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A mixed finite element method to solve the Stokes problem in the stream function and vorticity formulation

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ABSTRACT. The aim of this work is to present a new mixed finite element method to solve the two-dimensional Stokes problem in terms of the stream function and the vorticity. The main feature of the method is to overcome the difficulty associated with the lack of boundary conditions for the vorticity on a no-slip boundary, by means of an incorporated uncoupling technique of both variables.

Introduction

Let us begin by recalling the problem to solve: Given a field of volumetric forces f and denoting by ν the kinematic viscosity of the fluid occupying a bounded simply connected domain Ω of R^2 with boundary Γ, we wish to find the stream function ψ and the vorticity ω such that:

$$\begin{cases} -\nu\Delta\omega = \operatorname{curl} f & \text{in } \Omega \\ -\Delta\psi = \omega & \text{in } \Omega. \end{cases} \tag{0.1}$$

We consider the case where the following boundary conditions apply:

$$\begin{cases} \psi = g_0 & \text{on } \Gamma \\ \frac{\partial\psi}{\partial n} = g_1 & \text{on } \Gamma, \end{cases} \tag{0.2}$$

where g_0 and g_1 are functions determined from the given velocity on Γ (Cf. [8]).

Long since some advantages are recognized in using the stream function-vorticity formulation to solve computationally the two-dimensional incompressible Navier-Stokes equations. Among those let us mention the fact that the former leads to computed flow fields that implicitly satisfy the incompressibility condition; moreover the number of unknown functions is reduced from three to two and the vorticity is computed directly instead of being obtained from the velocity field.

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The main difficulty for treating system (0.1) is the fact that, whenever the velocity is given on Γ, there are two boundary conditions for ψ and none for ω.

Many authors contributed to the study of the Stokes system in terms of the stream function and the vorticity. In this context we refer in particular to the bibliography and comments given in the seventh chapter of Ciarlet's book [6], and also to Girault-Raviart [9] and Brezzi-Fortin [4].

In [11] the second author proposed and studied a finite element method for solving system (0.1)–(0.2) based on the direct computation of a harmonic component of the vorticity. In this paper we shall further exploit this approach, by presenting a new mixed method to solve the associated saddle-point problem.

An outline of the paper is as follows: In § 1 we recall some basic results due to Ruas [10, 11], and we introduce two related saddle-point problems. § 2 is devoted to the analysis of the continuous saddle-point problems. In § 3 we study the discrete version of the problems in § 2, obtained by the finite element technique. In § 4 and 5 we make a series of remarks concerning implementation and relationship between our method and other works. We conclude with some numerical illustrations given in § 6.

1. Variational and uncoupling techniques

Like in [11], in order to derive a variational formulation of system (0.1) and (0.2) we consider the set

$$X(\Omega) = \{\chi \in L^2(\Omega); \Delta\chi \in H^{-1}(\Omega)\}. \tag{1.1}$$

$X(\Omega)$ is a Hilbert space with the following inner product

$$(\chi, \xi)_X = (\chi, \xi)_0 + (\chi_0, \xi_0)_1, \tag{1.2}$$

where $(\cdot,\cdot)_0$ denotes the standard inner product of $L^2(\Omega)$ and χ_0 in $H_0^1(\Omega)$ is uniquely defined by

$$(\chi_0, v)_1 = \langle -\Delta\chi, v\rangle \quad (\forall v \in H_0^1(\Omega)). \tag{1.3}$$

$(\cdot,\cdot)_1$ and $\langle\cdot,\cdot\rangle$ denote the standard inner product of $H_0^1(\Omega)$ and the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ respectively. We further introduce the following closed subset of $X(\Omega)$:

$$X_H(\Omega) = \{\chi \in L^2(\Omega); \Delta\chi = 0\}. \tag{1.4}$$

Using standard arguments from the theory of Hilbert spaces, we may uniquely write every function $\chi \in X(\Omega)$ as a sum of the form:

$$\chi = \chi_0 + \chi_H, \quad \text{where } \chi_0 \in H_0^1(\Omega) \text{ is uniquely defined by (1.3).} \tag{1.5}$$

Note that χ_H is the harmonic function of $L^2(\Omega)$ whose trace over Γ coincides with that of χ.

Let us assume that $f \in L^2(\Omega)^2$, $g_i \in H^{3/2-i}(\Gamma)$ for $i = 0, 1$. Set $\omega = \omega_0 + \omega_H$, where $\omega_0 \in H_0^1(\Omega)$ and $\omega_H \in X_H(\Omega)$.

The variational form associated with the system (0.1)–(0.2) is:

$$
(VP) \quad \begin{cases} \text{Find } (\omega_0, \omega_H, \psi) \in H_0^1(\Omega) \times X_H(\Omega) \times H_{g_0}^1(\Omega) \text{ such that} \\ \text{i) } (\omega_0, \varphi)_1 = \dfrac{1}{\nu}(f, \operatorname{curl} \varphi)_0 & (\forall \varphi \in H_0^1(\Omega)) \\ \text{ii) } (\omega_H, \chi_H)_0 = -(\omega_0, \chi_H)_0 - \langle g_1, \chi_H \rangle_{1/2} + \left\langle g_0, \dfrac{\partial \chi_H}{\partial n} \right\rangle_{3/2} & (\forall \chi_H \in X_H(\Omega)) \\ \text{iii) } (\psi, \chi_0)_1 = (\omega_0 + \omega_H, \chi_0)_0 & (\forall \chi_0 \in H_0^1(\omega)) \end{cases}
$$

where $H_{g_0}^1(\Omega) = \{v \in H^1(\Omega); v_{|\Gamma} = g_0\}$ and $\langle \cdot, \cdot \rangle_s$ denotes the duality product between $H^{-s}(\Gamma)$ and $H^s(\Gamma)$ with $s \in R^+$.

Proposition 1. *Problem (VP) has a unique solution $(\omega_0, \omega_H, \psi)$, such that the field $(\psi, \omega_0 + \omega_H)$ is the unique solution of system* (0.1)–(0.2).

Proof. See [8, 11].

For the sake of simplicity but without loss of essential results we assume hence-forth that $g_0 = 0$ and $g_1 = 0$.

