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Geometry of contrast functions and conformal geometry

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Abstract. We give a necessary and sufficient condition for a pseudo-Riemannian manifold with a compatible affine connection to be projectively flat, dual-projectively flat or conformally flat in terms of the Batlett tensor, which is derived from forth-order derivatives of contrast function.

Introduction

In 1982, Nagaoka and Amari formulated information geometry, which has been applied to various fields of information sciences, for example, information theory, neural networks, system theory, and so on (cf. [2]). In information geometry, contrast functions play an essential role. The Kullback-Leibler divergence is an interesting example of contrast function, which is used in statistical inference (see, [1]). A geometric divergence introduced by Kurose [8] is also an example of contrast function. In this paper, we study the geometry of geometric divergences in affine differential geometry.

In general, a contrast function ρ induces a dualistic geometrical structure on M (cf. [4] and [8]). The second-order derivatives of a given contrast function ρ of M induce a pseudo-Riemannian metric h on M. The third-order derivatives induce two torsion-free affine connections ∇ and ∇^* such that these are mutually dual with respect to h. In this case, the tensors ∇h and $\nabla^* h$ are symmetric. Then the triplets (M, ∇, h) and (M, ∇^*, h) are statistical manifolds.

The *Bartlett tensor* B of contrast function ρ, which was formulated by Eguchi [4], is defined from forth-order derivatives of ρ. The anti-symmetric part of B is the curvature tensor of the induced affine connection ∇. The Bartlett tensor B and the dual Bartlett tensor B^* correspond to the Bartlett corrections in likelihood ratio tests in statistics.

In this paper, we study the Bartlett tensors of contrast functions in the geometric divergences case. As applications, we give necessary and sufficient conditions for a statistical manifold to be projectively flat, dual-projectively flat or conformally flat in terms of the Bartlett tensor.

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Both concepts of conformal flatness of Riemannian metrics and projective flatness of affine connections were introduced by Weyl. They are characterized by vanishing of the conformal curvature tensors or the projective curvature ones, respectively. Ivanov [7] introduced the dual-projective flatness of affine connections, which are also characterized by vanishing of the dual-projective curvature tensors. In our recent paper [10], we introduced the conformal-projective flatness of statistical manifolds, which is a natural generalization of the projective flatness, the conformal flatness or the dual-projective flatness of statistical manifolds.

In § 4, we give a necessary and sufficient condition for an affine connection to be projectively flat or dual-projectively flat in terms of the Bartlett tensor. In § 5, we determine the Bartlett tensors B in the cases of geometric divergences as contrast functions on conformally-projectively flat statistical manifolds. We also give a necessary and sufficient condition for a Riemannian metric to be conformally flat.

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1. Contrast functions and statistical manifolds

We assume that all the objects are smooth throughout this paper. In this section, we recall several definitions and preliminary facts on contrast functions. For more details, see [4].

Let M be an n-dimensional manifold. Let ∇ be a torsion-free affine connection and h a pseudo-Riemannian metric on M. We call (M, ∇, h) a *statistical manifold* if ∇h is symmetric. For a statistical manifold (M, ∇, h), we can define another torsion-free affine connection ∇^* by

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h(Y, \nabla_X^* Z),$$

where X, Y and Z are arbitrary vector fields on M. It is straightforward to show that (M, ∇^*, h) is also a statistical manifold. We say that ∇^* is the *dual connection* of ∇ with respect to h and that (M, ∇^*, h) is the *dual statistical manifold* of (M, ∇, h).

Let ρ be a function on $M \times M$. Identifying the tangent space $T_{(p,q)}(M \times M)$ with the direct sum of $T_pM \oplus T_qM$, we use the following notation:

$$\rho[X_1 \dots X_i \mid Y_1 \dots Y_j](p) := (X_1, 0) \dots (X_i, 0)(0, Y_1) \dots (0, Y_j)\rho|_{(p,p)},$$

where $p \in M$ and $X_1, \dots, X_i$, $Y_1, \dots, Y_j (i, j \geq 0)$ are arbitrary vector fields on M. We call ρ a *contrast function* of M if

1) $\rho(p, p) = 0$ for an arbitrary point $p \in M$,
2) $\rho[X|] = \rho[|X] = 0$,
3) $h(X, Y) := -\rho[X|Y]$ is a pseudo-Riemannian metric on M.

Let ρ be a contrast function of M and h the pseudo-Riemannian metric induced by ρ. We define two torsion-free affine connections ∇ and ∇^* on M as follows:

$$h(\nabla_X Y, Z) = -\rho[XY|Z],$$

$$h(Y, \nabla^*_X Z) = -\rho[Y|XZ],$$

where X, Y and Z are arbitrary vector fields on M. It is easy to show that the triplets (M, ∇, h) and (M, ∇^*, h) are statistical manifolds. We say that the statistical manifold (M, ∇, h) and (M, ∇^*, h) are *induced* by the contrast function ρ.

We also define $(1,3)$-tensor fields B and B^* on M by the following equations:

$$h(B(X, Y)Z, V) = -\rho[XYZ - \nabla_X \nabla_Y Z|V],$$

$$h(V, B^*(X, Y)Z) = -\rho[V|XYZ - \nabla^*_X \nabla^*_Y Z],$$

where X, Y, Z and V are arbitrary vector fields on M. We call B the *Bartlett tensor* of contrast function ρ and B^* the *dual Bartlett tensor*.

