HIROSHIMA MATH. J.
**29** (1999), 161–174

# Carleson inequalities in classes of derivatives of harmonic Bergman functions with $0 < p \leq 1$

*Dedicated to Professor Jyunji Inoue on his sixtieth birthday*

Masahiro YAMADA

(Received January 23, 1998)
(Revised April 20, 1998)

ABSTRACT. We give a necessary and sufficient condition for a positive measure $\mu$ on the upper half-space of $\mathbf{R}^n$ to satisfy the inequalities

$$\left(\int |D^\alpha u|^q d\mu\right)^{1/q} \leq C\left(\int |D_y^m u|^p y^r dV\right)^{1/p}$$

for all $u$ in a subclass of a harmonic Bergman space when $0 < p \leq 1$ and $p \leq q$, where $D_y$ denotes the partial differentiation operator with respect to the last coordinate $y$. We also show that the Bergman norm is comparable to derivative norms and harmonic conjugation is bounded on the harmonic Bergman space $b^p$ when $0 < p \leq 1$.

## 1. Introduction

Let $H$ be the upper half-space of the $n$-dimensional Euclidean space $\mathbf{R}^n (n \geq 2)$, that is, $H = \{z = (x, y) \in \mathbf{R}^n; y > 0\}$, where we have written a point $z \in \mathbf{R}^n$ as $z = (x, y)$ with $x = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. For $0 < p < \infty$, let $b^p = b^p(H, dV)$ be the class of all harmonic functions $u$ on $H$ such that

$$\|u\|_p = \left(\int_H |u|^p dV\right)^{1/p} < \infty$$

where $dV$ denotes the Lebesgue volume measure on $H$. The class $b^p$ is called the harmonic Bergman space. Recently, properties of functions in the harmonic Bergman space $b^p$ for $1 \leq p < \infty$ have been studied by Ramey and Yi [9], and several important results have been given. Our aim is to investigate properties in the harmonic Bergman space $b^p$ when $p \leq 1$.

In this paper, we study conditions on a $\sigma$-finite positive Borel measure $\mu$ on

1991 *Mathematics Subject Classification*: 46E30.

*Key words and phrases*: Bergman space, Carleson inequality, harmonic function, harmonic conjugation.

$H$ for which there is a constant $C$ satisfying $\int |u|^p d\mu \leq C \int |D_y u|^p y^r dV$ for all $u$ in a subclass of $b^p$ when $p \leq 1$, where $D_y$ denotes the partial derivative with respect to $y$ and $r > -1$. (Our consideration is more general.) Such inequalities on the unit disk in the complex plane were studied by Stegenga [10]. It was proved that when $r \geq 1$ a finite positive Borel measure $\nu$ on the unit disk satisfies the inequality $\int |f|^2 d\nu \leq C \int |f'|^2 (1-|\zeta|)^r dA$ for all holomorphic functions $f$ if and only if there is a constant $K$ such that $\nu(S(I)) \leq K|I|^r$ for any interval $I$ in the unit circle, where $dA$ denotes the Lebesgue area measure, $|I|$ denotes the normalized arc length of $I$, and $S(I) = \{\zeta : \zeta/|\zeta| \in I, 1-|I| < |\zeta| < 1\}$. It was also proved that when $0 \leq r < 1$ such measures are those satisfying $\nu(\cup S(I_j)) \leq K\,\mathrm{Cap}(\cup I_j)$ for all finite disjoint collections of intervals $\{I_j\}$, where Cap is an appropriate Bessel capacity (if $r < 0$ any finite Borel measure satisfies this inequality). It is known that these characterizations can be generalized to the case of $p > 1$ (see also [10]). When $p \leq 1$, the characterization in Ahern and Jevtić [1] is simpler. Indeed, $\nu$ satisfies the inequality $\int |f|^p d\nu \leq C \int |f'|^p (1-|\zeta|)^r dA$ if and only if $\nu(S(I)) \leq K|I|^{2-p+r}$ when $p \leq 1$. In the proof of the case $p \leq 1$, a Hausdorff capacity was used in stead of the Bessel capacity. When $p > 1$ investigations for several variables are given in [3]. In these investigations, necessary and sufficient conditions were not obtained completely, and it was also shown that, in general, the above condition is not necessary, in contrast to the result on the unit disk. In case $p \leq 1$, no necessary and sufficient conditions are known.

In § 3, we give a necessary and sufficient condition for a measure $\mu$ on the upper half-space $H$ to satisfy the inequality $\int |u|^p d\mu \leq C \int |D_y u|^p y^r dV$ for all $u$ in a subclass of $b^p$ when $p \leq 1$ (see Theorem 1). § 2 is devoted to some preliminary lemmas for this investigation in § 3. In the proofs of characterizations of measures on the unit disk to satisfy such inequalities in [10] and [1], capacity estimates are used. However, in the proof of Theorem 1 in § 3, we use integral representations for harmonic functions.

In § 4, we study properties of functions in the harmonic Bergman space $b^p$ when $p \leq 1$. All results described in § 4 were proved in [9] when $p \geq 1$. In [9], it was shown that if $p \geq 1$ and $u \in b^p$ then there exist unique harmonic conjugates $u_1, \ldots, u_{n-1}$ of $u$ that belong to $b^p$. Using the ideas used in the proof of Theorem 1, we show that these conjugation results are also valid in the case of $p \leq 1$. Therefore, harmonic conjugation is bounded on the harmonic Bergman space $b^p$ for all $0 < p < \infty$ and all dimensions $n$. It is well known that such conjugation result is not valid in the theory of Hardy spaces (see [5, pp. 102–123] and [4, pp. 167–172]). Moreover, we show that when $p \leq 1$ the Bergman norm is comparable to several "derivative norms" as in [9]. These results are consequences of Theorem 1 and the boundedness of harmonic conjugation.