2. A saddle-point for the Stokes problem

While problems $(VP\text{-i})$ and $(VP\text{-iii})$ are classical, $(VP\text{-ii})$ is a problem with a constraint $\omega_H \in X_H(\Omega)$. In order to avoid this difficulty we introduce an equivalent saddle-point problem assuming that ω_0 is a solution of $(VP\text{-i})$:

$$
(SP1) \quad \begin{cases} \text{Find } (\omega_H, \lambda) \in L^2(\Omega) \times H_0^2(\Omega) \text{ such that} \\ (\omega_H, v)_0 - (v, \Delta\lambda)_0 = -(\omega_0, v)_0 \quad (\forall v \in L^2(\Omega)) \text{ and} \\ \qquad (\omega_H, \Delta\mu)_0 = 0 \qquad (\forall \mu \in H_0^2(\Omega)). \end{cases}
$$

Now we set $V = L^2(\Omega)$, $M = H_0^2(\Omega)$ and we define the two bilinear forms

$$a(u, v) = (u, v)_0 \text{ for } u, v \in V \text{ and } b(u, \mu) = -(u, \Delta\mu)_0 \text{ for } u \in V,\ \mu \in M.$$

We can rewrite problem $(SP1)$ as follows:

$$
\begin{cases} \text{Find } (\omega_H, \lambda) \in V \times M \text{ such that} \\ a(\omega_H, v) + b(v, \lambda) = -(\omega_0, v)_0 \quad (\forall v \in V) \text{ and} \\ \qquad b(\omega_H, \mu) = 0 \qquad (\forall \mu \in M). \end{cases}
$$

Theorem 1. *Problem* $(SP1)$ *has a unique solution* (ω_H, λ) *in* $V \times M$. *Moreover* ω_H *is the unique solution of problem* $(VP\text{-}ii)$.

Proof. See [8].

Remark 1. The main difficulty in solving problem $(SP1)$ is the approximation of the space $H_0^2(\Omega)$, by finite element spaces. We shall next give another characterization of the solution of this problem in $(SP2)$.

2.1. An ill-posed mixed problem

In order to avoid the difficulty mentioned in Remark 1 we use a result due to Brezzi and Raviart, namely, Theorem 2 of [5]. In other words we introduce a second saddle-point problem and we show that under some hypothesis this problem has the same solution as $(SP1)$.

In order to replace $X_H(\Omega)$ suitably, we introduce the set

$$\tilde{X}_H(\Omega) = \{u \in H^1(\Omega); (\nabla u, \nabla v)_0 = 0 \ (\forall v \in H_0^1(\Omega))\} \tag{2.1}$$

which is a closed subspace of $H^1(\Omega)$. Let us consider the following problem:

$$(CP) \qquad \begin{cases} \text{Find } \omega_H \in \tilde{X}_H(\Omega) \text{ such that} \\ (\omega_H, v)_0 = -(\omega_0, v)_0 \quad (\forall v \in \tilde{X}_H(\Omega)). \end{cases}$$

Problem (CP) can be viewed as a constrained problem, whose constraint is $\omega_H \in \tilde{X}_H(\Omega)$. It can be changed into an unconstrained one, at the cost of adding a Lagrange multiplier. More specifically, problem (CP) is equivalent to the following saddle-point problem:

$$(SP2) \qquad \begin{cases} \text{Find}(\omega_H, \lambda) \in \tilde{V} \times \tilde{M} \text{ such that} \\ a(\omega_H, v) + \tilde{b}(v, \lambda) = -(\omega_0, v)_0 \quad (\forall v \in \tilde{V}) \text{ and} \\ \qquad \tilde{b}(\omega_H, \mu) = 0 \qquad (\forall \mu \in \tilde{M}) \end{cases}$$

where $\tilde{V} = H^1(\Omega)$, $\tilde{M} = H_0^1(\Omega)$ and $\tilde{b}(v, \mu) = (\nabla v, \nabla \mu)_0$ for $v \in \tilde{V}$, $\mu \in \tilde{M}$.

Remark 2. Since the bilinear form $a(\cdot,\cdot)$ is not a priori $H^1(\Omega)$-elliptic, problem $(SP2)$ is not well-posed at least in general.

Theorem 2. *If the first argument* ω_H *of the solution* (ω_H, λ) *of problem* $(SP1)$ *belongs to the space* $H^1(\Omega)$, *then* (ω_H, λ) *is the unique solution of problem* $(SP2)$.

Proof. If the first argument of the solution (ω_H, λ) of problem $(SP1)$ belongs to $H^1(\Omega)$, then (ω_H, λ) is a solution of problem $(SP2)$ because $H_0^2(\Omega)$ is dense in $H_0^1(\Omega)$ and we have $(\nabla v, \nabla \mu)_0 = -(v, \Delta \mu)_0$ $(\forall v \in \tilde{V}, \forall \mu \in M)$.

Let us prove that problem $(SP2)$ has a unique solution. Suppose that $(\varpi_H, \bar{\lambda}) \in H^1(\Omega) \times H_0^1(\Omega)$ is a solution of the following problem:

$$\begin{cases} \text{Find}(\varpi_H, \bar{\lambda}) \in \tilde{V} \times \tilde{M} \text{ such that} \\ a(\varpi_H, v) + \tilde{b}(v, \bar{\lambda}) = 0 \quad (\forall v \in \tilde{V}) \text{ and} \\ \qquad\qquad \tilde{b}(\varpi_H, \mu) = 0 \quad (\forall \mu \in \tilde{M}). \end{cases}$$

Taking $v = \varpi_H$ we obtain $\tilde{b}(\varpi_H, \bar{\lambda}) = 0$ and hence $a(\varpi_H, \varpi_H) = 0$ that is $\varpi_H = 0$. Therefore $(\nabla v, \nabla \bar{\lambda})_0 = 0$ $(\forall v \in H^1(\Omega))$. On the other hand for some $\tilde{\beta} > 0$

$$\operatorname*{Sup}_{v \in H^1(\Omega)} \frac{(\nabla v, \nabla \mu)_0}{\|v\|_{H^1(\Omega)}} \geq \tilde{\beta} \|\mu\|_{H_0^1(\Omega)} \quad (\forall \mu \in H_0^1(\Omega))$$

which yields $\bar{\lambda} = 0$.