PROPOSITION 1.1. *Let ρ be a contrast function. Let ∇ and ∇^* be the induced affine connections by ρ. Then, the anti-symmetric part of B and B^* are the curvature tensors R of ∇ and R^* of ∇^*, respectively, that is, the following equations holds:*

$$R(X, Y)Z = B(Y, X)Z - B(X, Y)Z,$$

$$R^*(X, Y)Z = B^*(Y, X)Z - B^*(X, Y)Z.$$

2. Curvature tensors

In this section, we recall that definitions and preliminary facts on curvature tensors.

2.1. Projective curvature tensors

Let M be an n-dimensional manifold. We say that two affine connections ∇ and $\tilde{\nabla}$ on M are *projectively equivalent* if there exists a 1-form τ on M such that

$$\tilde{\nabla}_X Y = \nabla_X Y + \tau(Y)X + \tau(X)Y, \tag{2.1}$$

where X and Y are arbitrary vector fields of M. We say that an affine connection ∇ is *projectively flat* if ∇ is projectively equivalent to a flat affine connection in a neighbourhood of an arbitrary point of M.

Let Ric be the Ricci tensor of ∇. Suppose that ∇ is torsion-free and Ric is symmetric. We define the *projective curvature tensor* W_P by

$$W_P(X, Y)Z := R(X, Y)Z - \frac{1}{n-1}\,\mathrm{Ric}(Y, Z)X + \frac{1}{n-1}\,\mathrm{Ric}(X, Z)Y.$$

If affine connections ∇ and $\tilde{\nabla}$ are projectively equivalent, then their projective curvature tensors coincide. For $n \geq 3$, an affine connection ∇ is projectively flat if and only if its projective curvature tensor W_P vanishes (cf. [5]).

Let (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ be statistical manifolds. If a 1-form τ is given by $\tau = d\psi$, where ψ is a function, the relation (2.1) is known as (-1)-conformal equivalence relation of statistical manifolds, provided $\tilde{h} = e^{\psi}h$. We suppose that $n \geq 3$. A statistical manifold (M, ∇, h) is (-1)-conformally flat if and only if ∇ is projectively flat with symmetric Ricci tensor. Kurose [9] showed that, for a statistical manifold, if ∇ is projectively flat then the Ricci tensor of ∇ is symmetric. Hence, a statistical manifold (M, ∇, h) is (-1)-conformally flat if and only if ∇ is projectively flat.

2.2. Dual-projective curvature tensors

Let (M, ∇, h) be an n-dimensional statistical manifold. We say that two affine connections ∇ and $\tilde{\nabla}$ on M are *dual-projectively equivalent* if there exists a 1-form α on M such that

$$\tilde{\nabla}_X Y = \nabla_X Y - h(X, Y)\alpha^{\#}, \tag{2.2}$$

where X and Y are arbitrary vector fields and $\alpha^{\#}$ is the tangent vector field defined by $h(\alpha^{\#}, Y) := \alpha(Y)$.

We say that an affine connection ∇ is *dual-projectively flat* if ∇ is dual-projectively equivalent to a flat affine connection in a neighbourhood of an arbitrary point of M.

We define the *dual-projective curvature tensor* W_{dP} by

$$W_{dP}(X, Y)Z := R(X, Y)Z - h(Y, Z)M(X) + h(X, Z)M(Y),$$

where the tensor M is given as follows:

$$M(X) := -\mathrm{Ric}^{\#}(X) + \frac{\gamma}{n-1}X. \tag{2.3}$$

In (2.3), we denote by γ the trace of the Ricci tensor with respect to h and by $\mathrm{Ric}^{\#}(X)$ the Ricci operator defined by $h(\mathrm{Ric}^{\#}(X), Y) := \mathrm{Ric}(X, Y)$. If two affine connections ∇ and $\tilde{\nabla}$ are dual-projectively equivalent, then their dual-projective curvature tensors coincide. For $n \geq 3$, an affine connection ∇ is dual-projectively flat if and only if its dual-projective curvature tensor W_{dP} of ∇ vanishes (cf. [7]).

Let (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ be statistical manifolds. If the 1-form α is given by $\alpha = d\phi$, where ϕ is a function, the relation (2.2) is known as 1-conformal equivalence relation of statistical manifolds, provided $\tilde{h} = e^{\phi}h$. We suppose that $n \geq 3$. A statistical manifold (M, ∇, h) is 1-conformally flat if and only if ∇ is dual-projectively flat with symmetric Ricci tensor. Kurose [9] showed that, for a statistical manifold, if ∇ is dual-projectively flat then the Ricci tensor of ∇ is symmetric. Hence, a statistical manifold (M, ∇, h) is 1-conformally flat if and only if ∇ is dual-projectively flat.

2.3. Conformal curvature tensors

Let (M, g) and $(M, \tilde{g})$ be n-dimensional Riemannian manifolds. We say that two Riemannian metrics g and $\tilde{g}$ are *conformally equivalent* if there exists a function ϕ on M such that

$$\tilde{g}(X, Y) = e^{2\phi} g(X, Y).$$

We say that a Riemannian metric g is *conformally flat* if g is conformally equivalent to a flat Riemannian metric in a neighbourhood of an arbitrary point of M.