Throughout this paper, $C$ will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

## 2. Preliminary lemmas

Recall that a point $z \in H$ will be written as $z = (x, y)$ with $x \in \mathbf{R}^{n-1}$ and $y > 0$. We use the absolute value symbol $|\cdot|$ to denote the Euclidean norm in $\mathbf{R}^n$ or $\mathbf{R}^{n-1}$. For $z = (x, y)$, let $\bar{z} = (x, -y)$. The pseudohyperbolic metric $\rho$ in $H$ is defined by $\rho(z, w) = |z - w|/|z - \bar{w}|$. It is clear that $\rho$ is invariant under horizontal translations and dilations. Let $D_\varepsilon(w) = \{z \in H; \rho(z, w) < \varepsilon\}$ when $w = (s, t) \in H$ and $0 < \varepsilon < 1$. $D_\varepsilon(w)$ is a Euclidean ball whose center and radius are $\left(s, \dfrac{1+\varepsilon^2}{1-\varepsilon^2}t\right)$ and $\dfrac{2\varepsilon t}{1-\varepsilon^2}$ respectively. It follows that there is a constant $C = C_\varepsilon > 0$ such that $C^{-1}t^n \le V(D_\varepsilon(w)) \le Ct^n$ for all $w \in H$. The proof of (3) of Lemma 1 is parallel to that of Lemma 4.3.6 in [12].

LEMMA 1. *Let $0 < \varepsilon < 1$. Then, the following are true.*

(1) *If $z, w, \zeta$ are in $H$ and $\rho(z, w) < \varepsilon$, then $C^{-1}|z - \bar{\zeta}| \le |w - \bar{\zeta}| \le C|z - \bar{\zeta}|$ with a positive constant $C$ depending only on $\varepsilon$.*

(2) *If $z = (x, y)$, $w = (s, t)$ are in $H$ and $\rho(z, w) < \varepsilon$, then $C^{-1}y \le t \le C_y$ with a positive constant $C$ depending only on $\varepsilon$.*

(3) *If $0 < \varepsilon < 1/2$ then there exist a positive integer $N$ and a sequence $\{\zeta_j\}$ in $H$ satisfying the following conditions:* (a) $H = \bigcup D_\varepsilon(\zeta_j)$, (b) $D_{\varepsilon/4}(\zeta_i) \cap D_{\varepsilon/4}(\zeta_j) = \emptyset$ *if $i \ne j$,* (c) *any point in $H$ belongs to at most $N$ of the sets $D_{2\varepsilon}(\zeta_j)$.*

PROOF. (1) Suppose that $z, w, \zeta \in H$ and $\rho(z, w) < \varepsilon$. It is sufficient to prove that $|w - \bar{\zeta}| \le C|z - \bar{\zeta}|$. Since the condition $\rho(z, w) < \varepsilon$ implies the inequality $|w - z| < \varepsilon|w - \bar{z}|$, we have $|w - \bar{\zeta}| \le |w - z| + |z - \bar{\zeta}| < \varepsilon|w - \bar{z}| + |z - \bar{\zeta}| \le \varepsilon(|w - \bar{\zeta}| + |\bar{\zeta} - \bar{z}|) + |z - \bar{\zeta}|$. It follows that $(1 - \varepsilon)|w - \bar{\zeta}| < \varepsilon|\bar{\zeta} - \bar{z}| + |z - \bar{\zeta}| \le \varepsilon|\bar{\zeta} - z| + |z - \bar{\zeta}| = (1 + \varepsilon)|z - \bar{\zeta}|$. (2) In the first inequality in (1), if we put $\zeta = w$ then we have $2t = |w - \bar{w}| \ge C^{-1}|z - \bar{w}| \ge C^{-1}y$. (3) See the proof of Lemma 4.3.6 in [12].

For a function $u$ on $H$ and $\delta > 0$, let $\tau_\delta u$ denote the function on $H$ defined by $\tau_\delta u(x, y) = u(x, y + \delta)$, and let $\mathscr{T}^p = \{\tau_\delta u; u \in b^p, \delta > 0\}$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of nonnegative integers with order $\ell$, then $D^\alpha$ denotes the partial differentiation operator $\partial^\ell/\partial x_1^{\alpha_1} \dots \partial x_{n-1}^{\alpha_{n-1}} \partial y^{\alpha_n}$. The following lemma is stated in [2, Corollary 8.2] when $p \ge 1$.

LEMMA 2. *Let $0 < p \le 1$. Then, the following are true.*

(1) *For any $u \in b^p$, there is a constant $C > 0$ such that $|D^\alpha u(s, t)| \le C/t^{n/p+|\alpha|}$ for all $(s, t) \in H$.*

(2) *For any* $u \in b^p$, *there is a constant* $C > 0$ *such that* $|(D^\alpha \tau_\delta u)(s,t)| \le C/(t+\delta)^{n/p+|\alpha|}$ *for all* $(s,t) \in H$.

PROOF. (1) Let $w_0 = (0,1)$ and $0 < \varepsilon < 1$. Then, by Lemma 2 in [4, Section 9] there is a constant $C = C_\varepsilon > 0$ such that $|u(w_0)|^p \le C \int_{D_\varepsilon(w_0)} |u|^p dV$. For $w = (s,t) \in H$, replacing $u(z)$ by $u(tz + (s,0))$ in the inequality and making a change of variable, we have $|u(s,t)|^p \le C/t^n \int_{D_\varepsilon(w)} |u|^p dV$. Therefore, we obtain $|u(s,t)| \le C/t^{n/p}$. As in the proof of Corollary 8.2 in [2], we apply this inequality and Cauchy's estimates to $u$. Then the desired inequality follows. (2) follows from (1).