REMARK 3. i) In general if Ω is an open bounded domain of R^2 and Γ is of the C^2 class (Cf. [4, 9]) then $\omega_H \in H^1(\Omega)$.

ii) If ψ, a solution of problem (VP-iii), belongs to $H^3(\Omega)$ then $\lambda = -\psi$ (Cf. [1]).

iii) Since the finite element approximation of problem $(SP2)$ involves only the construction of finite-dimensional subspaces of the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$, problem $(SP2)$ is easier to approximate numerically than problem $(SP1)$.

3. Discretized Variational problems

Throughout this section, Ω will be a polygonal domain. In all the sequel, we shall assume that the conclusion of Theorem 2 holds.

Let $\{T_h\}_h$ be a quasi-uniform family of triangulations of $\bar{\Omega}$ with mesh step size $h < 1$. We introduce the finite element spaces defined by:

$$\begin{aligned} Y_h &= \{v_h \in L^2(\Omega); v_{h|K} \in P_k(K) \ (\forall K \in T_h)\} \\ V_h &= Y_h \cap C^0(\Omega) \\ M_h &= V_h \cap H_0^1(\Omega), \end{aligned}$$

where $P_k(K)$ denotes the space of polynomials of degree less than or equal to k, defined in K. Finally, we define

$$X_H^h = \{u_h \in V_h, (\nabla u_h, \nabla v_h)_0 = 0 \quad (\forall v_h \in M_h)\}. \tag{3.1}$$

We note that $V_h \subset H^1(\Omega)$ and $M_h \subset H_0^1(\Omega)$.

3.1. Approximation of problem (VP-i)

We discretize the formulation (VP-i) by

$$(P_{0h}) \qquad \begin{cases} \text{Find } \omega_0^h \in M_h \text{ such that} \\ \nu(\nabla\omega_0^h, \nabla\varphi_h)_0 = (f, \operatorname{curl} \varphi_h)_0 \quad (\forall\varphi_h \in M_h). \end{cases}$$

We note that problem (P_{0h}) has a unique solution (see Ciarlet [6]); moreover we have the following error estimate.

THEOREM 3. *If $\omega_0 \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ for k greater than or equal to one, then we have $\|\omega_0 - \omega_0^h\|_1 \leq Ch^k\|\omega_0\|_{k+1}$ where C is a constant independent of h.*

PROOF. See Ciarlet [6].

REMARK 4. If Ω is a bounded convex polygon then we have the L^2-estimate $\|\omega_0 - \omega_0^h\|_0 \leq Ch^{k+1}\|\omega_0\|_{k+1}$.

3.2. Approximation of problem ($SP2$) by a mixed finite element method.

The discretized problem associated with ($SP2$) is:

$$(SPD) \qquad \begin{cases} \text{Find}(\omega_H^h, \lambda_h) \in V_h \times M_h \text{ such that} \\ (\omega_H^h, v_h)_0 + (\nabla v_h, \nabla\lambda_h)_0 = -(\omega_0^h, v_h)_0 \quad (\forall v_h \in V_h) \text{ and} \\ \qquad (\nabla\omega_H^h, \nabla\mu_h)_0 = 0 \qquad (\forall\mu_h \in M_h). \end{cases}$$

THEOREM 4. *As well as problem ($SP2$), problem (SPD) has a unique solution $(\omega_H^h, \lambda_h) \in V_h \times M_h$.*

PROOF. The result is a simple application of Brezzi's theory [5] (see [8]).

REMARK 5. We note that $\lambda_h = -\psi_h$ where ψ_h is the unique solution of the following discrete problem:

$$\begin{cases} \text{Find } \psi_h \in M_h \text{ such that} \\ (\nabla\psi_h, \nabla\varphi_h)_0 = (\omega_0^h + \omega_H^h, \varphi_h)_0 \quad (\forall\varphi_h \in M_h). \end{cases}$$

THEOREM 5. *Assume that Ω is a bounded convex polygonal domain. For an integer k greater than or equal to two, we assume*

$$\psi \in H^{k+1}(\Omega) \cap H_0^2(\Omega), \quad \omega_0 \in H^k(\Omega) \cap H_0^1(\Omega) \text{ and } \omega_H \in H^k(\Omega).$$

Then we have

$$\|\omega_H - \omega_H^h\|_0 + \|\psi - \psi_h\|_1 \leq Ch^{k-1}\{\|\omega_H\|_k + \|\omega_0\|_k + \|\psi\|_{k+1}\} \text{ and}$$

$$\|\psi - \psi_h\|_0 \leq \tilde{C}h^k\{\|\omega_H\|_k + \|\omega_0\|_k + \|\psi\|_{k+1}\}$$

where C and $\tilde{C}$ are constants independent of h.

PROOF. In order to prove this theorem we use the result due to Brezzi and Raviart (Cf. [5]. Theorem 3), with a simple modification in the case where the right hand side of the discrete problem is an approximation of the one of the continuous problem.

In order to establish this result, we have to prove the two following error estimates

$$(3.2.1)\quad \|\omega_H - \omega_H^h\|_0 \leq A\left\{\operatorname*{Inf}_{v_h \in V_h} \|v_h - \omega_H\|_1 + \frac{1}{h}\operatorname*{Inf}_{\mu_h \in M_h} \|\lambda - \mu_h\|_1 + \|\omega_0 - \omega_0^h\|_0\right\}$$

$$(3.2.2)\qquad \|\lambda - \lambda_h\|_1 \leq B\left\{\|\omega_H - \omega_H^h\|_0 + \operatorname*{Inf}_{\mu_h \in M_h} \|\lambda - \mu_h\|_1 + \|\omega_0 - \omega_0^h\|_0\right\}.$$

We recall that $\lambda = -\psi$, $\lambda_h = -\psi_h$,

$$(3.2.3)\qquad \operatorname*{Inf}_{\mu_h \in M_h} \|\psi - \mu_h\|_1 \leq Ch^k\|\psi\|_{k+1},$$

$$(3.2.4)\qquad \operatorname*{Inf}_{v_h \in V_h} \|\omega_H - v_h\|_1 \leq Ch^{k-1}\|\omega_H\|_k \text{ and}$$

$$(3.2.5)\qquad \|\omega_0 - \omega_0^h\|_0 \leq Ch^k\|\omega_0\|_k.$$

Combining inequalities (3.2.1), (3.2.2), (3.2.3), (3.2.4) and (3.2.5) we obtain

$$\|\omega_H - \omega_H^h\|_0 + \|\psi - \psi_h\|_1 \leq Ch^{k-1}\{\|\omega_H\|_k + \|\omega_0\|_k + \|\psi\|_{k+1}\}.$$

Now let us prove inequalities (3.2.1) and (3.2.2).

