *for all* $0 < \theta < 1$.

PROOF. The proof is divided into four steps.

(i) Let g be an arbitrary element of $C^{\theta}(\bar{D})$, and φ an arbitrary element of $C_*^{1+\theta}(\partial D)$ such that

$$\varphi = a\varphi_1 + b\varphi_2, \quad \varphi_1 \in C^{1+\theta}(\partial D), \quad \varphi_2 \in C^{2+\theta}(\partial D).$$

First we show that the boundary value problem

$$\begin{cases} Au = g & \text{in } D, \\ Bu = \varphi & \text{on } \partial D \end{cases} \tag{0.1}$$

can be reduced to the study of an operator on the boundary.

To do so, we consider the Neumann problem

$$\begin{cases} Av = g & \text{in } D, \\ \dfrac{\partial v}{\partial \nu} = \varphi_1 & \text{on } \partial D. \end{cases} \tag{2.1}$$

By [13, Theorem 6.31], one can find a unique solution v in the space $C^{2+\theta}(\bar{D})$ of problem (2.1). Then it is easy to see that a function $u \in C^{2+\theta}(\bar{D})$ is a solution of problem (0.1) if and only if the function $w = u - v \in C^{2+\theta}(\bar{D})$ is a solution of the problem

$$\begin{cases} Aw = 0 & \text{in } D, \\ Bw = \varphi - Bv & \text{on } \partial D. \end{cases}$$

Here we remark that

$$Bv = a\frac{\partial v}{\partial \nu} + bv = a\varphi_1 + bv \quad \text{on } \partial D,$$

so that

$$Bw = \varphi - Bv = b(\varphi_2 - v) \in C^{2+\theta}(\partial D).$$

However we know that every solution $w \in C^{2+\theta}(\bar{D})$ of the homogeneous equation $Aw = 0$ can be expressed by means of a single layer potential in the following form (cf. [23, Theorem 8.2.4]):

$$w = \mathscr{P}\psi, \quad \psi \in C^{2+\theta}(\partial D),$$

where the operator $\mathscr{P} : C^{2+\theta}(\partial D) \longrightarrow C^{2+\theta}(\bar{D})$ is the Poisson operator, that is, the function $w = \mathscr{P}\psi$ is the unique solution of the Dirichlet problem

$$\begin{cases} Aw = 0 & \text{in } D, \\ w = \psi & \text{on } \partial D. \end{cases}$$

Thus we have the following:

PROPOSITION 2.2. *For given functions $g \in C^{\theta}(\bar{D})$ and $\varphi = a\varphi_1 + b\varphi_2 \in C_*^{1+\theta}(\partial D)$, there exists a solution $u \in C^{2+\theta}(\bar{D})$ of problem* (0.1) *if and only if there exists a solution $\psi \in C^{2+\theta}(\partial D)$ of the equation*

$$T\psi := B\mathscr{P}\psi = b(\varphi_2 - v) \qquad \text{on } \partial D. \tag{2.2}$$

Furthermore the solutions u and ψ are related as follows:

$$u = v + \mathscr{P}\psi,$$

where $v \in C^{2+\theta}(\bar{D})$ is the unique solution of problem (2.1).

(ii) We study the operator T in question. It is known (see [14, Chapter XX], [18, Chapter 3]) that the operator

$$T\psi = B\mathscr{P}\psi = a\frac{\partial}{\partial\nu}(\mathscr{P}\psi)\bigg|_{\partial D} + b\psi$$

is a first-order, *pseudo-differential operator* on the boundary ∂D.

The next lemma is an essential step in the proof of Theorem 2.1:

LEMMA 2.3. *If conditions* (H.1) *and* (H.2) *are satisfied, then there exists a* parametrix E *in the Hörmander class* $L^0_{1,1/2}(\partial D)$ *for* T *which maps* $C^{k+\theta}(\partial D)$ *continuously into itself for any nonnegative integer* k.

PROOF. By making use of [14, Theorem 22.1.3] just as in the proof of [24, Lemma 5.2], one can construct a parametrix $E \in L^0_{1,1/2}(\partial D)$ for T:

$$ET \equiv TE \equiv I \mod L^{-\infty}(\partial D).$$

The boundendness of $E : C^{k+\theta}(\partial D) \to C^{k+\theta}(\partial D)$ follows from an application of a Besov-space boundedness theorem [6, Theorem 1], since we have $C^{k+\theta}(\partial D) = B^{k+\theta}_{\infty,\infty}(\partial D)$. $\square$

(iii) We consider problem (0.1) in the framework of Sobolev spaces of L^p style, and prove an L^p version of Theorem 2.1.

If k is a positive integer and $1 < p < \infty$, we define the Sobolev space

$W^{k,p}(D)$ = the space of functions $u \in L^p(D)$ whose derivatives $D^\alpha u$, $|\alpha| \le k$, in the sense of distributions are in $L^p(D)$,

and let

$B^{k-1/p,p}(\partial D)$ = the space of the boundary values φ of functions $u \in W^{k,p}(D)$.

In the space $B^{k-1/p,p}(\partial D)$ we introduce a norm

$$|\varphi|_{B^{k-1/p,p}(\partial D)} = \inf\{\|u\|_{W^{k,p}(D)} : u \in W^{k,p}(D), u|_{\partial D} = \varphi\}.$$

The space $B^{k-1/p,p}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{B^{k-1/p,p}(\partial D)}$; more precisely it is a Besov space (see [4], [29]).

We introduce a subspace of $B^{1-1/p,p}(\partial D)$ which is an L^p version of $C^{1+\theta}_*(\partial D)$. We let

$$B^{1-1/p,p}_*(\partial D) = \{\varphi = a\varphi_1 + b\varphi_2 : \varphi_1 \in B^{1-1/p,p}(\partial D), \varphi_2 \in B^{2-1/p,p}(\partial D)\},$$

and define a norm

$$|\varphi|_{B_*^{1-1/p,p}(\partial D)} = \inf\{|\varphi_1|_{B^{1-1/p,p}(\partial D)} + |\varphi_2|_{B^{2-1/p,p}(\partial D)} : \varphi = a\varphi_1 + b\varphi_2\}.$$

It is easy to verify that the space $B_*^{1-1/p,p}(\partial D)$ is a Banach space with respect to the norm $|\cdot|_{B_*^{1-1/p,p}(\partial D)}$.

Then we can obtain the following L^p version of Theorem 2.1 (see [24, Theorem 1]):

THEOREM 2.4. *If conditions* (H.1), (H.2) *and* (H.3) *are satisfied, then the mapping*

$$(A, B) : W^{2,p}(D) \longrightarrow L^p(D) \oplus B_*^{1-1/p,p}(\partial D)$$

is an algebraic and topological isomorphism *for all* $1 < p < \infty$.

(iv) Now we remark that

$$\begin{cases} C^\theta(\bar{D}) \subset L^p(D), \\ C_*^{1+\theta}(\partial D) \subset B_*^{1-1/p,p}(\partial D). \end{cases}$$

Thus we find from Theorem 2.4 that problem (0.1) has a unique solution $u \in W^{2,p}(D)$ for any $g \in C^\theta(\bar{D})$ and any $\varphi = a\varphi_1 + b\varphi_2 \in C_*^{1+\theta}(\partial D)$. Furthermore, by virtue of Proposition 2.2 it follows that the solution u can be written in the form

$$u = v + \mathscr{P}\psi, \quad v \in C^{2+\theta}(\bar{D}), \quad \psi \in B^{2-1/p,p}(\partial D).$$

However Lemma 2.3 tells us that

$$\psi \in C^{2+\theta}(\partial D),$$

since we have by equation (2.2)

$$\psi \equiv E(T\psi) = E(b(\varphi_2 - v)) \mod C^\infty(\partial D).$$

Therefore we obtain that

$$u = v + \mathscr{P}\psi \in C^{2+\theta}(\bar{D}).$$

The proof of Theorem 2.1 is complete. □

Furthermore, by combining Proposition 2.2 and Lemma 2.3 we can obtain the following regularity theorem for problem (0.1) (see [24, Theorem 5.1]):

THEOREM 2.5. *If conditions* (H.1), (H.2) *and* (H.3) *are satisfied, then we have, for all* $s \in \mathbf{R}$ *and all* $p > 1$,

$$u \in L^p(D), \quad Au \in W^{s-2,p}(D), \quad Bu \in B_*^{s-1-1/p,p}(\partial D) \Rightarrow u \in W^{s,p}(D).$$

2.2. Selfadjointness of the operator $\mathfrak{A}$. First we show that the operator $\mathfrak{A}$, defined by formula (0.2), is a nonnegative, selfadjoint operator in the Hilbert space $L^2(D)$ (see [22, Theorems 7.3 and 7.4]):

THEOREM 2.6. *If conditions* (H.1), (H.2) *and* (H.3) *are satisfied, then the operator* $\mathfrak{A}$ *is nonnegative and selfadjoint in* $L^2(D)$.

