Classification of closed oriented 4-manifolds modulo connected sum with simply connected manifolds

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ABSTRACT. The fundamental group is not changed by taking connected sum with simply connected manifolds. The closed oriented 4-dimensional manifolds with finite presentable fundamental group π are classified modulo this operation by the quotient $H_4(B\pi; \mathbf{Z})/(\operatorname{Aut}\pi)_*$. The relation to Lusternik-Schnirelmann π_1-category and some stable decomposition theorems are also discussed.

1. Introduction

Let M be a closed oriented 4-manifold. Then, we have a map $f : M \to B\pi = K(\pi, 1)$ to the Eilenberg-MacLane complex given by subsequently attaching cells of dimension greater than two, where π denotes $\pi_1(M)$. The map is unique up to homotopy by the obstruction theory if we fix the induced isomorphism on the fundamental group. The map determines the oriented cobordism class in $\Omega_4(B\pi)$. On the other hand any element of $\Omega_4(B\pi)$ gives a closed oriented 4-manifold N with $\pi_1(N) = \pi$ and a map $g : N \to B\pi$ by Lemma 5. The manifolds M and N will be shown to be stably equivalent in the sense that we have closed simply connected manifolds M_0 and N_0 satisfying $M \# M_0$ and $N \# N_0$ are orientation preserving diffeomorphic to each other.

Because any topological 4-manifold is smoothable possibly after taking connected sum with some copies of $S^2 \times S^2$ and a closed simply connected manifold with quadratic form E_8 [3], it is natural to restrict ourselves to smooth manifolds in this stable equivalence.

Since stably equivalent manifolds have the isomorphic fundamental group, it suffices to prove the following theorem for the classification.

THEOREM 1. *Let π be a finite presentable group. The stable equivalence classes of connected closed oriented 4-dimensional manifolds with fundamental*

group π are bijective to the quotient $H_4(B\pi; \mathbf{Z})/(\mathrm{Aut}\,\pi)_*$ by the correspondence $M \mapsto f_*(\sigma)$, where σ is the fundamental homology class of M and $f : M \to B\pi$ is a map inducing isomorphism on π_1.

If the fundamental group is a non-trivial free group, then its classifying space is a bouquet of circles and $H_4(\vee S^1; \mathbf{Z}) = 0$ so that every oriented manifold is stably equivalent to $\# S^1 \times S^3$ as we showed in [8].

We get a corollary relating to the L-S π_1-category.

COROLLARY 2. *If the Lusternik-Schnirelmann π_1-category of a given connected closed oriented 4-manifold M is not 4, then M is stably equivalent to the boundary $\partial N(K^2)$ of the regular neighborhood of an embedded finite 2-complex K^2 realizing the fundamental group in $\mathbf{R}^5$.*

If the corresponding element $f_*(\sigma) \in H_4(B\pi; \mathbf{Z})$ is non-zero then $f^* : H^4(B\pi; \mathbf{Z}_m) \to H^4(M; \mathbf{Z}_m)$ is non-zero for some m and so, $\mathrm{cat}_{\pi_1}(M) = 4$ by [7]. Here $\mathbf{Z}_\infty = \mathbf{Z}$. Therefore, if the L-S π_1-category of a given closed oriented 4-manifold M is not 4, then it corresponds to the zero element of $H_4(B\pi; \mathbf{Z})$. On the other hand the induced map from the boundary $\partial N(K^2)$ of the regular neighborhood passes through $H_4(K^2; \mathbf{Z}) = 0$. So, it also corresponds to the zero element of $H_4(B\pi; \mathbf{Z})$ and the corollary follows from Theorem 1.

Also we get a stable decomposition theorem weaker than that of [4] and [6] as another corollary.

COROLLARY 3. *If the fundamental group of a given closed oriented 4-manifold M is a free product $G_1 * \cdots * G_n$, then $M \# M_0$ is diffeomorphic to $M_1 \# \cdots \# M_n$ with $\pi_1(M_1) = G_1, \ldots, \pi_1(M_n) = G_n$ for some simply connected 4-manifold M_0. The decomposition is stably unique.*

The corollary easily follows from Theorem 1 because $H_4(B(G_1 * \cdots * G_n);$ $\mathbf{Z}) = H_4(BG_1 \vee \cdots \vee BG_n; \mathbf{Z}) = H_4(BG_1; \mathbf{Z}) \oplus \cdots \oplus H_4(BG_n; \mathbf{Z})$. Remark that M_0 in [6] is restricted to be a connected sum of some copies of $S^2 \times S^2$ but the uniqueness statement in [6] is the same as ours.

