HIROSHIMA MATH. J.
29 (1999), 89–115

On the duality mapping of L^∞ spaces

Joseph Jude PEIRIS

(Received January 20, 1998)

ABSTRACT. Measure theoretic characterization of the duality mappings of the space $L^\infty(\Omega)$ and its product spaces $L^\infty(\Omega)^n$ is investigated. The duality mapping of $X = L^\infty(\Omega)$ (resp. $L^\infty(\Omega)^n$) is a multi-valued mapping from X into its dual space $X^* = ba(\Omega)$ (resp. $ba(\Omega)^n$) which assigns to each $v \in X$ a weakly-star compact convex subset of X^* defined by $F(v) = \{f \in X^* : \langle v, f\rangle = \|v\|^2 = \|f\|^2\}$. The structure of the values $F(v)$ is discussed in terms of their extremal points. The extremal points are characterized by means of the Jordan decomposition, Yosida-Hewitt decomposition and the use of 0-1 measures in $ba(\Omega)$. The structure theorems are obtained through the full application of Banach lattice theory, duality theory for general Banach spaces and Yosida-Hewitt theory. It is also shown that the results presented in this paper are applicable to the study of the dissipativity of quasilinear diffusion operators in function spaces of L^∞ type.

0. Introduction

The purpose of this paper is to discuss a measure theoretic characterization of the duality mapping of the Lebesgue space $L^\infty(\Omega)$ of essentially bounded measurable functions over an open domain Ω in $\mathbf{R}^d$. Although the Banach space $L^\infty(\Omega)$ is neither seperable nor reflexive, it is significant to study its precise geometric structure and topological properties. First it is a Banach lattice with respect to the natural ordering. Second it is the dual space of a separable Banach space $L^1(\Omega)$. Third its dual space $ba(\Omega)$ is an extremely large space of finitely additive bounded measures on the Lebesgue class $\mathscr{M}$ in Ω. It is therefore significant to study the precise geometric structure and topological properties of the spaces $L^\infty(\Omega)$ and $ba(\Omega)$. Our approach to these "bad" Banach spaces is based on the effective use of the duality mapping of $L^\infty(\Omega)$ and it is expected that the results obtained in this paper will be applied to general problems arised in nonlinear functional analysis as well as systems of nonlinear partial differential equations which can be naturally formulated in these important spaces.

The duality mapping of a Banach spaces X is a possibly multivalued

---

1991 *Mathematics Subject Classification*: 35K50
Key words and phrases: L^∞ spaces, $ba(\Omega)$, duality map, 0-1 measure, purely finitely additive measure

mapping from X into its dual space X^* which assigns to each v in X a subset of X^* defined by

$$F(v) = \{f \in X^* : \langle v, f \rangle = \|v\|^2 = \|f\|^2\},$$

where $\langle v, f \rangle$ stands for the value of $f \in X^*$ at the point $v \in X$. The mapping F is well-defined on all of X by the Hahn-Banach theorem and it is well-known ([1], [2], [21]) that $F(v)$ is weakly-star compact convex in X^* for each $v \in X$, and that F is weakly-star demi-closed in the sense that if v_n converges strongly to v in X, $f_n \in F(v_n)$ and f is a weak-star cluster point of the sequence $(f_n : n \uparrow \infty)$ then $f \in F(v)$. As treated in Diestel's book, the duality mappings not only play an crucial role in studying the geometry of Banach spaces, but also they are powerful tools to treat significant classes of nonlinear operators formulated in general Banach spaces.

In this paper we investigate the structure and detailed properties of the duality mapping F of the Lebesgue space $L^\infty(\Omega)$ under the assumption that Ω is a bounded domain in $\mathbf{R}^d$ such that the Lebesgue measure of the boundary $\partial\Omega$ is zero. This assumption is essential in connection with the representation theorems for the space $C(\overline{\Omega})$ of continuous functions over the closed domain $\overline{\Omega}$ in $\mathbf{R}^d$ and the fine properties of finitely additive bounded measures in $ba(\Omega)$. These problems are arised both in the investigations of quasilinear differential operators and in the study of weak-star derivatives of strongly absolutely continuous functions which take their values in the dual Banach space $L^\infty(\Omega)$. The results obtained in this paper suggest not only intrinsic properties possessed by the dual mappings of general non reflexive Banach spaces but also interesting applications to systems of nonlinear partial differential equations such as quasilinear reaction-diffusion systems.

Our work is mainly devoted to three problems. The first aim is to characterize the structure of the values $F(v), v \in L^\infty$, by means of 0-1 measures on the Lebesgue class $\mathscr{M}$ in Ω. The second purpose is to treat the duality mappings of product spaces $L^\infty(\Omega)^n$ equipped with the maximum norm. Thirdly we make an attempt to illustrate the use of such duality mappings by considering certain linear and quasilinear differential operators. In fact, our results will be applied to a broad class of reaction-diffusion systems formulated in a product space $L^\infty(\Omega)^n$ subject to natural boundary conditions in the forth coming paper [18]. Since the dual space $L^\infty(\Omega)^*$ is identified with the space $ba(\Omega)$ of finitely additive bounded measures on $\mathscr{M}$ which vanish on the sets of Lebesgue measure zero, it is necessary to apply fully the integration theory with respect to finitely additive measures and characterize the structure of the values $F(v)$, $v \in L^\infty(\Omega)$, in terms of finitely additive measure theory.

Three means are employed to treat the dual space $ba(\Omega)$ and investigate the duality mapping F of $L^\infty(\Omega)$. Since $ba(\Omega)$ is also a Banach lattice, we fully apply the lattice structure. The first means is the Jordan decomposition of measures in $ba(\Omega)$. A measure λ in $F(v)$ is represented as $\|\lambda\|\lambda = \|\lambda^+\|v^+ - \|\lambda^-\|v^-$, where $\lambda = \lambda^+ - \lambda^-$ is the Jordan decomposition of λ and v^+, $v^- \in ba(\Omega)^+$ are such that $v = v^+ - v^-$ in $L^\infty(\Omega)$, $v^+ \in F(v^+)$ and $v^- \in F(v^-)$, respectively. Hence our problem is reduced to the consideration of the value of F for nonnegative elements $v \in L^\infty(\Omega)^+$. The second means is the Hewitt-Yosida decomposition theorem which states that any measure $\lambda \in ba(\Omega)$ is decomposed as the sum of a countably additive measure λ_c and a purely finitely additive measure λ_p. By means of this decomposition, detailed properties of elements in $F(v)$ can be discussed along with various types of measures, their total variations and scalar products $\langle v, \lambda \rangle$. The third means is the use of 0-1 measures. A 0-1 measure (resp. 0-(-1) measure) is a fundamental type of measure which assumes only two values 0 and 1 (resp. -1). The extremal points of the weakly-star compact and convex set $F(v)$ are characterized in terms of 0-1 (resp. 0-(-1)) measures and the set $\operatorname{ext} F(v)$ of extremal points is represented by means of $\operatorname{ext} F(v^+)$ and $\operatorname{ext} F(v^-)$. In consequence, the structure of $F(v)$ is completely determined through the celebrated Krein-Milman theorem by 0-1 and 0-(-1) measures belonging to $F(v)$.

In the following sections it is shown that the above-mentioned fine properties are particularly useful in dealing with various nonlinear operators in $L^\infty(\Omega)$ and studying the geometry of the dual space $ba(\Omega)$. It is, however, interesting to note that there are substantial differences between the case of $L^\infty(\Omega)$ and the L^∞-space $l^\infty \equiv L^\infty(\mathbf{N})$ over an atomic measure space. In the case of l^∞, the Yosida-Hewitt decomposition is equivalent to the Dixmier decomposition, since $ba(\mathbf{N})$ is regarded as the third dual of the space c_0 of sequences converging to 0. Therefore it is seen that the countably additive parts λ_c correspond to elements in l^1 of absolutely summable sequences and purely finitely additive parts λ_p are understood to be annihilators of the closed subspace c_0 of l^∞. However, in the case of $L^\infty(\Omega)$, it is shown that any 0-1 measure is purely finitely additive. This is a contrast to the fact that a 0-1 measure λ in $ba(\mathbf{N})$ is countably additive if it is regarded as an element of l^1. The unit surface $S^*(0,1)$ of the dual space $ba(\Omega)$ is much more complicated than that of $ba(\mathbf{N})$, since the predual $L^1(\Omega)$ does not have the Radon-Nikodym property. In order to overcome this difficulty we connect the dual space $rca(\overline{\Omega})$ of $C(\overline{\Omega})$ to $ba(\Omega)$ via the Hahn-Banach theorem and interpret 0-1 measures in $ba(\Omega)$ as "point masses" in a generalized sense.

Furthermore, in this paper, we make an attempt to characterize the duality mappings of product L^∞ spaces. This can be done by applying the results obtained for the case of the duality mapping of $L^\infty(\Omega)$ and the extremal points of the values of the duality mapping of $L^\infty(\Omega)^n$ are completely represented as n-dimensional vectors of 0-1 measures in $ba(\Omega)$. These characterization theorems are useful for treating systems of nonlinear partial differential equations as shown in the subsequent sections.