PROOF. (1) Let $\mathfrak{A}^*$ be the adjoint operator of $\mathfrak{A}$. We shall show that:

$$\mathfrak{A}^* = \mathfrak{A}. \tag{2.3}$$

First we prove that the adjoint operator $\mathfrak{A}^*$ is an extension of the operator $\mathfrak{A}$:

$$\mathfrak{A} \subset \mathfrak{A}^*. \tag{2.4}$$

By Green's formula, we have for all functions u and v in $C^2(\bar{D})$

$$\iint_D (Au \cdot \bar{v} - u \cdot \overline{Av})\, dx = \int_{\partial D} \left(\frac{\partial u}{\partial \nu} \cdot \bar{v} - u \cdot \overline{\frac{\partial v}{\partial \nu}} \right) d\sigma. \tag{2.5}$$

However, if in addition the functions u and v satisfy the boundary conditions

$$a \frac{\partial u}{\partial \nu} + bu = 0 \qquad \text{on } \partial D,$$

$$a \frac{\partial v}{\partial \nu} + bv = 0 \qquad \text{on } \partial D,$$

then it follows that

$$\begin{pmatrix} \dfrac{\partial u}{\partial \nu} & u \\ \overline{\dfrac{\partial v}{\partial \nu}} & \bar{v} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{on } \partial D.$$

Thus we obtain that

$$\begin{vmatrix} \dfrac{\partial u}{\partial \nu} & u \\ \overline{\dfrac{\partial v}{\partial \nu}} & \bar{v} \end{vmatrix} = 0 \quad \text{on } \partial D, \tag{2.6}$$

since we have

$$(a, b) \neq (0, 0) \quad \text{on } \partial D.$$

Therefore, combining formulas (2.5) and (2.6) we find that for all functions u,

$v \in C^2(\bar{D}) \cap D(\mathfrak{A})$

$$\iint_D (Au \cdot \bar{v} - u \cdot \overline{Av})\, dx = 0,$$

or equivalently

$$(\mathfrak{A}u, v) = (u, \mathfrak{A}v).$$

However it is easy to see that the functions in $C^2(\bar{D}) \cap D(\mathfrak{A})$ are dense in the domain $D(\mathfrak{A})$.

Summing up, we have proved that

$$(\mathfrak{A}u, v) = (u, \mathfrak{A}v), \qquad u, v \in D(\mathfrak{A}).$$

This proves assertion (2.4).

Next we prove that

$$D(\mathfrak{A}^*) \subset D(\mathfrak{A}). \tag{2.7}$$

Let v be an arbitrary element of the domain $D(\mathfrak{A}^*)$. Theorem 2.4 with $p = 2$ tells us that the operator

$$\mathfrak{A} : D(\mathfrak{A}) \longrightarrow L^2(D)$$

is bijective. Thus there exists an element $v_0 \in D(\mathfrak{A})$ such that

$$\mathfrak{A}v_0 = \mathfrak{A}^* v.$$

Then, by assertion (2.4) it follows that for all $u \in D(\mathfrak{A})$

$$(\mathfrak{A}u, v - v_0) = (u, \mathfrak{A}^* v - \mathfrak{A}v_0) = 0.$$

This proves that

$$v = v_0 \in D(\mathfrak{A}),$$

since the operator $\mathfrak{A} : D(\mathfrak{A}) \longrightarrow L^2(D)$ is bijective.

Therefore we have proved assertion (2.7) and hence assertion (2.3).

(2) Finally it remains to show that the operator $\mathfrak{A}$ is nonnegative:

$$(\mathfrak{A}u, u) \geq 0, \quad u \in D(\mathfrak{A}). \tag{2.8}$$

It suffices to prove estimate (2.8) for all functions $u \in C^2(\bar{D}) \cap D(\mathfrak{A})$.

By conditions (H.1) and (H.2), we find that

$$u(x') = 0 \qquad \text{on } M = \{x' \in \partial D : a(x') = 0\},$$

and

$$\frac{\partial u}{\partial \nu}(x') = -\frac{b(x')}{a(x')}\, u(x') \qquad \text{on } \partial D \setminus M.$$

Hence we have by the divergence theorem

$$\begin{aligned}\iint_D Au\cdot\bar{u}\,dx &= \sum_{i,j=1}^N \iint_D a^{ij}(x)\frac{\partial u}{\partial x_i}(x)\cdot\overline{\frac{\partial u}{\partial x_j}(x)}\,dx + \iint_D c(x)|u(x)|^2\,dx \\ &\quad - \int_{\partial D}\frac{\partial u}{\partial \nu}(x')\cdot\overline{u(x')}\,d\sigma \\ &\geq -\int_{\partial D}\frac{\partial u}{\partial \nu}(x')\cdot\overline{u(x')}\,d\sigma \\ &\geq \int_{\partial D\setminus M}\frac{b(x')}{a(x')}|u(x')|^2\,d\sigma \\ &\geq 0.\end{aligned}$$

This proves estimate (2.8).

The proof of Theorem 2.6 is complete. □

2.3. Positivity of the resolvent K. First we let

$$W_B^{2,p}(D) = \{u \in W^{2,p}(D) : Bu = 0 \text{ on } \partial D\}.$$

By Theorem 2.1, we can introduce a continuous linear operator

$$K : L^p(D) \longrightarrow W_B^{2,p}(D)$$

as follows: For any $g \in L^p(D)$, the function $u = Kg \in W^{2,p}(D)$ is the unique solution of the problem

$$\begin{cases} Au = g & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Furthermore, by the Ascoli-Arzelà theorem we find that the operator K, considered as

$$K : C(\bar{D}) \longrightarrow C^1(\bar{D}),$$

is *compact*. Indeed it follows from an application of Sobolev's imbedding theorem that $W^{2,p}(D)$ is continuously imbedded into $C^{2-N/p}(\bar{D})$ for all $N < p < \infty$.

Then, by using Theorem 2.5 we can obtain the following:

Claim 2.1. *A function $u \in L^p(D)$, $1 < p < \infty$, is a solution of the problem*

$$\begin{cases} Au = \lambda u & \textit{in } D, \\ Bu = 0 & \textit{on } \partial D \end{cases}$$

if and only if it satisfies the operator equation

$$u = \lambda K u \qquad \text{in } C(\bar{D}). \tag{2.9}$$

For two functions u and v in $C(\bar{D})$, we write $u \le v$ if $u(x) \le v(x)$ for all $x \in \bar{D}$. Then it is easy to verify that the space $C(\bar{D})$ is an ordered Banach space with the linear ordering $\le$ and the positive cone

$$P = \{u \in C(\bar{D}) : u \ge 0 \text{ on } \bar{D}\}.$$

However we shall introduce another ordered Banach subspace of $C(\bar{D})$ for the fixed point equation (2.9) which combines the good properties of the resolvent K with the good properties of the natural ordering of $C(\bar{D})$.

In doing so, we need the following:

LEMMA 2.7. *Assume that hypotheses* (H.1), (H.2) *and* (H.3) *are satisfied. If* $v \in C^\theta(\bar{D})$ *and if* $v \ge 0$ *but* $v \not\equiv 0$ *on* $\bar{D}$, *then the function* $u = Kv \in C^{2+\theta}(\bar{D})$ *satisfies the following conditions*:

(a) $u(x') = 0$ *on* $M = \{x' \in \partial D : a(x') = 0\}$.

(b) $u(x') > 0$ *on* $\bar{D} \backslash M$.