After completing a proof of Theorem 1 in §2, we will discuss the case of topological manifolds and another stable decomposition theorem due to [4] in §3.

2. Proof of Theorem 1

For a topological space X the set $\mathbf{M}_n(X)$ consists of all the pairs of $\{M, f\}$ where M is a closed oriented smooth n-manifold and $f : M \to X$ is a map. We denote $\{M, f\} \sim \{N, g\}$ if there is an oriented cobordism $\{W, F\}$

such that $\partial W = M \cup -N$, $F|M = f$ and $F|N = g$. The oriented cobordism group $\Omega_n(X)$ is defined by the quotient $\mathbf{M}_n(X)/\sim$ [2]. The oriented cobordism class of $\{M, f\}$ will be denoted by $[M, f]$.

Since we know about the oriented cobordism group $\Omega_0 \cong \mathbf{Z}$, $\Omega_1 = 0$, $\Omega_2 = 0$, $\Omega_3 = 0$ and $\Omega_4 \cong \mathbf{Z}$ generated by CP^2 [9], we see the following lemma.

LEMMA 4. *Let X be a CW complex. The map $\mu : \Omega_4(X) \to H_4(X; \mathbf{Z})$ defined by $\mu([M, f]) = f_*(\sigma)$ is a surjection and the kernel is Ω_4, where σ is the fundamental homology class of M. Moreover, the restriction of μ on $\tilde{\Omega}_4(X) = \mathrm{Ker}(\Omega_4(X) \to \Omega_4(*))$ is an isomorphism.*

PROOF. We have a spectral sequence $E^2_{p,q} = H_p(X; \Omega_q) \Rightarrow \Omega_{p+q}(X)$, which is regular and hence convergent in the sense of [1]. By dimensional reasoning $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ and by comparing with the spectral sequence for $\Omega_{p+q}(*)$, we see easily that every element of $E^2_{0,4}$ and $E^2_{4,0}$ is a permanent cycle. Then, we get an exact sequence $0 \to E^2_{0,4} \to \Omega_4(X) \to E^2_{4,0} \to 0$. The map $\mu : \Omega_n(X) \to H_n(X; \mathbf{Z})$ induces a map from the spectral sequence for $\Omega_{p+q}(X)$ to the spectral sequence for $H_{p+q}(X; \mathbf{Z})$ and coincides with the map $\Omega_4(X) \to E^2_{4,0}$ for $n = 4$.

The following lemma is a key step.

LEMMA 5. *If π is finite presentable, any element ω of $\Omega_4(B\pi)$ gives a closed oriented 4-manifold N with $\pi_1(N) = \pi$ and a map $g : N \to B\pi$ such that g induces an isomorphism on π_1 and $[N, g] = \omega$.*

PROOF. As in Corollary 2 we have a closed oriented 4-manifold $\partial N(K^2)$ with fundamental group π so that the element $[\partial N(K^2), f_0]$ of $\Omega_4(B\pi)$ is zero and f_0 induces an isomorphism on π_1. In fact, $\{\partial N(K^2), f_0\}$ is the boundary of $\{N(K^2), F\}$. For a given element $\omega = [M, f]$ of $\Omega_4(B\pi)$ we take a connected sum $M \# \partial N(K^2)$ and a map $f' = f \# f_0$. Then the kernel of the induced map $f'_* : \pi_1(M \# \partial N(K^2)) \to \pi_1(B\pi)$ is generated by $f_*(\gamma_1)\gamma_1^{-1}, \ldots, f_*(\gamma_m)\gamma_m^{-1}$, where $\gamma_1, \ldots, \gamma_m$ are the generators of $\pi_1(M)$ and $f_*(\gamma_1), \ldots, f_*(\gamma_m)$ are considered as the elements of $\pi_1(\partial N(K^2))$. Since this 4-manifold $M \# \partial N(K^2)$ is oriented, it is easy to annihilate the kernel of the fundamental group by 1-dimensional surgery on the embedded circles representing these generators and get a new closed oriented 4-manifold N and a map $g : N \to B\pi$ which induces an isomorphism $g_* : \pi_1(N) \to \pi_1(B\pi) = \pi$ such that $[N, g] = [M, f] + [\partial N(K^2), f_0] = [M, f]$.