Error estimate (3.2.1)

By definition $\omega_H^h \in X_H^h$ and for every $v_h \in X_H^h$ we have

$$(3.2.6)\quad a(v_h - \omega_H^h, v_h - \omega_H^h) = a(v_h - \omega_H, v_h - \omega_H^h) + a(\omega_H - \omega_H^h, v_h - \omega_H^h).$$

Combining this with (SPD) and $(SP2)$, we obtain

$$a(\omega_H - \omega_H^h, v_h - \omega_H^h) = (\omega_0^h - \omega_0, v_h - \omega_H^h)_0 + \tilde{b}(v_h - \omega_H^h, \mu_h - \lambda) \quad (\forall \mu_h \in M_h).$$

Relation (3.2.6) thus becomes:

$$a(v_h - \omega_H^h, v_h - \omega_H^h) = a(v_h - \omega_H, v_h - \omega_H^h) + \tilde{b}(v_h - \omega_H^h, \mu_h - \lambda) + (\omega_0^h - \omega_0, v_h - \omega_H^h)_0.$$

Hence we have

$$\|v_h - \omega_H^h\|_0^2 \le \|v_h - \omega_H\|_0 \, \|v_h - \omega_H^h\|_0 + \|v_h - \omega_H^h\|_1 \, \|\mu_h - \lambda\|_1 + \|\omega_0 - \omega_0^h\|_0 \, \|v_h - \omega_H^h\|_0.$$

Now, using the inverse inequality (Cf. Ciarlet [6], Girault-Raviart [9] and Brezzi-Fortin [4]) $\|v_h\|_1 \le \frac{A}{h}\|v_h\|_0$ $(\forall v_h \in V_h)$ with A independent of h, we obtain:

$$\|v_h - \omega_H^h\|_0 \le \|v_h - \omega_H\|_0 + \frac{A}{h}\|\mu_h - \lambda\|_1 + \|\omega_0 - \omega_0^h\|_0.$$

Combining the above inequality with $\|\omega_H - \omega_H^h\|_0 \le \|\omega_H - v_h\|_0 + \|v_h - \omega_H^h\|_0$ we obtain

$$\|\omega_H - \omega_H^h\|_0 \le 2\|v_h - \omega_H\|_0 + \frac{A}{h}\|\mu_h - \lambda\|_1 + \|\omega_0 - \omega_0^h\|_0 \quad \text{and hence}$$

$$(3.2.7) \quad \|\omega_H - \omega_H^h\|_0 \le C \inf_{v_h \in X_H^h} \|v_h - \omega_H\|_0 + \frac{A}{h} \inf_{\mu_h \in M_h} \|\mu_h - \lambda\|_1 + \|\omega_0 - \omega_0^h\|_0.$$

Finally, using the discrete inf-sup condition: for some $\gamma > 0$

$$(*) \qquad \sup_{v_h \in V_h} \frac{|\tilde{b}(v_h, \mu_h)|}{\|v_h\|_1} \ge \gamma \|\mu_h\|_1 \quad (\forall \mu_h \in M_h)$$

we have (Cf. Girault-Raviart [9], Brezzi-Fortin [4])

$$(3.2.8) \qquad \inf_{v_h \in X_H^h} \|\omega_H - v_h\|_1 \le C \inf_{\varphi_h \in V_h} \|\omega_H - \varphi_h\|_1$$

which together with inequality (3.2.7) gives (3.2.1).

Error estimate (3.2.2)

From $(SP2)$ and (SPD), we obtain for every $v_h \in V_h$

$$a(\omega_H - \omega_H^h, v_h) + \tilde{b}(v_h, \lambda - \lambda_h) = (\omega_0^h - \omega_0, v_h)_0.$$

Then we get for every $\mu_h \in M_h$

$$\tilde{b}(v_h, \lambda_h - \mu_h) = a(\omega_H - \omega_H^h, v_h) + (\omega_0 - \omega_0^h, v_h)_0 + \tilde{b}(v_h, \lambda - \mu_h).$$

Using again the discrete inf-sup condition $(*)$ and the inclusion $\tilde{V} \subset V$ we obtain

$$\|\mu_h - \lambda_h\|_1 \le C\{\|\omega_0 - \omega_0^h\|_0 + \|\lambda - \mu_h\|_1 + \|\omega_H - \omega_H^h\|_0\}.$$

Combining the above inequality with $\|\lambda - \lambda_h\|_1 \leq \|\lambda - \mu_h\|_1 + \|\mu_h - \lambda_h\|_1$ we obtain

$$\|\lambda - \lambda_h\|_1 \leq \tilde{C}\left\{ \|\omega_0 - \omega_0^h\|_0 + \|\omega_H - \omega_H^h\|_0 + \operatorname*{Inf}_{\mu_h \in M_h} \|\lambda - \mu_h\|_1 \right\}.$$

Error estimate for $(\lambda - \lambda_h)$ in the space $L^2(\Omega)$.