(c) *For the conormal derivative* $\partial u / \partial \nu$ *of* u, *we have*

$$\frac{\partial u}{\partial \nu}(x') < 0 \qquad \text{on } M.$$

Moreover the operator K *is* positive, *that is, the operator* K *maps the positive cone* P *into itself.*

PROOF. (1) First, since the function $u = Kv \in C^{2+\theta}(\bar{D})$ satisfies the condition

$$Au = v \ge 0 \qquad \text{in } D,$$

it follows from an application of the weak maximum principle (see Appendix, Theorem 7.1) that the function u may take its negative minimum only on the boundary ∂D.

However we have the following:

CLAIM 2.2. *The function* $u = Kv$ *does not take its negative minimum on the boundary* ∂D. *In other words, the function* u *is nonnegative on* $\bar{D}$.

PROOF. Assume to the contrary that there exists a point $x'_0 \in \partial D$ such that

$$u(x'_0) < 0.$$

If $a(x'_0) = 0$, then we have, by conditions (H.1) and (H.2),

$$0 = Bu(x'_0) = a(x'_0)\frac{\partial u}{\partial \nu}(x'_0) + b(x'_0)u(x'_0) = b(x'_0)u(x'_0) < 0.$$

This is a contradiction.

If $a(x_0') > 0$, then it follows that

$$\begin{cases} Au(x) = v(x) \geq 0 & \text{in } D, \\ u(x_0') = \min_{x \in \bar{D}} u(x) < 0, & \\ u(x) > u(x_0') & \text{in } D. \end{cases}$$

Thus it follows from an application of the boundary point lemma (see Theorem 7.3) that

$$\frac{\partial u}{\partial \nu}(x_0') < 0,$$

so that

$$0 = Bu(x_0') = a(x_0')\frac{\partial u}{\partial \nu}(x_0') + b(x_0')u(x_0') \leq a(x_0')\frac{\partial u}{\partial \nu}(x_0') < 0.$$

This is also a contradiction. □

(2) Furthermore we have the following:

CLAIM 2.3. *The function $u = Kv$ is strictly positive in D.*

PROOF. In view of Claim 2.2, we assume to the contrary that there exists a point $x_0 \in D$ such that

$$u(x_0) = 0.$$

Then we obtain from the strong maximum principle (see Theorem 7.2) that

$$u(x) \equiv 0 \qquad \text{in } D,$$

so that

$$v(x) \equiv Au(x) = 0 \qquad \text{in } D.$$

This contradicts the condition that v is not the zero function in D. □

(3) PROOF OF LEMMA 2.7. If there exists a point $x_0' \in \partial D$ such that

$$u(x_0') = 0,$$

then we have by Claim 2.3

$$\begin{cases} Au(x) = v \geq 0 & \text{in } D, \\ u(x_0') = \min_{x \in \bar{D}} u(x) = 0, & \\ u(x) > 0 & \text{in } D. \end{cases}$$

Thus it follows from an application of the boundary point lemma that

$$\frac{\partial u}{\partial \nu}(x_0') < 0.$$

This implies that

$$a(x_0') = 0,$$

since we have

$$0 = Bu(x_0') = a(x_0')\frac{\partial u}{\partial \nu}(x_0') = 0.$$

Conversely, if $a(x_0') = 0$, then by conditions (H.1) and (H.2) it follows that $b(x_0') > 0$. This implies that

$$u(x_0') = 0,$$

since we have $0 = Bu(x_0') = b(x_0')u(x_0')$.

Summing up, we have proved that

$$u(x) > 0 \Leftrightarrow x \in \bar{D}\backslash M;$$
$$u(x') = 0 \Leftrightarrow x' \in M.$$

Assertion (c) is an immediate consequence of the boundary point lemma, since the function u attains its minimum 0 at the set M.

Finally, in order to prove the positivity of K, let v be an arbitrary function in $C(\bar{D})$ such that $v(x) \geq 0$ and $v(x) \not\equiv 0$ on $\bar{D}$. Then, by using Friedrichs' mollifiers we can find a sequence $\{v_j\} \subset C^1(\bar{D})$ satisfying the conditions

$$\begin{cases} v_j(x) \geq 0 & \text{on } \bar{D}, \\ v_j \longrightarrow v & \text{in } C(\bar{D}). \end{cases}$$

Hence we have, by assertions (a) and (b),

$$\begin{cases} Kv_j \in C^2(\bar{D}), & \\ Kv_j(x) \geq 0 & \text{on } \bar{D}, \end{cases}$$

and so

$$Kv(x) = \lim_j Kv_j(x) \geq 0 \qquad \text{on } \bar{D}.$$

The proof of Lemma 2.7 is complete. □

Now we introduce an ordered Banach space which is associated with the operator $K : C(\bar{D}) \to C^1(\bar{D})$. If we let

$$e = K1,$$

then it follows from an application of Lemma 2.7 that the function e belongs to

$C^{2+\theta}(\bar{D})$ and satisfies the conditions

$$\begin{cases} e(x) > 0 & \text{on } \bar{D}\backslash M, \\ e(x') = 0 & \text{on } M, \\ \dfrac{\partial e}{\partial \nu}(x') < 0 & \text{on } M. \end{cases}$$

By using the function $e = K1$, we can introduce a subspace of $C(\bar{D})$ as follows:

$$C_e(\bar{D}) = \{u \in C(\bar{D}) : \text{there exists a constant } c > 0 \text{ such that } -ce \leq u \leq ce\}.$$

The space $C_e(\bar{D})$ is given a norm by the formula

$$\|u\|_e = \inf\{c > 0 : -ce \leq u \leq ce\}.$$

If we let

$$P_e = C_e(\bar{D}) \cap P = \{u \in C_e(\bar{D}) : u \geq 0 \text{ on } (\bar{D})\},$$

then it is easy to verify that the space $C_e(\bar{D})$ is an ordered Banach space having the positive cone P_e with nonempty interior. Indeed, every function $u \in C_e(\bar{D})$ which satisfies the conditions

$$\begin{cases} u(x) > 0 & \text{on } \bar{D}\backslash M, \\ u(x') = 0 & \text{on } M, \\ \dfrac{\partial u}{\partial \nu}(x') < 0 & \text{on } M \end{cases}$$

belongs to the interior of P_e.

The next proposition tells us that one may consider the fixed point equation (2.9) in the ordered Banach space $C_e(\bar{D})$ in the proof of Theorems 2 and 3 in Section 4:

PROPOSITION 2.8. *The operator K maps $C(\bar{D})$* compactly *into $C_e(\bar{D})$. Moreover, K is strongly positive, that is, if $v \in P$ and $v \not\equiv 0$ on $\bar{D}$, then the function Kv is an interior point of P_e.*

PROOF. (i) First, by the positivity of K we find that K maps $C(\bar{D})$ into $C_e(\bar{D})$. Indeed, since we have $-\|v\| \leq v(x) \leq \|v\|$ on $\bar{D}$ for all $v \in C(\bar{D})$, it follows that

$$-\|v\|K1(x) \leq Kv(x) \leq \|v\|K1(x) \qquad \text{on } \bar{D}.$$

This proves that $-ce \leq Kv \leq ce$ with $c = \|v\|$.

(ii) Next we prove that $K : C(\bar{D}) \to C_e(\bar{D})$ is compact. To do so, we let

$$C^1_B(\bar{D}) = \{u \in C^1(\bar{D}) : Bu = 0 \text{ on } \partial D\}.$$

Since K maps $C(\bar{D})$ compactly into $C^1_B(\bar{D})$, it suffices to show that the inclusion mapping

$$\imath : C^1_B(\bar{D}) \longrightarrow C_e(\bar{D}) \tag{2.10}$$

is continuous.

(ii-a) We verify that $\imath$ maps $C^1_B(\bar{D})$ into $C_e(\bar{D})$.

