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Self-similar radial solutions to a parabolic system modelling chemotaxis via variational method

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1. Introduction

In the previous paper [2] the first author studied the positive self-similar radial solutions

$$u(x,t) = \frac{1}{t}\varphi\left(\frac{|x|}{\sqrt{t}}\right), \quad v(x,t) = \psi\left(\frac{|x|}{\sqrt{t}}\right)$$

concerning the system of parabolic differential equations

$$\text{(KS)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v) & \text{in } \boldsymbol{R}^2, \quad t > 0, \\ \varepsilon \dfrac{\partial v}{\partial t} = \Delta v + \alpha u & \text{in } \boldsymbol{R}^2, \quad t > 0, \end{cases}$$

where α, χ and ε are positive constants. This system is one of the mathematical model by [1] describing chemotactic aggregation of cellular slime molds which move preferentially towards relatively high concentrations of a chemical substance secreted by the amoebae themselves. At place x and time t, $u(x,t)$ means the cell density of the cellular slime molds, and $v(x,t)$ the concentration of the chemical substance. Substitute $u = \varphi/t$ and $v = \varphi$ in (KS) and note φ and ψ are radially symmetric in x. Then $(\varphi(r), \psi(r))$ with $r = |x|/\sqrt{t}$ satisfies

$$\text{(KSO)} \qquad \begin{cases} (\varphi' - \chi\varphi\psi')' + \dfrac{1}{r}(\varphi' - \chi\varphi\psi') + \dfrac{r}{2}\varphi' + \varphi = 0 \\ \psi'' + \dfrac{1}{r}\psi' + \dfrac{\varepsilon r}{2}\psi' + \alpha\varphi = 0 \\ \varphi'(0) = \psi'(0) = 0. \end{cases}$$

From the first equation in (KSO) we have

$$\{2r(\varphi' - \chi\varphi\psi') + r^2\varphi\}' = 0 \quad \text{for } r > 0,$$

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which leads to

$$2r(\varphi' - \chi\varphi\psi') + r^2\varphi = 0.$$

Dividing this equation by $2r\varphi$, we have

$$(\log \varphi - \chi\psi)' + \frac{r}{2} = 0.$$

Hence

$$\varphi = \lambda e^{-r^2/4} e^{\chi\psi},$$

where $\lambda = \varphi(0)e^{-\chi\psi(0)} > 0$. Substituting this into the second equation in (KSO), we have

$$\psi'' + \left(\frac{1}{r} + \frac{\varepsilon r}{2}\right)\psi' + \alpha\lambda e^{-r^2/4} e^{\chi\psi} = 0.$$

Transform as

$$\chi\psi \to \psi$$

and put

$$\mu = \lambda\alpha\chi.$$

Then we have

$$\psi'' + \left(\frac{1}{r} + \frac{\varepsilon r}{2}\right)\psi' + \mu e^{-r^2/4} e^{\psi} = 0.$$

Since $v(x, t) = \psi(|x|\sqrt{t})$ is the concentration of the chemical substance, we have

$$v(x, t) > 0 \quad \text{and} \quad \iint_{R^2} v(x, t)\, dx < \infty,$$

and so we may assume

$$\int_0^\infty r\psi(r)\, dr < \infty.$$

Thus our problem is reduced to finding positive solutions ψ on $[0, \infty)$ of

$$\text{(1.1)} \qquad \begin{cases} \psi'' + \left(\dfrac{1}{r} + \dfrac{\varepsilon r}{2}\right)\psi' + \mu e^{-r^2/4} e^{\psi} = 0, \\ \psi'(0) = 0, \end{cases}$$

with the condition

$$\text{(1.2)} \qquad \int_0^\infty r\psi(r)\, dr < \infty.$$

Let us define μ_ε by $\mu_\varepsilon = \mu$, if $\varepsilon = 1$ and $\mu_\varepsilon = \mu \log \varepsilon/(\varepsilon - 1)$, if $\varepsilon \neq 1$. Let $\psi(0) = a$. The theorem of the previous paper [2] can be rewritten with a modification as

THEOREM 1 ([2]). *Let $0 < \mu_\varepsilon < 1/e$. Then there exists an $0 < a_* < 1$ such that the equation* (1.1) *with $\psi(0) = a_*$ admits a positive solution with* (1.2). *Furthermore there exists a μ^* such that if $\mu > \mu^*$, there are no positive solutions of* (1.1).