Section 1 contains some basic facts on the dual space $ba(\Omega)$ which is a Dedekind complete Banach space. In section 2 the duality mapping F and normalized duality mapping F_0 of $L^\infty(\Omega)$ are introduced. We here review basic facts on the Jordan decomposition of measures in $ba(\Omega)$ and briefly review the Hewitt-Yosida theory. In section 3 we discuss 0-1 measures in $rca(\overline{\Omega})$ and $ba(\Omega)$ and give fundamental results which play an important role in the subsequent sections. It is shown here that any 0-1 measure is purely finitely additive. Section 4 concerns the geometric structure of the normalized duality mapping F_0 of $L^\infty(\Omega)$. Here extremal points of the values $F_0(v)$ are characterized in terms of the Jordan decomposition and 0-1 measure. Moreover, in this section, we exhibit how the characterization theorems may be applied to quasilinear partial differential operators. Section 5 deals with the duality mapping F of the product space $L^\infty(\Omega)^n$. We here give characterization theorems for the values $\boldsymbol{F}(\boldsymbol{v}), \boldsymbol{v} = (v_1, \dots, v_n) \in L^\infty(\Omega)$ and $\mathrm{ext}\,\boldsymbol{F}(\boldsymbol{v})$ in terms of $F(v_i)$ and $\mathrm{ext}\,F(v_i)$. Finally, we treat a typical quasilinear differential operators in $L^\infty(\Omega)^n$ by applying the characterization theorems.

1. The dual space of $L^\infty(\Omega)$

Let Ω be an open bounded set in $\mathbf{R}^d$ such that $\partial\Omega$ is Lebesgue measure zero, namely $m(\partial\Omega) = 0$. It should be note that here the above assumption allows us to identify $L^\infty(\Omega)$ with $L^\infty(\overline{\Omega})$. Let $\mathcal{M}$ be the class of all Lebesgue measurable subsets of Ω. We denote by $ba(\Omega)$ the set of all finitely additive bounded measures on $\mathcal{M}$ which vanish on sets of Lebesgue measure zero. In this section we focus our attention on the Lebesgue measure space $(\Omega, \mathcal{M}, m)$ and outline the main points of the duality theory for the L^∞ space over $(\Omega, \mathcal{M}, m)$.

1.1. Properties of $ba(\Omega)$

Basic to the duality theory for L^∞ spaces is the following representation theorem:

Representation Theorem for $L^\infty(\Omega)$ [2, p. 296]. *There is an isometric isomorphism between $L^\infty(\Omega)^*$ and $ba(\Omega)$ such that the corresponding elements $\phi \in L^\infty(\Omega)^*$ and $\lambda \in ba(\Omega)$ satisfy the identity*

$$\langle \phi, u \rangle = \int_\Omega u(t)\lambda(dt), \quad \textit{for } u \in L^\infty(\Omega),$$

where the right hand side is Radon's integral.

In this subsection we first investigate that the space $ba(\Omega)$ is a Dedekind complete Banach lattice and then apply the Hewitt-Yosida theorem to $ba(\Omega)$.

The space $ba(\Omega)$ is a linear space over $\mathbf{R}$ in the sense that $(\alpha\phi)(E) = \alpha\phi(E)$ and $(\lambda+\mu)(E) = \lambda(E)+\mu(E)$ for $\lambda, \mu \in ba(\Omega)$, $\alpha \in \mathbf{R}$ and $E \in \mathscr{M}$. Also, $ba(\Omega)$ is partially ordered in the sense that $\lambda \geq \mu$ if and only if $(\lambda-\mu)(E) \geq 0$ for any $E \in \mathscr{M}$. Given $\lambda, \mu \in ba(\Omega)$ the join and the meet of λ and μ are defined by

$$(\lambda \wedge \mu)(E) = \inf\{\lambda(T) + \mu(E \backslash T) : T \subset E \text{ and } T \in \mathscr{M}\},$$

$$(\lambda \vee \mu)(E) = -((-\lambda) \wedge (-\mu))(E).$$

These elements are well-defined in $ba(\Omega)$ and it is shown in [4, p. 7] and [2, p. 162] that the system $(ba(\Omega), \wedge, \vee)$ is a Dedekind complete lattice as stated below:

PROPOSITION 1.1.1. (a) *For* $u, v, w \in ba(\Omega)$, $(u \vee v) \wedge w = (u \wedge w) \vee (v \wedge w)$ *and* $(u \wedge v) \vee w = (u \vee w) \wedge (v \vee w)$. (b) *Every subset of the partially ordered collection of additive scalar valued set functions on a field which has a upper bound (lower bound) has a least upper bound (greatest lower bound).*

We next define the positive part, negative part and the absolute value of an element λ in $ba(\Omega)$ by

$$\lambda^+ = \lambda \vee 0,\ \lambda^- = (-\lambda) \vee 0 \quad \text{and} \quad |\lambda| = \lambda \vee (-\lambda),$$

respectively. Then for $\lambda \in ba(\Omega)$,

$$\lambda = \lambda^+ - \lambda^-,\ \lambda^+ \wedge \lambda^- = 0,\ |\lambda| = \lambda^+ + \lambda^-. \tag{1.1.1}$$

The first relation in (1.1.1) is nothing but the Jordan decomposition of λ. Hence λ^+ and λ^- are the positive and negative variation of λ, respectively. For $E \in \mathscr{M}$, the total variation of $v(\lambda, E)$ of λ on E is defined by $v(\lambda, E) = \lambda^+(E) + \lambda^-(E)$. Moreover, the total variation $v(\lambda, \Omega)$ is defined to be the norm $\|\lambda\|$ of $\lambda \in ba(\Omega)$. Clearly, $\lambda, \mu \in ba(\Omega)$ and $|\lambda| \leq |\mu|$ imply $\|\lambda\| \leq \|\mu\|$.

Combining these facts, one can formulate the following

THEOREM 1.1.2. *The space* $(ba(\Omega), \|\cdot\|, \wedge, \vee)$ *forms a Dedekind complete Banach lattice.*

Finally, the set $\{\lambda : \lambda \in ba(\Omega) \text{ and } \lambda \geq 0\}$ is called the *positive cone* of $ba(\Omega)$ and denoted by $ba(\Omega)^+$.

1.2. Decomposition of measures in $ba(\Omega)$

In the space $ba(\Omega)$ there are two kinds of measures. A measure $\lambda \in ba(\Omega)$ is said to be *countably additive* if any pairwise disjoint sequence of sets $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{M}$ the relation $\lambda(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \lambda(A_n)$ holds. On the other hand, $\lambda \in ba(\Omega)$ is said to be *purely finitely additive*, if any countably additive measure $\sigma \in ba(\Omega)$ satisfying $0 \leq \sigma \leq v(\lambda, \cdot)$ is identically zero.

Henceforth we write c.a. measure for countably additive measure and p.f.a. measure for purely finitely additive measure. On c.a. and p.f.a. measures in $ba(\Omega)$, some of the basic facts are listed in the following

PROPOSITION 1.2.1. *Let μ_1, μ_2, $\lambda \in ba(\Omega)$ and let $\lambda = \lambda^+ - \lambda^-$ be the Jordan decomposition of λ. Then:*
(a) *If μ_1 and μ_2 are c.a. and $\mu_1 \leq \lambda \leq \mu_2$, then so is λ.*
(b) *If μ_1 and μ_2 are both c.a. (resp. p.f.a.), then the following elements are all c.a. (resp. p.f.a.) $\mu_1 + \mu_2$, $\alpha\mu_1$, $\mu_1 \vee \mu_2$, and $\mu_1 \wedge \mu_2$*
(c) *If λ is p.f.a. then both λ^+ and λ^- are p.f.a.*

For the proof we refer to Hewitt-Yosida [20]. We now state the Hewitt-Yosida decomposition theorem which plays a central role in our argument.

THEOREM 1.2.2. *Any $\mu \in ba(\Omega)$ is uniquely decomposed as the sum of a c.a. measure μ_c and a p.f.a. measure μ_p in the sense that $\mu = \mu_c + \mu_p$.*

1.3. Decomposition of Variation of λ in $ba(\Omega)$

In this section we give a decomposition theorem for the total variations of elements $\lambda \in ba(\Omega)$ by means of the Hewitt-Yosida decomposition. To this end we need the following lemma.

LEMMA 1.3.1. *If $\lambda \in ba(\Omega)$ is represented as $\lambda = \lambda_1 - \lambda_2$ for some $\lambda_1, \lambda_2 \in ba(\Omega)^+$ satisfying $\lambda_1 \wedge \lambda_2 = 0$, then $\lambda^+ = \lambda_1$ and $\lambda^- = \lambda_2$.*

PROOF. It is known ([4]) that the relation $\lambda + v = \lambda \wedge v + \lambda \vee v$ holds for any $\lambda, v \in ba(\Omega)$. This fact implies $\lambda^+ = \lambda \vee 0 = (\lambda_1 - \lambda_2) \vee 0 = (\lambda_1 \vee \lambda_2) - \lambda_2 = \lambda_1 - (\lambda_1 \wedge \lambda_2) = \lambda_1$. Hence $\lambda_1 = \lambda^+$. Similarly, we have $\lambda_2 = \lambda^-$. □

The aimed decomposition theorem is stated as follows:

THEOREM 1.3.2. *Let $\lambda \in ba(\Omega)$ and let $\lambda = \lambda_c + \lambda_p$ be its Yosida-Hewitt decomposition. Then $\|\lambda\| = \|\lambda_c\| + \|\lambda_p\|$.*

PROOF. The proof is given in the same way as in [10, p. 75]. Let $\lambda \in ba(\Omega)$ and let $\lambda = \lambda^+ - \lambda^-$. Applying Theorem 1.2.2. to λ^+ and λ^- respectively, we get the decompositions of λ^+ and λ^- as $\lambda^+ = \lambda_c^+ + \lambda_p^+$ and $\lambda^- = \lambda_c^- +$

λ_p^-. Hence we can write $\lambda = (\lambda_c^+ + \lambda_p^-) - (\lambda_c^- + \lambda_p^-)$. Rearranging the terms in the right-hand side, we get $\lambda = (\lambda_c^+ - \lambda_c^-) + (\lambda_p^+ - \lambda_p^-)$. Now we prove the terms $(\lambda_c^+ - \lambda_c^-)$ and $(\lambda_p^+ - \lambda_p^-)$ are Jordan decompositions of λ^+ and λ^-, respectively. Clearly $\lambda_c^+, \lambda_c^-, \lambda_p^+$ and λ_p^- are non negative. Since $\lambda_c^+ \le \lambda^+$ and $\lambda_c^- \le \lambda^-$, we have $0 \le \lambda_c^+ \wedge \lambda_c^- \le \lambda^+ \wedge \lambda^- = 0$. This implies $\lambda_c^+ \wedge \lambda_c^- = 0$. Similarly $\lambda_p^+ \wedge \lambda_p^- = 0$. Then by Lemma 1.3.1 we obtain the desired result. □

2. The duality mapping of $\mathbf{L}^\infty(\boldsymbol{\Omega})$

The so-called duality mapping of $L^\infty(\Omega)$ is a multi-valued mapping from $L^\infty(\Omega)$ into $ba(\Omega)$ which assigns to each $u \in L^\infty(\Omega)$ the subset $F(u)$ of $ba(\Omega)$ defined by $F(u) = \{\lambda \in ba(\Omega) : \langle u, \lambda \rangle = \|u\|^2 = \|\lambda\|^2\}$, where $\langle u, \lambda \rangle = \int_\Omega u(x)\lambda(dx)$.

