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Asymptotic analysis of a phase field model with memory for vanishing time relaxation

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Abstract. A phase-field model accounting for memory effects is considered. This model consists of a hyperbolic integrodifferential equation coupled with a parabolic differential inclusion. The latter relation rules the evolution of the phase field and contains a time relaxation parameter which happens to be very small in the applications. A well-posed initial and boundary value problem for the evolution system is introduced and the asymptotic behavior of its solution as the time relaxation goes to zero is analyzed rigorously. Convergence results and error estimates are obtained under suitable assumptions ensuring that the limit problem has a unique solution.

1. Introduction

Consider a two-phase system which occupies a bounded domain $\Omega \subset \mathbf{R}^3$ until a given time $T > 0$. Denote by ϑ its relative temperature (fixed in order that $\vartheta = 0$ is the equilibrium temperature between the two phases) and by χ the so-called phase-field, that is, an order parameter which could represent the local proportion of one phase. To describe the evolution of the pair (ϑ, χ), we have recently introduced and studied the following system (see [7–9])

$$\partial_t(\vartheta + \lambda\chi + \varphi * \vartheta + \psi * \chi) - \Delta(k * \vartheta) = g \quad \text{in } \Omega \times (0, T) \tag{1.1}$$

$$\mu\partial_t\chi - \nu\Delta\chi + \beta(\chi) \ni \gamma(\chi) + \lambda\vartheta \quad \text{in } \Omega \times (0, T) \tag{1.2}$$

coupled with the boundary and initial conditions

$$\partial_n(k * \vartheta) = h \quad \text{and} \quad \partial_n\chi = 0 \quad \text{on } \partial\Omega \times (0, T) \tag{1.3}$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{1.4}$$

Here $*$ denotes the usual time convolution product over $(0, T)$, defined by

$$(a * b)(t) = \int_0^t a(s)b(t - s)\, ds, \quad t \in [0, T] \tag{1.5}$$

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while ∂_t and ∂_n indicate the partial time derivative and the standard outward normal derivative, respectively. Moreover, λ, μ, and ν are positive constants, φ, ψ, $k :]0,+\infty[\to \mathbf{R}$ are time-dependent memory kernels, $\beta : \mathbf{R} \to 2^{\mathbf{R}}$ is a maximal monotone operator, and γ is a Lipschitz continuous function. We further remind that the source term g depends both on the heat supply and on the past histories of ϑ and datum h may rely on the values attained by ϑ for negative times.

Equation (1.1) comes out from the energy balance when the internal energy linearly depends on ϑ, χ and on their evolutions, while the constitutive assumption on the heat flux is the linearized version of the Gurtin-Pipkin law (proposed in [13] and recently reviewed in [14–15]). On the other hand, equation (1.2) is an extended version of the phase-field relationship introduced in [11] and [6] on the basis of the Ginzburg-Landau theory of phase transitions.

The corresponding system with the Fourier law for the heat flux and without any memory term (i.e., (1.1–2) with k proportional to the Dirac mass and $\varphi = \psi = 0$) has been extensively studied. General existence results and asymptotic analyses can be found in [17], [10], and in the review paper [16], along with a list of references.

Coming back to (1.1–4), well-posedness and regularity of solutions were essentially dealt with in [7–9] (see also [1], where the long time stability is investigated as well). Besides, in [9] we showed that there exists a sequence of problems (1.1–4) whose solutions converge to a solution to a hyperbolic phase relaxation problem as the interfacial energy coefficient ν tends to zero, provided that $\gamma \equiv 0$. The limit problem is formally obtained from (1.1–4) by setting $\nu = 0$ and, as far as we know, the related uniqueness of solutions is still an open issue.

Here we are going to examine carefully the asymptotic behavior of the solution to (1.1–4) as the relaxation time μ goes to zero. From the physical standpoint, the limit problem is interesting in itself, as pointed out in [6] within the framework of the Fourier law (for an existence theorem see [19]). Moreover, investigating asymptotics as $\mu \searrow 0$ seems even more important than the analysis done in [9] for μ is much smaller than ν in the actual application [11]. For these reasons, we study the question in detail and prove strong convergences for the whole sequence of the variational solutions $(\vartheta_\mu, \chi_\mu)$ as $\mu \searrow 0$. Also, strengthening a bit the regularity requirements on the data, some error estimates of orders $O(\mu)$ and $O(\mu^{1/2})$ are obtained. All these results are achieved under suitable conditions on λ, β, and γ which imply that the solution to the limit problem is unique.

A plan of the paper follows. After some preliminaries, in the next section an equivalent formulation of problem (1.1–4) is introduced on account of [9]. Then, referring to that formulation, the main results are stated. Section 3 is concerned with the uniqueness for the limit problem. Section 4 contains the proof of the convergence theorem. Finally, in Section 5 the error estimates are derived.

2. Main results

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ be a bounded, open, and connected set with boundary $\Gamma := \partial\Omega$ of class $C^{2,1}$ and let $T > 0$. Set

$$Q := \Omega \times]0, T[, \quad \Sigma := \Gamma \times]0, T[$$

and let the following constants and functions fulfill the assumptions listed below.

$$\lambda, \nu \in]0, \infty[\tag{2.1}$$

$$\varphi \in W^{1,1}(0, T), \quad \psi \in L^1(0, T) \tag{2.2}$$

$$k \in W^{2,1}(0, T), \quad k(0) > 0 \tag{2.3}$$

$$\gamma \in C^{0,1}(\mathbf{R}), \quad \gamma(0) = 0. \tag{2.4}$$

Moreover, we consider a maximal monotone operator

$$\beta : \mathbf{R} \to 2^{\mathbf{R}} \quad \text{with } \beta(0) \ni 0. \tag{2.5}$$

Next, we set for convenience

$$V = H^1(\Omega), \quad H = L^2(\Omega), \quad \text{and} \quad W = H^2(\Omega) \tag{2.6}$$

and define the operator $A : V \to V'$ by means of

$${}_{V'}\langle Au, v\rangle_V := \int_\Omega \nabla u \cdot \nabla v \quad \forall u, v \in V. \tag{2.7}$$

In the framework of the Hilbert triplet (V, H, V'), we introduce the (formal) problem of finding (ϑ, χ) such that

$$\partial_t(\vartheta + \lambda\chi + \varphi * \vartheta + \psi * \chi) + A(k * \vartheta) = f \text{ in } V', \quad \text{a.e. in } (0, T) \tag{2.8}$$

$$\mu\partial_t\chi + \nu A\chi \in H \text{ a.e. in } (0, T) \quad \text{and}$$

$$\mu\partial_t\chi + \nu A\chi + \rho = \gamma(\chi) + \lambda\vartheta \text{ in } V', \quad \text{a.e. in } (0, T)$$

$$\text{and for some } \rho \text{ fulfilling } \rho \in \beta(\chi) \quad \text{a.e. in } Q \tag{2.9}$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0 \tag{2.10}$$

where μ is any strictly positive constant and the right hand sides of (2.8) and (2.10) are given.

As mentioned in the Introduction, the basic theory for (2.8–10) is developed in [7–8] where existence, uniqueness, and regularity results are proved in a number of functional settings. Though the structure of (2.9) is more general as far as nonlinearity is concerned, the kernels φ and ψ are not considered there. On the contrary, in [9] these kernels are introduced and the theory is completed along with the study of the asymptotic behavior as $\nu \searrow 0$, using an alternative formulation of the problem which we recall briefly.

The state variable ϑ is substituted with the integrated enthalpy w specified by

$$w := 1 * e \quad \text{where } e := \vartheta + \lambda\chi + \varphi * \vartheta + \psi * \chi. \tag{2.11}$$

Since (2.11) gives

$$\vartheta + \varphi * \vartheta = w_t - \lambda\chi - \psi * \chi \tag{2.12}$$

it turns out that ϑ can be expressed in terms of w_t and χ by introducing the so-called resolvent Φ of φ. This function is, by definition, the solution of $\Phi + \varphi * \Phi = \varphi$ (see, e.g., [12, Ch. 2, Sect. 3]) and allows us to rewrite (2.12) as

$$\vartheta = (w_t - \lambda\chi - \psi * \chi) - \Phi * (w_t - \lambda\chi - \psi * \chi).$$

Moreover, setting

$$\kappa_1 := k(0)\Phi + k' * \Phi - k' \tag{2.13}$$

$$\kappa_2 := \lambda k + k * \psi - \lambda k * \Phi - k * \Phi * \psi \tag{2.14}$$

$$\Psi := \psi - \lambda\Phi - \Phi * \psi \tag{2.15}$$

we note that (2.2–3) imply

$$\Phi, \kappa_1, \kappa_2 \in W^{1,1}(0,T) \quad \text{and} \quad \Psi \in L^1(0,T). \tag{2.16}$$

Then, equations (2.8–9) formally become

$$w_{tt} + k(0)Aw = f - \kappa_1 * Aw - \kappa_2 * A\chi$$

$$\mu\chi_t + \nu A\chi + \beta(\chi) \ni \gamma(\chi) - \lambda^2\chi + \lambda(w_t - \Phi * w_t - \Psi * \chi)$$

in V', a.e. in $(0,T)$, with the expected meaning for the latter. There, taking $\mu = 0$ leads to the elliptic differential inclusion

$$\nu A\chi + \beta(\chi) - \gamma(\chi) + \lambda^2\chi \ni \lambda(w_t - \Phi * w_t - \Psi * \chi)$$

where the operator A reduces to $-\Delta$ with the Neumann homogeneous boundary conditions. Since we are interested in the asymptotic behavior as

$\mu \searrow 0$, it seems quite natural to ask for a strong monotonicity condition on the left hand side which is surely satisfied if the Lipschitz constant of γ is smaller that λ^2. From the physical point of view, this corresponds to require the latent heat to be large enough. For the sake of convenience, we introduce a new graph α, related to β, γ, and λ by

$$\alpha(z) := \beta(z) - \gamma(z) + \lambda^2 z \quad \forall z \in \mathbf{R} \tag{2.17}$$

and state directly on α the conditions we need. We assume

$$\alpha = \partial j : \mathbf{R} \to 2^{\mathbf{R}} \quad \text{with } \alpha(0) \ni 0 \tag{2.18}$$

$$j : \mathbf{R} \to [0, +\infty] \text{ is proper, convex, lower semicontinuous and } j(0) = 0 \tag{2.19}$$

$$(\eta_1 - \eta_2)(z_1 - z_2) \geq \ell(z_1 - z_2)^2 \ \forall z_i \in D(\alpha), \quad \forall \eta_i \in \alpha(z_i), \quad i = 1, 2 \tag{2.20}$$

for some $\ell > 0$, where $D(\alpha)$ is the effective domain of α. In the sequel, for $z \in D(\alpha)$, the symbol $\alpha^0(z)$ denotes the element of $\alpha(z)$ having minimum modulus.