We show an existence of another solution with a large initial value, that is,

THEOREM 2. *Let $0 < \mu_\varepsilon < 1/e$. Then there exists an $1 < a^*$ such that the equation* (1.1) *with $\psi(0) = a^*$ admits a positive solution with* (1.2). *Furthermore $\psi(0)$ tends to infinity as $\mu_\varepsilon \to 0$.*

Our objective of this paper is to prove Theorem 2 with the aid of continuity of the solution with respect to the initial data and the variational method of an elliptic equation with the Dirichlet boundary condition on $\{|x| < R\}$. Furthermore in Appendix we shall prove Theorem 1. From Theorems 1 and 2 we can guess an existence of the global branch of $(\psi(0), \mu_\varepsilon)$ which starts from $(0,0)$, turns at some point (a_c, μ_c) and $\psi(0)$ tends to infinity as $\mu_\varepsilon \to 0$. Here ψ is a solution of (1.1) with (1.2). Furthermore we can expect the positive solutions of (KS) with small data which depends on μ_c, tend to the branch of solutions obtained by Theorem 1, as $t \to \infty$, and the other positive solutions blow up at finite time. But these problems are open for us.

2. Preliminaries

Put

$$I(\varepsilon) = \int_0^\infty \frac{1}{s} e^{-\varepsilon s^2/4} ds \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} d\tau.$$

Then we recall the following lemma in [2] of which proof is simplified

LEMMA 1. *$I(\varepsilon)$ is represented as*

$$I(\varepsilon) = \begin{cases} \dfrac{\log \varepsilon}{\varepsilon - 1} & \text{if } \varepsilon \neq 1, \\ 1 & \text{if } \varepsilon = 1. \end{cases}$$

PROOF. When $\varepsilon = 1$ it is easy that $I(1) = 1$. Thus we prove only in the case $\varepsilon \neq 1$. Since

$$I(\varepsilon) = \frac{2}{\varepsilon - 1} \int_0^\infty \frac{1}{s} (e^{-s^2/4} - e^{-\varepsilon s^2/4})\, ds = \frac{2}{\varepsilon - 1} I_0.$$

Let us calculate I_0.

$$I_0 = \int_0^\infty \frac{1}{s}\, ds \int_1^\varepsilon \frac{d}{dt}\{-e^{-ts^2/4}\}\, dt = \lim_{\rho\to\infty} \int_0^\rho \frac{s}{4}\, ds \int_1^\varepsilon e^{-ts^2/4} dt$$
$$= \lim_{\rho\to\infty} \int_1^\varepsilon dt \int_0^\rho \frac{s}{4} e^{-ts^2/4} ds = \lim_{\rho\to\infty} \int_1^\varepsilon \frac{1}{2t}(1 - e^{-t\rho^2/4})\, dt$$
$$= \lim_{\rho\to\infty} \frac{1}{2}[(\log t)(1 - e^{-t\rho^2/4})]_1^\varepsilon - \lim_{\rho\to\infty} \frac{\rho^2}{8} \int_1^\varepsilon (\log t) e^{-t\rho^2/4} dt$$
$$= \frac{\log \varepsilon}{2}.$$

Thus the proof is complete.

Let us denote the solution ψ of (1.1) with $\psi(0) = a$ by $\psi(r; a)$.

LEMMA 2. *Let* $\mu_\varepsilon = \mu I(\varepsilon)$. *Then for* $r > 0$,
(i) $\psi'(r; a) < 0$,
(ii) $\psi'(r; a) > -\dfrac{\mu e^a r}{2}$,
(iii) $\psi(r; a) > a - \mu_\varepsilon e^a$.

PROOF. Proof of (i): Since ψ satisfies (1.1), we have

$$(p(r)\psi')' + \mu p(r) e^{-r^2/4} e^\psi = 0,$$

where $p(r) = re^{\varepsilon r^2/4}$. Integrating from 0 to r the above equation, we have

$$p(r)\psi' = -\mu \int_0^r s e^{(\varepsilon-1)s^2/4} e^\psi ds, \tag{2.1}$$

which together with $p(r) > 0$ yields the assertion of (i).

Proof of (ii): Since we have $a = \psi(0; a) > \psi(r; a)$ by (i), it follows from (2.1) that

$$p(r)\psi' > -\mu e^a \int_0^r s e^{(\varepsilon-1)s^2/4} ds$$
$$> -\mu e^a e^{\varepsilon r^2/4} \int_0^r s e^{-s^2/4} ds$$
$$> -\frac{\mu e^a e^{\varepsilon r^2/4} r^2}{2},$$

which leads us to (ii).

Proof of (iii): From (2.1) it follows that

$$\psi(r;a) - \psi(0;a) = -\mu \int_0^r \frac{1}{s} e^{-\varepsilon s^2/4} ds \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^{\psi} d\tau. \tag{2.2}$$

Thus we have from (2.2)

$$\psi(r;a) - a > -\mu e^a \int_0^r \frac{1}{s} e^{-\varepsilon s^2/4} ds \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} d\tau$$

$$> -\mu e^a \int_0^\infty \frac{1}{s} e^{-\varepsilon s^2/4} ds \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} d\tau = -\mu_\varepsilon e^a,$$

which implies (iii). The proof is complete.