Let u be an arbitrary function in the space $C^1_B(\bar{D})$. Since we have, for some neighborhood U of M in ∂D,

$$\begin{cases} b > 0 & \text{in } U, \\ \dfrac{\partial e}{\partial \nu} < 0 & \text{in } U, \end{cases}$$

it follows that

$$\frac{u}{e} = \frac{\left(-\dfrac{a}{b}\right)\dfrac{\partial u}{\partial \nu}}{\left(-\dfrac{a}{b}\right)\dfrac{\partial e}{\partial \nu}} = \frac{\dfrac{\partial u}{\partial \nu}}{\dfrac{\partial e}{\partial \nu}} \quad \text{in } U \setminus M.$$

Hence there exists a constant $c_1 > 0$ such that

$$|u(x')| \leq c_1 e(x') \quad \text{in } U.$$

Thus, by using Taylor's formula we can find a neighborhood W of U in D and a constant $c_2 > 0$ such that

$$|u(x)| \leq c_2 e(x) \qquad \text{in } W.$$

On the other hand, since we have, with some constant $\alpha > 0$,

$$e(x) \geq \alpha \qquad \text{on } \bar{D} \setminus W,$$

we can find a constant $c_3 > 0$ such that

$$\left| \frac{u(x)}{e(x)} \right| \leq c_3 \qquad \text{on } \bar{D} \setminus W.$$

Therefore we have, with $c = \max\{c_2, c_3\}$,

$$-ce(x) \leq u(x) \leq ce(x) \qquad \text{on } \bar{D}.$$

This proves that $u \in C_e(\bar{D})$.

(ii-b) Now we assume that

$$\begin{cases} u_j \in C^1_B(\bar{D}) & \\ u_j \longrightarrow u & \text{in } C^1_B(\bar{D}), \\ u_j \longrightarrow v & \text{in } C_e(\bar{D}). \end{cases}$$

Then there exists a sequence $\{c_j\}$, $c_j \to 0$, such that

$$\|u_j - v\| \leq c_j \|e\|,$$

so that $u_j \to v$ in $C(\bar{D})$. Hence it follows that $u = v$. By the closed graph theorem, this proves that the mapping ι is continuous.

(iii) It remains to prove the strong positivity of K.

(iii-a) We show that, for any $v \geq 0$ but $v \not\equiv 0$ on $\bar{D}$, there exist constants $\beta > 0$ and $\gamma > 0$ such that

$$\beta e(x) \leq Kv(x) \leq \gamma e(x) \qquad \text{on } \bar{D}. \tag{2.11}$$

By the positivity of K, one may modify the function v in such a way that $v \in C^1(\bar{D})$. Furthermore, since the functions $u = Kv$ and $e = K1$ vanish only on the set M, it suffices to prove that there exists a neighborhood W of M in D such that

$$\beta e(x) \leq u(x) \qquad \text{in } W. \tag{2.12}$$

Recall that we have, in a neighborhood U of M in ∂D,

$$\begin{cases} u = \left(-\dfrac{a}{b}\right) \dfrac{\partial u}{\partial \nu} & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} < 0 & \text{in } U, \end{cases}$$

and also

$$\begin{cases} e = \left(-\dfrac{a}{b}\right) \dfrac{\partial e}{\partial \nu} & \text{in } U, \\ \dfrac{\partial e}{\partial \nu} < 0 & \text{in } U. \end{cases}$$

Thus we have, for β sufficiently small,

$$\begin{cases} u(x') - \beta e(x') \geq 0 & \text{in } U, \\ \dfrac{\partial}{\partial \nu}(u - \beta e)(x') < 0 & \text{in } U. \end{cases}$$

Therefore, by using Taylor's formula we can find a neighborhood W of M in D such that

$$u(x) - \beta e(x) \geq 0 \qquad \text{in } W.$$

This proves estimate (2.12).

(iii-b) Finally we show that the function $u = Kv$ is an interior point of P_e. Take

$$\varepsilon = \frac{\beta}{2},$$

where β is the same constant as in estimate (2.11). Then, for any function $w \in C_e(\bar{D})$ satisfying

$$\|w - Kv\|_e < \varepsilon,$$

we have by estimate (2.11)

$$w \leq Kv + \varepsilon e \leq (\gamma + \varepsilon)e,$$

and also

$$w \geq Kv - \varepsilon e \geq \frac{\varepsilon}{2} e.$$

This implies that $w \in P_e$, that is, the function Kv is an interior point of P_e.

The proof of Proposition 2.8 is complete. □

Now we consider the resolvent K as an operator in the ordered Banach space $C_e(\bar{D})$, and characterize the eigenvalues and positive eigenfunctions of K.

First Proposition 2.8 tells us that the operator

$$K : C_e(\bar{D}) \longrightarrow C_e(\bar{D})$$

is strongly positive and compact. This implies that K has a countable number of positive eigenvalues, μ_j, which may accumulate only at 0. Hence they may be arranged in a decreasing sequence

$$\mu_1 \geq \mu_2 \geq \dots,$$

where each eigenvalue is repeated according to its multiplicity.

The next theorem is an immediate consequence of Theorem 1.1:

THEOREM 2.9. *The resolvent K, considered as an operator $K : C_e(\bar{D}) \to C_e(\bar{D})$, has the following spectral properties*:

(1) *The largest eigenvalue μ_1 is* simple, *i.e.*, $\mu_1 > \mu_2$, *and it has a* positive *eigenfunction ψ_1*.

(2) No *other eigenvalues, μ_1, $j \geq 2$, have positive eigenfunctions.*

2.4. End of Proof of Theorem 0. By Claim 2.1 and assertion (2.10), it is easy to see that

$$\mathfrak{A}u = \lambda u \quad \text{in } L^2(D) \Leftrightarrow Ku = \frac{1}{\lambda} u \qquad \text{in } C_e(\bar{D}).$$

Therefore Theorem 0 follows by combining Theorem 2.6 and Theorem 2.9. □

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Theorem 1 follows from an application of local static bifurcation theory from a simple eigenvalue (Theorem 1.2). We shall apply Theorem 1.2 with

$$X = C_B^{2+\theta}(\bar{D}) = \{u \in C^{2+\theta}(\bar{D}) : Bu = 0 \text{ on } \partial D\},$$
$$Y = C^{\theta}(\bar{D}),$$
$$F(t,x) := Au - \lambda u + G(\lambda, u).$$

First Theorems 2.6 tells us that $\mathfrak{A}$ is a nonnegative, selfadjoint operator in the Hilbert space $L^2(D)$. Hence we have, for each $\lambda > 0$, the following orthogonal decomposition:

$$L^2(D) = N(\mathfrak{A} - \lambda I) \oplus R(\mathfrak{A} - \lambda I). \tag{3.1}$$

However it follows from an application of the regularity theorem for problem (0.1) (Theorem 2.5) that

$$N(\mathfrak{A} - \lambda I) = \{u \in C_B^{2+\theta}(\bar{D}) : (A - \lambda)u = 0 \text{ in } D\},$$

and also

$$R(\mathfrak{A} - \lambda I) \cap C^{\theta}(\bar{D}) = \{(A - \lambda)u : u \in C_B^{2+\theta}(\bar{D})\}.$$

Thus, by restricting decomposition (3.1) to the space $C^{\theta}(\bar{D})$ and also by taking $\lambda = \lambda_1$ we obtain the orthogonal decomposition

$$\begin{aligned} C^{\theta}(\bar{D}) &= \{u \in C_B^{2+\theta}(\bar{D}) : (A - \lambda_1)u = 0 \text{ in } D\} \oplus \{(A - \lambda_1)u : u \in C_B^{2+\theta}(\bar{D})\} \\ &= N(F_u(\lambda_1, 0)) \oplus R(F_u(\lambda_1, 0)). \end{aligned} \tag{3.2}$$

By virtue of decomposition (3.2), it is easy to verify conditions (2) and (3) of Theorem 1.2. Indeed, by Theorem 0 we find that the null space $N(F_u(\lambda_1, 0))$ is one dimensional, spanned by the eigenfunction ψ_1.

Therefore Theorem 1 follows by applying Theorem 1.2 with $X = C_B^{2+\theta}(\bar{D})$, $Y = C^{\theta}(\bar{D})$ and $F(t,x) := Au - \lambda u + G(\lambda, u)$.

The proof of Theorem 1 is complete. □

We remark that Examples 1 and 2 follow from a straightforward application of Corollaries 1.3 and 1.4, respectively.