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection of g. We define the *conformal curvature tensor* W_C by

$$\begin{aligned} g(W_C(X, Y)Z, U) := {} & g(R(X, Y)Z, U) \\ & - \frac{1}{n-2}\{g(Y, Z)\,\mathrm{Ric}(X, U) - g(X, Z)\,\mathrm{Ric}(Y, U) \\ & + \mathrm{Ric}(Y, Z)g(X, U) - \mathrm{Ric}(X, Z)g(Y, U)\} \\ & + \frac{\mathrm{tr}_g(\mathrm{Ric})}{(n-1)(n-2)}\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}, \end{aligned}$$

where R is the Riemannian curvature tensor of ∇ and Ric the Ricci tensor of ∇. If Riemannian metrics g and $\tilde{g}$ on M are conformally equivalent, then their conformal curvature tensors coincide.

For $n \geq 4$, a Riemannian metric g is conformally flat if and only if its conformal curvature tensor W_C vanishes (cf. [6]).

2.4. Conformal-projective curvature tensors

Let (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ be statistical manifolds. We say that (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are *conformally-projectively equivalent* (or *generalized conformally equivalent*) if there exist two functions ϕ and ψ on M such that

$$\tilde{h}(X, Y) = e^{\phi+\psi} h(X, Y),$$

$$h(\tilde{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - d\phi(Z) h(X, Y) + d\psi(Y) h(X, Z) + d\psi(X) h(Y, Z). \tag{2.4}$$

We say that a statistical manifold (M, ∇, h) is *conformally-projectively flat* if (M, ∇, h) is conformally-projectively equivalent to a flat statistical manifold in a neighbourhood of an arbitrary point of M (cf. [10]).

Let ∇^* be the dual connection of ∇ with respect to h. We define the *conformal-projective curvature tensor* W_{CP} by

$$\begin{aligned} h(W_{CP}(X, Y)Z, U) := {} & h(R(X, Y)Z, U) - \frac{1}{n-2}\{h(Y, Z)A(X, U) \\ & - h(X, Z)A(Y, U) + A^*(Y, Z)h(X, U) - A^*(X, Z)h(Y, U)\} \\ & + \frac{\mathrm{tr}_h(\mathrm{Ric})}{(n-1)(n-2)}\{h(Y, Z)h(X, U) - h(X, Z)h(Y, U)\}, \end{aligned} \tag{2.5}$$

where the tensors A and A^* are defined by

$$A(Y, Z) := \frac{1}{n}\{\mathrm{Ric}(Y, Z) + (n-1)\mathrm{Ric}^*(Y, Z)\}, \tag{2.6}$$

$$A^*(Y, Z) := \frac{1}{n}\{(n-1)\,\mathrm{Ric}(Y, Z) + \mathrm{Ric}^*(Y, Z)\}. \tag{2.7}$$

In (2.6) and (2.7), we denote by Ric and Ric^* the Ricci tensors of ∇ and ∇^*, respectively. If two statistical manifolds (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are conformally-projectively equivalent, then their conformal-projective curvature tensors coincide (Kurose 1995). In (2.5), if $\nabla = \nabla^*$, then W_{CP} is the conformal curvature tensor.

For a statistical manifold (M, ∇, h), if ∇ is projectively flat or dual-projectively flat and the Ricci tensor of ∇ is symmetric, then (M, ∇, h) is conformally-projectively flat.

3. Geometric divergences

In this section, we recall several definitions and preliminary facts on centroaffine immersions of codimension two and on geometric divergences. For more details, see [10] and [11].

Let M be an n-dimensional manifold and f an immersion from M into $\mathbf{R}^{n+2}$. Denote by D the standard flat affine connection of $\mathbf{R}^{n+2}$ and by η the radial vector field of $\mathbf{R}^{n+2}$. An immersion $f: M \to \mathbf{R}^{n+2}$ is called a *centroaffine immersion of codimension two* if there exists, at least locally, a vector field ξ along f such that, at each point $x \in M$, the tangent space $T_{f(x)}\mathbf{R}^{n+2}$ is decomposed as the direct sum of the span $\mathbf{R}\{\eta_{f(x)}\}$, the tangent space $f_*(T_xM)$ and the span $\mathbf{R}\{\xi_x\}$. We call ξ a *transversal vector field.*

For simplicity, we often omit the term "codimension two" from now on.

For a given centroaffine immersion $\{f, \xi\}$, the induced connection ∇ and the affine fundamental forms h, T are determined by

$$D_X f_* Y = T(X, Y)\eta + f_*(\nabla_X Y) + h(X, Y)\xi,$$

the transversal connection forms μ, τ and the affine shape operator S are determined by

$$D_X \xi = \mu(X)\eta - f_*(SX) + \tau(X)\xi. \tag{3.1}$$

Since the connection D is flat, we have fundamental equations for centroaffine immersions of codimension two.