Since $F(0) = \{0\}$ in $ba(\Omega)$, we mainly treat the *normalized duality mapping* F_0 of $L^\infty(\Omega)$ which is defined by $F_0(0) = S^*(0,1)$, the unit surface of $ba(\Omega)$, and

$$F_0(u) = \{\lambda \in ba(\Omega) : \langle u, \lambda \rangle = \|u\|, \|\lambda\| = 1\} \quad \text{for } u \neq 0.$$

The aim of this section is to discuss the precise structure of the values $F_0(u)$, $u \in L^\infty(\Omega)$, and topological properties of the normalized duality mapping F_0. To this end, we begin by treating the Jordan decomposition of the scalar product $\langle u, \lambda \rangle$.

2.1. Jordan decomposition of $\langle u, \lambda \rangle$

We here discuss the Jordan decompositions of scalar products $\langle u, \lambda \rangle$ for $u \in L^\infty(\Omega)$ and $\lambda \in ba(\Omega)$. Our first result in this section is the following.

THEOREM 2.1.1. *Let $u \in L^\infty(\Omega)\backslash\{0\}$ and $\lambda \in F_0(u)$. Let $u = u^+ - u^-$ and $\lambda = \lambda^+ - \lambda^-$. Then we have*

$$(2.1.1) \qquad \langle u\chi_E, \lambda \rangle = \langle u^+\chi_E, \lambda^+ \rangle + \langle u^-\chi_E, \lambda^- \rangle \quad \text{for } E \in \mathscr{M}.$$

If in particular, $E = \Omega$, then

$$(2.1.2) \quad \langle u^+, \lambda^+ \rangle = \|u^+\|\,\|\lambda^+\| = \|u\|\,\|\lambda^+\|, \langle u^-, \lambda^- \rangle = \|u^-\|\,\|\lambda^-\| = \|u\|\,\|\lambda^-\|.$$

Moreover, if $\|u^\pm\| < \|u\|$ then $\lambda^\pm = 0$, respectively.

PROOF. We first prove the following identity.

$$(2.1.3) \qquad \langle u\chi_E, \lambda \rangle = \langle |u|\chi_E, v(\lambda, \cdot) \rangle = \|u\| v(\lambda, E), \quad \text{for any } E \in \mathscr{M}.$$

In view of the definition of F_0, we can write $\|u\| = \langle u, \lambda\rangle = \langle u\chi_E, \lambda\rangle + \langle u\chi_{\Omega\backslash E}, \lambda\rangle$. Hence $\langle u\chi_E, \lambda\rangle \le \langle |u|\chi_E, v(\lambda, \cdot)\rangle \le \langle \|u\|\chi_E, v(\lambda, \cdot)\rangle$ and $\langle u\chi_{\Omega\backslash E}, \lambda\rangle \le \langle |u|\chi_{\Omega\backslash E}, v(\lambda, \cdot)\rangle \le \langle \|u\|\chi_{\Omega\backslash E}, v(\lambda, \cdot)\rangle$. Since $\|\lambda\| = 1$, these estimates together imply

$$
\begin{aligned}
\|u\| &= \langle u, \lambda\rangle = \langle u\chi_E, \lambda\rangle + \langle u\chi_{\Omega\backslash E}, \lambda\rangle \\
&\le \langle |u|\chi_E, v(\lambda, \cdot)\rangle + \langle |u|\chi_{\Omega\backslash E}, v(\lambda, \cdot)\rangle \le \langle \|u\|\chi_E, v(\lambda, \cdot)\rangle + \langle \|u\|\chi_{\Omega\backslash E}, v(\lambda, \cdot)\rangle \\
&= \|u\|v(\lambda, E) + \|u\|v(\lambda, \Omega\backslash E) = \|u\|v(\lambda, \Omega) \\
&= \|u\|\,\|\lambda\| = \|u\|.
\end{aligned}
$$

Comparing the corresponding terms in this estimate, we obtain the desired identity (2.1.3). Since $u = u^+ - u^-$ and $\lambda = \lambda^+ - \lambda^-$, we have

$$
\langle u\chi_E, \lambda\rangle = \langle u^+\chi_E, \lambda^+\rangle - \langle u^+\chi_E, \lambda^-\rangle - \langle u^-\chi_E, \lambda^+\rangle + \langle u^-\chi_E, \lambda^-\rangle \tag{2.1.4}
$$

But from (2.1.3) we see that

$$
\begin{aligned}
\langle u\chi_E, \lambda\rangle &= \langle |u|\chi_E, v(\lambda, \cdot)\rangle \\
&= \langle (u^+ + u^-)\chi_E, \lambda^+ + \lambda^-\rangle \\
&= \langle u^+\chi_E, \lambda^+\rangle + \langle u^+\chi_E, \lambda^-\rangle + \langle u^-\chi_E, \lambda^+\rangle + \langle u^-\chi_E, \lambda^-\rangle.
\end{aligned}
$$

Comparing the right hand sides of these two relations yields $\langle u^+\chi_E, \lambda^-\rangle + \langle u^-\chi_E, \lambda^-\rangle = 0$. Hence from (2.1.4) we obtain the first assertion (2.1.1). Putting $E = \Omega$ in equation (2.1.1) gives $\langle u, \lambda\rangle = \|u\| = \langle u^+, \lambda^+\rangle + \langle u^-, \lambda^-\rangle$. Since $\langle u^+, \lambda^+\rangle \le \|u^+\|\,\|\lambda^+\|, \langle u^-, \lambda^-\rangle \le \|u^-\|\,\|\lambda^-\|$ and $\|\lambda^+\| + \|\lambda^-\| = \|\lambda\| = 1$, we have

$$
\begin{aligned}
\langle u, \lambda\rangle &= \|u\| = \langle u^+, \lambda^+\rangle + \langle u^-, \lambda^-\rangle \\
&\le \|u^+\|\,\|\lambda^+\| + \|u^-\|\,\|\lambda^-\| \le \|u\|\,\|\lambda^+\| + \|u\|\,\|\lambda^-\| \\
&\le \|u\|(\|\lambda^+\| + \|\lambda^-\|) \le \|u\|.
\end{aligned}
$$

Comparing the corresponding terms of this series of estimate we obtain (2.1.2). Finally, suppose $\|u^+\| < \|u\|$ and $\|\lambda^+\| > 0$. Then $\|u^+\|\,\|\lambda^+\| \le \|u\|\,\|\lambda^+\|$. But from (2.1.2) $\|u^+\|\,\|\lambda^+\| = \|u\|\,\|\lambda^+\|$, and so $\|u\|\,\|\lambda^+\| < \|u\|\,\|\lambda^+\|$. Since $\|u\| > 0$, this is a contradiction. Hence we obtain the last assertion. □

The following immediate consequence of Theorem 2.1.1 deduces a remarkable property of $F_0(u)$.

COROLLARY 2.1.2. *Let* $u \in L^\infty(\Omega)\backslash\{0\}$ *and let* $\lambda \in F_0(u)$. *If* $u \in L^\infty(\Omega)^+$, *then* $\lambda \geq 0$. *If* $-u \in L^\infty(\Omega)^+$, *then* $\lambda \leq 0$.

REMARK 2.1.3. This result states that the duality mapping $F_0(u)$ is order preserving in the sense that $u_2 - u_1 \in L^\infty(\Omega)^+$ implies $F_0(u_2 - u_1) \subseteq ba(\Omega)^+$.

We now state the main result of this subsection.

THEOREM 2.1.4. *Let* $u \in L^\infty(\Omega)\backslash\{0\}$, $\lambda \in F_0(u)$, and $\lambda = \lambda^+ - \lambda^-$. *Then* λ *is written as* $\lambda = \|\lambda^+\|v^+ - \|\lambda^-\|v^-$, *where* $v^+ \in F_0(u^+)$, $v^- \in F_0(u^-)$ *and* $\|\lambda^+\|v^+ \wedge \|\lambda^-\|v^- = 0$.