4. Proof of Theorems 2 and 3

This section is devoted to the proof of Theorems 2 and 3. We transform problem (0.4) into operator equation (0.10) in the ordered Banach space

$C(\bar{D})$. By condition $(\mathrm{R})_\sigma$, it follows that the map H, defined by $H(u) = K(F(u))$, leaves invariant the ordering (Lemma 4.1). In the case of an increasing map, it suffices to verify that H maps *two* points of a bounded, closed and convex set into itself in order to apply Schauder's fixed point theorem. This essential step in the proof of Theorem 2 is done in Lemma 4.2. The proof of Theorem 3 is based on a uniqueness theorem of fixed points of strongly increasing and strongly sublinear mappings in ordered Banach spaces (Theorem 4.4).

4.1. Proof of Theorem 2. (1) First we replace the function $c(x)$ by the function $c(x) + \omega$, where $\omega > 0$ is the same constant as in condition $(\mathrm{R})_\sigma$, and consider instead of problem (0.4) the following problem:

$$\begin{cases} (A + \omega)u = \omega u + F(u) & \text{in } D, \\ Bu = 0 & \text{on } \partial D, \end{cases} \tag{$(0.4)_\omega$}$$

where $F(u)$ is the Nemytskii operator of $f(x, \xi)$ defined by the formula

$$Fu(x) = f(x, u(x)), \qquad x \in \bar{D}.$$

It is clear that problem (0.4) is equivalent to problem $(0.4)_\omega$. Furthermore, since $f \in C^\theta(\bar{D} \times [0, \sigma])$, it is easy to verify that problem $(0.4)_\omega$ is equivalent to an operator equation

$$u = K_\omega(\omega u + F(u)) \qquad \text{in } C(\bar{D}), \tag{4.1}$$

just as in Subsection 2.3. Here $K_\omega : C(\bar{D}) \to C^1(\bar{D})$ is the compact operator introduced in Subsection 2.3 with c replaced by $c + \omega$.

(2) We let

$$H_\omega(u) = K_\omega(\omega u + F(u)), \qquad u \in C(\bar{D}).$$

The next lemma asserts that the map H_ω leaves invariant the ordering of the ordered Banach space $C(\bar{D})$:

Lemma 4.1. *The operator $H_\omega : [\phi, \psi] \to C(\bar{D})$ is increasing. Here $[\phi, \psi]$ is the order interval defined by the formula*

$$[\phi, \psi] = \{u \in C(\bar{D}) : \phi \le u \le \psi\}.$$

Proof. Let u and v be arbitrary functions in $C(\bar{D})$ satisfying $\phi \le u \le v \le \psi$ on $(\bar{D})$. Then we have

$$\begin{aligned} &\omega(v(x) - u(x)) + (Fv(x) - Fu(x)) \\ &\quad = \begin{cases} 0 & \text{if } v(x) = u(x), \\ \left(\omega + \dfrac{Fv(x) - Fu(x)}{v(x) - u(x)}\right)(v(x) - u(x)) & \text{if } v(x) > u(x), \end{cases} \end{aligned}$$

and so by condition $(\mathrm{R})_\sigma$

$$\omega(v-u)+(Fu-Fv)\geq 0 \qquad \text{on } \bar{D}.$$

However Lemma 2.7 tells us that $K_\omega : C(\bar{D}) \to C(\bar{D})$ is positive. Thus it follows that

$$H_\omega(v)-H_\omega(u)=K_\omega(\omega(v-u)+(F(v)-F(u)))\geq 0 \quad \text{on } \bar{D},$$

or equivalently,

$$H_\omega(u)\leq H_\omega(v) \qquad \text{on } \bar{D}.$$

This proves that H_ω is increasing. □

Moreover we have the following:

LEMMA 4.2. *The operator H_ω maps the order interval $[\phi, \psi]$ into itself.*

PROOF. Let u be an arbitrary function $C(\bar{D})$ satisfying $\phi \leq u \leq \psi$ on $\bar{D}$. Then it follows from an application of Lemma 4.1 that

$$H_\omega(\phi)\leq H_\omega(u)\leq H_\omega(\psi) \qquad \text{on } \bar{D}.$$

Hence, in order to prove the lemma it suffices to show that

$$\phi \leq H_\omega(\phi),\ H_\omega(\psi)\leq \psi \qquad \text{on } \bar{D}.$$

If we let

$$v=H_\omega(\psi)=K_\omega(\omega\psi+F(\psi)),$$

then we have

$$\begin{cases} (A+\omega)v=\omega\psi+F(\psi) & \text{in } D, \\ Bv=0 & \text{on } \partial D. \end{cases}$$

But, since ψ is a supersolution of problem (0.4), it follows that

$$\begin{aligned}(A+\omega)(v-\psi) &= \omega\psi+F(\psi)-(A+\omega)\psi \\ &= -(A\psi-F(\psi))\leq 0 \qquad \text{in } D,\end{aligned}$$

and

$$B(v-\psi)=-B\psi\leq 0 \qquad \text{on } \partial D.$$

Thus, using the maximum principle as in the proof of Lemma 2.7 we find that

$$H_\omega(\psi)=v\leq\psi \qquad \text{on } \bar{D}.$$

Indeed, if the function $v - \psi$ takes its positive maximum m at an interior point $x_0 \in D$, then we have

$$(A + \omega)(v - \psi)(x_0) \geq \omega m > 0.$$

This contradicts the condition: $(A + \omega)(v - \psi) \leq 0$ in D. On the other hand, if $v - \psi$ takes the positive maximum m at a boundary point $x_0' \in \partial D$, then it follows from an application of the boundary point lemma (Lemma 7.3) that

$$\frac{\partial}{\partial \nu}(v - \psi)(x_0') > 0.$$

Hence we have, by conditions (H.1) and (H.2),

$$B(v - \psi)(x_0') = a(x_0')\frac{\partial}{\partial \nu}(v - \psi)(x_0') + b(x_0')m > 0.$$

This contradicts the condition: $B(v - \psi) \leq 0$ on ∂D.

Similarly we can prove that

$$\phi \leq H_\omega(\phi) \quad \text{on } \bar{D}.$$

The proof of Lemma 4.2 is complete. □

(3) Now we need an extension of Brouwer's fixed point theorem to the infinitedimensional case, due to Schauder (see [3, Theorem 2.4.3], [19, Proposition 3.60]):

THEOREM 4.3 (Schauder's fixed point theorem). *A compact mapping f of a closed bounded convex set K in a Banach space X into itself has a fixed point $x \in K : f(x) = x$.*

Since $K_\omega : C(\bar{D}) \to C^1(\bar{D})$ is compact, it follows from Lemma 4.2 that the mapping $H_\omega : [\phi, \psi] \to [\phi, \psi]$ is compact. Furthermore the order interval $[\phi, \psi]$ is closed, bounded and convex in the space $C(\bar{D})$. Therefore, applying Schauder's fixed point theorem we can find a solution $u \in [\phi, \psi]$ of equation (4.1).

Now the proof of Theorem 2 is complete. □

4.2. Proof of Theorem 3. (1) Our proof of Theorem 3 is based on the following uniqueness theorem of fixed points of strongly increasing and strongly sublinear mappings in ordered Banach spaces (see [1, Theorem 24.2]):

THEOREM 4.4. *Let (E, Q) be an ordered Banach space having the positive cone Q with nonempty interior. If σ is a positive number, we let*

$$\bar{Q}_\sigma = \{u \in Q : \|u\| \leq \sigma\}.$$

Assume that a mapping $f : \bar{Q}_\sigma \longrightarrow E$ satisfies the following two conditions:

(A) *f is* strongly increasing, *that is, if u, $v \in \bar{Q}_\sigma$ and if $u \le v$ and $v \neq u$, then $f(v) - f(u)$ is an interior point of Q.*

(B) *f is* strongly sublinear, *that is, $f(0) \ge 0$ and if $u \in \bar{Q}_\sigma$ and $u \neq 0$, then $f(\tau u) - \tau f(u)$ is an interior point of Q for every $0 < \tau < 1$.*

Then the mapping f has at most one positive fixed point.