Gauss:

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY - T(Y, Z)X + T(X, Z)Y, \tag{3.2}$$

Codazzi:

$$\begin{aligned}
(\nabla_X T)(Y, Z) + \mu(X)h(Y, Z) &= (\nabla_Y T)(X, Z) + \mu(Y)h(X, Z), \\
(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) &= (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z), \\
(\nabla_X S)(Y) - \tau(X)SY + \mu(X)Y &= (\nabla_Y S)(X) - \tau(Y)SX + \mu(Y)X,
\end{aligned} \tag{3.3}$$

Ricci:

$$\begin{aligned}
T(X, SY) - T(Y, SX) &= (\nabla_X \mu)(Y) - (\nabla_Y \mu)(X) + \tau(Y)\mu(X) - \tau(X)\mu(Y), \\
h(X, SY) - h(Y, SX) &= (\nabla_X \tau)(Y) - (\nabla_Y \tau)(X).
\end{aligned}$$

We change a transversal vector field ξ to $\tilde{\xi} = \phi^{-1}(\xi + a\eta + f_* U)$, where ϕ, a and U are a nonzero function, a function and a tangent vector field on M, respectively. Then the induced objects change as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y - h(X, Y)U, \tag{3.4}$$

$$\tilde{T}(X, Y) = T(X, Y) - ah(X, Y), \tag{3.5}$$

$$\tilde{h}(X, Y) = \phi h(X, Y), \tag{3.6}$$

$$\tilde{\tau}(X) = \tau(X) - X(\log \phi) + h(X, U), \tag{3.7}$$

$$\tilde{S}X = \phi^{-1}\{SX + \tau(X)U - aX - \nabla_X U + h(X, U)U\}. \tag{3.8}$$

If h is nondegenerate everywhere, we say that the immersion f is *non-degenerate*. When h is nondegenerate, we can take a transversal vector field ξ such that τ vanishes because of equation (3.7). We say that $\{f, \xi\}$ is *equiaffine* if τ vanishes. In this case, ∇h is symmetric because of equation (3.3), then the triplet (M, ∇, h) is a statistical manifold. We say that the nondegenerate centroaffine immersion $\{f, \xi\}$ *realizes* the statistical manifold (M, ∇, h) in $\mathbf{R}^{n+2}$.

By equations (3.5), (3.6) and (3.8), we can take a function a such that affine fundamental forms of $\{f, \xi\}$ satisfy the following equation:

$$\mathrm{tr}_h\{T(X, Y) + h(SX, Y)\} = 0.$$

In this case, we say that $\{f, \xi\}$ is *pre-normalized.*

PROPOSITION 3.1. *Let (M, ∇, h) be a simply connected n-dimensional statistical manifold. Suppose that the Ricci tensor of ∇ is symmetric and $n \geq 3$. Then the followings hold:*

(1) *The connection ∇ is projectively flat if and only if there exists a centroaffine immersion $\{f, \xi\}$ such that it realizes (M, ∇, h) in $\mathbf{R}^{n+2}$ and ξ is a parallel vector field.*
(2) *The connection ∇ is dual-projectively flat if and only if there exists a centroaffine immersion $\{f, \xi\}$ such that it realizes (M, ∇, h) in some affine hyperplane of $\mathbf{R}^{n+2}$.*
(3) *The statistical manifold (M, ∇, h) is conformally-projectively flat if and only if there exists a centroaffine immersion $\{f, \xi\}$ such that it realizes (M, ∇, h) in $\mathbf{R}^{n+2}$.*

The second statement in Proposition 3.1 is Theorem 5.2 in [7] and the third one is the main result of [10]. Then we have only to show the first statement.

In order to prove Proposition 3.1, we need the following lemmas, which are given as Theorem 4.1 and Lemma 4.2 in [11].

LEMMA 3.2. *Let (M, ∇, h) be a statistical manifold of dimension $n \geq 3$. The connection ∇ is projectively flat if and only if $S = (\mathrm{tr}\, S/n)I$.*

LEMMA 3.3. *Let k be a function on M. Assume $S = kI$ and $n \geq 2$. Then* *the following equation holds:*

$$dk - k\tau + \mu = 0.$$

PROOF OF PROPOSITION 3.1. Suppose that $\{f, \xi\}$ is a centroaffine immersion which realizes the statistical manifold (M, ∇, h) and ξ is a parallel vector field. By equation (3.1), the affine shape operator S vanishes. Then the Gauss equation (3.2) is

$$R(X, Y)Z = -T(Y, Z)X + T(X, Z)Y. \tag{3.9}$$

Contracting the equation (3.9), we obtain

$$T(Y, Z) = -\frac{1}{n-1}\operatorname{Ric}(Y, Z). \tag{3.10}$$

Substituting (3.10) into (3.9), we can show that the projective curvature tensor vanishes. This implies that the connection ∇ is projectively flat.

Conversely, we assume that the connection ∇ is projectively flat. Since the Ricci tensor of ∇ is symmetric, the given statistical manifold (M, ∇, h) is conformally-projectively flat. Hence, there exists a centroaffine immersion $\{f, \xi\}$ such that it realizes (M, ∇, h) in $\mathbf{R}^{n+2}$.