Since $\psi(r;a)$ is monotone decreasing and bounded from below with $a - \mu_\varepsilon e^a$, if $a - \mu_\varepsilon e^a > 0$, then we have $\lim_{r\to\infty} \psi(r;a) > 0$. The inequality $a - \mu_\varepsilon e^a > 0$ is equivalent to the inequality $0 < \mu_\varepsilon < ae^{-a}$. Since the maximum of ae^{-a} is $1/e$, if $0 < \mu_\varepsilon < 1/e$, the line $y = \mu_\varepsilon$ and the curve $y = ae^{-a}$ intersect at α_{μ_ε} and $\beta_{\mu_\varepsilon} (\alpha_{\mu_\varepsilon} < \beta_{\mu_\varepsilon})$. Then if

$$\alpha_{\mu_\varepsilon} < a < \beta_{\mu_\varepsilon},$$

then

$$a - \mu_\varepsilon e^a > 0.$$

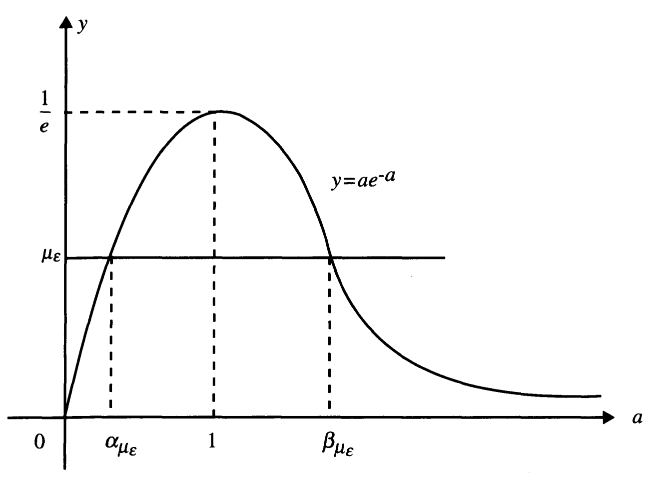

Fig. 1

Thus we have

Lemma 3. *Let* $0 < \mu_\varepsilon < 1/e$ *and put* $\psi(\infty; a) = \lim_{r\to\infty} \psi(r; a)$. *If* $\alpha_{\mu_\varepsilon} < a < \beta_{\mu_\varepsilon}$, *then* $\psi(\infty; a) > 0$ *holds.*

Lemma 4. *Let* $c_{\mu,\varepsilon} = \max\{\mu, \mu/\varepsilon\}$ *and* $\kappa_\varepsilon = \min\{1, \varepsilon\}$. *Then*

$$\psi(r; a) < \psi(\infty; a) + e^a c_{\mu,\varepsilon} e^{-\kappa_\varepsilon r^2/4} \quad (r > 0).$$

Proof. From (2.1) we have

$$\begin{aligned}\psi(r; a) &= \psi(\infty; a) + \mu \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} ds \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} e^\psi d\tau \\ &< \psi(\infty; a) + e^a \mu \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} ds \int_0^s \tau e^{(\varepsilon-1)\tau^2/4} d\tau.\end{aligned}$$

If $\varepsilon \geqq 1$, then

$$\begin{aligned}\psi(r; a) &< \psi(\infty; a) + e^a \mu \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} e^{(\varepsilon-1)s^2/4} ds \int_0^s \tau\, d\tau \\ &= \psi(\infty; a) + \frac{e^a \mu}{2} \int_r^\infty s e^{-s^2/4} ds = \psi(\infty; a) + e^a \mu e^{-r^2/4}.\end{aligned}$$

If $0 < \varepsilon < 1$, then

$$\begin{aligned}\psi(r; a) &< \psi(\infty; a) + e^a \mu \int_r^\infty \frac{1}{s} e^{-\varepsilon s^2/4} ds \int_0^s \tau\, d\tau \\ &= \psi(\infty; a) + \frac{e^a \mu}{2} \int_r^\infty s e^{-\varepsilon s^2/4} ds \\ &= \psi(\infty; a) + \frac{e^a \mu}{\varepsilon} e^{-\varepsilon r^2/4}.\end{aligned}$$

The proof is complete.