We need to generalize the classical Aubin-Nitsche's duality argument like in [5]. We may write

$$\|\lambda - \lambda_h\|_0 = \operatorname*{Sup}_{\varphi \in L^2(\Omega)} \frac{|(\varphi, \lambda - \lambda_h)_0|}{\|\varphi\|_0}.$$

Since problem $(SP1)$ is regular (Cf. [5]), any solution of the problem: find $(y, \zeta) \in V \times M$ such that $a(v, y) + b(v, \zeta) = 0$ $(\forall v \in V)$ and $b(y, \mu) = (\varphi, \mu)_0$ $(\forall \mu \in M)$ where φ is given in $L^2(\Omega)$, satisfies

$$\begin{cases} a(v, y) + \tilde{b}(v, \zeta) = 0 \quad (\forall v \in \tilde{V}) \text{ and} \\ \qquad\qquad \tilde{b}(y, \mu) = (\varphi, \mu)_0 \quad (\forall \mu \in \tilde{M}). \end{cases}$$

Hence we obtain

$$a(\omega_H - \omega_H^h, y) + \tilde{b}(\omega_H - \omega_H^h, \zeta) = 0. \tag{3.2.9}$$

On the other hand, using $(SP2)$ and (SPD) we have

$$a(\omega_H - \omega_H^h, y_h) + \tilde{b}(y_h, \lambda - \lambda_h) = (\omega_0^h - \omega_0, y_h)_0 \quad (\forall y_h \in V_h) \tag{3.2.10}$$

and

$$\tilde{b}(\omega_H - \omega_H^h, \zeta_h) = 0 \quad (\forall \zeta_h \in M_h). \tag{3.2.11}$$

Therefore, combining (3.2.9), (3.2.10) and (3.2.11) we get for all $y_h \in V_h$ and all $\zeta_h \in M_h$

$$\begin{aligned} (\varphi, \lambda - \lambda_h)_0 &= \tilde{b}(y, \lambda - \lambda_h) \\ &= a(\omega_H - \omega_H^h, y - y_h) + \tilde{b}(\omega_H - \omega_H^h, \zeta - \zeta_h) \\ &\quad + \tilde{b}(y - y_h, \lambda - \lambda_h) + (\omega_0^h - \omega_0, y_h)_0. \end{aligned}$$

Next, assuming that $y_h \in Z_h(\varphi) = \{v_h \in V_h; \tilde{b}(v_h, \mu_h) = (\varphi, \mu_h)_0 \ (\forall \mu_h \in M_h)\}$ we have for all $\mu_h \in M_h$

$$\tilde{b}(y - y_h, \lambda - \lambda_h) = \tilde{b}(y - y_h, \lambda - \mu_h).$$

Hence we obtain, by denoting

$$F(\varphi) = \operatorname*{Inf}_{\substack{y_h \in Z_h(\varphi) \\ \zeta_h, \mu_h \in M_h}} \{\|\omega_H - \omega_H^h\|_0 \|y - y_h\|_0 + \|\omega_H - \omega_H^h\|_1 \|\zeta - \zeta_h\|_1$$

$$+ \|\omega_0 - \omega_0^h\|_0 \|y\|_0 + \|y - y_h\|_1 \|\lambda - \mu_h\|_1 + \|\omega_0 - \omega_0^h\|_0 \|y - y_h\|_0\},$$

$$\|\lambda - \lambda_h\|_0 \le C \operatorname*{Sup}_{\varphi \in L^2(\Omega)} \frac{F(\varphi)}{\|\varphi\|_0} \tag{3.2.12}$$

and the desired inequality follows from (3.2.12) as in [5].

In order to extend the result of Theorem 5 to the case $k = 1$, we have:

THEOREM 6. *We suppose that* $\psi \in H^4(\Omega) \cap H_0^2(\Omega)$, $\omega_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $\omega_H \in H^2(\Omega)$. *Then we have*

$$\|\omega_H - \omega_H^h\|_0 + \|\psi - \psi_h\|_1 \le C\{h\|\omega_H\|_2 + h^2\|\omega_0\|_2 + h\|\psi\|_2 + h^{1/2}|Ln(h)|\,\|\psi\|_4\}$$

and the quantity $\|\psi - \psi_h\|_0$ *is at least of the same order, where C is a constant independent of h.*

PROOF. In this proof we use essentially a result due to Scholz given in [14]. As in proof of Theorem 5 we have

$$a(v_h - \omega_H^h, v_h - \omega_H^h) = a(v_h - \omega_H, v_h - \omega_H^h) + \tilde{b}(v_h - \omega_H^h, \mu_h - \lambda)$$

$$+ (\omega_0^h - \omega_0, v_h - \omega_H^h)_0 \quad (\forall v_h \in X_H^h, \forall \mu_h \in M_h).$$

Hence we have

$$\|v_h - \omega_H^h\|_0 \le \|v_h - \omega_H\|_0 + \frac{|\tilde{b}(v_h - \omega_H^h, \mu_h - \lambda)|}{\|v_h - \omega_H^h\|_0} + \|\omega_0 - \omega_0^h\|_0.$$

The relation above is valid for $\mu_h \in M_h$, in particular for $\mu_h = P_0^h \lambda$ where P_0^h denotes the projection operator which satisfies:

$$\begin{cases} P_0^h v \in M_h, \\ (\nabla(v - P_0^h v), \nabla\varphi_h)_0 = 0 \quad (\forall \varphi_h \in M_h, \forall v \in H_0^1(\Omega)). \end{cases}$$

Using a lemma due to Scholz [14] we get

$|\tilde{b}(\lambda - P_0^h \lambda, v_h - \omega_H^h)| \le C^* \sqrt{h}\,|Ln(h)|\,\|\Delta\lambda\|_{L^\infty(\Omega)}\,\|v_h - \omega_H^h\|_0$ and hence

$$\|\omega_H - \omega_H^h\|_0 \le C \operatorname*{Inf}_{v_h \in X_H^h} \|v_h - \omega_H\|_1 + C^*\sqrt{h}\,|Ln(h)|\,\|\Delta\lambda\|_{L^\infty(\Omega)} + \|\omega_0 - \omega_0^h\|_0.$$

Using (3.2.8) we obtain

$$\|\omega_H - \omega_H^h\|_0 \le C \operatorname*{Inf}_{v_h \in V_h} \|v_h - \omega_H\|_1 + C^*\sqrt{h}\,|Ln(h)|\,\|\Delta\lambda\|_{L^\infty(\Omega)} + \|\omega_0 - \omega_0^h\|_0. \tag{3.2.13}$$