Thus we end up with the equivalent version of problem (2.8–10) (see [9, Sect. 2] and observe that ξ takes the place of $\rho - \gamma(\chi) + \lambda^2\chi$)

$$w_{tt} + k(0)Aw = f - \kappa_1 * Aw - \kappa_2 * A\chi \quad \text{in } V', \quad \text{a.e. in } (0, T) \tag{2.21}$$

$$\mu\chi_t + \nu A\chi + \xi = \lambda(w_t - \Phi * w_t - \Psi * \chi) \quad \text{in } V', \quad \text{a.e. in } (0, T) \tag{2.22}$$

$$\chi \in D(\alpha) \quad \text{and} \quad \xi \in \alpha(\chi) \quad \text{a.e. in } Q \tag{2.23}$$

$$w(0) = 0, \quad w_t(0) = e_0, \quad \text{and} \quad \chi(0) = \chi_0 \tag{2.24}$$

where $e_0 := \vartheta_0 + \lambda\chi_0$, and [9, Thm. 2.4] ensures that the following result holds.

Proposition 2.1. *Let $\mu > 0$ and assume* (2.1–3) *and* (2.18–20). *Moreover, let f_μ, $e_{0,\mu}$, and $\chi_{0,\mu}$ satisfy*

$$f_\mu \in L^1(0, T; H) + W^{1,1}(0, T; V') \tag{2.25}$$

$$e_{0,\mu} \in H, \quad \chi_{0,\mu} \in V, \quad \text{and} \quad j(\chi_{0,\mu}) \in L^1(\Omega). \tag{2.26}$$

Then there exists a unique triplet $(w_\mu, \chi_\mu, \xi_\mu)$ with

$$w_\mu \in W^{2,1}(0, T; V') \cap C^1([0, T]; H) \cap C^0([0, T]; V) \tag{2.27}$$

$$\chi_\mu \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W) \tag{2.28}$$

$$\xi_\mu \in L^2(0, T; H) \tag{2.29}$$

which fulfills (2.21–24) *with* $f = f_\mu$, $e_0 = e_{0,\mu}$, *and* $\chi_0 = \chi_{0,\mu}$, *i.e.*,

$$\partial_t^2 w_\mu + k(0)Aw_\mu = f_\mu - \kappa_1 * Aw_\mu - \kappa_2 * A\chi_\mu \quad \textit{in } V', \quad \textit{a.e. in } (0,T) \tag{2.30}$$

$$\mu\partial_t\chi_\mu + vA\chi_\mu + \xi_\mu = \lambda(\partial_t w_\mu - \Phi * \partial_t w_\mu - \Psi * \chi_\mu) \quad \textit{in } V', \quad \textit{a.e. in } (0,T) \tag{2.31}$$

$$\chi_\mu \in D(\alpha) \quad \textit{and} \quad \xi_\mu \in \alpha(\chi_\mu) \quad \textit{a.e. in } Q \tag{2.32}$$

$$w_\mu(0) = 0, \quad \partial_t w_\mu(0) = e_{0,\mu}, \quad \textit{and} \quad \chi_\mu(0) = \chi_{0,\mu}. \tag{2.33}$$

In addition, if

$$f_\mu \in W^{1,1}(0,T;H) + W^{2,1}(0,T;V') \quad \textit{with } f_\mu(0) \in H \tag{2.34}$$

$$e_{0,\mu} \in V \tag{2.35}$$

then w_μ *enjoys the regularity conditions*

$$w_\mu \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V). \tag{2.36}$$

Furthermore, if

$$\chi_{0,\mu} \in W \quad \textit{and} \quad \partial_n \chi_{0,\mu|_\Gamma} = 0 \tag{2.37}$$

$$\chi_{0,\mu} \in D(\alpha) \textit{ a.e. in } \Omega \quad \textit{and} \quad \alpha^0(\chi_{0,\mu}) \in H \tag{2.38}$$

then χ_μ *and* ξ_μ *also satisfy*

$$\chi_\mu \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \tag{2.39}$$

$$\xi_\mu \in L^\infty(0,T;H). \quad \square \tag{2.40}$$

As stated in the Introduction, the aim of this paper is to study the asymptotic behavior of the solution of the above problem as $\mu \searrow 0$. Therefore, from now on, we let μ vary, say, in $(0,1)$ and denote by $(w_\mu, \chi_\mu, \xi_\mu)$ the unique solution to problem (2.30–33) given by Proposition 2.1. This is the solution that corresponds to the data $f_\mu, e_{0,\mu}, \chi_{0,\mu}$ satisfying (2.25–26).

Here are the basic conditions allowing us to pass to the limit in (2.30–33) as $\mu \searrow 0$. Let

$$f_\mu \to f \quad \text{in } L^1(0,T;H) + W^{1,1}(0,T;V') \tag{2.41}$$

$$e_{0,\mu} \to e_0 \quad \text{in } H \tag{2.42}$$

$$\mu^{1/2}\chi_{0,\mu} \to 0 \quad \text{in } H \tag{2.43}$$

$$\mu^{1/2}\|\chi_{0,\mu}\|_V + \mu\|j(\chi_{0,\mu})\|_{L^1(\Omega)} \le C \tag{2.44}$$

for some $C > 0$ and any $\mu \in (0,1)$. Observe that no convergence is required for $\{\chi_{0,\mu}\}$ since in the limit problem the third initial condition of (2.24) must be removed.

THEOREM 2.2. *Let the structural assumptions* (2.1–3) *and* (2.18–20) *hold. Moreover, let f_μ, $e_{0,\mu}$, $\chi_{0,\mu}$, f, and e_0 satisfy* (2.41–44). *Then there exists one triplet (w,χ,ξ) such that the strong, weak, or weak* convergences listed below hold.*

$$w_\mu \to w \quad \text{in } C^1([0,T];H) \cap C^0([0,T];V) \tag{2.45}$$

$$\chi_\mu \to \chi \quad \text{in } L^2(0,T;V) \tag{2.46}$$

$$\mu^{1/2}\chi_\mu \to 0 \quad \text{in } L^2(0,T;H) \tag{2.47}$$

$$\chi_\mu \rightharpoonup \chi \quad \text{in } L^2(0,T;W) \tag{2.48}$$

$$\mu\chi_\mu \rightharpoonup 0 \quad \text{in } H^1(0,T;H) \tag{2.49}$$

$$\xi_\mu \rightharpoonup \xi \quad \text{in } L^2(0,T;H) \tag{2.50}$$

$$\mu^{1/2}\chi_\mu \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;V). \tag{2.51}$$

In addition, the triplet (w,χ,ξ) solves the problem

$$w_{tt} + k(0)Aw = f - \kappa_1 * Aw - \kappa_2 * A\chi \quad \text{in } V', \quad \text{a.e. in } (0,T) \tag{2.52}$$

$$\nu A\chi + \xi = \lambda(w_t - \Phi * w_t - \Psi * \chi) \quad \text{in } V', \quad \text{a.e. in } (0,T) \tag{2.53}$$

$$\chi \in D(\alpha) \quad \text{and} \quad \xi \in \alpha(\chi) \quad \text{a.e. in } Q \tag{2.54}$$

$$w(0) = 0 \quad \text{and} \quad w_t(0) = e_0. \ \square \tag{2.55}$$

Note that Theorem 2.2 ensures, in particular, the existence of a solution to problem (2.52–55) whenever f and u_0 are as in the statement below. Indeed, it is sufficient to choose approximating data fulfilling (2.25–26) and (2.41–44). For instance, one can take $f_\mu = f$, $e_{0,\mu} = e_0$, and $\chi_{0,\mu} = 0$.

THEOREM 2.3. *Let the structural assumptions* (2.1–3) *and* (2.18–20) *hold and let f and e_0 satisfy*

$$f \in L^1(0,T;H) + W^{1,1}(0,T;V') \tag{2.56}$$

$$e_0 \in H. \tag{2.57}$$

Then problem (2.52–55) *has a unique solution.* $\square$

From now on, (w, χ, ξ) denotes the solution to problem (2.52–55) corresponding to data f and e_0.

By strengthening the assumptions on the sequence of data, we can deduce uniform bounds in stronger norms and prove error estimates.