LEMMA 6. *If two pairs $\{M, f\}$ and $\{N, g\}$ represent the same element of $\Omega_4(B\pi)$ such that the induced maps on the fundamental group are isomorphic, then we have an oriented cobordism $\{W, F\}$ such that $\partial W = M \cup -N$ and both $M \subset W$ and $N \subset W$ induce isomorphisms on π_1.*

PROOF. The proof is almost the same as that of Lemma 5. Let $\{W', F'\}$

be a cobordism between $\{M, f\}$ and $\{N, g\}$. The finite generators of $\pi_1(W')$ and their induced elements considered in $\pi_1(M)$ are used to get the generators of the kernel of the map $F'_* : \pi_1(W') \to \pi_1(B\pi)$. The 1-dimensional surgery is performed in the 5-dimensional oriented manifold W' and get a new cobordism $\{W, F\}$ between $\{M, f\}$ and $\{N, g\}$ which induces an isomorphism on π_1. Then, $M \subset W$ and $N \subset W$ induce isomorphisms on π_1.

LEMMA 7. *Assume the 5-dimensional oriented cobordism W between M and N satisfies the condition that $\partial W = M \cup -N$ and both $M \subset W$ and $N \subset W$ induce isomorphisms on π_1. Then, there are M_0 and N_0 which are connected sums of some copies of $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$ such that $M \# M_0$ is orientation preserving diffeomorphic to $N \# N_0$.*

PROOF. We can simplify the handle decomposition of W relative to M so that it has only 2-handles and 3-handles as in the usual proof of s-cobordism theorem in higher dimension. Then, the feet of 2-handles are isotopic to the trivial one because it should represent the zero element in π_1 by the condition. So, the middle level manifold is a connected sum of M and some copies of $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$. By thinking from the other direction it is also diffeomorphic to a connected sum of N and some copies of $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$.

Now we are in a position to prove Theorem 1. A closed oriented 4-manifold M with fundamental group π carries a classifying map $f : M \to B\pi$ to the Eilenberg-MacLane complex. The pair determines the oriented cobordism class $[M, f]$ of $\Omega_4(B\pi)$ and then an element $\mu([M, f])$ of $H_4(B\pi; \mathbf{Z})$. Stably euqivalent manifolds determine the same element of $H_4(B\pi; \mathbf{Z})$ because $\mu([M \# M_0, f \# f_0]) = f_*(\sigma)$ where M_0 is a closed simply connected manifold, $f_0 : M_0 \to B\pi$ is a collapsing map to one point and σ is the fundamental homology class of the oriented manifold M. We may change the map f by any automorphism of the fundamental group and the indeterminacy comes in.

On the other hand take any element of $H_4(B\pi; \mathbf{Z})$. Then, it gives an element of $\tilde{\Omega}_4(B\pi) = \text{Ker}(\Omega_4(B\pi) \to \Omega_4(*))$ by Lemma 4. It comes from a closed oriented 4-manifold N with $\pi_1(N) = \pi$ and a map $g : N \to B\pi$ by Lemma 5. Let $\{M, f\}$ be another pair with $\pi_1(M) = \pi$ and a map $f : M \to B\pi$ such that $\mu([M, f]) = \mu([N, g])$. Then for some k, ℓ we have $[M \# kCP^2, f'] = [N \# \ell CP^2, g']$ in $\Omega_4(B\pi)$ by Lemma 4 and that Ω_4 is generated by CP^2, where f', g' are maps sending CP^2's to one point. Therefore, the manifolds M and N are stably equivalent by Lemmas 6 and 7, that is, $M \# M_0$ and $N \# N_0$ are orientation preserving diffeomorphic to each other by taking $M_0 = kCP^2 \# k_1 S^2 \times S^2 \# k_2 S^2 \tilde{\times} S^2$ and $N_0 = \ell CP^2 \# \ell_1 S^2 \times S^2 \# \ell_2 S^2 \tilde{\times} S^2$ for some $k, k_1, k_2, \ell, \ell_1$ and ℓ_2.