By Lemma 3.2, the traceless part of affine shape operator S vanishes. Then we have $S = (\operatorname{tr} S/n)I$. Therefore, by equation (3.8), we may assume $S = 0$. By Lemma 3.3, we obtain $\mu = 0$ since $S = 0$ and $\tau = 0$. Then the transversal vector field ξ is parallel since $D_X\xi = 0$. □

Let $\{f, \xi\} : M \to \mathbf{R}^{n+2}$ be a centroaffine immersion. Denote by $\mathbf{R}_{n+2}$ the dual space of $\mathbf{R}^{n+2}$, by η^* the radial vector field of $\mathbf{R}_{n+2} - \{0\}$, and by $\langle\ ,\ \rangle$ the pairing of $\mathbf{R}_{n+2}$ and $\mathbf{R}^{n+2}$. We assume that $\{f, \xi\}$ is nondegenerate and equiaffine. For a centroaffine immersion $\{f, \xi\}$, we define the *conormal maps* v and $w : M \to \mathbf{R}_{n+2}$ by

$$\langle v(x), \xi_x\rangle = 1, \quad \langle w(x), \xi_x\rangle = 0, \tag{3.11}$$

$$\langle v(x), \eta_{f(x)}\rangle = 0, \quad \langle w(x), \eta_{f(x)}\rangle = 1, \tag{3.12}$$

$$\langle v(x), f_*X_x\rangle = 0, \quad \langle w(x), f_*X_x\rangle = 0, \tag{3.13}$$

for each $x \in M$. The derivatives of the maps v and w are given as follows:

$$\langle v_*X, \xi\rangle = 0, \qquad \langle w_*X, \xi\rangle = -\mu(X), \tag{3.14}$$

$$\langle v_*X, \eta\rangle = 0, \qquad \langle w_*X, \eta\rangle = 0, \tag{3.15}$$

$$\langle v_*X, f_*Y\rangle = -h(X, Y), \qquad \langle w_*X, f_*Y\rangle = -T(X, Y). \tag{3.16}$$

The pair $\{v, w\}$ is a centroaffine immersion from M into $\mathbf{R}_{n+2}$ since h is nondegenerate and $v(x)$ and $w(x)$ are linearly independent at each point x of M. We call the pair $\{v, w\}$ the *dual map* of $\{f, \xi\}$.

For the dual map $\{v, w\}$, the objects $\nabla^*, T^*, h^*, S^*, \mu^*$ and τ^* are defined by

$$D_X v_* Y = T^*(X, Y)\eta^* + v_*(\nabla^*_X Y) + h^*(X, Y)w,$$

$$D_X w = \mu^*(X)\eta^* - v_*(S^* X) + \tau^*(X)w.$$

The induced objects satisfy the following relations.

$$T^*(X, Y) = -h(SX, Y), \tag{3.17}$$

$$h^*(X, Y) = h(X, Y), \tag{3.18}$$

$$Zh(X, Y) = h(\nabla_Z X, Y) + h(X, \nabla^*_Z Y), \tag{3.19}$$

$$\tau^*(X) = 0, \tag{3.20}$$

$$h(S^* X, Y) = -T(X, Y). \tag{3.21}$$

Equation (3.19) implies that two connections ∇ and ∇^* are mutually dual with respect to h.

We define a function ρ on $M \times M$ for $\{f, \xi\}$ by

$$\rho(p, q) := \langle v(q), f(p) - f(q) \rangle,$$

where p and q are arbitrary points in M. We call the function ρ the *geometric divergence* of $\{f, \xi\}$.

We now show that the geometric divergence is a contrast function of M.

PROPOSITION 3.4. *The derivatives of geometric divergence are given as follows:*

$$\rho[X|Y] = -h(X, Y), \tag{3.22}$$

$$\rho[XY|Z] = -h(\nabla_X Y, Z), \tag{3.23}$$

$$\rho[Y|XZ] = -h(Y, \nabla^*_X Z), \tag{3.24}$$

$$\rho[XYZ|U] = -h(\nabla_X \nabla_Y Z, U) + h(Y, Z)h(SX, U) - T(Y, Z)h(X, U), \tag{3.25}$$

$$\rho[U|XYZ] = -h(U, \nabla^*_X \nabla^*_Y Z) + h(Y, Z)h(S^* X, U) - T^*(Y, Z)h(X, U). \tag{3.26}$$

PROOF. By the definition of geometric divergences, we have

$$(Z,0)(0,U)\rho(p,q) = \langle v_* U_q, f_* Z_p\rangle,$$

$$(Y,0)(Z,0)(0,U)\rho(p,q) = (Y,0)\langle v_* U_q, f_* Z_p\rangle$$
$$= \langle v_* U_q, T(Y,Z)\eta_{f(p)} + f_*(\nabla_Y Z)_p + h(Y,Z)\xi_p\rangle,$$

$$(X,0)(Y,0)(Z,0)(0,U)\rho(p,q) = (X,0)\langle v_* U_q, T(Y,Z)\eta_{f(p)} + f_*(\nabla_Y Z)_p + h(Y,Z)\xi_p\rangle$$
$$= \langle v_* U_q, XT(Y,Z)\eta_{f(p)} + T(Y,Z)f_* X_p + T(X,\nabla_Y Z)\eta_{f(p)} + f_*(\nabla_X \nabla_Y Z)_p + h(X,\nabla_Y Z)\xi_p + Xh(Y,Z)\xi_p + h(Y,Z)\{\mu(X)\eta_{f(p)} - f_* SX\}\rangle.$$

Set $p = q$. From equations (3.11)–(3.16), we have equations (3.22), (3.23) and (3.25).