Let $w = (s,t) \in H$. The Poisson kernel $P_w$ is the function on $\mathbf{R}^{n-1}$ given by $P_w(x) = P(s-x,t) = \gamma_n t/(|s-x|^2 + t^2)^{n/2}$ ($\gamma_n$ is the positive constant $\gamma_n = 2/(nV(\mathbf{B}_n))$, where $\mathbf{B}_n$ denotes the unit ball in $\mathbf{R}^n$). The harmonic extension of this function to $H$ is $P(s-x, t+y)$. If $z = (x,y) \in H$, then we may write $P_w(z)$. We note that $P_w(z) = \gamma_n(t+y)/|w-\bar{z}|^n$, $|D_z^\alpha P_w(z)| \le C/|w-\bar{z}|^{n+|\alpha|-1}$, and $D_z^\alpha P_w(z) = (-1)^{\alpha_1+\cdots+\alpha_{n-1}} D_w^\alpha P_w(z)$. The following lemma is useful and stated in [9, Lemma 3.1]

LEMMA 3. *Let* $0 < c < 1$. *Then, there is a constant* $C > 0$ *depending on* $c$ *and* $n$ *such that*

$$\int_H \frac{y^{-c}}{|w-\bar{z}|^n} \, dV(z) = Ct^{-c}$$

*for all* $w = (s,t) \in H$.

Let $m$ be a nonnegative integer and let $c_m = (-2)^m/m!$. The following Lemma 4 is given in [2, Chapter 8] and [9], when $u \in b^p$ and $p \ge 1$. The proofs of (1) and (2) of Lemma 4 are parallel to the proofs of Theorem 8.22 in [2, Chapter 8] and Lemma 4.1 in [9] respectively, except only minor changes.

LEMMA 4. *Let* $0 < p \le 1$. *If* $u \in \mathscr{T}^p$, *then the following equalities hold.*

(1) $u(w) = -2 \int_H u(z) D_y P_w(z) \, dV(z)$ *for all* $w \in H$.

(2) $u(w) = -2c_m \int_H y^m (D_y^{m+1} u)(z) P_w(z) \, dV(z)$ *for all* $w \in H$, $m = 0, 1, 2, \ldots$.

PROOF. (1) We only show that $uD_yP_w$ is integrable, because the remainder of the proof is parallel to that of Theorem 8.22 in [2, Chapter 8]. Since $u \in \mathscr{T}^p$, (2) of Lemma 2 implies that there are constants $C$ and $\delta > 0$ such that $|D^\alpha u(z)| \le C/(y+\delta)^{n/p+|\alpha|}$. Thus, we have $|u(z)| \le C(y+\delta)^{-n/p+c}(y+\delta)^{-c} \le Cy^{-c}$ for some $0 < c < 1$. Therefore, we obtain $\int |uD_yP_w| \, dV \le C \int y^{-c}/|w-\bar{z}|^n dV = Ct^{-c}$, where the last equality follows from Lemma 3. Thus, $uD_yP_w$ is integrable. (2) Similarly, we have $|D_y^{m+1} u(z)| \le Cy^{-c-m}/(y+\delta)$.

Therefore, $\int |y^m P_w D_y^{m+1} u|\, dV \le C \int y^{-c}(y+t)/\{(y+\delta)|w-\bar{z}|^n\}\, dV$. Since $(y+t)/(y+\delta)$ is bounded for $y>0$, we see that $y^m P_w D_y^{m+1} u$ is integrable.

The following Lemma 5 is a consequence of Lemma 4, and we omit the proof (note that arguments similar to those in the proof of Lemma 4 guarantee that the right-hand side of the equality in Lemma 5 is well defined and the remainder of the proof is parallel to that of Lemma 4.6 in [9]).

LEMMA 5. *Let* $0 < p \le 1$. *If* $u \in \mathscr{T}^p$, *then*

$$u(w) = -2c_{m+k} \int_H y^{m+k} (D_y^m u)(z) D_y^{k+1} P_w(z)\, dV(z)$$

*for all* $m, k \ge 0$ *and* $w \in H$.

## 3. Carleson inequalities

Let $B_t(s)$ denote the ball in $\mathbf{R}^{n-1}$ with center $s \in \mathbf{R}^{n-1}$ and radius $t > 0$. When no confusion arises we may write $B_t$ in stead of $B_t(s)$. For each ball $B_t$ in $\mathbf{R}^{n-1}$ set $S(B_t) = \{(x,y); x \in B_t, y < 2t\}$. We now state our main result in this section.

THEOREM 1. *Suppose that* $0 < p \le 1$, $p \le q$ *and* $r > -1$. *Let* $\mu$ *be a σ-finite positive Borel measure on* $H$, *and let* $\ell$ *and* $m$ *be nonnegative integers. Then, the following* (1) $\sim$ (3) *are equivalent.*

(1) *There is a constant* $C > 0$ *such that*

$$\left( \int_H |D^\alpha u|^q d\mu \right)^{1/q} \le C \left( \int_H |D_y^m u|^p y^r dV \right)^{1/p}$$

*for all* $u \in \mathscr{T}^p$ *and for all multi-indices* $\alpha$ *of order* $\ell$.

(2) *There is a constant* $C > 0$ *such that*

$$\left( \int_H |D_y^\ell u|^q d\mu \right)^{1/q} \le C \left( \int_H |D_y^m u|^p y^r dV \right)^{1/p}$$

*for all* $u \in \mathscr{T}^p$.

(3) *There is a constant* $K > 0$ *such that* $\mu(S(B_t)) \le K t^{(n+r)q/p + (\ell - m)q}$ *for all balls* $B_t \subset \mathbf{R}^{n-1}$.