Lemma 5. *Put*

$$h(t) = t e^{(\varepsilon-1)t^2/4} \int_t^\infty \frac{1}{s} e^{-\varepsilon s^2/4} ds \quad \text{and} \quad c = \max\{a, b\}.$$

Then

(i) $\int_0^\infty h(r)\, dr = I(\varepsilon)$,

(ii) $|\psi(r; a) - \psi(r; b)| \leqq |a - b| \exp\left(\mu e^c \int_0^r h(t)\, dt\right)$,

(iii) $|\psi(\infty; a) - \psi(\infty; b)| \leqq |a - b| \exp(\mu_\varepsilon e^c)$.

Proof. Proof of (i): Since

$$\int_0^\infty h(r)\,dr = \int_0^\infty re^{(\varepsilon-1)r^2/4}dr \int_r^\infty \frac{1}{s}e^{-\varepsilon s^2/4}ds$$
$$= \int_0^\infty \frac{1}{s}e^{-\varepsilon s^2/4}ds \int_0^s re^{(\varepsilon-1)r^2/4}dr = I(\varepsilon),$$

we have (i).

Proof of (ii): Make use of (2.2). Then we have, by change of the order of the integral,

$$\psi(r;a) = a - \mu \int_0^r \frac{1}{s}e^{-\varepsilon s^2/4}ds \int_0^s te^{(\varepsilon-1)t^2/4}e^{\psi(t;a)}dt$$
$$= a - \mu \int_0^r e^{\psi(t;a)}te^{(\varepsilon-1)t^2/4}dt \int_t^r \frac{1}{s}e^{-\varepsilon s^2/4}ds.$$

Thus it follows that

$$|\psi(r;a) - \psi(r;b)| \leqq |a-b| + \mu \int_0^r |e^{\psi(t;a)} - e^{\psi(t;b)}|te^{(\varepsilon-1)t^2/4}dt \int_t^r \frac{1}{s}e^{-\varepsilon s^2/4}ds.$$

Since

$$|e^{\psi(t;a)} - e^{\psi(t;b)}| \leqq e^c|\psi(t;a) - \psi(t;b)|$$

with $c = \max\{a, b\}$, we have

$$|\psi(r;a) - \psi(r;b)| \leqq |a-b| + \mu e^c \int_0^r |\psi(t;a) - \psi(t;b)|h(t)\,dt,$$

from which together with the Gronwall inequality (ii) holds.

Proof of (iii): By letting **r** tend to infinity in the both sides of (ii), we have (iii). The proof is complete.

3. The Dirichlet problem of an elliptic equation

In this section we consider the Dirichlet problem of finding the radial solution of the

$$\text{(DP)} \qquad \begin{cases} \nabla(e^{\varepsilon/4|x|^2}\nabla v) + \mu e^{(\varepsilon-1)/4|x|^2}e^v = 0 & \text{in } B_R \stackrel{\text{def}}{=} \{x \in \boldsymbol{R}^2 \mid |x| < R\} \\ v = 0 & \text{on } \partial B_R \end{cases}$$

by the aid of the Mountain Path Theorem and the Principle of symmetric criticality.

PROPOSITION 1. *For small R there exists a radially symmetric positive solution $v(x)$ of the Dirichlet problem (DP).*

First we recall the Palais-Smale condition.

DEFINITION 1. *Let* X *be a Banach space and* $J \in C^1(X, \boldsymbol{R})$. *Then we say* J *satisfies the Palais-Smale condition, if any sequence* $\{x_n\} \subset X$ *such that*

$$|J(x_n)| \leqq c \text{ for some } c, \tag{3.1}$$

$$J'(x_n) \to 0 \text{ in } X' \text{ as } n \to \infty, \tag{3.2}$$

has a strongly convergent subsequence.

THEOREM A (Mountain Path Theorem, e.g. see [6]). *Let* X *be a Banach space,* $J \in C^1(X, \boldsymbol{R})$, $U_\rho = \{x \in X \mid \|x\|_X < \rho\}$ *and* $e \in X \backslash \bar{U}_\rho$ *be such that* $J(0) = 0$ *and*

$$\inf_{\|x\|=\rho} J(x) \geqq \alpha \text{ for some } \alpha > 0, \tag{3.3}$$

$$J(e) < 0. \tag{3.4}$$

Let $\Gamma = \{\gamma \in C([0, 1], X) | \gamma(0) = 0, \gamma(1) = e\}$. *If* J *satisfies the Palais-Smale condition, then*

$$c = \inf_{\gamma \in \Gamma} \sup_{x \in \gamma([0,1])} J(x)$$

is a critical value of J.