THEOREM 2.4. *Let the structural assumptions* (2.1–3) *and* (2.18–20) *hold and let* f_μ, $e_{0,\mu}$, $\chi_{0,\mu}$, f, *and* e_0 *satisfy* (2.41–44). *Assume moreover*

$$\|f_\mu\|_{W^{1,1}(0,T;H)+W^{2,1}(0,T;V')} + \|f_\mu(0)\|_H + \|e_{0,\mu}\|_V \leq C' \tag{2.58}$$

for some $C' > 0$ *and any* $\mu \in (0,1)$. *Then*

$$\|w_\mu\|_{W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V)} \leq C_1 \tag{2.59}$$

for some $C_1 > 0$ *and any* $\mu \in (0,1)$. *If, in addition,* (2.37–38) *are fulfilled and there exists* $\xi_{0,\mu} \in H$ *such that*

$$\xi_{0,\mu} \in \alpha(\chi_{0,\mu}) \text{ a.e. in } \Omega, \quad \|\chi_{0,\mu}\|_H + \mu^{-1/2}\|\nu A\chi_{0,\mu} + \xi_{0,\mu} - \lambda e_{0,\mu}\|_H \leq C'' \tag{2.60}$$

for some $C'' > 0$ *and any* $\mu \in (0,1)$, *then*

$$\mu^{1/2}\|\partial_t\chi_\mu\|_{L^\infty(0,T;H)} + \|\partial_t\chi_\mu\|_{L^2(0,T;V)} + \|\chi_\mu\|_{L^\infty(0,T;W)} \leq C_2 \tag{2.61}$$

for some $C_2 > 0$ *and any* $\mu \in (0,1)$. □

REMARK 2.5. In view of Theorems 2.2 and 2.3, it is straightforward to verify that (2.59) and (2.61) yield

$$\begin{aligned} w_\mu &\overset{*}{\rightharpoonup} w \quad \text{in } W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V) \\ \mu^{1/2}\chi_\mu &\overset{*}{\rightharpoonup} 0 \quad \text{in } W^{1,\infty}(0,T;H) \\ \chi_\mu &\rightharpoonup \chi \quad \text{in } H^1(0,T;V) \\ \chi_\mu &\overset{*}{\rightharpoonup} \chi \quad \text{in } L^\infty(0,T;W) \end{aligned}$$

besides (2.45–51). Hence, by compactness we recover the further strong convergence

$$\chi_\mu \to \chi \quad \text{in } C^0([0,T];V).$$

THEOREM 2.6. *Let the structural assumptions* (2.1–3) *and* (2.18–20) *hold and let* f_μ, $e_{0,\mu}$, $\chi_{0,\mu}$, f, *and* e_0 *satisfy* (2.41–44), (2.58), (2.37–38), *and* (2.60). *Then*

$$\|w_\mu - w\|_{C^1([0,T];H) \cap C^0([0,T];V)} + \|\chi_\mu - \chi\|_{L^2(0,T;V)} \leq C_3(\mu + \varepsilon_\mu) \tag{2.62}$$

$$\|\chi_\mu - \chi\|_{C^0([0,T];V)} \leq C_4(\mu^{1/2} + \varepsilon_\mu) \tag{2.63}$$

where

$$\varepsilon_\mu := \|e_{0,\mu} - e_0\|_H + \|f_\mu - f\|_{L^1(0,T;H)+W^{1,1}(0,T;V')} \tag{2.64}$$

for some $C_3, C_4 > 0$ *and any* $\mu \in (0,1)$. $\square$

REMARK 2.7. Observe that (2.41–42) and (2.58) entail

$$f \in W^{1,1}(0,T;H) + W^{2,1}(0,T;V'), \quad f(0) \in H, \quad \text{and} \quad e_0 \in V.$$

On the other hand, one could wonder about the existence of sequences $\{\chi_{0,\mu}\}$ and $\{\xi_{0,\mu}\}$ fulfilling (2.37–38) and (2.60) as well as (2.44). To this concern, let us point out that (cf. (2.20)) the unique solution $\chi_{0,\mu} \in W$ of the elliptic problem

$$-\Delta\chi_{0,\mu} + \alpha(\chi_{0,\mu}) \ni \lambda e_{0,\mu} \quad \text{a.e. in } \Omega \tag{2.65}$$

$$\partial_n \chi_{0,\mu} = 0 \quad \text{a.e. on } \Gamma \tag{2.66}$$

and $\xi_{0,\mu} := \lambda e_{0,\mu} + \Delta\chi_{0,\mu}$ yield a proper example. Indeed, multiplying (2.65) by $\chi_{0,\mu}$ and integrating by parts, with the help of (2.66), (2.18–20), and of the definition of subdifferential, we obtain

$$\int_\Omega |\nabla\chi_{0,\mu}|^2 + \frac{\ell}{2}\|\chi_{0,\mu}\|_H^2 + \frac{1}{2}\int_\Omega |j(\chi_{0,\mu})| \le \lambda\|e_{0,\mu}\|_H\|\chi_{0,\mu}\|_H$$

and consequently (2.60) and (2.44) follow just from the boundedness of $\|e_{0,\mu}\|_H$ given by (2.42).

REMARK 2.8. Our convergence results and error estimates have been expressed in terms of (w_μ, χ_μ) and (w, χ). Coming back to the original variable ϑ in place of w (cf. (2.11–16)) and putting

$$\vartheta_\mu := \partial_t w_\mu - \lambda\chi_\mu - \Phi * \partial_t w_\mu - \Psi * \chi_\mu$$

it turns out that (owing to the Young theorem quoted below)

$$\vartheta_\mu \rightarrow \vartheta \quad \text{in } L^2(0,T;H)$$

$$1 * \vartheta_\mu \rightarrow 1 * \vartheta \quad \text{in } C^0([0,T];V)$$

in the framework of Theorem 2.2, while (2.62–63) imply

$$\|\vartheta_\mu - \vartheta\|_{L^2(0,T;H)} + \|1 * \vartheta_\mu - 1 * \vartheta\|_{C^0([0,T];V)} \le C_5(\mu + \varepsilon_\mu)$$

$$\|\vartheta_\mu - \vartheta\|_{C^0([0,T];H)} \le C_6(\mu^{1/2} + \varepsilon_\mu)$$

for some $C_5, C_6 > 0$ and any $\mu \in (0,1)$. $\square$

The next sections are devoted to the proofs of the above theorems. In carrying out them, we use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing between V' and V and $|\cdot|_V$ for the L^2-norm of the gradient, namely

$$|v|_V^2 := \int_\Omega |\nabla v|^2, \quad v \in V. \tag{2.67}$$

Setting also

$$Q_t := \Omega \times (0, t) \qquad \text{for } t \in (0, T] \tag{2.68}$$

we recall the formulas (which hold whenever they make sense)

$$(a * b)_t = a(0)b + a_t * b \quad \text{and} \quad (a * b)_t = b(0)a + a * b_t \tag{2.69}$$

the estimate coming from the Schwarz inequality and the well-known Young theorem

$$\|b * v\|_{L^\infty(0,T;X)} \le \|b\|_{L^2(0,T)} \|v\|_{L^2(0,T;X)} \tag{2.70}$$

$$\|b * v\|_{L^p(0,T;X)} \le \|b\|_{L^1(0,T)} \|v\|_{L^p(0,T;X)} \tag{2.71}$$

where X is a real Banach space and $p \in [1, \infty]$, and the elementary inequality

$$2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \quad \forall a, b \in \mathbf{R}, \quad \forall \varepsilon > 0. \tag{2.72}$$

Finally, we denote by the same symbol c, with possible subscripts, different constants depending only on the coefficients λ, $k(0)$, ν, on the norms of the kernels that are involved, and on the final time T, while further dependences are specified explicitly.

3. Proof of Theorem 2.3

Let (w_1, χ_1, ξ_1) and (w_2, χ_2, ξ_2) be two solutions to (2.52–55) and set

$$w := w_1 - w_2, \quad \chi := \chi_1 - \chi_2, \quad \xi := \xi_1 - \xi_2.$$

Writing down (2.52–55) for both triplets and taking the differences lead to

$$w_{tt} + k(0)Aw = -\kappa_1 * Aw - \kappa_2 * A\chi \tag{3.1}$$

$$\nu A\chi + \xi = \lambda(w_t - \Phi * w_t - \Psi * \chi) \tag{3.2}$$

$$w(0) = 0 \quad \text{and} \quad w_t(0) = 0 \tag{3.3}$$

with the equations fulfilled in V', a.e. in $(0, T)$. Now we test (3.1) by $2\partial_t w^\varepsilon$, where $\varepsilon > 0$ and w^ε is the V-valued solution of

$$w^\varepsilon(t) + \varepsilon A w^\varepsilon(t) = w(t) \text{ in } V', \quad \forall t \in [0, T].$$

Then we integrate over $(0,t)$ for an arbitrary $t \in]0,T]$ and take $\varepsilon \searrow 0$. Thanks to [8, Appendix] and owing to (2.69) and (3.3), we obtain

$$\|w_t(t)\|_H^2 + k(0)|w(t)|_V^2$$
$$= -2\langle(\kappa_1 * Aw)(t), w(t)\rangle - 2\langle(\kappa_2 * A\chi)(t), w(t)\rangle + \sum_{j=1}^{4} I_j(t) \tag{3.4}$$

where

$$I_1(t) = 2\kappa_1(0)\int_0^t \langle Aw(s), w(s)\rangle\, ds$$

$$I_2(t) = 2\int_0^t \langle(\kappa_1' * Aw)(s), w(s)\rangle\, ds$$

$$I_3(t) = 2\kappa_2(0)\int_0^t \langle A\chi(s), w(s)\rangle\, ds$$

$$I_4(t) = 2\int_0^t \langle(\kappa_2' * A\chi)(s), w(s)\rangle\, ds.$$

Let us treat each term on the right hand side of (3.4) separately and denote by η and σ arbitrary positive numbers. Taking (2.7), (2.16), (2.67), and (2.70–72) into account, we infer

$$2|\langle(\kappa_1 * Aw)(t), w(t)\rangle| \le \eta|w(t)|_V^2 + \frac{1}{\eta}\|\kappa_1\|_{L^2(0,T)}^2 \int_0^t |w(s)|_V^2 ds \tag{3.5}$$

$$2|\langle(\kappa_2 * A\chi)(t), w(t)\rangle| \le \eta|w(t)|_V^2 + \frac{1}{\eta}\|\kappa_2\|_{L^2(0,t)}^2 \int_0^t |\chi(s)|_V^2 ds \tag{3.6}$$

$$|I_1(t)| \le 2|\kappa_1(0)|\int_0^t |w(s)|_V^2 ds \tag{3.7}$$

$$|I_2(t)| \le \int_0^t |w(s)|_V^2 ds + \int_0^t |(\kappa_1' * w)(s)|_V^2 ds \le (1 + \|\kappa_1'\|_{L^1(0,T)}^2)\int_0^t |w(s)|_V^2 ds \tag{3.8}$$

$$|I_3(t)| \le \sigma\int_o^t |\chi(s)|_V^2 ds + \frac{|\kappa_2(0)|^2}{\sigma}\int_0^t |w(s)|_V^2 ds \tag{3.9}$$

$$|I_4(t)| \le \sigma\int_0^t |(\kappa_2' * \chi)(s)|_V^2 ds + \frac{1}{\sigma}\int_0^t |w(s)|_V^2 ds$$
$$\le \sigma\|\kappa_2'\|_{L^1(0,T)}^2 \int_0^t |\chi(s)|_V^2 ds + \frac{1}{\sigma}\int_0^t |w(s)|_V^2 ds. \tag{3.10}$$