We note that in case $(n+r)q/p + (\ell - m)q = 0$ (or equivalently, $n + r = p(m - \ell)$), $\mu$ satisfies the above inequalities if and only if $\mu$ is a finite measure. In fact, in this case, condition (3) of Theorem 1 is reduced to $\mu(S(B)) \le K$ for all balls $B$. For each compact set $E \subset H$, we can choose a ball $B$ satisfying $E \subset S(B)$. Therefore, we have $\mu(E) \le K$ for all compact sets $E \subset H$, and thus $\mu$ is finite. Similarly, we can see that in case

$(n+r)q/p+(\ell-m)q<0$, $\mu$ satisfies the above inequalities if and only if $\mu=0$. In the inequality in (2) of Theorem 1, if $m\geq\ell$, then, of course, we can replace $D_y^\ell u$ and $D_y^m u$ by $u$ and $D_y^{m-\ell}u$ respectively. Similarly, if $m<\ell$, then we can replace $D_y^\ell u$ and $D_y^m u$ by $D_y^{\ell-m}u$ and $u$ respectively.

We give a sufficient condition for a measure $\mu$ to satisfy the inequality.

PROPOSITION 2. *Under the assumptions on $p,q,r,\ell$ and $m$ in Theorem 1, let $k$ be a nonnegative integer such that $p(n+k)-2n>0$. Suppose that there is a constant $K>0$ such that*

$$\int_H \frac{1}{|w-\bar{z}|^{q(n+\ell+k)}}\,d\mu(z)\leq Kt^{(n+r)q/p-q(n+m+k)}$$

*for all $w=(s,t)\in H$. Then, there is a constant $C>0$ such that*

$$\left(\int_H |D^\alpha u|^q d\mu\right)^{1/q}\leq C\left(\int_H |D_y^m u|^p y^r dV\right)^{1/p}$$

*for all $u\in\mathscr{T}^p$ and for all multi-indices $\alpha$ of order $\ell$.*

PROOF. Let $k$ be a nonnegative integer such that $p(n+k)-2n>0$. Let $u\in\mathscr{T}^p$. Then, Lemma 5 implies that

$$u(z)=-2c_{k+m}\int_H t^{m+k}(D_t^m u)(w)D_t^{k+1}P_z(w)\,dV(w)$$

for all $z\in H$. We will estimate $|D^\alpha u|$. The remark before Lemma 3 implies that

$$\begin{aligned}|D_z^\alpha u(z)|&\leq C\int_H t^{m+k}|D_t^m u(w)||D_z^\alpha D_t^{k+1}P_z(w)|\,dV(w)\\&\leq C\int_H \frac{t^{m+k}}{|w-\bar{z}|^{n+\ell+k}}|D_t^m u(w)|\,dV(w).\end{aligned}$$

Let $0<\varepsilon<1/2$. Then, by (3) of Lemma 1, we can choose a positive integer $N$ and a sequence $\{\zeta_j\}$ in $H$ such that $H=\cup_j D_\varepsilon(\zeta_j)$ and any point in $H$ belongs to at most $N$ of the sets $D_{2\varepsilon}(\zeta_j)$. We shall write $\zeta_j=(\zeta_j,\eta_j)$ with $\xi_j\in\mathbf{R}^{n-1}$ and $\eta_j>0$. Using (1) and (2) of Lemma 1, we have

$$\begin{aligned}|D^\alpha u(z)|&\leq C\sum_j\int_{D_\varepsilon(\zeta_j)}\frac{t^{m+k}}{|w-\bar{z}|^{n+\ell+k}}|D_t^m u(w)|\,dV(w)\\&\leq C\sum_j\frac{\eta_j^{m+k}}{|\zeta_j-\bar{z}|^{n+\ell+k}}\int_{D_\varepsilon(\zeta_j)}|D_t^m u(w)|\,dV(w).\end{aligned}$$

Since $D_t^m u$ is harmonic, a result in the proof of (1) of Lemma 2 (or Lemma 2 in [4, Section 9]) implies that $|D_t^m u(w)|^p \le C/t^n \int_{D_\varepsilon(w)} |D_t^m u|^p dV$. Moreover, since $D_\varepsilon(w) \subset D_{2\varepsilon}(\zeta_j)$ if $w \in D_\varepsilon(\zeta_j)$, (2) of Lemma 1 implies that

$$\begin{aligned}
|D^\alpha u(z)| &\le C \sum_j \frac{\eta_j^{m+k}}{|\zeta_j - \bar{z}|^{n+\ell+k}} \int_{D_\varepsilon(\zeta_j)} \left( \frac{1}{t^n} \int_{D_\varepsilon(w)} |D_t^m u|^p dV \right)^{1/p} dV(w) \\
&\le C \sum_j \frac{\eta_j^{m+k}}{|\zeta_j - \bar{z}|^{n+\ell+k}} \eta_j^n \left( \frac{1}{\eta_j^n} \int_{D_{2\varepsilon}(\zeta_j)} |D_t^m u|^p dV \right)^{1/p} \\
&= C \sum_j \left( \frac{\eta_j^{p(n+m+k)-n}}{|\zeta_j - \bar{z}|^{p(n+\ell+k)}} \int_{D_{2\varepsilon}(\zeta_j)} |D_t^m u|^p dV \right)^{1/p} \\
&\le C \left( \sum_j \frac{\eta^{p(n+m+k)-n_j}}{|\zeta_j - \bar{z}|^{p(n+\ell+k)}} \int_{D_{2\varepsilon}(\zeta_j)} |D_t^m u|^p dV \right)^{1/p},
\end{aligned}$$

where the last inequality follows from Jensen's inequality. Thus, the choice of $\{\zeta_j\}$, (1) and (2) of Lemma 1 imply that

$$|D^\alpha u(z)| \le C N^{1/p} \left( \int_H \frac{t^{p(n+m+k)-n}}{|w - \bar{z}|^{p(n+\ell+k)}} |D_t^m u(w)|^p dV(w) \right)^{1/p}.$$

We note that the right-hand side of this inequality is finite. In fact, since $u \in \mathcal{T}^p$, (2) of Lemma 2 implies that $|D_t^m u(w)|^p \le C/t^{n+pm}$. Moreover, since $|w - \bar{z}| \ge t$, the condition $p(n+k) - 2n > 0$ implies that the integrand is dominated by $1/|w - \bar{z}|^{n+(n+p\ell)}$. Thus, the integrability of the function $1/|w - \bar{z}|^{n+c} (c > 0)$ guarantees that the right-hand side of the inequality is finite.