THEOREM B (Principle of symmetric criticality, Palais [5], e.g. see [7]). *Let* G *be a topological group on a Hilbert space* X *which acts on* X *continuously, that is,*

$$G \times X \to X : [g, x] \to gx$$

is continuous map such that

$$1 \cdot x = x,$$

$$(gh)x = g(hx),$$

$$x \mapsto gx \text{ is linear.}$$

Furthermore assume $\|gx\| = \|x\|$. *Let* $J \in C^1(X, \boldsymbol{R})$ *satisfy* $J \circ g = J$ *for every* $g \in G$. *If* x *is a critical point of* J *restricted to* $\{x \in X \mid gx = x, \forall g \in G\}$, *then* x *is a critical point of* J.

Let H be a Hilbert space defined by

$$H = \{v \in W_0^{1,2}(B_R) \mid v(x) = v(|x|)\}$$

with the inner product

$$(u, v)_H = \int_{B_R} e^{\varepsilon|x|^2/4} \nabla u \cdot \nabla v \, dx.$$

Put

$$J(v) = \frac{1}{2}\|v\|_H^2 - \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4}(e^v - 1)\, dx, \quad v \in H. \tag{3.5}$$

Then $J \in C^1(H, \boldsymbol{R})$ and furthermore J satisfies the assumption in Theorem B with the orthogonal transformation group $O(2)$ as G, where $H = \{v \in W_0^{1,2}(B_R) \mid v(gx) = v(x),\ \forall g \in O(2)\}$.

Let us recall here the Trudinger-Moser inequality in two dimensional case.

THEOREM C (The Trudinger-Moser Inequality [3]). *Let Ω be a domain in $\boldsymbol{R}^2$ such that*

$$|\Omega| = \int_\Omega dx < \infty.$$

Let $u \in W_0^{1,2}(\Omega)$ and

$$\int_\Omega |\nabla u|^2 dx \leqq 1.$$

Then if $\alpha \leqq 8\pi$, there exists a positive constant c such that

$$\int_\Omega e^{\alpha|u|^2} dx \leqq c|\Omega|.$$

COROLLARY. *Let Ω be the same as in Theorem C. Let $u \in W_0^{1,2}(\Omega)$. Then there exists a positive constant c such that*

$$\int_\Omega e^{|u|} dx \leqq c|\Omega| \exp(\|\nabla u\|_2^2/16\pi).$$

As for the proof of the corollary, for example see [4].

PROOF OF PROPOSITION 1. We show an existence of the weak solution of (DP) only in the case of $\varepsilon \geqq 1$, because in the case $0 < \varepsilon < 1$ it is shown in the similar way. Since we make use of the Mountain Path Theorem, we show the functional $J(v)$ in (3.5) satisfies (3.3) and (3.4) in addition to the Palais-Smale condition.

Choose $\rho > 0$ arbitrarily and put

$$U = \{v \in H \mid \|v\|_H < \rho\}.$$

Step 1. *If R is small, then*

$$J(v) > \frac{\rho^2}{4} \quad \textit{for } v \in \partial U.$$

In fact

$$\begin{aligned} J(v) &\geqq \frac{1}{2}\|v\|_H^2 - \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4}(e^{|v|}-1)\,dx \\ &\geqq \frac{\rho^2}{2} - \mu e^{(\varepsilon-1)R^2/4} \int_{B_R} (e^{|v|}-1)\,dx \\ &\geqq \frac{\rho^2}{2} + \mu e^{(\varepsilon-1)R^2/4}\pi R^2(1 - ce^{\rho^2/16\pi}). \end{aligned}$$

Here we used the Trudinger-Moser inequality and the fact that $\|\nabla v\|_2 \leqq \|v\|_H$. If we take R so small that

$$\mu e^{(\varepsilon-1)R^2/4}\pi R^2(1 - ce^{\rho^2/16\pi}) > -\frac{\rho^2}{4},$$

we have

$$J(v) > \frac{\rho^2}{2} - \frac{\rho^2}{4} = \frac{\rho^2}{4}.$$

Step 2. *There exists $v^* \in H \setminus \bar{U}$ such that $J(v^*) < 0$.*
Let b be a positive constant which is determined later and put

$$v^*(x) = b - \frac{b}{R}|x|.$$

Then $v^* \in H$ and

$$\begin{aligned} J(v^*) &= \frac{1}{2}\int_{B_R} e^{\varepsilon|x|^2/4}|\nabla v^*|^2 dx - \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4}(e^{v^*}-1)\,dx \\ &\leqq \frac{1}{2}e^{\varepsilon R^2/4}\cdot\frac{b^2}{R^2}\cdot\pi R^2 + \mu\pi R^2 - \mu e^b \int_{B_R} e^{-b|x|/R}dx \\ &= \frac{\pi e^{\varepsilon R^2/4}b^2}{2} + \mu\pi R^2 - 2\pi\mu e^b \int_0^R re^{-br/R}dr \\ &= \frac{\pi e^{\varepsilon R^2/4}b^2}{2} + \mu\pi R^2 + 2\pi\mu R^2\left(\frac{1}{b}+\frac{1}{b^2}\right) - \frac{2\pi\mu e^b R^2}{b^2}. \end{aligned}$$

On the other hand

$$\|v^*\|_H = \frac{2b}{R}\sqrt{\pi(e^{\varepsilon R^2/4} - 1)/\varepsilon}.$$

Therefore if we choose b large enough, then

$$J(v^*) < 0 \quad \text{and} \quad \|v^*\|_H > \rho,$$

which implies our claim.