Similarly, we have

$$(U,0)(0,Y)(0,Z)\rho(p,q) = (0,Y)\langle v_* Z_q, f_* U_p\rangle,$$
$$= \langle T^*(Y,Z)\eta^*_{v(q)} + v_*(\nabla^*_Y Z)_q + h^*(Y,Z)w(q), f_* U_p\rangle,$$

$$(U,0)(0,X)(0,Y)(0,Z)\rho(p,q) = (0,X)\langle T^*(Y,Z)\eta^*_{v(q)} + v_*(\nabla^*_Y Z)_q + h^*(Y,Z)w(q), f_* U_p\rangle$$
$$= \langle XT^*(Y,Z)\eta_{v(q)} + T^*(Y,Z)v_* X_q + T^*(X,\nabla^*_Y Z)\eta_{v(q)} + v_*(\nabla^*_X \nabla^*_Y Z)_p + h^*(X,\nabla^*_Y Z)w(q) + Xh^*(Y,Z)w(q) + h^*(Y,Z)\{\mu^*(X)\eta_{v(q)} - v_* S^* X\}, f_* U_p\rangle.$$

Setting $p = q$, we have equations (3.24) and (3.26). □

By the above arguments and Proposition 3.1, we have the following corollary. This has been proved as Theorem 5.3 in [10].

Corollary 3.5. *Let* (M,∇,h) *be a simply connected conformally-projectively flat statistical manifold of dimension* n (≥ 3) *and* $\{f,\xi\}$ *a centroaffine immersion which realizes the statistical manifold* (M,∇,h) *into*

$\mathbf{R}^{n+2}$. *Then there exists a contrast function* ρ *which induces the statistical manifold* (M, ∇, h). *Moreover,* ρ *is given as the geometric divergence of* $\{f, \xi\}$.

4. Projectively flat or dual-projectively flat cases

In this section, we give a condition on the Bartlett tensor B of contrast function that an affine connection on a statistical manifold be projectively flat or dual-projectively flat.

THEOREM 4.1. *Let* (M, ∇, h) *be a simply connected n-dimensional statistical manifold and* (M, ∇^*, h) *its dual statistical manifold. We denote by* Ric *and* Ric* *the Ricci tensors of* ∇ and ∇^*, *respectively. Suppose that* $n \geq 3$ *and* Ric *is symmetric. Then the connection* ∇ *is projectively flat if and only if there exists a contrast function which induces the statistical manifold* (M, ∇, h) *and the Bartlett tensor* B *is given by*

$$B(X, Y)Z = -\frac{1}{n-1} \operatorname{Ric}(Y, Z)X, \tag{4.1}$$

or equivalently, B^* *is given by*

$$B^*(X, Y)Z = -h(Y, Z)\left\{-(\operatorname{Ric}^*)^{\#}(X) + \frac{\gamma}{n-1}X\right\},$$

where γ *is the trace of Ricci tensor with respect to* h *and* $(\operatorname{Ric}^*)^{\#}$ *is the Ricci operator defined by* $h((\operatorname{Ric}^*)^{\#}(X), Y) := \operatorname{Ric}^*(X, Y)$.

We note that if the Bartlett tensor B satisfies equation (4.1), the difference between the curvature tensor R of ∇ to the anti-symmetric part of B is the projective curvature tensor, that is, the projective curvature tensor W_P is given by

$$W_P(X, Y)Z = R(X, Y)Z + B(X, Y)Z - B(Y, X)Z.$$

Similarly, the dual-projective curvature tensor W_{dP} is given by

$$W_{dP}(X, Y)Z = R^*(X, Y)Z + B^*(X, Y)Z - B^*(Y, X)Z.$$

PROOF OF THEOREM 4.1. Suppose that the tensor B is given by $B(X, Y)Z = -(n-1)^{-1} \operatorname{Ric}(Y, Z)X$. By the definition of the projective curvature tensor and Proposition 1.1, we calculate

$$\begin{aligned} W_P(X, Y)Z &= R(X, Y)Z - \frac{1}{n-1} \operatorname{Ric}(Y, Z)X + \frac{1}{n-1} \operatorname{Ric}(X, Z)Y \\ &= R(X, Y)Z + B(X, Y)Z - B(Y, X)Z \\ &= 0, \end{aligned}$$

where W_P is the projective curvature tensor of ∇. This implies that the connection ∇ is projectively flat.

Similarly, if $B^*(X, Y) \quad Z = -(n-1)^{-1}h(Y,Z) \quad \{-(\mathrm{Ric}^*)^{\#}(X) + \gamma(n-1)^{-1}X\}$, then the dual-projective curvature tensor W_{dP} of ∇^* vanishes. This implies that the connection ∇^* is dual-projectively flat. Hence, the connection ∇ is projectively flat.