Raising the inequality to the $q$-th power and integrating with respect to $\mu$ in the variable $z$, we have

$$\int_H |D^\alpha u(z)|^q d\mu(z) \le C \int_H \left( \int_H \frac{t^{p(n+m+k)-n}}{|w - \bar{z}|^{p(n+\ell+k)}} |D_t^m u(w)|^p dV(w) \right)^{q/p} d\mu(z).$$

Thus, Minkowski's integral inequality implies that

$$\int_H |D^\alpha u(z)|^q d\mu(z) \le C \left[ \int_H |D_t^m u(w)|^p \left( \int_H \frac{t^{q(n+m+k)-nq/p}}{|w - \bar{z}|^{q(n+\ell+k)}} d\mu(z) \right)^{p/q} dV(w) \right]^{q/p}.$$

Therefore, by hypothesis we obtain

$$\int_H |D^\alpha u|^q\, d\mu \le CK\left(\int_H |D_t^m u|^p t^r dV\right)^{q/p}.$$

This completes the proof.

In order to give a necessary condition for a measure $\mu$ to satisfy the inequality in (2) of Theorem 1, we need the following lemma.

Lemma 6. *Let $k$ be a nonnegative integer. Then, there exist constants $0 < \sigma \le 1$ and $C > 0$ such that $|D_y^k P_w(z)| \ge C/t^{n+k-1}$ for all $w = (s, t) \in H$ and $z \in S(B_{\sigma t}(s))$.*

Proof. Let $w = (s, t) \in H$. Without loss of generality we may assume that $s = 0$. If $z = (x, y) \in H$ and $|x|/(y + t) < 1$ then,

$$P_w(z) = C\frac{1}{(y+t)^{n-1}}\frac{1}{\{1+|x/(y+t)|^2\}^{n/2}} = C\sum_{j=0}^{\infty}\frac{\Gamma(n/2+j)}{j!\Gamma(n/2)}\frac{(-1)^j|x|^{2j}}{(y+t)^{2j+n-1}}.$$

Therefore, we have

$$D_y^k P_w(z) = C\sum_{j=0}^{\infty}\frac{\Gamma(n/2+j)}{j!\Gamma(n/2)}\frac{(2j+n+k-2)!}{(2j+n-2)!}\frac{(-1)^{j+k}|x|^{2j}}{(y+t)^{2j+n+k-1}}.$$

Now, we define a function $g$ on $[0, 1)$ by

$$g(\lambda) = \sum_{j=0}^{\infty}\frac{\Gamma(n/2+j)}{j!\Gamma(n/2)}\frac{(2j+n+k-2)!}{(2j+n-2)!}(-1)^j\lambda^{2j}.$$

Then

$$D_y^k P_w(z) = C\frac{(-1)^k}{(y+t)^{n+k-1}}g\left(\frac{|x|}{y+t}\right).$$

Since $g(0) \neq 0$ and $g$ is continuous on $[0, 1)$ (in fact, $g(\lambda) = \lambda^{2-n}(\lambda^{n+k-2}/\{1+\lambda^2\}^{n/2})^{(k)}$), there exist constants $0 < \sigma \le 1$ and $C_0 > 0$ such that $|g(\lambda)| \ge C_0$ if $\lambda < \sigma$, where the constants $\sigma$ and $C_0$ depend only on $n$ and $k$. Let $z = (x, y) \in S(B_{\sigma t}(0))$. Then clearly $|x|/(y + t) < \sigma$ and $y < 2\sigma t$. Therefore, we obtain

$$|D_y^k P_w(z)| \ge \frac{CC_0}{(y+t)^{n+k-1}} > \frac{CC_0}{\{(2\sigma+1)t\}^{n+k-1}} = \frac{C'}{t^{n+k-1}},$$

where the constant $C'$ depends only on $n$, $k$, and $\sigma$.

PROPOSITION 3. *Under the assumptions on $p, q, r, \ell$ and $m$ in Theorem* 1, *suppose that there is a constant $C > 0$ such that*

$$\left(\int_H |D_y^\ell u|^q d\mu\right)^{1/q} \leq C\left(\int_H |D_y^m u|^p y^r dV\right)^{1/p}$$

*for all $u \in \mathscr{T}^p$. Then, there is a constant $K > 0$ such that $\mu(S(B_t)) \leq Kt^{(n+r)q/p+(\ell-m)q}$ for all balls $B_t \subset \mathbf{R}^{n-1}$.*

PROOF. Let $s \in \mathbf{R}^{n-1}$ and $t > 0$. Put $w = (s, t)$ and let $k$ be a nonnegative integer such that $(n+k-1)p > n+2r+2$. Then, we see that $D_y^k P_w \in \mathscr{T}^p$ and

$$\int_H |D_y^{m+k} P_w(z)|^p y^r dV(z) \leq Ct^{-(n+m+k-1)p+n+r}.$$

In fact, since $|D_y^k P_w(z)|^p \leq C/|w-\bar{z}|^{(n+k-1)p}$, the choice of $k$ and the integrability of the function $1/|w-\bar{z}|^{n+c} (c > 0)$ guarantee that $D_y^k P_w \in \mathscr{T}^p$. Moreover, if $-1 < r < 0$ then Lemma 3 implies that