Step 3. *J satisfies the Palais-Smale condition.*

Let $\{v_n\} \subset H$ satisfy

(i) $J(v_n)$ is bounded,

(ii) $J'(v_n) \to 0$ in H' as $n \to \infty$.

Then we have only to show $\{v_n\}$ has a strongly convergent subset. We show first $\{v_n\}$ is bounded. Note that

$$J'(v)h = \int_{B_R} e^{\varepsilon|x|^2/4}\nabla v \cdot \nabla h \, dx - \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4} e^v h \, dx \quad \text{for } h \in H.$$

Since

$$J'(v)v = \|v\|_H^2 - \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4} e^v v \, dx$$

and

$$\frac{te^t}{4} - e^t + 1 \geqq -(e^3/4 - 1) \quad \text{for all } t \in \boldsymbol{R},$$

we have

$$\begin{aligned} J(v) - \frac{1}{4}J'(v)v &= \frac{1}{4}\|v\|_H^2 + \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4}\left(\frac{1}{4}ve^v - e^v + 1\right) dx \\ &\geqq \frac{1}{4}\|v\|_H^2 - \mu(e^3/4 - 1)\int_{B_R} e^{(\varepsilon-1)|x|^2/4} dx \\ &\geqq \frac{1}{4}\|v\|_H^2 - \pi\mu(e^3/4 - 1)R^2 e^{(\varepsilon-1)R^2/4}, \end{aligned}$$

and therefore

$$\|v_n\|_H^2 \leqq 4|J(v_n)| + \|J'(v_n)\|_{H'} \cdot \|v_n\|_H + \pi\mu(e^3 - 4)R^2 e^{(\varepsilon-1)R^2/4}.$$

From this inequality together with the assumptions (3.1) and (3.2) in the Palais-Smale condition it follows that $\{v_n\}$ is bounded in $W_0^{1,2}(B_R)$. Since $W_0^{1,2}(B_R)$ is compactly embedded in $L^2(B_R)$, $\{v_n\}$ has a subsequence $\{v_{n_k}\}$ convergent in $L^2(B_R)$. Then

$$
\begin{aligned}
\|v_{n_k} - v_{n_l}\|_H^2 &= \int_{B_R} e^{\varepsilon|x|^2/4} |\nabla v_{n_k} - \nabla v_{n_l}|^2 dx \\
&= \int_{B_R} e^{\varepsilon|x|^2/4} \nabla v_{n_k} (\nabla v_{n_k} - \nabla v_{n_l})\, dx - \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4} e^{v_{n_k}} (v_{n_k} - v_{n_l})\, dx \\
&\quad - \int_{B_R} e^{\varepsilon|x|^2/4} \nabla v_{n_l} (\nabla v_{n_k} - \nabla v_{n_l})\, dx + \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4} e^{v_{n_l}} (v_{n_k} - v_{n_l})\, dx \\
&\quad + \mu \int_{B_R} e^{(\varepsilon-1)|x|^2/4} (e^{v_{n_k}} - e^{v_{n_l}})(v_{n_k} - v_{n_l})\, dx \\
&\leqq |J'(v_{n_k})(v_{n_k} - v_{n_l})| + |J'(v_{n_l})(v_{n_k} - v_{n_l})| \\
&\quad + \mu e^{(\varepsilon-1)R^2/4} \int_{B_R} (e^{|v_{n_k}|} + e^{|v_{n_l}|}) |v_{n_k} - v_{n_l}|\, dx.
\end{aligned}
$$

Note that from (ii) in the above assumption for any δ there exists an integer N such that if $n \geqq N$, then

$$
|J'(v_n)h| \leqq \delta \|h\|_H \leqq \frac{1}{4} \|h\|_H^2 + \delta^2 \quad \text{for all } h \in H.
$$