PROOF. First we prove the result for the case $\lambda^+ = 0$ and then discuss the general case. Since $\|\lambda\| = 1$, we have $\|\lambda^-\| = 1$ and $\langle u^-, \lambda^- \rangle = \|u^-\|$ by the second identity in (2.1.2) this shows that $\lambda^- \in F_0(u^-)$. Therefore, putting $v^- = \lambda^-$ and taking any element v^+ of $F_0(u^+)$, we obtain a desired representation for λ. Similarly, in the case of $\lambda^- = 0$, we obtain a desired representation by setting $v^+ = \lambda^+$ and choosing an arbitrary element v^- of $F_0(u^-)$. Now we assume both λ^+ and λ^- are non zero. Let $v^+ = \lambda^+/\|\lambda^+\|$ and $v^- = \lambda^-/\|\lambda^-\|$. From Theorem 2.1.1 we infer that $\langle u^\pm, v^\pm \rangle = \|u^\pm\|$ and $v^\pm \in F_0(u^\pm)$, respectively. Now $\|\lambda^+\|v^+ \wedge \|\lambda^-\|v^- = \|\lambda^+\|(\lambda^+/\|\lambda^+\|) \wedge \|\lambda^-\|(\lambda^-/\|\lambda^-\|) = \lambda^+ \wedge \lambda^- = 0$. This completes the proof. □

2.2. Hewitt-Yosida decomposition of $\langle u, \lambda \rangle$

In this subsection we give a decomposition theorem for the scalar product $\langle u, \lambda \rangle$.

PROPOSITION 2.2.1. *Let* $u \in L^\infty(\Omega)\backslash\{0\}$ *and,* $\lambda \in F_0(u)$ *and let* $\lambda = \lambda_c + \lambda_p$ *be the Hewitt-Yosida decomposition of* λ. *Then we have*

(a) $\|u\| = \langle u, \lambda_c \rangle + \langle u, \lambda_p \rangle$,

(b) $\langle u, \lambda_c \rangle = \langle |u|, v(\lambda_c, \cdot) \rangle = \|u\| \, \|\lambda_c\|$,

(c) $\langle u, \lambda_p \rangle = \langle |u|, v(\lambda_p, \cdot) \rangle = \|u\| \, \|\lambda_p\|$.

PROOF. Since, $\|\lambda_c\| + \|\lambda_p\| = \|\lambda\|$ by Theorem 1.3.2, we have $\|u\| = \langle u, \lambda \rangle = \langle u, \lambda_c + \lambda_p \rangle \leq \langle |u|, v(\lambda_c, \cdot) \rangle + \langle |u|, v(\lambda_p, \cdot) \rangle \leq \|u\| \, \|\lambda_c\| + \|u\| \, \|\lambda_p\| \leq \|u\|(\|\lambda_c\| + \|\lambda_p\|) \leq \|u\|$. Using the same idea as in the Theorem 2.1.1 the desired equalities are obtained by comparing the corresponding terms in the above series of estimates. □

3. 0-1 measures in $C(\bar{\Omega})^*$ and $L^\infty(\Omega)^*$

Our objective here is to show that 0-1 measures in $ba(\Omega)$ are all p.f.a.. To this end we consider the space $rca(\bar{\Omega})$ of regular countably additive set functions defined on $\bar{\Omega}$. We denote by $C(\bar{\Omega})$ the space of set of all continuous functions defined on $\bar{\Omega}$. Since $\bar{\Omega}$ is compact, it is known that there is an isometric isomorphism between $rca(\bar{\Omega})$ and the dual space of $C(\bar{\Omega})^*$ as in the case of $ba(\Omega)$.

Representation Theorem for $C(\bar{\Omega})^*$ ([2, p. 265]) *There exists an isometric isomorphism between $C(\bar{\Omega})^*$ and $rca(\bar{\Omega})$ such that the corresponding elements $\phi \in C(\bar{\Omega})^*$ and $\lambda \in rca(\bar{\Omega})$ satisfies the identity*

$$\langle \phi, u \rangle = \int_{\Omega} u(t)\lambda(dt) \quad \text{for } u \in C(\bar{\Omega}).$$

Through out this section we denote the class of all Borel measurable sets in $\bar{\Omega}$ by $\mathscr{B}(\bar{\Omega})$.

3.1. 0-1 measures in $rca(\bar{\Omega})$ and $ba(\bar{\Omega})$

In this section we introduce the notion of 0-1 measures on $\mathscr{M}$ and that of 0-1 measure on $\mathscr{B}(\Omega)$. In what follows, these 0-1 measures play an essential role.

Definition 3.1.1. Let (S, Σ, μ) be a measure space. A measure on Σ is said to be a 0-1 measure, if either $\lambda(E) = 1$ or $\lambda(E^c) = 1$ for any $E \in \Sigma$.

We generically denote by δ and ω 0-1 measures in $rca(\bar{\Omega})$ and $ba(\Omega)$, respectively. In order to discuss the 0-1 measures, we introduce two sets $\mathscr{B}(\delta)$ and $\mathscr{M}(\omega)$ defined by

$$\mathscr{B}(\delta) = \{E \in \mathscr{B}(\bar{\Omega}) : \delta(E) = 1\} \quad \text{and} \quad \mathscr{M}(\omega) = \{E \in \mathscr{M} : \omega(E) = 1\}.$$

We first need the following lemma.

Lemma 3.1.2. *The class $\mathscr{M}(\omega)$ and $\mathscr{B}(\delta)$ form the bases of ultrafilters in $\mathscr{M}$ and $\mathscr{B}(\bar{\Omega})$, respectively.*

Our first result is stated as follows:

Theorem 3.1.3 (a) *For any $\delta \in rca(\bar{\Omega})$ there is a unique point $a \in \bar{\Omega}$ such that for all neighbourhoods U of a, $U \cap \bar{\Omega} \in \mathscr{B}(\delta)$ and $\bigcap \mathscr{B}(\delta) = \{a\}$.* (b) *For any $\omega \in ba(\Omega)$ there exists a unique point $a \in \bar{\Omega}$ such that for any neighbourhood U of a, $U \cap \Omega \in \mathscr{M}(\omega)$, $\bigcap \overline{\mathscr{M}(\omega)} = \{a\}$,* and *$\bigcap \mathscr{M}(\omega) = \emptyset$, where $\overline{\mathscr{M}(\omega)} = \{\bar{A} : A \in \mathscr{M}(\omega)\}$.*

Proof. (a) Let δ be a 0-1 measure in $rca(\bar{\Omega})$. Since Ω is bounded in $\mathbf{R}^d$ there exists a closed cube C_1 which contains Ω. Hence $\delta(\bar{\Omega} \cap C_1) = \delta(\bar{\Omega}) = 1$. Divide C_1 into $2^d =$ numbers of closed cubes by using planes parallel to planes spanned by two coordinate axes, where the diameter of any cube is one half of the diameter of C_1. Since δ is 0-1 measure we should have one and only one cube, say C_2, which contains the support of δ, namely, $\delta(C_2 \cap \bar{\Omega}) = 1$ and $\mathrm{diam}(C_2) = \frac{1}{2}\mathrm{diam}(C_1)$. Now we divide C_2 into 2^d numbers of closed cubes in the same way as before and find a cube, say C_3, with $\mathrm{diam}(C_3) = \frac{1}{4}\mathrm{diam}(C_1)$ and $\delta(\bar{\Omega} \cap C_3) = 1$. Continuing this process one find a sequence of closed cubes $\{C_n\}_{n=1}^\infty$ such that $C_n \supseteq C_{n+1}$ and $\mathrm{diam}(C_n) = 2^{-n+1}\mathrm{diam}(C_1)$ for all n. By Baire's category theorem, $\bigcap_{n=1}^\infty C_n \neq \emptyset$ and $\bigcap C_n$ is a singleton set such that $\bigcap_{n=1}^\infty C_n = \{a\}$ for some $a \in \mathbf{R}^d$. Now we demonstrate that any neighbourhood of U of a, $U \cap \bar{\Omega}$ is in $\mathscr{B}(\delta)$. Suppose U is any open neighbourhood of a, then by the construction of cubes we should have at least one cube contained in U. That is, $C_n \cap \bar{\Omega} \subset U \cap \bar{\Omega}$ and $\delta(C_n \cap \bar{\Omega}) = 1$. Hence $\delta(U \cap \bar{\Omega}) = 1$. To show that $\bigcap \mathscr{B}(\delta) = \{a\}$, we proceed as follows: Let $\{U_n\}$ be a sequence of open neighbourhoods of a with $\bigcap_{n=1}^\infty U_n = \{a\}$. Then it follows from the previous result that $U_n \cap E \in \mathscr{B}(\delta)$ for any $E \in \mathscr{B}(\delta)$ and $n \geq 1$. Since δ is countable additive, $\delta(\{a\} \cap E) = \lim_{n\to\infty} \delta(U_n \cap E) = 1$, and so $a \in E$ and $\{a\} \in \mathscr{B}(\delta)$. Thus we have $\bigcap \mathscr{B}(\delta) = \{a\}$, this completes the proof of assertion (a). In the same way as in the proof of (a), we may prove the first half of assertion (b). So, it suffices to show that $\bigcap \mathscr{M}(\omega) = \emptyset$. Let $a \in \Omega$. Since the Lebesgue measure of a singleton set $\{a\}$ is zero, $\omega(\{a\}) = 0$. Therefore $\Omega \backslash \{a\} \in \mathscr{M}(\omega)$ for any $a \in \Omega$ and $\bigcap \mathscr{M}(\omega) \subset \bigcap_{a \in \Omega} (\Omega \backslash \{a\}) = \emptyset$. This completes the proof of (b). □

Definition 3.1.4. To any 0-1 measure $\delta \in rca(\bar{\Omega})$ or $\omega \in ba(\Omega)$ there corresponds a unique point $a \in \bar{\Omega}$. The singleton set $\{a\}$ is called the *essential support* of δ or ω.

We now give a remarkable result on p.f.a. measures in $ba(\Omega)$.