Now we test (3.2) with χ and integrate over $(0,t)$ as before. Hence we have

$$\nu \int_0^t |\chi(s)|_V^2 ds + \int_0^t \langle \xi(s), \chi(s) \rangle \, ds = \int_0^t \lambda \langle w_t(s), \chi(s) \rangle \, ds$$
$$- \int_0^t \lambda \langle (\Phi * w_t)(s), \chi(s) \rangle \, ds - \int_0^t \lambda \langle (\Psi * \chi)(s), \chi(s) \rangle \, ds. \tag{3.11}$$

Thanks to (2.54) and (2.20), it turns out that

$$\int_0^t \langle \xi(s), \chi(s) \rangle \, ds \geq \ell \int_0^t \|\chi(s)\|_H^2 ds$$

while the three terms on the right hand side are treated as follows

$$\int_0^t \langle w_t(s), \chi(s) \rangle \, ds \leq \sigma \int_0^t \|\chi(s)\|_H^2 ds + \frac{1}{4\sigma} \int_0^t \|w_t(s)\|_H^2 ds$$
$$\int_0^t \langle -(\Phi * w_t)(s), \chi(s) \rangle \, ds \leq \sigma \int_0^t \|\chi(s)\|_H^2 ds + \frac{1}{4\sigma} \int_0^t \|(\Phi * w_t)(s)\|_H^2 ds$$
$$\leq \sigma \int_0^t \|\chi(s)\|_H^2 ds + \frac{1}{4\sigma} \|\Phi\|_{L^1(0,T)}^2 \int_0^t \|w_t(s)\|_H^2 ds$$
$$\int_0^t \langle -(\Psi * \chi)(s), \chi(s) \rangle \, ds \leq \sigma \int_0^t \|\chi(s)\|_H^2 ds + \frac{1}{4\sigma} \|\Psi\|_{L^1(0,t)}^2 \int_0^t \|\chi(s)\|_H^2 ds.$$

Adding (3.11) to (3.4) and using (3.5–10) and the above estimates, we get

$$\|w_t(t)\|_H^2 + k(0)|w(t)|_V^2 + \nu \int_0^t |\chi(s)|_V^2 ds + \ell \int_0^t \|\chi(s)\|_H^2 ds$$
$$\leq 2\eta |w(t)|_V^2 + c(\eta, \sigma) \int_0^t (|w(s)|_V^2 + \|w_t(s)\|_H^2) \, ds$$
$$+ (\eta^{-1} \|\kappa_2\|_{L^2(0,t)}^2 + \sigma + \sigma \|\kappa_2'\|_{L^1(0,T)}^2) \int_0^t |\chi(s)|_V^2 ds$$
$$+ \lambda (3\sigma + (4\sigma)^{-1} \|\Psi\|_{L^1(0,t)}^2) \int_0^t \|\chi(s)\|_H^2 ds \tag{3.12}$$

where $c(\eta, \sigma)$ depends only on η, σ, λ, and the norms of the kernels. We now choose first $\eta = k(0)/4$ and σ such that

$$\sigma + \sigma \|\kappa_2'\|_{L^1(0,T)}^2 \leq \frac{\nu}{4} \quad \text{and} \quad 3\lambda\sigma \leq \frac{\ell}{4}$$

and, consequently, $\delta > 0$ according to

$$\frac{1}{\eta}\|\kappa_2\|^2_{L^2(0,\delta)} \le \frac{\nu}{4} \quad \text{and} \quad \frac{\lambda}{4\sigma}\|\Psi\|^2_{L^1(0,\delta)} \le \frac{\ell}{4}.$$

Therefore, (3.12) yields

$$\|w_t(t)\|^2_H + \frac{k(0)}{2}|w(t)|^2_V + \frac{\nu}{2}\int_0^t |\chi(s)|^2_V ds + \frac{\ell}{2}\int_0^t \|\chi(s)\|^2_H ds$$

$$\le c\int_0^t (|w(s)|^2_V + \|w_t(s)\|^2_H)\, ds$$

provided $t \le \delta$. Thus, the Gronwall lemma implies $w(t) = 0$ and $\chi(t) = 0$ for any $t \in [0,\delta]$, i.e., $w_1(t) = w_2(t)$ and $\chi_1(t) = \chi_2(t)$ for any $t \in [0,\delta]$, and a comparison in (3.2) gives $\xi_1(t) = \xi_2(t)$ for a.a. $t \in (0,\delta)$.

Therefore, we can easily conclude our proof by unique continuation. We define

$$t_* := \sup\{t \in [0,T] : (w_1,\chi_1,\xi_1) \equiv (w_2,\chi_2,\xi_2) \text{ on } (0,t)\}$$

and argue by contradiction assuming $t_* < T$. Then we can apply the above procedure for $t \in (t_*,T]$ by exploiting the fact that the integrands of the convolution products in (3.1–2) vanish in a prescribed time interval. Thus, we find some $t^* > t_*$ such that $w_1 = w_2$, $\chi_1 = \chi_2$, and $\xi_1 = \xi_2$ in (t_*,t^*) and this contradicts the definition of t_*.

4. Proof of Theorem 2.2

It is convenient to split the proof into several steps because of some technicalities which are essentially due to assumptions (2.2) and (2.16). To be more precise, these minimal requirements entail a careful treatment of the terms involving the kernel Ψ.

First a priori estimate. We test (2.30) with $2\partial_t w^\varepsilon_\mu$, where w^ε_μ is the V-valued function defined by

$$w^\varepsilon_\mu(s) + \varepsilon A w^\varepsilon_\mu(s) = w_\mu(s) \quad \text{in } V', \quad \forall s \in [0,T].$$

Then we integrate from 0 to $t \in [0,T]$ and take $\varepsilon \searrow 0$ as in Section 3, applying [8, Appendix] and (2.33). We can argue exactly as in the first part of the uniqueness proof (see (3.5–10)) taking $\sigma = 1$ at once. However, here we have to deal with the source term coming from equation (2.30). This can be done by splitting f_μ into $f_{\mu,1} + f_{\mu,2}$, with $f_{\mu,1} \in L^1(0,T;H)$ and $f_{\mu,2} \in W^{1,1}(0,T;V')$, and estimating the integrals this way

$$2\int_0^t \langle f_{\mu,1}(s), \partial_t w_\mu(s)\rangle\, ds \le 2\int_0^t \|f_{\mu,1}(s)\|_H \|\partial_t w_\mu(s)\|_H ds \tag{4.1}$$

$$2\int_0^t \langle f_{\mu,2}(s), \partial_t w_\mu(s)\rangle\, ds = 2\langle f_{\mu,2}(t), w_\mu(t)\rangle - 2\int_0^t \langle \partial_t f_{\mu,2}(s), w_\mu(s)\rangle\, ds$$

$$\le \eta\|w_\mu(t)\|_V^2 + \frac{1}{\eta}\|f_{\mu,2}(t)\|_{V'}^2 + 2\int_0^t \|\partial_t f_{\mu,2}(s)\|_{V'} \|w_\mu(s)\|_V ds. \tag{4.2}$$

Then, as $\|w_\mu(t)\|_V^2 = |w_\mu(t)|_V^2 + \|w_\mu(t)\|_H^2$ and $\|w_\mu(t)\|_H^2 \le T\int_0^t \|\partial_t w_\mu(s)\|_H^2\, ds$ (since $w_\mu(0) = 0$), we deduce that

$$\|\partial_t w_\mu(t)\|_H^2 + k(0)\|w_\mu(t)\|_V^2$$

$$\le \|e_{0,\mu}\|_H^2 + k(0)T\int_0^t \|\partial_t w_\mu(s)\|_H^2 ds + 3\eta\|w_\mu(t)\|_V^2 + \frac{1}{\eta}\|f_{\mu,2}\|_{C^0([0,T];V')}^2$$

$$+ c(\eta)\int_0^t \|w_\mu(s)\|_V^2 ds + \left(\frac{1}{\eta}\|\kappa_2\|_{L^2(0,T)}^2 + 1 + \|\kappa_2'\|_{L^1(0,T)}^2\right)\int_0^t |\chi_\mu(s)|_V^2 ds$$

$$+ 2\int_0^t (\|f_{\mu,1}(s)\|_H \|\partial_t w_\mu(s)\|_H + \|\partial_t f_{\mu,2}(s)\|_{V'} \|w_\mu(s)\|_V)\, ds \tag{4.3}$$

where $c(\eta)$ is a constant depending only on η and on the quantities $\|\kappa_2\|_{L^2(0,T)}, |\kappa_1(0)|, \|\kappa_1'\|_{L^1(0,T)}, |\kappa_2(0)|$.