In the proof of Theorem 3 we shall apply Theorem 4.4 with

$$E = C_e(\bar{D}),$$
$$Q = P_e = C_e(\bar{D}) \cap P = \{u \in C_e(\bar{D}) : u \ge 0\},$$
$$f = H_\omega.$$

(2) If σ is a positive number, we let

$$(\overline{P_e})_\sigma = \{u \in P_e : \|u\|_e \le \sigma\}.$$

It suffices to prove Theorem 3 in the space $(\overline{P_e})_\sigma$ for every $\sigma > 0$. Indeed, if u_1 and u_2 are two positive solutions of problem (0.4), then one can find a constant $\sigma > 0$ such that $\|u_1\|_e$, $\|u_2\|_e \le \sigma$, so that u_1, $u_2 \in (\overline{P_e})_\sigma$.

If we take a constant $\omega = \omega(\sigma) > 0$ given in condition $(R)_\sigma$, then we have the following:

LEMMA 4.5. *The operator H_ω maps $(\overline{P_e})_\sigma$ into P_e.*

PROOF. Let u be an arbitrary function in $(\overline{P_e})_\sigma$. Then we have, by condition $(\mathrm{R})_\sigma$ with $\xi = u$ and $\eta = 0$ and condition (S2),

$$F(u) \ge F(0) - \omega u \ge -\omega u \qquad \text{on } \bar{D},$$

so that

$$\omega u + F(u) \ge 0 \qquad \text{on } \bar{D}.$$

Hence it follows from an application of Proposition 2.8 that

$$H_\omega(u) = K_\omega(\omega u + F(u)) \in P_e. \quad \square$$

Moreover the next lemma asserts that the map H_ω is a strongly increasing selfmap of the space $C_e(\bar{D})$:

LEMMA 4.6. *The operator $H_\omega : (\overline{P_e})_\sigma \longrightarrow P_e$ is strongly increasing.*

PROOF. Lemma 4.6 follows by combining Lemma 4.1 and Proposition 2.8. $\square$

On the other hand the next lemma verifies the strong sublinearity of the map H_ω:

LEMMA 4.7. *The operator* $H_\omega : (\overline{P_e})_\sigma \longrightarrow P_e$ *is strongly sublinear.*

PROOF. Let u be an arbitrary function in $(\overline{P_e})_\sigma$ but $u \not\equiv 0$. Then we have by condition (S)

$$\begin{cases} f(x, \tau u(x)) \geq \tau f(x, u(x)) & \text{if } u(x) > 0, \\ f(x, \tau u(x)) = f(x, 0) \geq 0 & \text{if } u(x) = 0. \end{cases}$$

This implies that

$$\begin{aligned} &\omega\tau u + F(\tau u) - \tau(\omega u + F(u)) \\ &\quad = F(\tau u) - \tau F(u) \geq 0 \text{ and } \not\equiv 0 \qquad \text{on } \bar{D}. \end{aligned}$$

Hence it follows from an application of Proposition 2.8 that the function

$$H_\omega(\tau u) - \tau H_\omega(u) = K_\omega(\omega\tau u + F(\tau u) - \tau(\omega u + F(u)))$$

is an interior point of P_e. □

(3) Combining Lemmas 4.5, 4.6 and 4.7, we have proved that the mapping $H_\omega : (\overline{P_e})_\sigma \to P_e$ satisfies conditions (A) and (B) of Theorem 4.4 with $E = C_e(\bar{D})$ and $Q = P_e$. Therefore Theorem 3 follows from an application of the same theorem.

The proof of Theorem 3 is complete. □

5. Proof of Theorem 4

In this section we prove Theorem 4. Theorem 4 follows from an application of Theorem 2 if we construct explicitly super- and subsolutions of problem (0.4). First, by using the positive eigenfunction ψ_1 of problem (0.1) we have a subsolution $\phi_\varepsilon = \varepsilon\psi_1$ for $\varepsilon > 0$ sufficiently small. On the other hand, in order to construct a supersolution of problem (0.4) we make good use of the positivity lemma (Lemma 5.1) and the existence and uniqueness theorem for problem (0.1) (Theorem 2.1), just as in the proof of [12, Theorem 2.2].

(I) First we construct a subsolution of problem (0.4).

By condition (0.5), we can find a constant $c_1 > 0$ such that

$$f(x, \xi) \geq \lambda_1 \xi, \qquad x \in \bar{D}, 0 < \xi < c_1. \tag{5.1}$$

On the other hand, Theorem 0 tells us that the linearized boundary value problem

$$\begin{cases} A\varphi = \lambda_1 \varphi & \text{in } D, \\ B\varphi = 0 & \text{on } \partial D \end{cases}$$

has a positive eigenfunction $\psi_1 \in C^{2+\theta}(\bar{D})$. If we let

$$\phi_\varepsilon = \varepsilon\psi_1$$

for $\varepsilon > 0$ sufficiently small, we may assume that

$$\max_{\bar{D}} \phi_\varepsilon < c_1.$$

Then we have by condition (5.1)

$$\begin{cases} A\phi_\varepsilon - f(x,\phi_\varepsilon) \le \lambda_1\phi_\varepsilon - \lambda_1\phi_\varepsilon = 0 & \text{in } D, \\ B\phi_\varepsilon = 0 & \text{on } \partial D. \end{cases}$$

This proves that the function $\phi_\varepsilon \in C^{2+\theta}(\bar{D})$ is a subsolution of problem (0.4).

(II) In order to construct a supersolution of problem (0.4), we make use of the following lemma (see [15, Theorem 2.16]):

LEMMA 5.1 (The positivity lemma). *Let $T : C_e(\bar{D}) \to C_e(\bar{D})$ be a strongly positive, compact linear operator and λ_0 the largest eigenvalue of T. Then, for any given positive function $g \in C_e(\bar{D})$ the equation*

$$\lambda v - Tv = g$$

has a unique positive solution $v \in C_e(\bar{D})$ for each $\lambda > \lambda_0$.

(III) By condition (0.6), we can find constants $c_2 > 0$ and $0 < d < \lambda_1$ such that

$$f(x,\xi) \le (\lambda_1 - d)\xi, \qquad x \in \bar{D}, \xi > c_2.$$

Hence, if we let

$$k = 1 + \max\{|f(x,\xi)| : x \in \bar{D}, 0 \le \xi \le c_2\},$$

then we have

$$f(x,\xi) \le (\lambda_1 - d)\xi + k, \qquad x \in \bar{D}, \xi \ge 0. \tag{5.2}$$

We show that the boundary value problem

$$\begin{cases} A\psi = (\lambda_1 - d)\psi + k & \text{in } D, \\ B\psi = 0 & \text{on } \partial D \end{cases} \tag{5.3}$$

has a positive solution $\psi \in C^{2+\theta}(\bar{D})$.

First it is easy to see that $\psi \in C^{2+\theta}(\bar{D})$ is a solution of problem (5.3) if and only if it satisfies the following operator equation:

$$\psi = (\lambda_1 - d)K\psi + Kk \qquad \text{in } C_e(\bar{D}). \tag{5.4}$$

However we remark that the largest eigenvalue $(\lambda_1 - d)/\lambda_1$ of the operator

$(\lambda_1 - d)K$ is less than 1, and that the function Kk is positive on $\bar{D}$. Thus, using Lemma 5.1 and Theorem 2.1 we can find a positive solution $\psi \in C^{2+\theta}(\bar{D})$ of equation (5.4), or equivalently, a solution of problem (5.3).

Then we have by condition (5.2)

$$\begin{cases} A\psi - f(x, \psi) \geq (\lambda_1 - d)\psi + k - ((\lambda_1 - d)\psi + k) = 0 & \text{in } D, \\ B\psi = 0 & \text{on } \partial D. \end{cases}$$

This proves that the function $\psi \in C^{2+\theta}(\bar{D})$ is a supersolution of problem (0.4).

(IV) One may assume that the super- and subsolutions ψ, ϕ_ε satisfy the condition

$$\phi_\varepsilon \leq \psi \qquad \text{on } \bar{D}.$$

Indeed it suffices to note that the functions $\phi_\varepsilon = \varepsilon\psi_1$ and ψ behave like the function $e = K1$ (see the proof of Proposition 2.8). Furthermore, if we take a constant $\sigma > 0$ such that

$$\max_{\bar{D}} \phi_\varepsilon, \ \max_{\bar{D}} \psi \leq \sigma,$$

then it follows that the functions ψ and ϕ_ε are respectively super- and subsolutions of problem (0.4) such that $0 \leq \phi_\varepsilon(x) \leq \psi(x) \leq \sigma$ on $\bar{D}$.