$$\begin{aligned}\int_H |D_y^{m+k} P_w(z)|^p y^r dV(z) &\leq C\int_H \frac{y^r}{|w-\bar{z}|^{(n+m+k-1)p}} dV(z) \\ &\leq Ct^{-(n+m+k-1)p+n} \int_H \frac{y^r}{|w-\bar{z}|^n} dV(z) \\ &= Ct^{-(n+m+k-1)p+n+r},\end{aligned}$$

because $(n+m+k-1)p - n > 2r+2 > 0$. If $r \geq 0$ then the choice of $k$ and Lemma 3 also imply that

$$\begin{aligned}\int_H |D_y^{m+k} P_w(z)|^p y^r dV(z) &\leq Ct^{-(n+m+k-1)p+n+2r} \int_H \frac{y^r}{|w-\bar{z}|^{n+2r}} dV(z) \\ &\leq Ct^{-(n+m+k-1)p+n+2r} \int_H \frac{y^{-r}}{|w-\bar{z}|^n} dV(z) \\ &= Ct^{-(n+m+k-1)p+n+r}.\end{aligned}$$

Therefore, we obtain the above assertions.

Put $u = D_y^k P_w$. Then, the above assertions and Lemma 6 imply that

$$\begin{aligned}
Ct^{(n+r)q/p-(n+m+k-1)q} &\geq \left(\int_H |D_y^m u(z)|^p y^r dV(z)\right)^{q/p} \\
&\geq C\int_H |D_y^\ell u(z)|^q \, d\mu(z) \\
&\geq C\int_{S(B_{\sigma t}(s))} |D_y^{\ell+k} P_w(z)|^q d\mu(z) \\
&\geq Ct^{-(n+\ell+k-1)q}\int_{S(B_{\sigma t}(s))} d\mu(z).
\end{aligned}$$

Thus, we obtain $\mu(S(B_{\sigma t}(s))) \leq Ct^{(n+r)q/p+(\ell-m)q}$, where $C$ is independent of $s$ and $t$. Since $s$ and $t$ are arbitrary, we can replace $t$ by $t/\sigma$. This implies that $\mu(S(B_t(s))) \leq C(t/\sigma)^{(n+r)q/p+(\ell-m)q}$ for all $s \in \mathbf{R}^{n-1}$ and $t > 0$.

Proof of Theorem 1. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) was already shown in Proposition 3. We will show (3) $\Rightarrow$ (1). Let $c = (n+r)q/p + (\ell - m)q$ and suppose that $\mu(S(B_\eta)) \leq K\eta^c$ for all balls $B_\eta \subset \mathbf{R}^{n-1}$. By Proposition 2, it is sufficient to prove that there exists a nonnegative integer $k$ such that $p(n+k) - 2n > 0$ and $\int_H 1/|w-\bar{z}|^\gamma d\mu(z) \leq Ct^{c-\gamma}$ for all $w = (s,t) \in H$, where $\gamma = q(n+\ell+k)$. Let $w \in H$. Without loss of generality we may assume that $w = (0,t)$, and $k$ will be determined later. Let $S_j = S(B_{2^j t}(0))$ $(j \geq 0)$. Clearly, if $z \notin S_{j-1}$, then $|w-\bar{z}| \geq 2^{j-1}t (j \geq 1)$. Therefore,

$$\begin{aligned}
\int_H \frac{1}{|w-\bar{z}|^\gamma} d\mu(z) &\leq t^{-\gamma}\int_{S_0} d\mu + t^{-\gamma}\sum_{j=1}^\infty \frac{1}{2^{\gamma(j-1)}}\int_{S_j\setminus S_{j-1}} d\mu \\
&\leq Ct^{c-\gamma} + C't^{c-\gamma}\sum_{j=1}^\infty \frac{1}{(2^{\gamma-c})^j}.
\end{aligned}$$

Since $\gamma - c = q(n+m+k) - (n+r)q/p$, we can choose an integer $k$ such that $\gamma - c > 0$ and $p(n+k) - 2n > 0$. It follows that $\int_H 1/|w-\bar{z}|^\gamma d\mu(z) \leq Ct^{c-\gamma}$.

## 4. Derivative norms and harmonic conjugates of $b^p$-functions

When $p \geq 1$, properties of the harmonic Bergman space $b^p$ have been studied by Ramey and Yi [9]. We show that some of these properties are also valid for $0 < p < 1$. For each $\delta > 0$, set $\Omega_\delta = \{z \in H; y > \delta\}$ and denote by $\chi_\delta$ the characteristic function of $\Omega_\delta$. We use the expression $A \approx B$ meaning that there is a constant $C > 0$ such that $C^{-1}A \leq B \leq CA$. We show that the Bergman norm is comparable to "derivative norms". The following theorem is a consequence of Theorem 1.

THEOREM 4. *Let $0 < p \le 1$ and $\ell$ be a nonnegative integer. Then*

$$\|u\|_p \approx \sum_{|\alpha|=\ell} \|y^\ell D^\alpha u\|_p \approx \|y^\ell D_y^\ell u\|_p$$

*for all $u \in b^p$.*

PROOF. Let $u \in b^p$. We note that $\|y^\ell D^\alpha \tau_\delta u - y^\ell D^\alpha u\|_p \to 0 (\delta \to 0)$. In fact, since

$$\begin{aligned}\|y^\ell D^\alpha \tau_\delta u\|_p^p &= \int_\delta^\infty (y-\delta)^{\ell p} \int_{\partial H} |D^\alpha u(x,y)|^p \, dx\, dy \\ &= \int_0^\infty (y-\delta)^{\ell p} \int_{\partial H} \chi_\delta(x,y) |D^\alpha u(x,y)|^p dx\, dy,\end{aligned}$$

the monotone convergence theorem implies that $\|y^\ell D^\alpha \tau_\delta u\|_p \to \|y^\ell D^\alpha u\|_p$. By the definition of $\tau_\delta$, we have $y^\ell D^\alpha \tau_\delta u(x,y) \to y^\ell D^\alpha u(x,y)$ for each $(x,y) \in H$. Hence, Egoroff's Theorem implies that $\|y^\ell D^\alpha \tau_\delta u - y^\ell D^\alpha u\|_p \to 0$.