Next, letting σ be an arbitrary positive number, we perform the scalar product of (2.31) and of $\exp(-2\sigma s)\chi_\mu(s)$, $s \in [0, t]$, at the same instant s and then integrate with respect to s. Thus, thanks to (2.18–20) (note that $0 \in \alpha(0)$) we obtain

$$\frac{\mu}{2}e^{-2\sigma t}\|\chi_\mu(t)\|_H^2 + \nu\int_0^t e^{-2\sigma s}|\chi_\mu(s)|_V^2 ds + \ell\int_0^t e^{-2\sigma s}\|\chi_\mu(s)\|_H^2 ds$$

$$\le \frac{\mu}{2}\|\chi_{0,\mu}\|_H^2 - \sigma\mu\int_0^t e^{-2\sigma s}\|\chi_\mu(s)\|_H^2 ds$$

$$+ \lambda\int_0^t \|e^{-\sigma s}(\partial_t w_\mu - \Phi * \partial_t w_\mu - \Psi * \chi_\mu)(s)\|_H \|e^{-\sigma s}\chi_\mu(s)\|_H ds. \tag{4.4}$$

Hence, introducing the general notation

$$\bar{v}(t) := e^{-\sigma t}v(t) \quad \forall t \in [0, T] \tag{4.5}$$

for functions v from the interval $[0, T]$ to a Banach space X, one can easily realize that (cf. (2.71–72))

$$\frac{\mu}{2}\|\overline{\chi_\mu}(t)\|_H^2 + \nu\int_0^t |\overline{\chi_\mu}(s)|_V^2 ds + \ell\int_0^t \|\overline{\chi_\mu}(s)\|_H^2 ds$$

$$\leq \frac{\mu}{2}\|\chi_{0,\mu}\|_H^2 + \frac{2\lambda^2}{\ell}(1+\|\Phi\|_{L^1(0,T)}^2)\int_0^t \|\partial_t w_\mu(s)\|_H^2 ds + \frac{\ell}{4}\int_0^t \|\overline{\chi_\mu}(s)\|_H^2 ds$$

$$+\lambda\int_0^t \|(\overline{\Psi}*\overline{\chi_\mu})(s)\|_H\|\overline{\chi_\mu}(s)\|_H ds \tag{4.6}$$

since $\overline{\Psi*\chi_\mu} = \overline{\Psi}*\overline{\chi_\mu}$. Now we observe that the Schwarz inequality and (2.71) imply

$$\lambda\int_0^t \|(\overline{\Psi}*\overline{\chi_\mu})(s)\|_H\|\overline{\chi_\mu}(s)\|_H ds \leq \lambda\|\overline{\Psi}\|_{L^1(0,T)}\int_0^t \|\overline{\chi_\mu}(s)\|_H^2 ds$$

and fix σ in order to have

$$\lambda\|\overline{\Psi}\|_{L^1(0,T)} = \lambda\int_0^T e^{-\sigma s}|\Psi(s)|\,ds \leq \frac{\ell}{4}. \tag{4.7}$$

This is always possible provided we take σ sufficiently large. Hence we have

$$e^{-2\sigma T}\left(\frac{\mu}{2}\|\chi_\mu(t)\|_H^2 + \nu\int_0^t |\chi_\mu(s)|_V^2 ds + \frac{\ell}{2}\int_0^t \|\chi_\mu(s)\|_H^2 ds\right)$$

$$\leq \frac{\mu}{2}\|\chi_{0,\mu}\|_H^2 + \frac{2\lambda^2}{\ell}\left(1+\|\Phi\|_{L^1(0,T)}^2\right)\int_0^t \|\partial_t w_\mu(s)\|_H^2 ds. \tag{4.8}$$

Multiply (4.8) by M and add it to (4.3). Then choose η and M according to

$$3\eta = \frac{k(0)}{2}, \quad Me^{-2\sigma T}\nu = 2\left(\frac{1}{\eta}\|\kappa_2\|_{L^2(0,T)}^2 + 1 + \|\kappa_2'\|_{L^1(0,T)}^2\right)$$

we finally get

$$\|\partial_t w_\mu(t)\|_H^2 + \frac{k(0)}{2}\|w_\mu(t)\|_V^2 + \frac{M\mu}{2}e^{-2\sigma T}\|\chi_\mu(t)\|_H^2 + \frac{M}{2}e^{-2\sigma T}\min\{\nu,\ell\}\int_0^t \|\chi_\mu(s)\|_V^2 ds$$

$$\leq c(\|e_{0,\mu}\|_H^2 + \mu\|\chi_{0,\mu}\|_H^2 + \|f\|_{C^0([0,T];V')}) + c\int_0^t (\|\partial_t w_\mu(s)\|_H^2 + \|w_\mu(s)\|_V^2)\,ds$$

$$+2\int_0^t (\|f_{\mu,1}(s)\|_H\|\partial_t w_\mu(s)\|_H + \|\partial_t f_{\mu,2}(s)\|_{V'}\|w_\mu(s)\|_V)\,ds$$

for any $t\in[0,T]$ and $\mu\in(0,1)$. Then an extended version of the Gronwall lemma (see [3] or combine the two lemmas in [5, pp. 156–157]) enables us to conclude that

$$\|w_\mu\|^2_{C^1([0,T];H)\cap C^0([0,T];V)} + \mu\|\chi_\mu\|^2_{C^0([0,T];H)} + \|\chi_\mu\|^2_{L^2(0,T;V)}$$
$$\leq c(\|e_{0,\mu}\|^2_H + \mu\|\chi_{0,\mu}\|^2_H + \|f_{\mu,1}\|^2_{L^1(0,T;H)} + \|f_{\mu,2}\|^2_{W^{1,1}(0,T;V')}). \quad (4.9)$$

Also, by comparison in (2.30), on account of (2.16) we deduce that

$$\|\partial_t^2 w_\mu - f_{\mu,1}\|^2_{L^\infty(0,T;V')}$$
$$\leq c(\|e_{0,\mu}\|^2_H + \mu\|\chi_{0,\mu}\|^2_H + \|f_{\mu,1}\|^2_{L^1(0,T;H)} + \|f_{\mu,2}\|^2_{W^{1,1}(0,T;V')}). \quad (4.10)$$

REMARK 4.1. After the proof of (4.9), a natural comment concerns the possibility of applying the same argument to prove uniqueness, thus proposing a variation to the method developed in Section 3.

Second a priori estimate. Assume for a moment that the graph α is a Lipschitz continuous function. Then $\xi_\mu = \alpha(\chi_\mu)$, whence, in particular, $\xi_\mu \in L^2(0,T;V)$ and (cf. (2.18))

$$\int_0^t \langle \partial_t \chi_\mu(s), \xi_\mu(s)\rangle\, ds = \int_\Omega j(\chi_\mu(t)) - \int_\Omega j(\chi_{0,\mu})$$

$$\int_0^t \langle A\chi_\mu(s), \xi_\mu(s)\rangle\, ds = \iint_{Q_t} \alpha'(\chi_\mu)|\nabla\chi_\mu|^2 \geq 0.$$

Therefore, testing (2.31) with ξ_μ, one obtains

$$\mu\|j(\chi_\mu(t))\|_{L^1(\Omega)} + \int_0^t \|\xi_\mu(s)\|^2_H ds \leq \mu\|j(\chi_{0,\mu})\|_{L^1(\Omega)} + \int_0^t (F_\mu(s), \xi_\mu(s))_H ds$$

where F_μ stands for the right hand side of (2.31) and belongs to $L^2(0,T;H)$ thanks to (2.16) and (2.27–28). The resulting inequality still holds when α satisfies (2.18–20). To check that, it suffices to approximate α by its Yosida regularization (see [7–8]). Then, on account of (4.9) we infer

$$\mu\|j(\chi_\mu)\|_{L^\infty(0,T;L^1(\Omega))} + \|\xi_\mu\|^2_{L^2(0,T;H)}$$
$$\leq c(\mu\|j(\chi_{0,\mu})\|_{L^1(\Omega)} + \|e_{0,\mu}\|^2_H + \mu\|\chi_{0,\mu}\|^2_H$$
$$+ \|f_{\mu,1}\|_{L^1(0,T;H)} + \|f_{\mu,2}\|_{W^{1,1}(0,T;V')}). \quad (4.11)$$

Third a priori estimate. As we have just noted, the right hand side of (2.31) is a function $F_\mu \in L^2(0,T;H)$. Recalling (2.7) and (2.28), we see that we can use the H-valued function $A\chi_\mu$ as a test function in (2.31). Arguing as above, with the help of (2.26), (2.16), (4.9), and of well-known elliptic regularity results, it is not difficult to get

$$\|\chi_\mu\|^2_{L^2(0,T;W)} + \mu\|\chi_\mu\|^2_{L^\infty(0,T;V)}$$
$$\le c(\|e_{0,\mu}\|^2_H + \mu\|\chi_{0,\mu}\|^2_V + \|f_{\mu,1}\|^2_{L^1(0,T;H)} + \|f_{\mu,2}\|^2_{W^{1,1}(0,T;V')}). \tag{4.12}$$

Besides, a comparison in (2.31) leads to

$$\mu^2\|\partial_t\chi_\mu\|^2_{L^2(0,T;H)}$$
$$\le c(\|e_{0,\mu}\|^2_H + \mu\|\chi_{0,\mu}\|^2_V + \|f_{\mu,1}\|^2_{L^1(0,T;H)} + \|f_{\mu,2}\|^2_{W^{1,1}(0,T;V')}). \tag{4.13}$$

Weak convergences and first consequences. Let us split f into the sum $f_1 + f_2$ with $f_1 \in L^1(0,T;H)$ and $f_2 \in W^{1,1}(0,T;V')$. Then, without any loss of generality (cfr. (2.41)), we can assume that

$$f_{\mu,1} \to f_1 \quad \text{in } L^1(0,T;H) \quad \text{and} \quad f_{\mu,2} \to f_2 \text{ in } W^{1,1}(0,T;V'). \tag{4.14}$$

Thanks to (4.9–13), (2.42–44), and (4.14), it is clear that the estimates listed below hold for some constant C and for any $\mu \in (0,1)$, namely,

$$\|w_\mu\|_{W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)} \le C \tag{4.15}$$

$$\|\partial_t^2 w_\mu - f_{\mu,1}\|_{L^\infty(0,T;V')} \le C \tag{4.16}$$

$$\|\chi_\mu\|_{L^2(0,T;W)} \le C \tag{4.17}$$

$$\|\xi_\mu\|_{L^2(0,T;H)} \le C \tag{4.18}$$

$$\mu^{1/2}\|\chi_\mu\|_{L^\infty(0,T;V)} \le C \tag{4.19}$$

$$\mu\|\chi_\mu\|_{H^1(0,T;H)} \le C \tag{4.20}$$

$$\mu\|j(\chi_\mu)\|_{L^\infty(0,T;L^1(\Omega))} \le C \tag{4.21}$$

with C obviously depending on an upper bound for the norms of the data.