Theorem 3.1.5. *Any 0-1 measure in $ba(\Omega)$ is purely finitely additive.*

Proof. Let B_n denote a closed ball in $\mathbf{R}^d$ with center at origin and radius n, where n is a positive integer. Let ω be any 0-1 measure in $ba(\Omega)$ and $\sigma \in ba(\Omega)$ a countably additive measure in $ba(\Omega)$ such that $0 \leq \sigma \leq \omega$. Choose n_0 such that $\omega(\Omega \cap B_{n_0}) = 1$. The essential support of ω lies in $\bar{\Omega}$. Let a be the point determined by Theorem 3.1.3. Then $\omega(U \cap \Omega) = 1$ for any neighbourhood U of a. Referring to the proof of Theorem 3.1.3, we consider the closed cubes C_n, such that $\mathrm{diam}\ (C_n) = 2^{-n+1}\mathrm{diam}(B_{n_0})$, $\omega(C_n \cap \Omega) = 1$ and $\bigcap C_n = \{a\}$. Let $\Omega_n = C_n \cap \Omega$ and $N_n = \Omega \backslash \Omega_n$. Then $\omega(N_n) = \omega(\Omega) - \omega(\Omega_n) = 1 - 1 = 0$. Further $N_n \uparrow$ as $n \to \infty$ and $\bigcup N_n = \Omega \backslash \{a\}$. Since $0 \leq \sigma(N_n) \leq \omega(N_n) = 0$ for any n and σ is countably additive, we have $\sigma(\bigcup N_n) = \lim_{n\to\infty} \sigma(N_n) = 0$. Now $\Omega \backslash \{a\} \subset N_n \subset \Omega$ implies that $\sigma(\Omega) \geq \sigma(\bigcup N_n) \geq$

$\sigma(\Omega) - \sigma(\{a\})$. Since $\sigma(\{a\}) = 0$ this gives $\sigma(\Omega) \geq 0 \geq \sigma(\Omega)$. It follows that $\sigma(\Omega) = 0$. This means that ω is p.f.a. □

3.2. Hahn-Banach Extension

In the previous section we have discussed some typical properties of 0-1 measures in $ba(\Omega)$ and $rca(\bar{\Omega})$. In this section we make an attempt to treat 0-1 measures in the spaces via the Hahn-Banach Theorem. We first refer to the following theorem in [20] which is very useful for constructing new 0-1 measures.

THEOREM 3.2.1. *Let $\mathscr{E}$ be a subfamily of $\mathscr{M}$ which has the measure theoretic finite intersection property: For any finite family $E_1, E_2, \ldots .E_n$ of $\mathscr{E}$, the intersection is not Lebesgue measure zero. Then there exists a* 0-1 *measure ω on $\mathscr{M}$ such that $\omega(E) = 1$ for any $E \in \mathscr{E}$.*

THEOREM 3.2.2. *If $\bar{\Omega}$ is compact and ω is a 0-1 measure in $ba(\Omega)$, then $\omega|_{C(\bar{\Omega})} = \delta_a$, where a is the essential support of ω.*

PROOF. Let a be the point given by Theorem 3.1.3 and consider a decreasing sequence $\{C_n\}$ of closed cubes such that $\operatorname{diam}(C_n) = 2^{-n+1} \operatorname{diam}(B_{n_0})$, $\omega(C_n \cap \Omega) = 1$ and $\bigcap C_n = \{a\}$. Let $u \in C(\bar{\Omega})$. Then

$$\int_{\bar{\Omega}} u(s)\, d\omega(s) = \int_{C_n \cap \Omega} u(s)\, d\omega(s).$$

Since u is continuous over $\bar{\Omega}$ we have

$$\lim_{n \to \infty} \int_{C_n \cap \Omega} u(s)\, d\omega(s) = u(a).$$

This shows that

$$\omega|_{C(\bar{\Omega})} = \delta_a.$$ □

Of concerning above theorem we have following remark.

REMARK 3.2.3. It should be noted that the Hahn-Banach extension of any point mass δ_a on $C(\bar{\Omega})$ to $L^\infty(\Omega)$ is not always a 0-1 measure. Infact for $a \in \Omega$, Define two families $\mathscr{A}$ and $\mathscr{B}$ of open rectangles in $\mathbf{R}^d$ by

$$\mathscr{A} = \left\{ \left(a - \frac{1}{n}, a \right) : n \geq 1 \right\}, \quad \mathscr{B} = \left\{ \left(a, a + \frac{1}{n} \right) : n \geq 1 \right\},$$

where $\left(a - \frac{1}{n}, a \right) = \prod_{i=1}^{d} \left(a_i - \frac{1}{n}, a_i \right)$ and $\left(a, a + \frac{1}{n} \right) = \prod_{i=1}^{d} \left(a_i, a_i + \frac{1}{n} \right)$. Then define $\mathscr{E}_1$ and $\mathscr{E}_2$ as

$$\mathscr{E}_1 = \{\bar{\Omega} \cap A : A \in \mathscr{A}\}, \ \mathscr{E}_2 = \{\bar{\Omega} \cap B : B \in \mathscr{B}\}.$$

Then both $\mathscr{E}_i$ and $\mathscr{E}_2$ have the measure theoretic finite intersection property mentioned in Theorem 3.2.1. Therefore one finds ω_1 and $\omega_2 \in ba(\Omega)$ such that $\omega_1(E_1) = 1$ for any $E_1 \in \mathscr{E}_1$ and $\omega_2(E_2) = 1$ for $E_2 \in \mathscr{E}_2$. Further, it is obvious that $\omega_1 \neq \omega_2$ and $\omega_1|_{C(\bar{\Omega})} = \omega_2|_{C(\bar{\Omega})} = \delta_a$. Let $\alpha = 2^{-1}(\omega_1 + \omega_2)$. Then $\|\alpha\| = 1$ and $\alpha|_{C(\bar{\Omega})} = \delta_a$, but α is not a 0-1 measure. □

The following is already given in [20, p. 60], although it is a useful fact for the treatment of nonlinear operators in L^∞ spaces.

THEOREM 3.2.4. *Let $\omega \in ba(\Omega)$ be a 0-1 measure in $ba(\Omega)$. Then ω is multiplicative in the sense that*

$$\langle uv, \omega \rangle = \langle u, \omega \rangle \langle v, \omega \rangle \quad u, v \in L^\infty(\Omega).$$

4. Geometric structure of the duality mapping

In this section we discuss the precise structure of the values $F_0(u), u \in L^\infty$.

4.1. Extremal Points of $F_0(u)$

In this subsection we make an attempt to characterize the extremal points of $F_0(u)$ in terms of 0-1 measure and discuss the precise geometrical structure of the convex sets $F_0(u), u \in L^\infty(\Omega)$.

Our first result is the following characterization theorem for the extremal points of the values $F_0(u), u \in L^\infty(\Omega)$.

THEOREM 4.1.1. *Let $u \in L^\infty(\Omega)^+$ and $\lambda \in F_0(u)$. Then, λ is an extremal point of $F_0(u)$ if and only if λ is a 0-1 measure.*

PROOF. Suppose first that λ is a 0-1 measure. Let $\alpha, \beta > 0$, $\alpha + \beta = 1, \lambda_0, \lambda_1 \in F_0(u)$ and let $\lambda = \alpha\lambda_0 + \beta\lambda_1$. We note that $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$ by Corollary 2.1.2. Now let E be an arbitrary element of $\mathscr{M}$. If $\lambda(E) = 0$, then $\lambda_0(E) = \lambda_1(E) = 0$. Assume that $\lambda(E) = 1$. If $0 \leq \lambda_0(E) < 1$, then $\|\lambda_1\| \geq \lambda_1(E) = \beta^{-1}(1 - \alpha\lambda_0(E)) > \beta^{-1}(1 - \alpha) = 1$, which contradicts the fact that $\|\lambda_1\| = 1$. Hence $\lambda_0(E)$ must be 1 and $\lambda_1(E) = 1$ in a similar manner. This means that $\lambda = \lambda_0 = \lambda_1$ and so that λ is an extremal point.

Conversely, suppose that λ is an extremal point of $F_0(u)$. To the contrary, we assume that there is $E_0 \in \mathscr{M}$ such that $0 < \lambda(E_0) < 1$. Since $u \geq 0$, we infer from Corollary 2.1.2 that $\lambda \geq 0$. We then define two bounded additive set functions λ_1 and λ_2 on $\mathscr{M}$ by

$$\lambda_1(E) = \lambda(E \cap E_0) \quad \text{and} \quad \lambda_2(E) = \lambda(E \cap E_0^c),$$

where E is an arbitrary element in $\mathscr{M}$. Then $\lambda_1, \lambda_2 > 0$. Since $E = (E \cap E_0) \cup (E \cap E_0^c)$ for any set E in $\mathscr{M}$ we have $\lambda(E) = \lambda(E \cap E_0) + \lambda(E \cap E_0^c)$. That is $\lambda(E) = \lambda_1(E) + \lambda_2(E)$ for $E \in \mathscr{M}$. Since $\lambda \geq 0$, we have $\|\lambda\| = \|\lambda_1\| + \|\lambda_2\| = 1$. Set, $v_1 = \lambda_1/\|\lambda_1\|$ and $v_2 = \lambda_2/\|\lambda_2\|$. Then $v_1, v_2 > 0$, $\|v_1\| = \|v_2\| = 1$ and $\lambda = \|\lambda_1\|v_1 + \|\lambda_2\|v_2$. Since $\|u\| = \langle u, \lambda \rangle = \langle u, \lambda_1 + \lambda_2 \rangle = \langle u, \lambda_1 \rangle + \langle u, \lambda_2 \rangle \leq \|\lambda_1\| \, \|u\| + \|\lambda_2\| \, \|u\| \leq \|u\|$. Comparing the corresponding terms of the above estimates we get $\|\lambda_1\| \, \|u\| = \langle \lambda_1, u \rangle$ and $\|\lambda_2\| \, \|u\| = \langle \lambda_2, u \rangle$. Namely $\|u\| = \langle u, \lambda_1 \|\lambda_1\|^{-1} \rangle = \langle u, \lambda_2 \|\lambda_2\|^{-1} \rangle$. Hence $\|u\| = \langle u, v_i \rangle$, for $i = 1, 2$. This implies that $v_i \in F_0(u)$ for $i = 1, 2$. Since λ is an extremal point and $\lambda = \|\lambda_1\|v_1 + \|\lambda_2\|v_2$, we should have the relations $v_1 = v_2 = \lambda$. Therefore $0 < \lambda(E_0) = v_2(E_0)\|\lambda_2\|^{-1}\lambda(E_0 \cap E_0^C) = 0$, which is a contradiction. This implies that λ cannot take any values between 0 and 1 and so that λ is a 0-1 measure. □

The above theorem states that for $u \in L^\infty(\Omega)^+$, the extremal points of $F_0(u)$ consists of only 0-1 measures. In order to treat the general case $u \in L^\infty(\Omega) \setminus \{0\}$, we need a fundamental theorem in functional analysis.