We show that derivative norms are dominated by Bergman norms. In (1) of Theorem 1, we put $q = p$ and $m = r = 0$. Then, there is a constant $C > 0$ such that $\int_H |D^\alpha v|^p d\mu \le C \int_H |v|^p dV$ for all $v \in \mathscr{T}^p$ if and only if there is a constant $K > 0$ such that $\mu(S(B_t)) \le K t^{n+\ell p}$ for all $B_t \subset \mathbf{R}^{n-1}$. Since $d\mu = y^{\ell p} dV$ satisfies this condition (in fact, $\mu(S(B_t)) = \int_0^{2t} y^{\ell p} \int_{B_t} dx\, dy = C t^{n+\ell p}$), we have $\|y^\ell D^\alpha \tau_\delta u\|_p \le C \|\tau_\delta u\|_p$. Letting $\delta \to 0$, we obtain $\|y^\ell D^\alpha u\|_p \le C \|u\|_p$. It follows that

$$\sum_{|\alpha|=\ell} \|y^\ell D^\alpha u\|_p \le C \|u\|_p.$$

Similarly, Theorem 1 also implies that $\|u\|_p \le C \|y^\ell D_y^\ell u\|_p$. Therefore, we conclude that

$$\|y^\ell D_y^\ell u\|_p \le \sum_{|\alpha|=\ell} \|y^\ell D^\alpha u\|_p \le C \|u\|_p \le C' \|y^\ell D_y^\ell u\|_p.$$

This completes the proof.

Given a harmonic function $u$ on $H$, recall that functions $u_1, \dots, u_{n-1}$ are called harmonic conjugates of $u = u_n$ if

$$\sum_{j=1}^n D_j u_j = 0 \quad \text{and} \quad D_i u_j = D_j u_i \quad (1 \le i, j \le n),$$

where $D_j = \partial/\partial x_j$ $(1 \le j \le n-1)$ and $D_n = D_y = \partial/\partial y$.

In [9], it was shown that harmonic conjugation is bounded on the harmonic Bergman space $b^p$ when $p \ge 1$. We show that this conjugation result is also valid in the case of $p \le 1$. That conjugation is bounded on the Bergman space on the unit disk for $p \leq 1$ was observed in [6]. An analogous result holds for the upper half-space in all dimensions.

THEOREM 5. *Let* $0 < p \leq 1$ *and* $u \in b^p$. *Then, there exist harmonic conjugates* $u_1, \dots, u_{n-1}$ *of* $u$ *such that* $u_j \in b^p$. *Moreover, they are uniquely determined and*

$$\|u\|_p \approx \sum_{j=1}^{n-1} \|u_j\|_p.$$

PROOF. Let $u \in b^p$ and $\delta > 0$. Let $k$ be a nonnegative integer such that $p(n+k) - 2n > 0$. Then, harmonic functions $v_j^\delta$ on $H$ $(1 \leq j \leq n-1)$ can be defined by

$$v_j^\delta(z) = 2c_k \int_H t^k \tau_\delta u(w) D_{s_j} D_t^k P_z(w)\, dV(w),$$

where $w = (s, t) = (s_1, \dots, s_{n-1}, t)$ and $z = (x_1, \dots, x_{n-1}, y)$. In fact, since $|\tau_\delta u(w)| \leq Ct^{-c}$ for some $0 < c < 1$ and $|D_{s_j} D_t^k P_z(w)| \leq Ct^{-k}/|w - \bar{z}|^n$, the absolute value of the integrand is dominated by $t^{-c}/|w - \bar{z}|^n$. Therefore, Lemma 3 implies that the right-hand side of the equality is well defined and harmonic on $H$. Since $D_{x_j} P_z(w) = -D_{s_j} P_z(w)$ $(1 \leq j \leq n-1)$, differentiating through the integral, we have $D_i v_j^\delta = D_j v_i^\delta$ for all $1 \leq i, j \leq n-1$. Moreover, since $D_y P_z(w) = D_t P_z(w)$, Lemma 5 implies that

$$D_y v_j^\delta(z) = D_{x_j}\left(-2c_k \int_H t^k \tau_\delta u(w) D_t^{k+1} P_z(w)\, dV(w)\right) = D_{x_j} \tau_\delta u(z).$$

Therefore, we obtain $D_n v_j^\delta = D_j \tau_\delta u$. Similarly, we can also show that (see [9])

$$\sum_{j=1}^{n-1} D_j v_j^\delta + D_n \tau_\delta u = 0.$$

Thus, these functions $v_1^\delta, \dots, v_{n-1}^\delta$ are harmonic conjugates of $\tau_\delta u$.