In the same way as in the proof of Theorem 5, we establish

$$\|\lambda - \lambda_h\|_1 \le \tilde{C}\left\{\|\omega_0 - \omega_0^h\|_0 + \|\omega_H - \omega_H^h\|_0 + \operatorname*{Inf}_{\mu_h \in M_h} \|\lambda - \mu_h\|_1\right\}. \tag{3.2.14}$$

We recall that

$$\operatorname*{Inf}_{\mu_h \in M_h} \|\psi - \mu_h\|_1 \le Ch\|\psi\|_2, \tag{3.2.15}$$

$$\operatorname*{Inf}_{v_h \in V_h} \|\omega_H - v_h\|_1 \le Ch\|\omega_H\|_2, \tag{3.2.16}$$

$$\|\omega_0 - \omega_0^h\|_0 \le Ch^2\|\omega_0\|_2 \quad \text{and} \tag{3.2.17}$$

$$\|\Delta\psi\|_{L^\infty(\Omega)} \le C\|\psi\|_4. \tag{3.2.18}$$

Combining the inequalities (3.2.13), (3.2.14), (3.2.15), (3.2.16), (3.2.17) and (3.2.18) we obtain the desired estimate.

4. Implementation Aspects

In practice we use the discrete version of problems (VP-i), (VP-ii) and (VP-iii), in order to compute ω_0^h, ω_H^h and ψ_h. For this purpose we have to compute a basis of the space X_H^h.

In the following paragraph, we describe how to compute this basis. The intersection of the closure of the elements of T_h with Γ form a partition of the latter, say $\sum_h$, into straight segments. We denote by N_h the number of segments of $\sum_h$. We have $\sum_h = \bigcup_{K \in T_h} (\bar{K} \cap \Gamma)$. We denote by Γ_i, $i = 1, \dots, N_h$ the elements of $\sum_h$, numbered in a certain manner. Let us associate with $\sum_h$ the space S^h defined by

$$S^h = \left\{v \in C^0(\Gamma); v_{|\Gamma_i} \in P_k(\Gamma_i)\left(\forall \Gamma_i \in \textstyle\sum_h\right)\right\}.$$

We denote by $\{\sigma_i\}_{i=1}^{kN_h}$ the basis of S^h defined by

$$\begin{cases} \sigma_i \in S^h \\ \sigma_i(P_j) = \delta_{ij} \quad (\forall P_j \in E^h, 1 \le j \le kN_h), \end{cases}$$

where E^h is the numbered set of all the nodes having been used to define the degrees of freedom of V_h, that lie on Γ.

Now we consider the function $\chi_i \in X_H(\Omega)$ such that $\chi_i = \sigma_i$ on Γ, $1 \le i \le kN_h$. This function can be approximated by χ_i^h, namely, the solution of the problem:

$$\begin{cases} \text{Find } \chi_i^h \in V_{h,\sigma_i} \text{ such that} \\ (\nabla \chi_i^h, \nabla v_h)_0 = 0 \quad (\forall v_h \in M_h). \end{cases}$$

Note that $\chi_i^h \in X_H^h$, and moreover $\{\chi_i^h\}_{i=1}^{kN_h}$ defines a basis of X_H^h.

The following algorithm describes how to compute ω_0^h, ω_H^h and ψ_h.

Algorithm:

Step 1

For $i = 1, \ldots, kN_h$ compute χ_i^h:

$$\begin{cases} \text{Find } \chi_i^h \in V_{h,\sigma_i} \text{ such that} \\ (\nabla \chi_i^h, \nabla \varphi_h)_0 = 0 \quad (\forall \varphi_h \in M_h). \end{cases}$$

Step 2

Compute ω_0^h by solving the problem

$$\begin{cases} \text{Find } \omega_0^h \in M_h \text{ such that} \\ (\nabla \omega_0^h, \nabla \varphi_h)_0 = \dfrac{1}{\nu}(\operatorname{curl} f, \varphi_h)_0 \quad (\forall \varphi_h \in M_h). \end{cases}$$

Step 3

In order to compute $\omega_H^h = \sum_{i=1}^{kN_h} \alpha_i \chi_i^h$, we solve the linear system: solve $AX = b$ with

$$\begin{cases} A_{ij} = (\chi_i^h, \chi_j^h)_0 \quad 1 \le i, j \le kN_h \\ b_i = -(\omega_0^h, \chi_i^h)_0 + \oint_\Gamma g_0 \dfrac{\partial \chi_i^h}{\partial n} - \oint_\Gamma g_1 \chi_i^h \\ X = (\alpha_i)_{1 \le i \le kN_h}. \end{cases}$$

Step 4

Compute ψ_h:

$$\begin{cases} \text{Find } \psi_h \in V_{h,g_0} \text{ such that} \\ (\nabla \psi_h, \nabla \varphi_h)_0 = (\omega_0^h + \omega_H^h, \varphi_h)_0 \quad (\forall \varphi_h \in M_h) \end{cases}$$

where

$$V_{h,g} = \{v_h \in V_h; v_h = \textstyle\prod_h g \text{ on } \Gamma\} \quad \text{with } \textstyle\prod_h g(P) = g(P) \ (\forall P \in E^h).$$

5. Further Remarks

1) It may seem unpractical a priori to use a finite dimensional space of test functions which are (approximately) harmonic such as X_H^h, to solve the

Stokes problem. Note however that our aim is to solve the Navier-Stokes equations. In this context the computation of a basis for X_H^h together with the matrix associated with Step 3 represents just a small amount of the whole computational effort of the numerical solution. These computations followed by the factorization of the above mentioned matrix are actually just a pre-processing to be done once for all at the beginning of the iterative solution as described in [8].

2) The matrix A, in Step 3, is full and symmetric positive definite, therefore the linear system $AX = b$ can be solved by the Cholesky method. We can also use an iterative method like conjugate gradient for this kind of matrix.

However from our experience, the Cholesky method appeared to be particularly well adapted to the solution of $AX = b$. Actually several numerical results have been obtained in the case $k = 1$. All of them illustrate a good performance of the methodology [8, 13].