3. Further comments

We discuss first the topological manifolds and then Theorem 8 which is an improvement of Corollary 3 without uniqueness.   Moreover, we give some calculations of $H_4(B\pi; \mathbf{Z})/(\operatorname{Aut}\pi)_*$.

Since the topological 4-manifolds are stably smoothable [3] as remarked in the introduction, Theorem 1 and hence Corollaries 2 and 3 are valid also for the topological manifolds if we replace the word diffeomorphic with homeomorphic in the stable equivalence.   By the way to get a proof using relative topological handle decomposition due to [5] it suffices to admit the existence of a closed simply connected manifold with quadratic form E_8 [3].

Note that the stable diffeomorphism or homeomorphism used in [3] and others is somewhat different from our stable equivalence; there allows only connected sum with some copies of $S^2 \times S^2$.   We will sketch a proof of the following Theorem 8 due to [4].

THEOREM 8 [4].   *If the fundamental group of a given closed 4-manifold M is a free product $\pi = G_1 * G_2$, then for some $m \geq 0$ the connected sum $M \# mS^2 \times S^2$ is diffeomorphic to $M_1 \# M_2$ with $\pi_1(M_1) = G_1$ and $\pi_1(M_2) = G_2$.*

Outline of proof.   Take a map $f : M \to B\pi = BG_1 \cup [-1, 1] \cup BG_2$ which induces an isomorphism on π_1 and is transverse at 0.   We may assume that the preimage $V = f^{-1}(0)$ is connected.   Then, V is an orientable 3-manifold because $f^* : H^1(B\pi; \mathbf{Z}_2) \to H^1(M; \mathbf{Z}_2)$ is an isomorphism and hence $w_1(V) = 0$. If we choose an appropriate spin structure on V, the argument of [4] gives the result.   In fact, V is spin cobordant to S^3 for any spin structure on V because $\Omega_3^{spin} = 0$.   A modified spin cobordism can be constructed only with 2-handles. Attach 2-handles to M making these 2-handles fat at $V \times [-\varepsilon, \varepsilon]$.   Since the feet of these 2-handles are homotopic to one point in M, the surgered manifold $\hat{M}$ is diffeomorphic to a connected sum with some copies of $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ to M.   It is clear also that $\hat{M}$ is decomposed into the desired connected sum $M_1 \# M_2$ by the embedded S^3.   In case $w_2(\tilde{M}) = p^* w_2(M) \neq 0$ for the universal cover $p : \tilde{M} \to M$ the connected sum $M \# mS^2 \tilde{\times} S^2$ is diffeomorphic to $M \# mS^2 \times S^2$ for any m and there is no problem.   Otherwise we have to choose the spin structure on V so that the framing of 2-handles gives the connected sum only with the copies of $S^2 \times S^2$.   To determine the spin structure on V we choose an appropriate framing of the regular neighborhood of a representing circle for each element of $H_1(V; \mathbf{Z}_2)$.   The spin structure does not depend on the choice of representatives and we cannot find appropriate framings only when $w_2(\tilde{M}) \neq 0$ by the exact sequence

$$0 \longrightarrow H^2(B\pi; \mathbf{Z}_2) \xrightarrow{f^*} H^2(M; \mathbf{Z}_2) \xrightarrow{p^*} H^2(\tilde{M}; \mathbf{Z}_2)$$

coming from Serre spectral sequence of the fibration $\tilde{M} \xrightarrow{p} M \xrightarrow{f} B\pi$.

Some examples of the calculation of $A = H_4(B\pi; \mathbf{Z})/(\mathrm{Aut}\,\pi)_*$:

1) If π is a free group or a cyclic group or a surface group, $A = 0$.
2) If $\pi = \pi_1$ (*a closed aspherical 4-manifold*), then $A = \mathbf{Z}$ or $\mathbf{Z}/\pm 1$. In particular, $A = \mathbf{Z}/\pm 1$ for $\pi = \pi_1(S^1 \times a$ closed aspherical 3-manifold).
3) If $\pi = \mathbf{Z} \times \mathbf{Z}_p$ with p a prime number, then $|A| = 2$. This is because $H_4(B\pi; \mathbf{Z}) = \mathbf{Z}_p$ and $\mathrm{Aut}\,\pi$ identifies all the non-zero elements.

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