Conversely, suppose that (M, ∇, h) is a simply connected statistical manifold and ∇ is projectively flat. By Proposition 3.1, there exists a nondegenerate equiaffine centroaffine immersion $\{f, \xi\}$ which realizes (M, ∇, h) into $\mathbf{R}^{n+2}$. Since Ric is symmetric, (M, ∇, h) is conformally-projectively flat. By Corollary 3.5, a contrast function ρ is given as the geometric divergence of $\{f, \xi\}$:

$$\rho(p,q) = \langle v(q), f(p) - f(q) \rangle,$$

where v is the conormal map of $\{f, \xi\}$. Denote by S the affine shape operator of $\{f, \xi\}$ and by T the affine fundamental form in the direction of f. Since the connection ∇ is projectively flat, by Proposition 3.1, we have $S = 0$ and $T(Y,Z) = -(n-1)^{-1}\,\mathrm{Ric}(Y,Z)$. By the definition of the Bartlett tensor and equation (3.25), we have

$$\begin{aligned} h(B(X, Y)Z, U) &= T(Y,Z)h(X, U) \\ &= -\frac{1}{n-1}\,\mathrm{Ric}(Y,Z)h(X, U), \end{aligned}$$

which is the desired result since h is nondegenerate.

From equations (3.17) and (3.26), we have

$$h(B^*(X, Y)Z, U) = -h(Y,Z)h(S^*X, U). \tag{4.2}$$

The dual map $\{v, w\}$ of $\{f, \xi\}$ is a centroaffine immersion. Contracting the Gauss equation of the dual map, using $T^* = 0$, we have

$$\mathrm{Ric}^*(X, Y) = \mathrm{tr}\, S^* h(Y,Z) - h(S^*Y, Z), \tag{4.3}$$

$$\gamma^* = (n-1)\mathrm{tr}\, S^*, \tag{4.4}$$

where γ^* is the trace of Ric^* wich respect to h. From the Gauss equation and equations (3.17)–(3.21), γ and γ^* coincide. Substituting (4.4) to (4.3), we have

$$S^*Y = -(\mathrm{Ric}^*)^{\#}Y + \frac{1}{n-1}Y. \tag{4.5}$$

By equations (4.2) and (4.5), we have the desired result. □

THEOREM 4.2. *Let (M, ∇, h) be a simply connected n-dimensional statistical manifold and (M, ∇^*, h) its dual statistical manifold. We denote by* Ric *and*

Ric* *the Ricci tensors of* ∇ *and* ∇^*, *respectively. Suppose that* $n \geq 3$ *and* Ric *is symmetric. Then the connection* ∇ *is dual-projectively flat if and only if there exists a contrast function which induces the given statistical manifold* (M, ∇, h) *and the Bartlett tensor* B *is given by*

$$B(X, Y)Z = -h(Y, Z)\left\{-\mathrm{Ric}^{\#}(X) + \frac{\gamma}{n-1}X\right\}, \tag{4.6}$$

where γ *is the trace of Ricci tensor with respect to* h *and* $\mathrm{Ric}^{\#}$ *is the Ricci operator defined by* $h(\mathrm{Ric}^{\#}(X), Y) := \mathrm{Ric}(X, Y)$, *or equivalently,* B^* *is given by*

$$B^*(X, Y)Z = -\frac{1}{n-1}\mathrm{Ric}^*(Y, Z)X. \tag{4.7}$$

PROOF. It is analogous to Theorem 4.1. The tensor B in Theorem 4.2 corresponds to the tensor B^* in Theorem 4.1 since the given connection ∇ is dual-projectively flat. □

We note that if the Bartlett tensor B satisfies equation (4.6), the difference between the curvature tensor R of ∇ to the anti-symmetric part of B is the dual-projective curvature tensor. If the dual Bartlett tensor B^* satisfies equation (4.7), the difference between the curvature tensor R^* of ∇^* to the anti-symmetric part of B^* is the projective curvature tensor.

5. Conformally-projectively flat case

In this section, we determine the Bartlett tensor B of contrast function on a conformally-projectively flat statistical manifold in the case the contrast function is given as a geometric divergence. We also give a condition on the Bartlett tensor B of contrast function that a Riemannian metric be conformally flat.

THEOREM 5.1. *Let* (M, ∇, h) *be a simply connected n-dimensional conformally-projectively flat statistical manifold and* (M, ∇^*, h) *its dual statistical manifold. Let* Ric *and* Ric* *be the Ricci tensors of* ∇ *and* ∇^*, *respectively. Suppose that* $n \geq 3$ *and* Ric *is symmetric. We denote by* $\{f, \xi\}$ *a centroaffine immersion which realizes* (M, ∇, h) *into* $\mathbf{R}^{n+2}$ *and by* $\{v, w\}$ *the dual map of* $\{f, \xi\}$. *If the geometric divergence* ρ *is given as* $\rho(p, q) = \langle v(q), f(p) - f(q)\rangle$ *then the Bartlett tensor* B *is*

$$h(B(X, Y)Z, U) = -\frac{1}{n-2}\{h(Y, Z)A(X, U) + A^*(Y, Z)h(X, U)\}$$
$$+\frac{\gamma}{(n-1)(n-2)}h(Y, Z)h(X, U), \tag{5.1}$$

or equivalently, B^* *is*

$$h(B^*(X,Y)Z,U) = -\frac{1}{n-2}\{h(Y,Z)A^*(X,U) + A(Y,Z)h(X,U)\} + \frac{\gamma}{(n-1)(n-2)}h(Y,Z)h(X,U),$$

where the tensors A *and* A^* *are defined by*

$$A(Y,Z) := \frac{1}{n}\{\mathrm{Ric}(Y,Z) + (n-1)\mathrm{Ric}^*(Y,Z)\}, \tag{5.2}$$

$$A^*(Y,Z) := \frac{1}{n}\{(n-1)\mathrm{Ric}(Y,Z) + \mathrm{Ric}^*(Y,Z)\}, \tag{5.3}$$

and γ *is the trace of Ricci tensor with respect to* h.