We show that there is a constant $C > 0$ independent of $\delta$ and $j$ such that $\|v_j^\delta\|_p \leq C\|u\|_p$. As in the proof of Proposition 2, we can show that there is a constant $C > 0$ independent of $\delta$ and $j$ such that

$$|v_j^\delta(z)|^p \leq C \int_H \frac{t^{p(n+k)-n}}{|w - \bar{z}|^{p(n+k)}} |\tau_\delta u(w)|^p dV(w)$$

for all $z \in H$. Integrating this inequality with respect to $dV$, we have

$$\int_H |v_j^\delta(z)|^p dV(z) \leq C \int_H \left(\int_H \frac{t^{p(n+k)-n}}{|w - \bar{z}|^{p(n+k)}}\, dV(z)\right) |\tau_\delta u(w)|^p dV(w).$$

Since $p(n+k) - 2n > 0$, we have $|w - \bar{z}|^{p(n+k)} = |w - \bar{z}|^{n+p(n+k)-n-c+c} \geq |w - \bar{z}|^n t^{p(n+k)-n-c} y^c$ for some $0 < c < 1$. Hence, Lemma 3 implies that there is a constant $K > 0$ such that

$$\int_H \frac{t^{p(n+k)-n}}{|w-\bar{z}|^{p(n+k)}}\, dV(z) \le \int_H \frac{t^{p(n+k)-n}}{|w-\bar{z}|^n t^{p(n+k)-n-c} y^c}\, dV(z)$$
$$= t^c \int_H \frac{y^{-c}}{|w-\bar{z}|^n}\, dV(z) \le K$$

for all $w \in H$. Therefore, we obtain $\|v_j^\delta\|_p \le C\|\tau_\delta u\|_p \le C\|u\|_p$.

Now, we define functions $u_j^\delta$ on $\Omega_\delta$ by

$$u_j^\delta(x, y) = v_j^\delta(x, y-\delta).$$

Then, the definition of $u_j^\delta$ implies that

$$\int_{\Omega_\delta} |u_j^\delta|^p dV \le C \int_H |u|^p dV.$$

Fix $\delta_0 > 0$ and let $\delta_1, \delta_2 < \delta_0$. Then, $D_n(u_j^{\delta_1} - u_j^{\delta_2}) = D_j u - D_j u = 0$ on $\Omega_{\delta_0}$. Therefore, the function $u_j^{\delta_1} - u_j^{\delta_2}$ on $\Omega_{\delta_0}$ is independent of $y$. Put $f(x) = f(x, y) = u_j^{\delta_1}(x, y) - u_j^{\delta_2}(x, y)$ ($f$ is independent of $y$). Since $u_j^{\delta_1}, u_j^{\delta_2} \in b^p(\Omega_{\delta_0}, dV)$, Fubini's theorem implies that

$$\infty > \int_{\Omega_{\delta_0}} |f|^p dV = \int_{\delta_0}^\infty \int_{\partial H} |f(x)|^p dx dy = \int_{\delta_0}^\infty dy \int_{\partial H} |f(x)|^p dx.$$

Therefore, we have $0 = f = u_j^{\delta_1} - u_j^{\delta_2}$ on $\Omega_{\delta_0}$. Thus, we can define harmonic functions $u_j$ on $H$ by

$$u_j(x, y) = \lim_{\delta \to 0} u_j^\delta(x, y).$$

Clearly, these functions $u_1, \dots, u_{n-1}$ are harmonic conjugates of $u$ on $H$. Moreover, the monotone convergence theorem implies that

$$\int_H |u_j|^p dV = \lim_{\delta \to 0} \int_H \chi_\delta |u_j|^p dV = \lim_{\delta \to 0} \int_{\Omega_\delta} |u_j^\delta|^p dV \le C \int_H |u|^p dV.$$

Thus, we obtain $u_j \in b^p$ and

$$\sum_{j=1}^{n-1} \|u_j\|_p \le C\|u\|_p.$$

By Theorem 4, we also obtain

$$\|u\|_p \le C\|yD_y u\|_p = C\left\|\sum_{j=1}^{n-1} yD_j u_j\right\|_p \le C\sum_{j=1}^{n-1} \|u_j\|_p.$$

The proof of uniqueness of $u_j$ is similar to that of Theorem 6.1 in [9]. (We use Theorem 4 in stead of Theorem 4.4 in [9].)

By Theorems 4 and 5, we see that Bergman norms are also comparable to tangential derivative norms. In the proof of Theorem 6.2 in [9], if we replace Theorems 4.4 and 6.1 in [9] by Theorems 4 and 5 respectively, then the following Theorem 6 is obtained. Therefore, we omit the proof.

THEOREM 6. *Let* $0 < p \leq 1$ *and* $\ell$ *be a nonnegative integer. Then,*

$$\|u\|_p \approx \sum_{|\alpha|=\ell,\, \alpha_n=0} \|y^\ell D^\alpha u\|_p$$

*for all* $u \in b^p$.

## References

[1] P. Ahern and M. Jevtić, Inner multipliers of the Besov space, $0 < p \leq 1$, Rocky Mountain J. Math. **20** (1990), 753–764.

[2] S. Axler, P. Bourdon and W. Ramey, Harmonic Function Theory, Springer-Verlag, New York, 1992.

[3] C. Cascante and J. Ortega, Carleson measures on spaces of Hardy-Sobolev type, Can. J. Math. **47** (1995), 1177–1200.

[4] C. Fefferman and E. Stein, $H^p$-Spaces of several variables, Acta Math. **129** (1972), 137–193.

[5] J. Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.

[6] G. Hardy and J. Littlewood, Some properties of conjugate functions, J. Reine Angew. Math. **167** (1931), 405–423.

[7] D. Luecking, Embedding derivatives of Hardy spaces into Lebesgue spaces, Proc. London Math. Soc. **63** (1991), 595–619.

[8] T. Nakazi and M. Yamada, $(A_2)$-conditions and Carleson inequalities in Bergman spaces, Pacific J. Math. **173** (1996), 151–171.

[9] W. Ramey and H. Yi, Harmonic Bergman functions on half-spaces, Trans. Amer. Math. Soc. **348** (1996), 633–660.

[10] D. Stegenga, Multipliers of the Dirichlet space, Ill. J. Math. **24** (1980), 113–139.

[11] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, New Jersey, 1970.

[12] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York, 1990.

*Department of Mathematics*
*Faculty of Science*
*Hiroshima University*

Current address: Department of Mathematics, Faculty of Education, Gifu University, Yanagido 1-1, Gifu 501-1193, Japan

E-mail address: yamada33@cc.gifu-u.ac.jp