4.2. Krein-Milman Theorem

The celebrated Krein-Milman theorem is fundamental in the subsequent discussions.

THEOREM 4.2.1 (KREIN-MILMAN). *If K is a compact subset of a locally convex linear topological space X and E the set of its extremal points, the closed convex hull $\overline{co}(E)$ of E contains K. If K is convex, $\overline{co}(E) = \overline{co}(K)$ and $\overline{co}(E) = K$.*

Given a subset K of $ba(\Omega) = L^\infty(\Omega)^*$, we denote by $\operatorname{ext} K$ the set of all extremal points of K.

THEOREM 4.2.2. *If $u \in L^\infty(\Omega)^+$, then $F_0(u)$ contains at least one 0-1 measure, and $F_0(u)$ is a weakly-star convex hull of the 0-1 measures in $F_0(u)$.*

PROOF. Since $F_0(u)$ is a convex and weakly-star compact subset of $ba(\Omega)$, it is a weakly-star convex hull of $\operatorname{ext} F_0(u)$ by Krein-Milman's theorem. Theorem 4.1.1 then implies that $\operatorname{ext} F_0(u)$ the consist of 0-1 measures. This completes the proof. □

Now we prove the result for general case. Let $u \in L^\infty(\Omega)$, $u = u^+ - u^-$, and assume that $\|u^+\| > 0$ and $\|u^-\| > 0$. Moreover, let $E_0^+ = \{t : u(t) > 0\}$, $E_0^- = \{t : u(t) < 0\}$, $E^+ = \{t : u(t) \geq 0\}$ and $E^- = \{t : u(t) \leq 0\}$. Clearly, E_0^+ and E_0^- are disjoint. Employing these sets, we have the following lemma.

LEMMA 4.2.3. *If* $v^+ \in F_0(u^+)$ *and* $v^- \in F_0(u^-)$, *then* $v^+(E_0^+) = v^-(E_0^-) = 1$ *and* $v^+(E^-) = v^-(E^+) = 0$.

PROOF. Let $\phi^+ \in \operatorname{ext} F_0(u^+)$ and $\phi^- \in \operatorname{ext} F_0(u^-)$. We first show that $\phi^+(E_0^+) = \phi^-(E_0^-) = 1$. Suppose $\phi^+(E_0^+) = 0$. Then $\|u^+\| = \langle u^+, \phi^+\rangle = \langle u^+\chi_{E_0^+}, \phi^+\rangle = 0$. This is a contradiction. Similarly, it is impossible to assume that $\phi^-(E_0^-) = 0$. It follows that $\phi^+(E^+) = \phi^-(E^-) = 1$ and $\phi^+(E^-) = \phi^-(E^+) = 0$. Now let $v^+ \in F_0(u^+)$ and $v^- \in F_0(u^-)$. Then, by Theorem 4.2.3, there exist generalized sequences $\{\phi_\alpha^+\}$ and $\{\phi_\beta^-\}$ such that $\phi_\alpha^+ \in \operatorname{co}[\operatorname{ext} F_0(u^+)]$, $\phi_\beta^- \in \operatorname{co}[\operatorname{ext} F_0(u^-)]$ and $\{\phi_\alpha^+\}$ and $\{\phi_\beta^-\}$ converge, respectively, to v^+ and v^- in the weak-star topology of $ba(\Omega)$. Hence we have $\langle \chi_{E_0^+}, \phi_\alpha^+\rangle = \phi_\alpha^+(E_0^+) = 1, \langle \chi_{E_0^-}, \phi_\beta^-\rangle = \phi_\beta^+(E_0^-) = 1$, and consequently, $v^+(E^+) = \langle \chi_{E_0^+}, v^+\rangle = \lim_\alpha \langle \chi_{E_0^+}, \phi_\alpha^+\rangle = 1$ and $v^-(E_0^-) = \lim_\beta \phi_\beta^-(E_0^-) = 1$. Thus, the first assertion is obtained. The last assertion follows from the additivity of v^+, v^- and the fact that $v^+(\Omega) = v^-(\Omega) = 1$. □

PROPOSITION 4.2.4. *Let* $u \in L^\infty(\Omega)$, $u^+ \neq 0$ *and* $u^- \neq 0$. *If* $v^+ \in F_0(u^+)$ *and* $v^- \in F_0(u^-)$, *then* $v^+ \wedge v^- = 0$ *and* $\langle u^+, v^-\rangle = \langle u^-, v^+\rangle = 0$.

PROOF. Let $E \in \mathscr{M}$. Then by the definition of the meet $v^+ \wedge v^-$ and Lemma 4.2.3, we have

$$(v^+ \wedge v^-)(E) = \inf_{T \subset E, T \in \mathscr{M}} \{v^+(T) + v^-(E \cap T^c)\}$$

$$= \inf_{T \subset E, T \in \mathscr{M}} \{v^+(T \cap E_0^+) + v^-(T^c \cap E \cap E_0^-)\}.$$

But the right side turns out to be 0 if we take $T = E \cap E_0^-$. Thus the first assertion is obtained. The last assertion follows from Lemma 4.2.3 and the relations

$$\langle u^+, v^-\rangle = \langle u^+\chi_{E^-}, v^-\rangle = 0 \tag{4.2.1}$$

and (4.2.1) with u^+ and v^- replaced respectively by u^- and v^+. □

Using the result obtained above, we obtain a converse of Theorem 2.1.4.

THEOREM 4.2.5. *Let* $u \in L^\infty(\Omega)\backslash\{0\}$, $u = u^+ - u^-$, $v^+ \in F_0(u^+)$, *and* $v^- \in F_0(u^-)$. *Let* α, β *be any nonnegative numbers satisfying* $\alpha + \beta = 1$ *and* $\alpha\|u^+\| + \beta\|u^-\| = \|u\|$, *and define* $\lambda = \alpha v^+ - \beta v^-$. *Then* $\lambda \in F_0(u)$ *and, in this case,* $\lambda^+ = \alpha v^+$ *and* $\lambda^- = \beta v^-$.

PROOF. Let $v^+ \in F_0(u^+)$ and $v^- \in F_0(u^-)$. Then $v^+ \wedge v^- = 0$ by Proposition 4.2.4. We here observe that if $\lambda, v \in ba(\Omega)^+$ and $\lambda \wedge v = 0$, then $\alpha\lambda \wedge \beta v = 0$ for $\alpha, \beta \geq 0$. If we define $\lambda = \alpha v^+ - \beta v^-$ then by Lemma 1.3.1

$\alpha v^+ - \beta v^-$ gives the Jordan decomposition of λ, namely, $\lambda^+ = \alpha v^+$ and $\lambda^- = \beta v^-$. Hence $\|\lambda\| = \alpha\|v^+\| + \beta\|v^-\| = 1$. On the other hand, we see from Proposition 4.2.4 and the restriction on α, β that $\langle u, \lambda \rangle = \alpha\langle u^+, v^+ \rangle + \beta\langle u^-, v^- \rangle = \|u\|$. This shows that $\lambda \in F_0(u)$, and the proof is complete. □

Combining Theorem 4.2.5 with Theorem 2.1.4, we obtain the main result of this section.

THEOREM 4.2.6. *For* $u \in L^\infty(\Omega)\backslash\{0\}$, *we have*

$$F_0(u) = \bigcup_{\alpha,\beta}[\alpha F_0(u^+) + \beta F_0(-u^-)], \tag{4.2.2}$$

where the union is taken over all $\alpha, \beta \geq 0$ *satisfying* $\alpha + \beta = 1$ *and* $\alpha\|u^+\| + \beta\|u^-\| = \|u\|$. *Therefore we have:*

(i) *If* $\|u^-\| < \|u\|$ *then* $F_0(u) = F_0(u^+)$.
(ii) *If* $\|u^+\| < \|u\|$ *then* $F_0(u) = F_0(-u^-)$.
(iii) *If* $\|u^+\| = \|u^-\| = \|u\|$, *then* $F_0(u) = \mathrm{co}[F_0(u^+) \cup F_0(-u^-)]$ *and* $\mathrm{ext}\, F_0(u) = \mathrm{ext}\, F_0(u^+) \cup \mathrm{ext}\, F_0(-u^-)$.