Proof. By Proposition 3.1, there exists a nondegenerate equiaffine centroaffine immersion $\{f,\xi\} : M \to \mathbf{R}^{n+2}$. From equations (3.4)–(3.7), we may assume that $\{f,\xi\}$ is pre-normalized. By the assumption, the geometric divergence is given by

$$\rho(p,q) = \langle v(q), f(p) - f(q)\rangle,$$

where v is the conormal map of $\{f,\xi\}$. Contracting the Gauss equations of $\{f,\xi\}$ and the dual map $\{v,w\}$ of $\{f,\xi\}$, we get

$$\mathrm{Ric}(Y,Z) = \mathrm{tr}\,Sh(Y,Z) - h(SY,Z) - (n-1)T(Y,Z), \tag{5.4}$$

$$\mathrm{Ric}^*(Y,Z) = \mathrm{tr}\,S^*h(Y,Z) - h(S^*Y,Z) - (n-1)T^*(Y,Z). \tag{5.5}$$

Since $\{f,\xi\}$ is a pre-normalized centroaffine immersion, we have

$$2(n-1)\mathrm{tr}\,S = \gamma = \gamma^* = 2(n-1)\mathrm{tr}\,S^*, \tag{5.6}$$

where γ and γ^* are the trace of Ric and Ric^* with respect to h, respectively. Substituting equation (5.6) to (5.4) and (5.5), using (3.17) and (3.21), we calculate

$$\mathrm{Ric}(Y,Z) = \frac{\gamma}{2(n-1)}h(Y,Z) - h(SY,Z) - (n-1)T(Y,Z),$$

$$\mathrm{Ric}^*(Y,Z) = \frac{\gamma}{2(n-1)}h(Y,Z) + (n-1)h(SY,Z) + T(Y,Z).$$

We also calculate

$$\frac{1}{n}\{\mathrm{Ric}(Y,Z)+(n-1)\mathrm{Ric}^*(Y,Z)\}=\frac{\gamma}{2(n-1)}h(Y,Z)+(n-2)h(SY,Z),$$

$$\frac{1}{n}\{(n-1)\,\mathrm{Ric}(Y,Z)+\mathrm{Ric}^*(Y,Z)\}=\frac{\gamma}{2(n-1)}h(Y,Z)-(n-2)T(Y,Z).$$

Hence, we obtain

$$h(SY,Z)=\frac{1}{n-2}\left\{A(Y,Z)-\frac{\gamma}{2(n-1)}h(Y,Z)\right\}, \tag{5.7}$$

$$T(Y,Z)=\frac{1}{n-2}\left\{-A^*(Y,Z)+\frac{\gamma}{2(n-1)}h(Y,Z)\right\}. \tag{5.8}$$

By equations (3.24), (5.7) and (5.8), we obtain the desired result. Similarly, we can also obtain the condition of tensor B^*. □

We note that if the Bartlett tensor B satisfies equation (5.1), the difference between the curvature tensor R of ∇ to the anti-symmetric part of B is the conformal-projective curvature tensor.

Let (M,g) be a Riemannian manifold. Suppose that ∇^0 is the Levi-Civita connection of g. In this case, (M,∇^0,g) is a statistical manifold. If g is conformally flat, then (M,∇^0,g) is conformally-projectively flat. In the following corollary, we give a condition on the Bartlett tensor B of contrast function that a Riemannian metric be conformally flat.

COROLLARY 5.2. *Let (M,g) be a simply connected Riemannian manifold of dimension n (≥ 4). Then g is conformally flat if and only if there exists a contrast function which induces the given Riemannian manifold (M,g) and the Bartlett tensor B is given by*

$$h(B(X,Y)Z,U)=-\frac{1}{n-2}\{g(Y,Z)\,\mathrm{Ric}(X,U)+\mathrm{Ric}(Y,Z)g(X,U)\}$$
$$+\frac{\gamma}{(n-1)(n-2)}g(Y,Z)g(X,U), \tag{5.9}$$

where Ric *is the Ricci tensor of ∇^0 and γ is the trace of Ricci tensor with respect to g.*

PROOF. Since the Levi-Civita connection is self-dual, we easily obtain the Bartlett tensor B from Theorem 5.1. Conversely, suppose that the tensor B is given by (5.9). Then the conformal curvature tensor W_C is given by

$$W_C(X,Y)Z=R(X,Y)Z+B(X,Y)Z-B(Y,X)Z,$$

where R is the Riemannian curvature tensor of ∇. Hence, the conformal curvature tensor vanishes. This implies that the given metric g is conformally flat since $n \geq 4$. □

In Theorems 4.1, 4.2, 5.1 and Corollary 5.2, even if M is not simply connected, there exists a contrast function at least locally.

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