Then, because of well-known compactness results, there exists a triplet (w,χ,ξ) such that, at least for a subsequence of μ tending to 0,

$$w_\mu \overset{*}{\rightharpoonup} w \quad \text{in } W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V) \tag{4.22}$$

$$\chi_\mu \rightharpoonup \chi \quad \text{in } L^2(0,T;W) \tag{4.23}$$

$$\xi_\mu \rightharpoonup \xi \quad \text{in } L^2(0,T;H). \tag{4.24}$$

In addition, owing to (4.14), (4.16), (4.19–20), and (4.23), we have

$$\partial_t^2 w_\mu - f_{\mu,1} \overset{*}{\rightharpoonup} w_{tt} - f_1 \quad \text{in } L^\infty(0,T;V') \tag{4.25}$$

$$\mu^{1/2}\chi_\mu \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;V) \tag{4.26}$$

$$\mu\chi_\mu \rightharpoonup 0 \quad \text{in } H^1(0,T;H). \tag{4.27}$$

This list covers, in particular, (2.48–51). Indeed, if we show that (w,χ,ξ) is a solution to problem (2.52–55), the uniqueness result entails that the whole family $\{(w_\mu,\chi_\mu,\xi_\mu)\}$ is convergent both in the sense specified above and in the sense of further convergences that we are going to deduce throughout the proof.

Observe now that, since $\xi_\mu \in \partial j(\chi_\mu)$ a.e. in Q and j fulfills (2.19), the definition of subdifferential ensures that $j(\chi_\mu) \le \chi_\mu\xi_\mu$ a.e. in Q. Hence (4.17–18) imply

$$\iint_Q j(\chi_\mu) \le C.$$

Therefore, (2.19), (4.23), and the convexity and lower semicontinuity of the functional $J : L^2(Q) \to [0,+\infty]$,

$$J(v) := \iint_Q j(v) \quad \text{if } j(v) \in L^1(Q), \quad J(v) := +\infty \text{ otherwise}$$

yield

$$J(\chi) \le \liminf_{\mu\searrow 0} J(\chi_\mu) \quad \text{and} \quad j(\chi) \in L^1(Q). \tag{4.28}$$

Clearly, (w,χ,μ) fulfills (2.52–53) at least in the sense of V'-valued distributions. Let us check now that w belongs to $C^0([0,T];V) \cap C^1([0,T];H)$ and satisfies (2.55). Indeed, using (4.22), (2.33), the well-known compact inclusion of V into H, and the generalized Ascoli theorem (see, e.g., [18, Thm. 3.1, p. 57]), we deduce the strong convergence

$$w_\mu \to w \quad \text{in } C^0([0,T];H) \tag{4.29}$$

whence the first condition (2.55). On account of (4.25), a similar reasoning gives

$$\partial_t w_\mu - 1 * f_{\mu,1} \to w_t - 1 * f_1 \quad \text{in } C^0([0,T];V')$$

and, thanks to (4.14),

$$\partial_t w_\mu \to w_t \quad \text{in } C^0([0,T];V'). \tag{4.30}$$

Consequently, the second initial condition in (2.55) follows easily from (2.33) and (2.42). As w solves a Cauchy problem for a linear hyperbolic second order equation with right hand side in the space $L^1(0,T;H) + W^{1,1}(0,T;V')$ and initial data in $V \times H$, the related general theory (see, e.g., [2]) allows us to conclude that w belongs to $C^0([0,T];V) \cap C^1([0,T];H)$.

The remaining conditions stated in Theorem 2.2 are less trivial to prove and the sequel of the section is devoted to check them. As far as (2.54) is concerned, we make a remark at once. From (2.31) we derive the equality

$$\iint_{Q_t} \xi_\mu \chi_\mu = -\nu \iint_{Q_t} |\nabla \chi_\mu|^2 - \frac{\mu}{2}\|\chi_\mu(t)\|_H^2 + \frac{\mu}{2}\|\chi_{0,\mu}\|_H^2$$

$$+ \lambda \int_0^t \langle (\partial_t w_\mu(s) - \Phi * \partial_t w_\mu)(s), \chi_\mu(s)\rangle\, ds - \iint_{Q_t} \lambda(\Psi * \chi_\mu)\chi_\mu \tag{4.31}$$

for any $t \in (0,T]$ and $\mu \in (0,1)$. Then, from (4.31) we would like to infer

$$\limsup_{\mu \searrow 0} \iint_{Q_t} \xi_\mu \chi_\mu$$

$$\leq -\nu \iint_{Q_t} |\nabla \chi|^2 + \lambda \int_0^t \langle (w_t(s) - \Phi * w_t)(s), \chi(s)\rangle\, ds - \iint_{Q_t} \lambda(\Psi * \chi)\chi$$

since this inequality combined with (2.53) yields

$$\limsup_{\mu \searrow 0} \iint_{Q_t} \xi_\mu \chi_\mu \leq \iint_{Q_t} \xi\chi \tag{4.32}$$

so that (2.54) would follow from [4, Prop. 1.1, p. 42]. Let us examine the right hand side of (4.31). The first term can be treated by lower semicontinuity using (4.23). The second term is negative and the third one tends to 0 because of (2.43). The fourth term is easily handled owing to (4.30) and (4.23) and converges to the desired integral. Finally, the last term does not give any trouble provided that $\Psi \in W^{1,1}(0,T)$. In fact, in this case, from (4.17), (2.69), and (2.71) it follows that the sequence $\{\Psi * \chi_\mu\}$ is bounded in $H^1(0,T;W)$, whence it converges strongly in $L^2(0,T;H)$ by compactness. Since our assumption (2.2) ensures only that $\Psi \in L^1(0,T)$, we have to recover the strong convergence for $\{\chi_\mu\}$ in another way.

Strong convergences and conclusion. We take the difference between (2.30) and (2.52) and test the resulting equation with the function $2\partial_t(w_\mu^\varepsilon - w^\varepsilon)$, where w_μ^ε and w^ε are defined by

$$w_\mu^\varepsilon(t) + \varepsilon A w_\mu^\varepsilon(t) = w_\mu(t) \quad \text{and} \quad w^\varepsilon(t) + \varepsilon A w^\varepsilon(t) = w(t) \quad \forall t \in [0,T].$$

As $w_\mu - w \in C^0([0,T];V) \cap C^1([0,T];H)$, we can use [8, Appendix] and deduce an inequality similar to (4.3) (by controlling the full V-norm in terms of the seminorm $|\cdot|_V$ and of the H-norm of the time derivative). Hence, one easily finds out that

$$\begin{aligned}
&\|\partial_t(w_\mu - w)(t)\|_H^2 + \|(w_\mu - w)(t)\|_V^2 \\
&\leq R_{1,\mu} + c\int_0^t (\|\partial_t(w_\mu - w)(s)\|_H^2 + \|(w_\mu - w)(s)\|_V^2)\, ds \\
&+ c\int_0^t \|(f_{\mu,1} - f_1)(s)\|_H \|\partial_t(w_\mu - w)(s)\|_H ds \\
&+ c\int_0^t \|\partial_t(f_{\mu,2} - f_2)(s)\|_{V'} \|(w_\mu - w)(s)\|_V ds \\
&+ c\int_0^t \|(\chi_\mu - \chi)(s)\|_V^2 ds
\end{aligned} \tag{4.33}$$

where $R_{1,\mu}$ are real numbers tending to 0 as $\mu \searrow 0$ due to (2.41–42). Thanks to the generalized Gronwall lemma (cf., e.g., [3] or [5]), it is straightforward to verify that

$$\|\partial_t(w_\mu - w)(t)\|_H^2 + \|(w_\mu - w)(t)\|_V^2 \leq c_1(R_{2,\mu} + \|\chi_\mu - \chi\|_{L^2(0,t;V)}^2) \tag{4.34}$$

for any $t \in (0, T]$ and for some numerical sequence $\{R_{2,\mu}\}$ going to 0 as $\mu \searrow 0$.