Therefore Theorem 4 follows from an application of Theorem 2. □

6. Proof of Theorem 5

By Theorem 4, it suffices to prove that condition (0.6) is necessary for the existence of positive solutions of problem (0.4). Our proof is inspired by [7, Section 3].

(I) Theorem 2.6 tells us that the operator $\mathfrak{A}$, defined by formula (0.2), is a nonnegative, selfadjoint operator in $L^2(D)$, and has a compact resolvent. Hence we find that the first eigenvalue λ_1 of $\mathfrak{A}$ is characterized by the following formula:

$$\lambda_1 = \min\{(\mathfrak{A}u, u) : u \in D(\mathfrak{A}), \|u\| = 1\}, \tag{6.1}$$

where $\|\cdot\|$ is the norm on $L^2(D)$.

(I-a) First we show that

$$\lambda_1 < l_0. \tag{6.2}$$

Since the function $f(\xi)/\xi$ is strictly decreasing, it follows that

$$l_\infty < \frac{f(\xi)}{\xi} < l_0, \qquad 0 < \xi < \infty. \tag{6.3}$$

Let $u \in C^2(\bar{D})$ be a positive solution of problem (0.4):

$$\begin{cases} Au = f(u) & \text{in } D, \\ u > 0 & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Here we have used the fact that every positive solution of problem (0.4) is strictly positive in D (see Lemma 2.7). Then, since $u \in D(\mathfrak{A})$, we have by inequality (6.3) with $\xi = u(x)$

$$(\mathfrak{A}u, u) = (Au, u) = \int_D f(u) u \, dx < l_0 \int_D u^2 \, dx.$$

Hence inequality (6.2) follows by using formula (6.1).

(I-b) Next we show that

$$\lambda_1 < l_\infty. \tag{6.4}$$

If $u \in C^2(\bar{D})$ is a positive solution of problem (0.4), we let

$$d = \frac{f(\|u\|_\infty + 1)}{\|u\|_\infty + 1}, \tag{6.5}$$

where $\|u\|_\infty = \max_{\bar{D}} u$. We remark that $d > l_\infty$.

Now we consider the eigenvalue problem

$$\begin{cases} Au - du = \lambda u & \text{in } D, \\ Bu = 0 & \text{on } \partial D, \end{cases}$$

and let $\lambda_1(d)$ be its first eigenvalue. Then, by formula (6.1) we find that

$$\lambda_1(d) = \min\{((\mathfrak{A} - dI)u, u) : u \in D(\mathfrak{A}), \|u\| = 1\} = \lambda_1 - d.$$

Furthermore we have the following:

CLAIM 6.1. $\lambda_1(d) = \lambda_1 - d > 0$.

PROOF. First, by Theorem 0 one may assume that the first eigenvalue $\lambda_1(d)$ has a positive eigenfunction $\psi \in C^{2+\theta}(\bar{D})$:

$$\begin{cases} A\psi - d\psi = \lambda_1(d)\psi & \text{in } D, \\ \psi > 0 & \text{in } D, \\ B\psi = 0 & \text{on } \partial D. \end{cases}$$

Since the function $f(\xi)/\xi$ is strictly decreasing, it follows from formula (6.5) that

$$f(u(x)) > du(x), \qquad x \in D.$$

Hence we have

$$(\mathfrak{A}u, \psi) = (Au, \psi) = \int_D f(u)\psi\, dx > d \int_D u\psi\, dx. \tag{6.6}$$

On the other hand, by the selfadjointness of $\mathfrak{A}$ it follows that

$$(\mathfrak{A}u, \psi) = (u, \mathfrak{A}\psi) = (u, A\psi) = \int_D u(\lambda_1(d) + d)\psi\, dx. \tag{6.7}$$

Thus, combining formulas (6.6) and (6.7) we obtain that

$$\lambda_1(d) \int_D u\psi\, dx > 0.$$

This proves Claim 6.1, since we have $u > 0$, $\psi > 0$ in D. □

Summing up, we have proved inequality (6.4):

$$\lambda_1 > d > l_\infty.$$

The desired inequality (0.6) follows from inequalities (6.2) and (6.4).

(II) Finally we prove the uniqueness of positive solutions of problem (0.4) (cf. [7, Section 2]).

Let $u_i \in C^2(\bar{D})$, $i = 1, 2$, be two positive solutions of problem (0.4):

$$\begin{cases} Au_i = f(u_i) & \text{in } D, \\ u_i > 0 & \text{in } D, \\ Bu_i = 0 & \text{on } \partial D. \end{cases}$$

The next claim is an essential step in the proof of uniqueness of positive solutions (cf. [7, Lemma 1]):

Claim 6.2. $u_1/u_2,\ u_2/u_1 \in C(\bar{D})$.

Proof. Since the function $f(\xi)/\xi$ is strictly decreasing, we can find two nonnegative constants ω_i, $i = 1, 2$, such that

$$f(u_i) + \omega_i u_i \geq 0 \qquad \text{in } D.$$

Indeed it suffices to take

$$\omega_i = \max\left\{0, -\frac{f(\|u_i\|_\infty)}{\|u_i\|_\infty}\right\}, \qquad i = 1, 2.$$

Then the solutions u_i, $i = 1, 2$, are expressed in the following form:

$$u_i = K_{\omega_i}(f(u_i) + \omega_i u_i),$$
$$f(u_i) + \omega_i u_i \geq 0 \qquad \text{in } D,$$

where K_{ω_i} is the resolvent of the boundary value problem

$$\begin{cases} (A+\omega_i)u = \varphi & \text{in } D, \\ Bu = 0 & \text{on } \partial D. \end{cases}$$

Hence Claim 6.2 follows from the strong positivity of the resolvents K_{ω_i}, $i = 1, 2$ (see inequality (2.11)). $\square$

By Claim 6.2, we can apply Green's formula to obtain that

$$\begin{aligned}
&\int_D \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2} \right) (u_1^2 - u_2^2)\, dx \qquad (6.8) \\
&= \int_D \left(\frac{Au_1}{u_1} - \frac{Au_2}{u_2} \right) (u_1^2 - u_2^2)\, dx \\
&= -\int_D \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a^{ij} \frac{\partial u_1}{\partial x_j} \right) u_1\, dx + \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a^{ij} \frac{\partial u_1}{\partial x_j} \right) \left(\frac{u_2^2}{u_1} \right) dx \\
&\quad - \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a^{ij} \frac{\partial u_2}{\partial x_j} \right) u_2\, dx + \int_D \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a^{ij} \frac{\partial u_2}{\partial x_j} \right) \left(\frac{u_1^2}{u_2} \right) dx \\
&= \int_D \sum_{i,j=1}^N a^{ij} \frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_j}\, dx - \int_D \sum_{i,j=1}^N a^{ij} \frac{\partial u_1}{\partial x_i} \frac{\partial}{\partial x_j} \left(\frac{u_2^2}{u_1} \right) dx \\
&\quad - \int_{\partial D} \frac{\partial u_1}{\partial \nu} u_1\, d\sigma + \int_{\partial D} \frac{\partial u_1}{\partial \nu} \left(\frac{u_2^2}{u_1} \right) d\sigma \\
&\quad + \int_D \sum_{i,j=1}^N a^{ij} \frac{\partial u_2}{\partial x_i} \frac{\partial u_2}{\partial x_j}\, dx - \int_D \sum_{i,j=1}^N a^{ij} \frac{\partial u_2}{\partial x_i} \frac{\partial}{\partial x_j} \left(\frac{u_1^2}{u_2} \right) dx \\
&\quad - \int_{\partial D} \frac{\partial u_2}{\partial \nu} u_2\, d\sigma + \int_{\partial D} \frac{\partial u_2}{\partial \nu} \left(\frac{u_1^2}{u_2} \right) d\sigma.
\end{aligned}$$

Here we remark that the four integrals over ∂D in the last line of formula (6.8) vanish. Indeed it suffices to note that

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial \nu} & u_1 \\ \dfrac{\partial u_2}{\partial \nu} & u_2 \end{vmatrix} = 0 \qquad \text{on } \partial D,$$

since the solutions u_1 and u_2 satisfy the boundary conditions

$$\begin{pmatrix} \dfrac{\partial u_1}{\partial \nu} & u_1 \\ \dfrac{\partial u_2}{\partial \nu} & u_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{on } \partial D$$

and since $(a, b) \neq (0, 0)$ on ∂D.