Hence

$$
\|v_{n_k} - v_{n_l}\|_H^2 \leqq \frac{1}{2} \|v_{n_k} - v_{n_l}\|_H^2 + 2\delta^2 + \mu e^{(\varepsilon-1)R^2/4} \int_{B_R} (e^{|v_{n_k}|} + e^{|v_{n_l}|}) |v_{n_k} - v_{n_l}|\, dx
$$

$$
\|v_{n_k} - v_{n_l}\|_H^2 \leqq 4\delta^2 + 2\mu e^{(\varepsilon-1)R^2/4} \left\{ \left(\int_{B_R} e^{2|v_{n_k}|} dx \right)^{1/2} + \left(\int_{B_R} e^{2|v_{n_l}|} dx \right)^{1/2} \right\} \|v_{n_k} - v_{n_l}\|_2.
$$

Since from the corollary of Theorem C

$$
\int_{B_R} e^{2|v_{n_j}|} dx \leqq c\pi R^2 e^{4\|\nabla v_{n_j}\|_2^2} \leqq c\pi R^2 e^{4\|v_{n_j}\|_H^2}
$$

and $\{v_n\}$ is bounded in H, there exists a positive constant c such that

$$
\|v_{n_k} - v_{n_l}\|_H^2 \leqq 4\delta^2 + c \|v_{n_k} - v_{n_l}\|_2,
$$

from which we have

$$
\lim_{k,l \to \infty} \|v_{n_k} - v_{n_l}\|_H^2 = 0.
$$

Thus since $\{v_{n_k}\}$ is the Cauchy sequence in H, the sequence $\{v_n\}$ has a strongly convergent sequence.

Step 4. *There exists a radially symmetric solution of (DP).*

From Step 1 through Step 3 we see J satisfies the assumption of the Mountain Path Theorem. Consequently J has a critical point v_c in H which is also the critical point in $W_0^{1,2}(B_R)$ by Theorem B. Thus v_c is a weak solution of (DP). Since $e^{v_c} \in L^p(B_R)$ for any $1 \leqq p < \infty$, by the standard regularity theorem to an elliptic equation we see $v_c \in C^2(\overline{B_R})$. The radial symmetry of v_c is evident, because $v_c \in H$. Finally the positivity of v_c follows from the strong maximum principle.

The proof of Proposition 1 is complete.

4. Proof of Theorem 2

Let v_c be the classical solution of (DP) found in Proposition 1. Then since v_c is radially symmetric, it follows that

$$(re^{\varepsilon r^2/4}v_c')' + r\mu e^{(\varepsilon-1)r^2/4}e^{v_c} = 0, \quad v_c'(0) = 0,$$

and $v_c(R) = 0$.

LEMMA 6. *If we choose R small enough, then $v_c(0) > \beta_{\mu_\varepsilon}$, where β_{μ_ε} is the largest point of intersection $y = ae^a$ and $y = \mu_\varepsilon$. (see Fig 1)*

PROOF. Suppose $0 < v_c(0) \leqq \beta_{\mu_\varepsilon}$. Since

$$\begin{aligned} J(v_c) &< \frac{1}{2}\int_{B_R} e^{\varepsilon|x|^2/4}|\nabla v_c|^2 dx \\ &= \pi\int_0^R re^{\varepsilon r^2/4}(v_c')^2 dr, \end{aligned}$$

by (ii) of Lemma 2 we have

$$\begin{aligned} J(v_c) &< \pi\int_0^R re^{\varepsilon r^2/4}\frac{\mu^2 e^{2v_c}(0)}{4}r^2 dr \\ &\leqq \frac{\pi\mu^2 e^{2\beta_{\mu_\varepsilon}}e^{\varepsilon R^2/4}}{4}\int_0^R r^3 dr = \frac{\pi\mu^2 e^{2\beta_{\mu_\varepsilon}}R^4 e^{\varepsilon R^2/4}}{16}. \end{aligned}$$

If we take R small, we have

$$J(v_c) < \frac{\rho^2}{4},$$

which contradicts the result $J(v_c) > \rho^2/4$ in Step 1 of the proof of Proposition 1. Thus we have

$$v_c(0) > \beta_{\mu_\varepsilon}.$$

The proof is complete.

LEMMA 7. *Put*

$$Y_+ = \{a \in (1, \infty) \mid \psi(\infty; a) > 0\},$$

$$Y_- = \{a \in (1, \infty) \mid \psi(\infty; a) < 0\}.$$

Then Y_+ and Y_- are open sets.

PROOF. It follows from (iii) of Lemma 5 that Y_+ is open. On the other hand we see Y_- is open, since the solution of the initial value problem of (1.1) is continuous with respect to the initial data by (ii) of Lemma 5. The proof is complete.