Now, let us introduce the graph $\alpha_\#$ and the proper and lower semicontinuous function $j_\#$ by setting

$$\alpha_\#(z) := \alpha(z) - \frac{\ell}{2}z, \quad z \in D(\alpha), \quad \text{and} \quad j_\#(z) := j(z) - \frac{\ell}{4}z^2, \quad z \in \mathbf{R}.$$

Owing to (2.18–20), $j_\#$ is nonnegative and convex, $\alpha_\# = \partial j_\#$, and $\alpha_\#$ fulfills the same inequality (2.20) with $\ell/2$ in place of ℓ. Putting

$$\zeta_\mu := \xi_\mu - \frac{\ell}{2}\chi_\mu$$

from (2.32) we have $\zeta_\mu \in \alpha_\#(\chi_\mu)$ a.e. in Q, whence $j_\#(\chi_\mu) - j_\#(\chi) \leq \zeta_\mu(\chi_\mu - \chi)$ a.e. in Q. We rewrite this in the equivalent form

$$\frac{\ell}{2}(\chi_\mu - \chi)^2 \leq j_\#(\chi) - j_\#(\chi_\mu) + \xi_\mu(\chi_\mu - \chi) - \frac{\ell}{2}\chi(\chi_\mu - \chi) \quad \text{a.e. in } Q. \tag{4.35}$$

On the other hand, in virtue of (4.22–24) and (2.31), we can deduce that

$$\left(\mu\partial_t\chi_\mu + \nu A(\chi_\mu - \chi) + \xi_\mu - \frac{\ell}{2}\chi - G_\mu - G\right)(x,t) = 0 \quad \text{for a.a. } (x,t) \in Q \tag{4.36}$$

where

$$G_\mu := \lambda(\partial_t(w_\mu - w) - \Phi * \partial_t(w_\mu - w) - \Psi * (\chi_\mu - \chi))$$

$$G := -\nu A\chi - \frac{\ell}{2}\chi + \lambda(w_t - \Phi * w_t - \Psi * \chi)$$

both belong to $L^2(0,T;H)$. Letting $\sigma > 0$ and multiplying (4.36) by the function $\exp(-2\sigma t)(\chi_\mu - \chi)(x,t)$, with the help of (4.35) we obtain

$$\frac{\mu}{2}\partial_t(e^{-2\sigma t}\chi_\mu{}^2(x,t)) + \nu e^{-2\sigma t}(A(\chi_\mu - \chi))(x,t)(\chi_\mu - \chi)(x,t) + \frac{\ell}{2}e^{-2\sigma t}(\chi_\mu - \chi)^2(x,t)$$
$$\leq -\mu\sigma e^{-2\sigma t}\chi_\mu{}^2(x,t) + \mu e^{-2\sigma t}((\partial_t\chi_\mu)\chi)(x,t) + e^{-2\sigma t}(j_\#(\chi) - j_\#(\chi_\mu))(x,t)$$
$$+ e^{-2\sigma t}(G_\mu + G)(x,t)(\chi_\mu - \chi)(x,t) \quad \text{for a.a. } (x,t) \in Q. \tag{4.37}$$

Integrating (4.37) over Q_t and recalling (2.33) and the notation (4.5), calculations analogous to those of (4.4) and (4.6) lead to

$$\frac{\mu}{2}\|\overline{\chi_\mu}(t)\|_H^2 + \nu\int_0^t |(\overline{\chi_\mu} - \bar{\chi})(s)|_V^2 ds + \frac{\ell}{2}\int_0^t \|(\overline{\chi_\mu} - \bar{\chi})(s)\|_H^2 ds$$
$$\leq R_{3,\mu}(t) + \frac{8\lambda^2}{\ell}(1 + \|\Phi\|_{L^1(0,T)}^2)\int_0^t \|\partial_t(w_\mu - w)(s)\|_H^2 ds$$
$$+ \left(\frac{\ell}{8} + \lambda\|\overline{\Psi}\|_{L^1(0,T)}\right)\int_0^t \|(\overline{\chi_\mu} - \bar{\chi})(s)\|_H^2 ds \tag{4.38}$$

for any $t \in (0,T]$, with $R_{3,\mu}(t)$ defined by

$$R_{3,\mu} := \frac{\mu}{2}\|\chi_{0,\mu}\|_H^2 + \left(\iint_{Q_t} e^{-2\sigma s}(j_\#(\chi) - j_\#(\chi_\mu))(x,s)\,dx\,ds\right)^+$$
$$+ \iint_{Q_t} e^{-2\sigma s}(\mu(\partial_t\chi_\mu)\chi + G(\chi_\mu - \chi))(x,s)\,dx\,ds. \tag{4.39}$$

Then we can fix σ in order that (cf. (4.7) and (4.8)) $\lambda\|\overline{\Psi}\|_{L^1(0,T)} \leq \ell/8$ and get

$$\mu\|\chi_\mu(t)\|_H^2 + \int_0^t \|(\chi_\mu - \chi)(s)\|_V^2 ds \leq c_2(R_{3,\mu}(t) + \|\partial_t(w_\mu - w)\|_{L^2(0,t;H)}^2). \tag{4.40}$$

Let $J_\#^t : L^2(Q_t) \to [0,+\infty]$ be the functional whose finite values are defined by

$$J_\#^t(v) := \iint_{Q_t} e^{-2\sigma s} j_\#(v(x,s))\,dxds \textit{ if } v \in L^2(Q_t) \quad \text{and} \quad j_\#(v) \in L^1(Q_t).$$

As $J_\#^t$ is induced by $j_\#$, it is convex and lower semicontinuous. So, the weak convergence $\chi_\mu \rightharpoonup \chi$ in $L^2(Q)$ (see (4.23)) entails that (compare with (4.28))

$$\limsup_{\mu\searrow 0} \iint_{Q_t} e^{-2\sigma s}(j_\#(\chi) - j_\#(\chi_\mu))(x,s)\,dxds \le 0.$$

Hence, the second term in (4.39) tends to 0 as well as the other terms do because of (2.43), (4.27), and (4.23). Moreover, note that

$$e^{-2\sigma s}(j_\#(\chi) - j_\#(\chi_\mu))(x,s) \le j(\chi(x,s)) \quad \text{for a.a. } (x,s) \in Q$$

and consequently (4.28), (4.20), (4.17) allow us to infer that the sequence $\{R_{3,\mu}\}$ is bounded in $L^\infty(0,T)$. Thus, in addition to

$$R_{3,\mu(t)} \to 0 \quad \forall t \in (0,T] \tag{4.41}$$

by the Lebesgue dominated convergence theorem we have that

$$R_{3,\mu} \to 0 \quad \text{in } L^1(0,T) \quad \text{as } \mu \searrow 0. \tag{4.42}$$

We now multiply (4.40) by $2c_1$ and add the resulting inequality to (4.34). We obtain

$$R_{4,\mu}(t) \le 2c_1c_2 \int_0^t R_{4,\mu}(s)\,ds + 2c_1c_2R_{3,\mu}(t) + c_1R_{2,\mu} \tag{4.43}$$

where

$$R_{4,\mu}(t) := \|\partial_t(w_\mu - w)(t)\|_H^2 + \|(w_\mu - w)(t)\|_V^2 + 2c_1\mu\,\|\chi_\mu(t)\|_H^2 + c_1\,\|\chi_\mu - \chi\|_{L^2(0,t;V)}^2.$$

Since we do not know whether $\{R_{3,\mu}\}$ tends to 0 uniformly in $[0,T]$, we integrate (4.43) from 0 to $\tau \in (0,T]$ and, at this point, we apply the Gronwall lemma to deduce that

$$\int_0^\tau R_{4,\mu}(t)\,dt \le c(\|R_{3,\mu}\|_{L^1(0,T)} + R_{2,\mu}) \quad \forall \tau \in (0,T].$$

Then, owing to (4.42), the sequence $\{R_{4,\mu}\}$ goes to 0 in $L^1(0,T)$ so that, due to (4.43) and (4.41), $R_{4,\mu}(t) \to 0$ for any $t \in (0,T]$ and (2.46–47) are proved. Moreover, (2.45) follows plainly from (4.34).

Therefore, we are ready to get (4.32) (with the full limit and the equality sign) and the subsequent property (2.54). Finally, since all terms in (2.52–53) belong to $L^1(0,T;V')$, both equations are satisfied in V' almost everywhere in $(0,T)$ and we conclude.

5. Proof of Theorems 2.4 and 2.6

In this last section we let the generic constant c depend also on the norms of the data, i.e., on the constants C, C', and C'' appearing in (2.44), (2.58), and (2.60).

First of all, we observe that the right hand side of (2.30) belongs to $W^{1,1}(0,T;H) + W^{2,1}(0,T;V')$ and that its value at $t = 0$ belongs to H thanks to (2.58), (2.36), (2.28), (2.16), (2.69), and (2.71). Then it turns out that $w_\mu \in C^1([0,T];V) \cap C^2([0,T];H)$, which improves (2.36) and makes the next argument rigorous. Remarking that $\partial_t^2 w_\mu(0) = f_\mu(0)$, we differentiate (2.30) in time and test the obtained equation with $2\partial_t^2 w_\mu^\varepsilon$, where w_μ^ε is defined by

$$w_\mu^\varepsilon(t) + \varepsilon A w_\mu^\varepsilon(t) = w_\mu(t) \quad \forall t \in (0,T].$$