Therefore we find that

$$\begin{aligned}
&\int_D \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)(u_1^2 - u_2^2)\,dx \\
&= \int_D \sum_{i,j=1}^N a^{ij}\left(\frac{\partial u_1}{\partial x_i} - \frac{u_2}{u_1}\frac{\partial u_1}{\partial x_i}\right)\left(\frac{\partial u_1}{\partial x_j} - \frac{u_2}{u_1}\frac{\partial u_1}{\partial x_j}\right)dx \\
&+ \int_D \sum_{i,j=1}^N a^{ij}\left(\frac{\partial u_2}{\partial x_i} - \frac{u_1}{u_2}\frac{\partial u_2}{\partial x_i}\right)\left(\frac{\partial u_2}{\partial x_j} - \frac{u_1}{u_2}\frac{\partial u_2}{\partial x_j}\right)dx \\
&\geq 0.
\end{aligned}$$

This implies that $u_1 \equiv u_2$ in D, since the function $f(\xi)/\xi$ is strictly decreasing.

The proof of Theorem 5 is now complete. □

7. Appendix: The maximum principle

Let D be a bounded domain of Euclidean space $\mathbf{R}^N$, with boundary ∂D, and let A be a second-order elliptic differential operator with real coefficients such that

$$Au(x) = -\sum_{i,j=1}^N a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^N b^i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x),$$

where:

(1) $a^{ij} \in C(\mathbf{R}^N)$, $a^{ij} = a^{ji}$ and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^N a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad x \in \mathbf{R}^N, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbf{R}^N.$$

(2) $b^i \in C(\mathbf{R}^N)$, $1 \leq i \leq N$.

(3) $c \in C(\mathbf{R}^N)$ and $c(x) \geq 0$ in D.

First we have the following (see [23, Theorem 7.1.1]):

THEOREM 7.1 (The weak maximum principle). *Assume that a function $u \in C(\bar{D}) \cap C^2(D)$ satisfies one of the conditions*

$$Au \geq 0 \text{ and } c > 0 \quad \text{in } D;$$

$$Au > 0 \text{ and } c \geq 0 \quad \text{in } D.$$

Then the function u may take its negative minimum only on the boundary ∂D.

Secondly we have the following (see [17, Chapter 2, Section, 3, Theorem 6], [23, Theorem 7.2.1]):

THEOREM 7.2 (The strong maximum principle). *Assume that a function $u \in C(\bar{D}) \cap C^2(D)$ satisfies the condition*

$$Au \geq 0 \qquad \text{in } D.$$

Then, if the function u attains its nonpositive minimum at an interior point of D, then it is constant.

Now assume that D is a domain of class C^2, that is, each point of the boundary ∂D has a neighborhood in which ∂D is the graph of a C^2 function of $N-1$ of the variables $x_1, x_2, \ldots, x_N$. We consider a function $u \in C(\bar{D}) \cap C^2(D)$ which satisfies the condition

$$Au \geq 0 \qquad \text{in } D,$$

and study the conormal derivative $\partial u/\partial v$ at a point where the function u takes its nonpositive minimum.

The boundary point lemma reads as follows (see [17, Chapter 2, Section 3, Theorem 8], [23, Lemma 7.1.7]):

LEMMA 7.3 (The boundary point lemma). *Let D be a domain of class C^2. Assume that a function $u \in C(\bar{D}) \cap C^2(D)$ satisfies the condition*

$$Au \geq 0 \qquad \text{in } D,$$

and that there exists a point x_0' of the boundary ∂D such that

$$\begin{cases} u(x_0') = \min_{x \in \bar{D}} u(x) \leq 0, \\ u(x) > u(x_0'), x \in D. \end{cases}$$

Then the conormal derivative $(\partial u/\partial v)(x_0')$ of u at the point x_0', if it exists, satisfies the condition

$$\frac{\partial u}{\partial v}(x_0') < 0.$$

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References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. **18** (1976), 620–709.

[2] R. Airs, The mathematical theory of diffusion and reaction in permeable catalysts, Clarendon Press, Oxford, 1975.

[3] M. S. Berger, Nonlinearity and functional analysis, Academic Press, New York San Francisco London, 1977.

[4] J. Bergh and J. Löfström, Interpolation spaces, an introduction, Springer-Verlag, Berlin Heidelberg New York, 1976.

[5] T. Boddington, P. Gray and C. Robinson, Thermal explosions and the disappearance of criticality at small activation energies: exact results for the slab, Proc. R. Soc. London A **368** (1979), 441–461.

[6] G. Bourdaud, L^p-estimates for certain non-regular pseudo-differential operators, Comm. Partial Differential Equations **7** (1982), 1023–1033.

[7] H. Brezis and L. Oswald, Remarks on sublinear elliptic equations, Nonlinear Analysis TMA **10** (1986), 55–64.

[8] K. J. Brown, M. M. A. Ibrahim and R. Shivaji, S-shaped bifurcation curves, Nonlinear Analysis TMA **5** (1981), 475–486.

[9] S.-N. Chow and J. K. Hale, Methods of bifurcation theory, Springer-Verlag, New York Heidelberg Berlin, 1982.

[10] D. S. Cohen and T. W. Laetsch, Nonlinear boundary value problems suggested by chemical reactor theory, J. Differential Equations **7** (1970), 217–226.

[11] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, J. Functional Analysis **8** (1971), 321–340.

[12] D. G. de Figueiredo, Positive solutions of semilinear elliptic problems, Lecture Notes in Mathematics, No. 957, Springer-Verlag, Berlin Heidelberg New York, 1982, pp. 34–87.

[13] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, New York Berlin Heidelberg Tokyo, 1983.

[14] L. Hörmander, The analysis of linear partial differential operators III, Springer-Verlag, New York Berlin Heidelberg Tokyo, 1983.

[15] M. A. Krasnosel'skii, Positive solutions of operator equations, P. Noordhoff, Groningen, 1964.

[16] L. Nirenberg, Topics in nonlinear functional analysis, Courant Institute of Mathematical Sciences, New York University, New York, 1974.

[17] M. H. Protter and H. F. Weinberger, Maximum principles in differential equations, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

[18] S. Rempel and B.-W. Schulze, Index theory of elliptic boundary problems, Akademie-Verlag, Berlin, 1982.

[19] J. T. Schwartz, Nonlinear functional analysis, Gordon and Breach, New York, 1969.

[20] R. B. Simpson and D. S. Cohen, Positive solutions of nonlinear elliptic eigenvalue problems, J. Math. Mech. **19** (1970), 895–910.

[21] A. Spence and B. Werner, Non-simple turning points and cusps, IMA J. Num. Analysis **2** (1982), 413–427.

[22] K. Taira, On some degenerate oblique derivative problems, J. Fac. Sci. Univ. Tokyo Sect. IA **23** (1976), 259–287.

[23] K. Taira, Diffusion processes and partial differential equations, Academic Press, San Diego New York London Tokyo, 1988.

[24] K. Taira, Analytic semigroups and semilinear initial boundary value problems, London Mathematical Society Lecture Note Series, No. 223, Cambridge University Press, London New York, 1995.

[25] K. Taira, Bifurcation for nonlinear elliptic boundary value problems I, Collect. Math. **47** (1996), 207–229.

[26] K. Taira and K. Umezu, Bifurcation for nonlinear elliptic boundary value problems II, Tokyo J. Math. **19** (1996), 387–396.

[27] K. Taira and K. Umezu, Bifurcation for nonlinear elliptic boundary value problems III, Adv. Differential Equations **1** (1996), 709–727.

[28] K. Taira and K. Umezu, Positive solutions of sublinear elliptic boundary value problems, Nonlinear Analysis, TMA **29** (1997), 761–771.

[29] H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland, Amsterdam, 1978.

[30] H. Wiebers, S-shaped bifurcation curves of nonlinear elliptic boundary value problems, Math. Ann. **270** (1985), 555–570.

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