PROOF. Theorem 2.1.4 states that any element λ of $F_0(u)$ lies in the set $\|\lambda^+\|F_0(u^+) + \|\lambda^-\|F_0(-u^-)$, and so $F_0(u)$ is contained in the right-hand side of (4.2.2). The converse inclusion follows from Theorem 4.2.5. We now prove (i) through (iii). If $\|u^-\| < \|u\|$ then only $\alpha = 1$ and $\beta = 0$ must be taken; hence $F_0(u)$ coincides with $F_0(u^+)$. Similarly, if $\|u^+\| < \|u\|$ then $F_0(u) = -F_0(u^-) = F_0(-u^-)$. However in the case of $\|u^+\| = \|u^-\| = \|u\|$, we can take any non-negative numbers α, β with $\alpha + \beta = 1$. This implies that $F_0(u) = \mathrm{co}[F_0(u^+) \cup F_0(-u^-)]$. To get the last assertion of (iii) we first observe that the set of extremal points of the set $W \equiv F_0(u^+) \cup F_0(-u^-)$ is exactly the set of those of $F_0(u^+)$ and $F_0(-u^-)$, namely

$$\mathrm{ext}\, W = \mathrm{ext}\, F_0(u^+) \cup \mathrm{ext}\, F_0(-u^-). \tag{4.2.3}$$

In fact, it is clear that $\mathrm{ext}\, W \subset \mathrm{ext}\, F_0(u^+) \cup \mathrm{ext}\, F_0(-u^-)$. Conversely, suppose that ϕ is an extremal point of $F_0(u^+)$ and that ϕ is written as $\phi = \alpha\lambda + \beta v$ for some $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and some $\lambda, v \in W$. First, both λ and v can not belong to $F_0(-u^-)$ by Lemma 4.2.3. Next let $\lambda \in F_0(u^+)$, $v \in F_0(-u^-)$, and let $E_0^{\pm}$ be the sets specified as in Lemma 4.2.3 then $(\alpha\lambda + \beta v)(E_0^-) = -\beta < 0$ by Lemma 4.2.3. This contradicts the fact that ϕ is a 0-1 measure. Consequently, both v and λ must belong to $F_0(u^+)$. But, in this case, $\lambda = v = \phi$ since $\phi \in \mathrm{ext}\, F_0(u^+)$. Thus, $\mathrm{ext}\, F_0(u^+) \subset \mathrm{ext}\, W$. Similarly, $\mathrm{ext}\, F_0(-u^-) \subset \mathrm{ext}\, W$ and so we have (4.2.3). We then show that $\mathrm{ext}\, W = \mathrm{ext}[\mathrm{co}\, W]$. Since both the set W and its weakly-star closed convex hull are weakly- star compact, the extremal points of $\mathrm{co}[W]$ are points in W by [2, p. 440]. From this we see that $\operatorname{ext}[\operatorname{co} W] \subset \operatorname{ext} W$. Conversely, let $\lambda \in \operatorname{ext} W$. Then (4.2.3) states that λ belongs to $\operatorname{ext} F_0(u^+)$ or $\operatorname{ext} F_0(-u^-)$; we may assume without loss of generality that $\lambda \in F_0(u^+)$. Suppose now that $\lambda = \alpha\lambda_1 + (1-\alpha)\lambda_2$ for some $\alpha \in (0,1)$ and some $\lambda_1, \lambda_2 \in \operatorname{co} W$. Then we must have $\lambda_1 \in F_0(u^+)$. In fact, if $\lambda_1 \notin F_0(u^+)$, then $\lambda_1 = \alpha_1\mu_1 + (1-\alpha_1)v_1$ for some $\alpha_1 \in [0,1)$ and $\mu_1 \in F_0(u^+)$ and some $v_1 \in F_0(-u^-)$, while $\lambda_2 = \alpha_2\mu_2 + (1-\alpha_2)v_2$ for some $\alpha_2 \in [0,1]$, $\mu_2 \in F_0(u^+)$ and $v_2 \in F_0(-u^-)$. Let E_0^- be the set specified as in Lemma 4.2.3. Then Lemma 4.2.3 yields that $\lambda(E_0^-) = -\alpha(1-\alpha_1) - (1-\alpha)(1-\alpha_2) < 0$. This contradicts the assumption that $\lambda \in F_0(u^+)$. But, $\lambda \in F_0(u^+)$; hence it follows that $\lambda = \lambda_1 = \lambda_2$. This means that $\lambda \in \operatorname{ext}[\operatorname{co} W]$. Consequently, combining the above-mentioned shows the last assertion of (iii). □

4.3. Typical examples: $\Delta, \beta\Delta$

In this subsection we exhibit how the duality mapping F_0 of $L^\infty(\Omega)$, where Ω is a domain with smooth boundary, may be used to prove the dissipativity of some typical linear and quasilinear diffusion operators. To this end, we need the maximum principle stated in Gilbarg and Trudinger [5].

THEOREM 4.3.1. (STRONG MAXIMUM PRINCIPLE). *Let $u \in W^{1,2}(\Omega)$ and assume that $\Delta u \geq 0$ in Ω, where Δ is the Laplacian operator in Ω. Then, if the closure of a ball B is contained in Ω and $\sup_B u = \sup_\Omega u \geq 0$, then the function u must be a constant in Ω.*

LEMMA 4.3.2. *Let $u \in W^{1,p}(\Omega)$ for some $p > d$ and $\Delta u \in L^\infty(\Omega)$. If u has nonnegative maximum at some $a \in \Omega$, then there exists a 0-1 measure $\omega \in ba(\Omega)$ concentrated at a such that $\langle \Delta u, \omega \rangle \leq 0$.*

PROOF. Let B be a ball in Ω with center at a. Let $\varepsilon > 0$ and define $E_\varepsilon = \{x \in B : \Delta u \leq \varepsilon\}$. First we prove that $m(E_\varepsilon) > 0$. To this end, we suppose that $\Delta u > \varepsilon$ a.e. in B. We then select a ball B' in B with center at a whose compact closure is contained in B. Since the point a at which u takes nonnegative maximum is contained in B', we have $\sup_{x \in B'} u = \sup_{x \in B} u \geq 0$ and $\Delta u \geq 0$ a.e. in B'. From Theorem 4.3.1 one can conclude that the function u must be a constant function in B. Hence $\Delta u = 0$ in B. But this contradicts our assumption that $\Delta u > \varepsilon$. Therefore $m(E_\varepsilon) > 0$. Let $\mathscr{E} = \{E_\varepsilon : \varepsilon > 0\}$. Then it is easily seen that the intersection of any finite subcollection of $\mathscr{E}$ is a Lebesgue non null set. By Theorem 3.2.1 one finds a 0-1 measure ω with the property that $E \in \mathscr{E}$ $\omega(E) = 1$ for any $E \in \mathscr{E}$. Then $\langle \Delta u, \omega \rangle = \int_\Omega \Delta u \, d\omega = \int_{E_\varepsilon} \Delta u \, d\omega \leq \varepsilon$. Since this is true for any $\varepsilon > 0$, we have $\langle \Delta u, \omega \rangle \leq 0$. Hence the proof is complete. □

In view of this result, we show that Δ is dissipative in $L^\infty(\Omega)$ under Dirichlet boundary conditions.

THEOREM 4.3.3. *Let* $p > d$. *Let* $D(A) = \{u \in W^{2,p}(\Omega) : \Delta u \in L^\infty(\Omega)$ *and* $u|_{\partial\Omega} = 0\}$ *and define* $Au = \Delta u$ *for* $u \in D(A)$. *Then* A *is dissipative in* $L^\infty(\Omega)$.

PROOF. To show that A is dissipative we have to prove that for any pair $u, v \in D(A)$, there exists $\lambda \in F_0(u - v)$ such that $\langle Au - Av, \lambda \rangle \le 0$. Let $u, v \in D(A)$. Then $\Delta u, \Delta v \in L^\infty(\Omega)$ and $u, v|_{\partial\Omega} = 0$. Therefore $u - v = 0$ on $\partial\Omega$. We then assume that $u - v \ne 0$ in Ω. Hence $|u - v|$ must have positive maximum at some interior point of Ω. Suppose $u(a) - v(a)$ is the maximum of $|u - v|$ for some $a \in \Omega$. (Otherwise change the role of u and v and exploit the fact $F_0(v - u) = -F_0(u - v)$.) Then by Lemma 4.3.2 there exists a 0-1 measure ω in $ba(\Omega)$ concentrated at a such that $\langle \Delta(u - v), \omega \rangle \le 0$. Since $u - v \in C(\bar{\Omega})$ and $\omega|_{C(\bar{\Omega})} = \delta_a$, we have $\omega \in F_0(u - v)$. □

REMARK 4.3.4. Although Δ is dissipative in $L^\infty(\Omega)$, it is not strongly dissipative in $L^\infty(\Omega)$.

To show this we set $\bar{\Omega} = [-1, 1]$ and the differential operator $(d/dx)^2$ in $L^\infty(\bar{\Omega})$ subject to the Dirichlet boundary condition. We then consider the function $u(x)$ on $[-1, 1]$ defined by

$$u(x) = \begin{cases} x^4\left(-2 + \sin\dfrac{\pi}{x}\right) + 2, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

It is easy to see that u takes its maximum at 0. Then it follows from Theorem 4.3.2 that there is $\omega \in F_0(u)$ such that $\langle \Delta u, \omega \rangle \le 0$. But the second derivative, u'' behaves like $\sin\dfrac{1}{x}$. Hence using Theorem 3.2.1 we can construct 0-1 measures ω_1 and ω_2 in $F_0(u)$ such that $\langle \Delta u, \omega_1 \rangle < 0$ and $\langle \Delta u, \omega_2 \rangle > 0$. This means that Δ is not strongly dissipative in $L^\infty(\Omega)$.

As a second example we consider a quasilinear diffusion operator $\beta\Delta$. In order to formulate the operator we impose the following three conditions.

(i) $\beta \in C(\bar{\Omega} \times \mathbf{R}^d)$ and there exists $c > 0$ such that $\beta \ge c$ on $\bar{\Omega} \times \mathbf{R}^d$.

(ii) $a \in C^1(\partial\Omega)$ and $a > 0$ on $\partial\Omega$.