Moreover, let $f_\mu = f_{\mu,1} + f_{\mu,2}$ where $f_{\mu,1}$ and $f_{\mu,2}$ satisfy

$$\|f_{\mu,1}\|_{W^{1,1}(0,T;H)} + \|f_{\mu,2}\|_{W^{2,1}(0,T;V')} \leq c.$$

Thanks to [8, Appendix] and (2.33), for any $t \in (0,T]$ we easily get

$$\begin{aligned}
&\frac{1}{2}\|\partial_t^2 w_\mu(t)\|_H^2 + \frac{k(0)}{2}|\partial_t w_\mu(t)|_V^2 \\
&\quad \leq \frac{1}{2}\|f_\mu(0)\|_H^2 + \frac{k(0)}{2}|e_{0,\mu}|_V^2 \\
&\qquad + \int_0^t \|\partial_t f_{\mu,1}(s)\|_H \|\partial_t^2 w_\mu(s)\|_H ds \\
&\qquad + \|\partial_t f_{\mu,2}(t)\|_{V'} \|\partial_t w_\mu(t)\|_V + \|\partial_t f_{\mu,2}(0)\|_{V'} \|e_{0,\mu}\|_V \\
&\qquad + \int_0^t \|\partial_t^2 f_{\mu,2}(s)\|_{V'} \|\partial_t w_\mu(s)\|_V ds + |(\kappa_1(0) w_\mu + \kappa_1' * w)(t)|_V |\partial_t w_\mu(t)|_V \\
&\qquad + (|\kappa_1(0)| + \|\kappa_1'\|_{L^1(0,T)}) \int_0^t |\partial_t w_\mu(s)|_V^2 ds \\
&\qquad + (|\kappa_2(0)| + \|\kappa_2'\|_{L^1(0,T)}) \|A\chi_\mu\|_{L^2(0,t;H)} \|\partial_t^2 w_\mu\|_{L^2(0,t;H)}
\end{aligned}$$

whence, owing to (2.58), using the trivial inequality $ab \leq a(1 + b^2)$ for $a, b \geq 0$, and replacing the seminorm with the full V-norm by adding $\|\partial_t w_\mu(t)\|_H^2$ (which is already bounded because of (4.15)), we can write

$$\begin{aligned}
&\|\partial_t^2 w_\mu(t)\|_H^2 + \|\partial_t w_\mu(t)\|_V^2 \\
&\quad \leq c + \int_0^t c(1 + \|\partial_t f_{\mu,1}(s)\|_H) \|\partial_t^2 w_\mu(s)\|_H^2 ds \\
&\qquad + \int_0^t c(1 + \|\partial_t f_{\mu,2}(s)\|_{V'}) \|\partial_t w_\mu(s)_H^2\| ds
\end{aligned}$$

and (2.59) follows from an application of the Gronwall lemma.

Next, assuming (2.60), we work on (2.31) proceeding formally. In fact, a rigorous argument would need, e.g., the Yosida ε-regularization α^{ε} of α ($\varepsilon > 0$) and the approximation of the initial datum $\chi_{0,\mu}$ in (2.33) by the solution $\chi_{0,\mu}^{\varepsilon} \in W$ to the elliptic problem

$$-\nu\Delta\chi_{0,\mu}^{\varepsilon} + \alpha^{\varepsilon}(\chi_{0,\mu}^{\varepsilon}) = -\nu\Delta\chi_{0,\mu} + \xi_{0,\mu} \quad \text{a.e. in } \Omega$$

$$\partial_n\chi_{0,\mu}^{\varepsilon} = 0 \quad \text{a.e. on } \Gamma$$

which is well-defined because of the coerciveness property (cf. (2.20))

$$(\alpha^{\varepsilon}(z_1) - \alpha^{\varepsilon}(z_2))(z_1 - z_2) \geq \frac{\ell}{1+\varepsilon\ell}(z_1 - z_2)^2 \quad \forall z_1, z_2 \in \mathbf{R}, \quad \forall \varepsilon > 0.$$

By (2.37–38) it is not difficult to check the convergences

$$\chi_{0,\mu}^{\varepsilon} \to \chi_{0,\mu} \quad \text{in } V$$

$$A\chi_{0,\mu}^{\varepsilon} \rightharpoonup A\chi_{0,\mu}, \quad \alpha^{\varepsilon}(\chi_{0,\mu}^{\varepsilon}) \rightharpoonup \xi_{0,\mu} \quad \text{in } H$$

as $\varepsilon \searrow 0$, so that one can perform the computation below on the approximating scheme and then pass to the limit via the methods used, for instance, in [7–8].

Let us come to the formal estimate. First we have to recover the initial value of $\partial_t\chi_\mu$ from (2.31), (2.33) and to remark that the formal derivative α' is bounded from below by ℓ because of (2.20). Then we differentiate (2.31) in time and take the scalar product with $\exp(-2\sigma s)\partial_t\chi_\mu(s)$ at the same instant $s \in (0, T)$, σ being a positive parameter. Observing that $\partial_t(\Psi * \chi_\mu) = \Psi\chi_{0,\mu} + \Psi * \partial_t\chi_\mu$ and recalling (4.5), the integration from 0 to $t \in (0, T]$ and already familiar computations lead to

$$\frac{\mu}{2}\|\overline{\partial_t\chi_\mu}(t)\|_H^2 + \nu\int_0^t |\overline{\partial_t\chi_\mu}(s)|_V^2 ds + \ell\int_0^t \|\overline{\partial_t\chi}(s)\|_H^2 ds$$

$$\leq \frac{1}{2\mu}\|\nu A\chi_{0,\mu} + \xi_{0,\mu} - \lambda e_{0,\mu}\|_H^2 + \frac{\ell}{4}\int_0^t \|\overline{\partial_t\chi_\mu}(s)\|_H^2 ds$$

$$+ c\int_0^t (\|\partial_t^2 w_\mu(s)\|_H^2 + \|\partial_t w_\mu(s)\|_H^2 + \|\chi_{0,\mu}\|_H^2)\, ds$$

$$+ \lambda\|\overline{\Psi}\|_{L^1(0,T)}\int_0^t \|\overline{\partial_t\chi_\mu}(s)\|_H^2 ds.$$

Choosing a suitable σ, on account of (2.59–60) we get the bound

$$\mu\|\partial_t\chi_\mu\|_{L^\infty(0,T;H)}^2 + \|\partial_t\chi_\mu\|_{L^2(0,T;V)}^2 \leq c$$

whose combination with (4.17) yields (2.61).

Let us now prove Theorem 2.6. Estimate (2.62) can be derived by repeating the arguments used in the proof of (4.9). In fact, consider (2.30) and (2.52), take the difference, test it with an approximation of $2\partial_t(w_\mu - w)$, and integrate over $(0, t)$. Then, take the scalar product of the difference between (2.31) and (2.53) with $\exp(-2\sigma s)(\chi_\mu - \chi)(s)$, $s \in (0, t)$. Moreover, use (2.32), (2.54), and (2.20). The modification with respect to the procedure followed in the previous sections concerns the term coming from $\mu\partial_t\chi_\mu$, which can be handled on the right hand side this way

$$
\begin{aligned}
&-\mu\int_0^t \langle \overline{\partial_t\chi_\mu}(s), (\overline{\chi_\mu} - \bar{\chi})(s)\rangle ds \\
&\qquad \leq \frac{\ell}{4}\int_0^t \|(\bar{\chi}_\mu - \bar{\chi})(s)\|_H^2 ds + \frac{\mu^2}{\ell}\int_0^t \|\partial_t\chi_\mu(s)\|_H^2 ds. \qquad (5.1)
\end{aligned}
$$

Now, the first integral on the right hand side is controlled by the left hand side of the formula corresponding to (4.6), while the last term of (5.1) gives a further contribution. Moreover, we notice that, contrary to (4.9), the initial value $(\chi_\mu - \chi)(0)$ does not provide any contribution to the right hand side. Summing up, we obtain

$$
\begin{aligned}
&\|w_\mu - w\|^2_{C^1([0,T];H)\cap C^0([0,T];V)} + \|\chi_\mu - \chi\|^2_{L^2(0,T;V)} \leq c\mu^2\|\partial_t\chi_\mu\|^2_{L^2(0,T;H)} \\
&\qquad + c(\|e_{0,\mu} - e_0\|_H^2 + \|f_{\mu,1} - f_1\|^2_{L^1(0,T;H)} + \|f_{\mu,2} - f_2\|^2_{W^{1,1}(0,T;V')}) \qquad (5.2)
\end{aligned}
$$

from which (2.62) is easily achieved taking (2.61) and (2.64) into account.

Finally, we show (2.63). As before, we test the difference between (2.31) and (2.53) by $\exp(-2\sigma s)(\chi_\mu - \chi)(s)$, but we do not integrate the resulting equality with respect to s. Then, thanks to (2.71–72), we simply have

$$
\begin{aligned}
&\nu|(\overline{\chi_\mu} - \bar{\chi})(s)|_V^2 + \frac{\ell}{2}\|(\overline{\chi_\mu} - \bar{\chi})(s)\|_H^2 \\
&\qquad \leq \frac{\mu^2}{\ell}\|\partial_t\chi_\mu(s)\|_H^2 + \frac{1}{\ell}(1 + \|\Phi\|_{L^1(0,T)})^2\|w_\mu - w\|^2_{C^1([0,T];H)} \\
&\qquad + \|\overline{\Psi}\|_{L^1(0,T)}\|\overline{\chi_\mu} - \bar{\chi}\|^2_{C^0([0,T];H)} \quad \text{for a.a. } s \in (0, T). \qquad (5.3)
\end{aligned}
$$

Taking the essential supremum and fixing σ such that $|\overline{\Psi}\|_{L^1(0,T)} \leq \ell/4$, from (5.3) and (5.2) we infer that

$$
\begin{aligned}
&\|\chi_\mu - \chi\|^2_{C^0([0,T];V)} \leq c\mu^2\|\partial_t\chi_\mu\|^2_{L^2(0,T;H)} \\
&\qquad + c(\|e_{0,\mu} - e_0\|_H^2 + \|f_{\mu,1} - f_1\|^2_{L^1(0,T;H)} + \|f_{\mu,2} - f_2\|^2_{W^{1,1}(0,T;V')})
\end{aligned}
$$

for $\chi_\mu - \chi$ is continuous from $[0, T]$ to V. Therefore, since $\mu\|\partial_t\chi_\mu\|^2_{L^\infty(0,T;H)}$ is uniformly bounded thanks to (2.61), it turns out that (2.63) and Theorem 2.6 are completely proved.

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