PROOF OF THEOREM 2. Since $v_c(r)$ is monotone decreasing and $v_c(R) = 0$, we have $v_c(r) < 0$ for $r > R$, that is, $v_c(0) \in Y_-$. On the other hand if $\alpha_{\mu_\varepsilon} < a < \beta_{\mu_\varepsilon}$, then by Lemma 3 we have $a \in Y_+$. Thus $Y_+ \neq \emptyset$ and $Y_- \neq \emptyset$. Put

$$a^* = \inf Y_-.$$

Then

$$a^* \in [\beta_\mu, \infty).$$

From Lemma 7 it follows

$$a^* \notin Y_+ \quad \text{and} \quad a^* \notin Y_-.$$

Consequently, we see

$$\psi(\infty; a^*) = 0.$$

From Lemma 4 we have

$$\int_0^\infty r\psi(r; a^*)\, dr \leqq e^{a^*} c_{\mu,\varepsilon} \int_0^\infty re^{-\kappa_\varepsilon r^2/4} dr = 2c_{\mu,\varepsilon} e^{a^*}/\kappa_\varepsilon.$$

Since $a^* \geqq \beta_\mu$ and $\beta_\mu \to \infty$ as $\mu_\varepsilon \to 0$, we have $\psi(0) \to \infty$ as $\mu_\varepsilon \to 0$. Thus the proof is complete.

5. Appendix

Let us here prove Theorem 1. First we show the following

PROPOSITION 2 ([2]). *Let $\psi(r)$ be a solution of* (1.1). *If $\mu_\varepsilon \geqq 1$, then $\psi(\infty) < 0$. Hence there exists no positive solution of* (1.1).

PROOF. Since $\psi(r)$ is monotone decreasing, from (2.1) it follows

$$\psi'(r) < -\frac{\mu}{r} e^{-\varepsilon r^2/4} e^{\psi(r)} \int_0^r \tau e^{(\varepsilon-1)r^2/4} d\tau,$$

from which it follows that

$$(-e^{-\psi(r)})' < -\frac{\mu}{r}e^{-\varepsilon r^2/4}\int_0^r \tau e^{(\varepsilon-1)\tau^2/4}d\tau.$$

Integrating this from 0 to ∞, we have

$$e^{-\psi(0)} - e^{-\psi(\infty)} < -\mu I(\varepsilon) = -\mu_\varepsilon.$$

Thus we have

$$\psi(\infty) < -\log\,(\mu_\varepsilon + e^{-\psi(0)}),$$

which together with $\mu_\varepsilon \geqq 1$ implies $\psi(\infty) < 0$. The proof is complete.

LEMMA 8. *The inequality*

$$\psi(\infty; a) < a - \mu_\varepsilon e^{\psi(\infty;a)}$$

holds.

PROOF. From (2.2) it follows that

$$\begin{aligned}\psi(\infty; a) &= a - \mu\int_0^\infty \frac{1}{s}e^{-\varepsilon s^2/4}ds\int_0^s \tau\, e^{(\varepsilon-1)\tau^2/4}e^{\psi}d\tau \\ &< a - \mu e^{\psi(\infty;a)}\int_0^\infty \frac{1}{s}e^{-\varepsilon s^2/4}ds\int_0^s \tau e^{(\varepsilon-1)\tau^2/4}d\tau = a - \mu_\varepsilon e^{\psi(\infty;a)}.\end{aligned}$$

The proof is complete.

LEMMA 9. *Put*

$$Z_+ = \{a \in (0,1) \mid \psi(\infty; a) > 0\},$$

$$Z_- = \{a \in (0,1) \mid \psi(\infty; a) < 0\}.$$

Then Z_+ and Z_- are open sets.

PROOF. The proof is the same as Lemma 7.

PROOF OF THEOREM 1. If $\alpha_{\mu_\varepsilon} < a < 1$, then it follows from Lemma 3 that $a \in Z_+$. On the other hand since from Lemma 8

$$\psi(\infty; 0) < -\mu_\varepsilon e^{\psi(\infty;0)} < 0,$$

there exists an $0 < a < \alpha_\mu$ such that $\psi(\infty; a) < 0$. Thus $Z_+ \neq \emptyset$ and $Z_- \neq \emptyset$. Put

$$a_* = \sup Z_- < 1.$$

Then from the same reasoning as in the proof of Theorem 1 we see

$$a_* \notin Z_- \quad \text{and} \quad a_* \notin Z_+.$$

Thus

$$\psi(\infty; a_*) = 0.$$

From Lemma 4 it follows

$$\int_0^\infty r\psi(r; a_*)\, dr \leqq e^{a_*} c_{\mu,\varepsilon} \int_0^\infty re^{-\kappa_\varepsilon r^2/4} dr = 2c_{\mu,\varepsilon} e^{a_*}/\kappa_\varepsilon.$$

which together with Proposition 2 completes the proof of Theorem 1.

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