3) Similar techniques based on the use of the variational approach introduced in [10] are given in [1], [2], [3] and [12].

4) Finally let us point out the relationship between the three-field mixed method introduced in this paper and the two-field one considered by Ciarlet & Raviart for $k > 1$ in [7], whose study was improved and extended by Scholz in [14] to the case of piecewise linear approximations.

Let us consider the solution of a single Stokes problem, with homogeneous velocity boundary conditions for simplicity. Choosing in (SPD) a test function $v_h = \chi_h \in X_H^h$, it is readily seen that:

$$(\omega_H^h, \chi_h)_0 + (\omega_0^h, \chi_h)_0 - (\nabla\psi_h, \nabla\chi_h)_0 = 0 \quad (\forall \chi_h \in X_H^h).$$

Moreover

$$(\omega_H^h, \varphi_h)_0 + (\omega_0^h, \varphi_h)_0 - (\nabla\psi_h, \nabla\varphi_h)_0 = 0 \quad (\forall \varphi_h \in M_h).$$

Then, since V_h is clearly the direct sum of X_H^h and M_h, we have:

$$(\omega_H^h, v_h)_0 + (\omega_0^h, v_h)_0 - (\nabla\psi_h, \nabla v_h)_0 = 0 \quad (\forall v_h \in V_h).$$

This means that in the case of equal order approximations of all the unknown functions, the uncoupled three-unknown problem we studied here is computationally equivalent to the coupled two-unknown mixed problem of Ciarlet & Raviart [7] for $k > 1$, and Scholz [14] for $k = 1$. Actually, the same conclusion applies to the case of nonhomogeneous boundary conditions, provided that a suitable modification is performed in the expression of the right hand side b_i appearing in Step 3 described in the previous section.

As a consequence, the same error estimates as those obtained by the third author, which incidentally can be extended to the case where $k > 1$, apply to the present methodology and conversely. However, as pointed out in the first remark of this section, the uncoupled procedure adopted here is aimed at being used to solve the Navier-Stokes equations, and in this case both techniques will function differently. We are persuaded that in this case our approach is globally superior, and for more details on its implementation we refer to Ghadi [8].

Another advantage of the present approach is the fact that it lends itself naturally to a set of approximation spaces of the unknown fields other than equal order. As shown in [12], a significant improvement in accuracy can be expected from well-chosen combinations of discretizations. We elaborate a little more on such possibilities in the next section.

6. Numerical Experiments

Just to illustrate the approach mentioned at the end of the previous section, we conclude this paper with a brief study of that class of methods. More specifically we first describe the corresponding approximate problem, and then we give numerical results obtained in the framework of a particular test-problem. Among these we include the equal order method with $k = 1$, for which convergence results were given in this paper.

As a by-product, we try to determine experimentally rates of convergence, that have not been formally derived, such as those of the stream function and vorticity in the L^∞-norm, or yet stream function errors in the L^2-norm. Additionally, this allows us to compare the experimented approximation schemes, in the light of both computational effort and accuracy.

The class of approximation methods that we consider are defined as follows: Let Ω be approximated by the union of the elements of its triangulation T_h in the usual way, and ∂K denote the boundary of a triangle $K \in T_h$. For a given integer $\ell \geq k$ we define the spaces,

$$V_h^\ell = \{v_h \in H^1(\Omega); v_{h|K} \in P_l(K)\ (\forall K \in T_h), v_{h|L} \in P_k(L)\ (\forall L \in \partial K \cap \Gamma)\}$$

and $M_h^\ell = V_h^\ell \cap H_0^1(\Omega)$.

Now we first determine ω_0^h satisfying (P_{0h}) and next the pair (ω_H^h, λ_h), followed by ψ_h, by solving successively:

$$(SPD^\ell) \qquad \begin{cases} \text{Find } (\omega_H^h, \lambda_h) \in V_h^\ell \times M_h^\ell \text{ such that} \\ (\omega_H^h, v_h)_0 + (\nabla v_h, \nabla \lambda_h)_0 = -(\omega_0^h, v_h)_0 \quad (\forall v_h \in V_h^l) \\ \qquad (\nabla \omega_H^h, \nabla \mu_h)_0 = 0 \quad (\forall \mu_h \in M_h^l). \end{cases}$$

and for $j \geq k$,

$$(VPD^j) \qquad \begin{cases} \text{Find } \psi_h \in M_h^j \text{ such that} \\ (\nabla\psi_h, \nabla\varphi_h)_0 = (\omega_0^h + \omega_H^h, \varphi_h)_0 \quad (\forall \varphi_h \in M_h^j). \end{cases}$$

Like (SPD), (SPD^ℓ) has a unique solution. Note however that we don't necessarily have $\lambda_h = -\psi_h$. Anyway, instead of (SPD^ℓ) we solve:

$$(CPD^\ell) \qquad \begin{cases} \text{Find } \omega_H^h \in X_H^{h,\ell} \text{ such that} \\ (\omega_H^h, \chi_H^h)_0 = -(\omega_0^h, \chi_H^h)_0 \quad (\forall \chi_H^h \in X_H^{h,\ell}) \end{cases}$$

where

$$X_H^{h,\ell} = \{\chi_H^h \in V_h^\ell; (\chi_H^h, \mu_h)_1 = 0 \ (\forall \mu_h \in M_h^\ell)\}.$$

Remark that the determination of a basis of $X_H^{h,\ell}$ followed by the computation of the matrix associated with problem (CPD^l) goes like Step 1 of §4. Note that it is also a $kN_h \times kN_h$ matrix here.

Now, seeking to remain realistic, we only checked the a priori most promising schemes (i.e. with $\ell, j = k$ or $\ell, j = k+1$), in which the polynomials of the highest degree involved are quadratic ($\ell, j \leq 2$). Moreover one should keep in mind that the ultimate goal of such schemes, is the solution of the incompressible Navier-Stokes equations by a straightforward modification of the procedure presented in this paper (Cf. [8]). That is why for all the schemes experimented, the zero trace component of the vorticity ω is represented by piecewise linear functions. Indeed most of the computational effort necessary for solving these non linear equations by our method, is related to the solution of an advection-diffusion equation for ω_0, since the corresponding matrix is the only one that changes at every iteration (Cf. [8]).

The test-problem we solved is:

$$\psi = (1 - x_1^2 - x_2^2)^2 \quad \text{and} \quad \Omega = \{(x_1, x_2) \in R^2; x_1^2 + x_2^2 < 1\}.$$

As pointed out before, the above test-problem was solved by means of the above scheme with $k = 1$ and $\ell, j \in \{1, 2\}$. This yields four classes of methods denoted in the sequel by the $P_l - P_j$ methods. Note that, as already pointed out in the previous section, the $P_1 - P_1$ method is nothing but the mixed method analysed in §3.

Incidentally, due to symmetry, the computational domain taken for our test is only the quarter of disk defined by $x_1 > 0$ and $x_2 > 0$. In doing so the boundary conditions on the lines given by $x_1 = 0$ and $x_2 = 0$ are respectively:

$$\frac{\partial\psi}{\partial x_1} = \frac{\partial\omega}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial x_2} = \frac{\partial\omega}{\partial x_2} = 0.$$

Note that this symmetry allows us to restrict the computation of the basis functions of the space $X_H^{h,l}$ necessary to compute ω_H^h, to those associated with the nodes situated on the curved boundary of the quarter of the disk. As a matter of fact, these functions must satisfy homogeneous Neumann boundary conditions on the above straight lines (Cf. [8]).

Finally, before presenting the numerical results, we mention a word about the meshes used in the test. A quasi-uniform family of meshes depending on a single integer parameter p was generated. Each mesh of the family, which has $p^2 + 2p + 1$ linear interpolation nodes and $4p^2 + 4p + 1$ quadratic interpolation nodes, is constructed in such a way that the polar coordinates of its vertices are either (0, 0) or $\left(\frac{m}{p}, \frac{q\pi}{4m}\right)$ for $m = 1, 2, \ldots, p$ and $q = 0, 1, \ldots, 2m$. In all the cases the chosen values of p are of the form 2^n, for n ranging from 2 up to 6.

The numerical results are summarized in Tables 1 through 3. In Table 1 we display the error of computed values of ω measured in the L^2-norm and in the L^∞-norm, for $\ell = 1$ and $\ell = 2$, respectively. In Tables 2 and 3 the errors of the approximate stream function expressed in the norms of $L^2(\Omega)$ and $L^\infty(\Omega)$, respectively, are given for the four experimented schemes.

As seen from the displayed values, in this test the schemes based on a quadratic approximation of ω_H in the interior of Ω, showed much more accurate than the case where this vorticity component is approximated by piecewise linear functions. Indeed, the observed order of convergence for the vorticity in the former case is one in the L^∞-norm and two in the L^2-norm.

On the other hand, the sequence of linear approximate vorticities does not even converge in the L^∞-norm, while it seems to converge linearly in the L^2-norm. As for the stream function, one observes in Table 2, convergence rates are slightly worse than quadratic for both schemes $P_1 - P_1$ and $P_1 - P_2$, whereas it is slightly better than quadratic for the $P_2 - P_1$ and $P_2 - P_2$-schemes. Also one infers from Table 2 that the $P_l - P_2$-scheme is more accurate than the $P_l - P_1$-scheme for $\ell = 1$ for $\ell = 2$, in terms of this norm, as expected. As for the L^∞-errors of the approximate stream function, from Table 3 the three schemes $P_1 - P_1$, $P_1 - P_2$ and $P_2 - P_1$ appear to have comparable convergence rates, better than linear but worse than quadratic. On the other hand the $P_2 - P_2$-scheme is undoubtedly superior with respect to this norm, since its observed convergence rate is a little better than quadratic. Notice that here, curiously enough, the $P_2 - P_1$-scheme performs no better than the $P_1 - P_1$-scheme, which in turn is quite naturally less accurate than the $P_1 - P_2$-scheme. This seems to be a convincing argument to rule out the $P_2 - P_1$-scheme, as it is more costly than the other two, at least as far as vorticity computations are concerned.

We will extend shortly the above comparative study to the case of other classes of domains and boundary conditions. Moreover we intend to further exploit natural extensions of the schemes defined in this section, to the solution of the Navier-Stokes equations.

Table 1. Errors of ω measured in the L^2 and the L^∞-norms.

p	$P_1(\omega_H)$		$P_2(\omega_H)$	
	L^2	L^∞	L^2	L^∞
4	0.51710E+0	0.27740E+1	0.12472E+0	0.79930E+0
8	0.35881E+0	0.42971E+1	0.31432E−1	0.35572E+0
16	0.21580E+0	0.52419E+1	0.78783E−2	0.16812E+0
32	0.11840E+0	0.57613E+1	0.19726E−2	0.81426E−1
64	0.62018E−1	0.60335E+1	0.51367E−3	0.39922E−1

Table 2. Error of ψ measured in the L^2-norm.

p	$P_1 - P_1$	$P_1 - P_2$	$P_2 - P_1$	$P_2 - P_2$
4	0.93352E−2	0.71399E−2	0.64203E−2	0.35488E−2
8	0.34280E−2	0.28937E−2	0.16378E−2	0.85130E−3
16	0.10889E−2	0.95361E−3	0.40990E−3	0.21137E−3
32	0.30935E−3	0.27499E−3	0.99790E−4	0.49285E−4
64	0.82610E−4	0.73942E−4	0.19992E−4	0.94694E−5

Table 3. Error of ψ measured in the L^∞-norm.

p	$P_1 - P_1$	$P_1 - P_2$	$P_2 - P_1$	$P_2 - P_2$
4	0.18554E−1	0.12003E−1	0.16707E−1	0.60973E−2
8	0.64413E−2	0.49587E−2	0.62592E−2	0.13729E−2
16	0.19308E−2	0.16408E−2	0.20595E−2	0.33232E−3
32	0.53312E−3	0.47077E−3	0.64751E−3	0.76616E−4
64	0.14032E−3	0.12597E−3	0.23178E−3	0.18339E−4

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