We then define $D(A)$ to be the set,

$$\{v \in L^\infty(\Omega) : v \in W^{2,p}(\Omega) \text{ for } p > d,\ \beta(\cdot, \nabla v)\Delta v \in L^\infty(\Omega),$$
$$\partial v/\partial \nu + av = 0 \text{ on } \partial\Omega\}$$

and formulate a quasilinear diffusion operator

$$Av = \beta(\cdot, \nabla u)\Delta v \quad v \in D(A),$$

where $\partial/\partial\nu$ is the derivative with respect to the outward normal direction.

Applying Theorem 4.3.3 and Theorem 3.2.4 one can show that A is dissipative in $L^\infty(\Omega)$. More precisely, we obtain the following.

THEOREM 4.3.5. *The operator A is m-dissipative in $L^\infty(\Omega)$ and the resolvent $(I - \lambda A)^{-1}$ is order-preserving in $L^\infty(\Omega)$ for $\lambda > 0$.*

PROOF. Let $u, v \in D(A)$. Suppose $u \neq v$. Since $u - v \in C(\overline{\Omega})$, $|u - v|$ must have the positive maximum. We may assume that there is $a \in \overline{\Omega}$ such that $u(a) - v(a)$ is the maximum of $|u - v|$. (Otherwise change the role of u and v.) Now the boundary condition for the operator A prevents the point a from belonging to $\partial\Omega$, and so $a \in \Omega$. Then $\nabla u(a) = \nabla v(a)$ and, by the argument in the proof of Theorem 4.3.3, there is a 0-1 measure $\omega_a \in F_0(u - v)$ which has the essential support at a and satisfies $\langle \Delta(u - v), \omega_a \rangle \leq 0$. From this and Theorem 3.2.4 it follows that $\langle Au - Av, \omega_a \rangle = \langle \beta(\cdot, \nabla u)\Delta u, \omega_a \rangle - \langle \beta(\cdot, \nabla v)\Delta v, \omega_a \rangle = \beta(a, \nabla u(a))\langle \Delta u, \omega_a \rangle - \beta(a, \nabla v(a))\langle \Delta v, \omega_a \rangle = \beta(a, \nabla u(a)) \langle \Delta(u - v), \omega_a \rangle \leq 0$. Hence A is dissipative in $L^\infty(\Omega)$.

In order to prove that A is m-dissipative in $L^\infty(\Omega)$, it suffices to show that $R(I - \lambda A) = L^\infty(\Omega)$ for some $\lambda > 0$. To this end, we find a sufficiently large $\lambda > 0$ and a solution $u \in D(A)$ of the equation $u - \lambda\beta(x, \nabla u)\Delta u = v$ for any $v \in L^\infty(\Omega)$. This equation is transformed into the equation $u = \Gamma_v u$, where

$$\Gamma_v u = (I - \sqrt{\lambda}\Delta)^{-1}\left(u + \frac{u - v}{\sqrt{\lambda}\beta(\cdot, \nabla u)}\right).$$

For an appropriate $\alpha \in (1/2, 1)$ we introduce a space $Y = D((-\Delta)^\alpha)$. Then it is shown that for a sufficiently large $\lambda > 0$ the mapping Γ_v maps Y into itself and the Schauder fixed point theorem is applied to solve $u = \Gamma_v u$. For the precise proof we refer to [18, Theorem 5]. Therefore the resolvent $(I - \lambda A)^{-1}$ exists for $\lambda > 0$. We then show that each $(I - \lambda A)^{-1}$ is order-preserving.

Let $f, g \in L^\infty(\Omega)$ and $\lambda > 0$. Then there exist $u, v \in D(A)$ such that $u - \lambda Au = f$ and $v - \lambda Av = g$. Suppose that $f \geq g$ in $L^\infty(\Omega)$. Suppose that there is a point $b \in \overline{\Omega}$ such that

$$v(b) - u(b) = \sup\{v(x) - u(x) : x \in \overline{\Omega}\} > 0$$

Just as the discussion for the dissipativity of A we see that there is a 0-1 measure $\omega_b \in ba(\Omega)$ centered at b such that

$$\langle Av - Au, \omega_b \rangle \leq 0$$

Since $\langle u - v, \omega_b \rangle < 0$, we obtain

$$\langle f - g, \omega_b \rangle = \langle u - \lambda Au - v + \lambda v, \omega_b \rangle < \lambda\langle Av - Au, \omega_b \rangle \leq 0.$$

This contradicts the assumption that $f \geq g$. Therefore $(I - \lambda A)^{-1}$ is order preserving. □

5. Duality mappings of $L^{\infty}(\Omega)^n$

This section is devoted to the precise study of the duality mappings of product L^{∞} spaces. Let X^* denotes the dual space of a Banach space X. For $v \in X$ and $f \in X^*$ the value of f at v is written as $\langle v, f \rangle$. In this section we give a representation theorem for $\mathbf{X}^*$, the dual space of the product space $\mathbf{X} = X_1 \times X_2 \times \cdots \times X_n$, where X_i, for $i = 1, \ldots, n$ are given Banach spaces. We represent a generic element $\boldsymbol{v}$ of $\mathbf{X}$ as $(v_1, v_2, \ldots, v_n)$, where $v_i \in X_i$ for $i = 1, \ldots, n$. The following result is basic to the subsequent discussions.

THEOREM 5.1.1 (REPRESENTATION THEOREM). *Let* $\boldsymbol{v} \in \mathbf{X}$ *and* $\|\boldsymbol{v}\| = \bigvee_{i=1}^{n} \|v_i\|$. *Then for any* $\boldsymbol{h} \in \mathbf{X}^*$ *there exists unique* $\boldsymbol{f} = (f_1, f_2, \ldots, f_n) \in X_1^* \times X_2^* \times \cdots X_n^*$ *such that*

$$\langle \boldsymbol{v}, \boldsymbol{h} \rangle = \sum_{i=1}^{n} \langle v_i, f_i \rangle$$

for any $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbf{X}$. *Moreover* $\|\boldsymbol{h}\| = \sum_{i=1}^{n} \|f_i\|$ *and* $\mathbf{X}^* = X_1^* \times X_2^* \times \cdots \times X_n^*$.

5.1. Properties of the duality mapping of $L^{\infty}(\Omega)^n$

Let $\mathbf{X} = X_1 \times X_2 \times \cdots \times X_n$ be a Banach space and $\mathbf{X}^*$ its dual space. The duality mapping $\boldsymbol{F}(\boldsymbol{v})$ is defined by

$$\boldsymbol{F}(\boldsymbol{v}) = \{\boldsymbol{f} \in \mathbf{X}^* : \langle \boldsymbol{v}, \boldsymbol{f} \rangle = \|\boldsymbol{v}\|^2 = \|\boldsymbol{f}\|^2\}, \quad \boldsymbol{v} \in \mathbf{X},$$

and the normalized duality mapping $\boldsymbol{F}_0$ of $\mathbf{X}$ is defined by $\boldsymbol{F}_0(\mathbf{0}) = \boldsymbol{S}^*(0, 1)$, the unit surface of $\mathbf{X}$, and

$$\boldsymbol{F}_0(\boldsymbol{v}) = \{\boldsymbol{f} \in \mathbf{X}^* : \langle \boldsymbol{v}, \boldsymbol{f} \rangle = \|\boldsymbol{v}\|, \|\boldsymbol{f}\| = 1\}, \quad \boldsymbol{v} \in \mathbf{X} \backslash \{\mathbf{0}\}.$$

Now we give a proposition of nontrivial result which we use later extensively. The following three product spaces are isometrically isomorphic. Also, for the product spaces $X \times Y$, $X \times Y \times Z$, we employ the max norm $\|(x, y)\| = \|x\| \wedge \|y\|$, $\|(x, y, z)\| = \|x\| \wedge \|y\| \wedge \|z\|$ for $x \in X$, $y \in Y$ and $z \in Z$. Also, for the product spaces $X^* \times Y^*$, $X^* \times Y^* \times Z^*$ we employ the norms by $\|(f, g)\| = \|f\| + \|g\|$, $\|(f, g, h)\| = \|f\| + \|g\| + \|h\|$ for $f \in X^*$, $g \in Y^*$ and $h \in Z^*$.

PROPOSITION 5.1.2. *Let* X, Y, Z *be Banach spaces and let* X^*, Y^*, Z^* *be their respective duals. Then we have followings:*

(i) $(X \times Y \times Z) \simeq (X \times Y) \times Z \simeq X \times (Y \times Z)$,
(ii) $(X \times Y \times Z)^* \simeq (X \times Y)^* \times Z^* \simeq X^* \times (Y \times Z)^* \simeq X^* \times Y^* \times Z^*$.

Let $\mathbf{X}$ be a product Banach space with the maximum norm. Then $\boldsymbol{F}_0$ has the following basic properties:

PROPOSITION 5.1.3. *Let $f \in \boldsymbol{F}_0(v)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ and $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$. Then $\langle v_i, f_i \rangle = \|v_i\| \, \|f_i\|$ and $\|v_i\| < \|\boldsymbol{v}\|$ implies $f_i = 0$ for $i = 1, \ldots, n$.*

PROOF. From the definition of the normalized duality mapping it follows that

$$\bigvee_{i=1}^{n} \|v_i\| = \langle v, f \rangle = \sum_{i=1}^{n} \langle v_i, f_i \rangle \leq \sum_{i=1}^{n} \|v_i\| \, \|f_i\| \leq \left(\bigvee_{i=1}^{n} \|v_i\| \right) \left(\sum_{i=1}^{n} \|f_i\| \right).$$

Since $\sum_{i=1}^{n} \|f_i\| = 1$, the above estimates together imply $\langle v_i, f_i \rangle = \|f_i\| \, \|v_i\|$ for $i = 1, \ldots, n$. Next, let $\|v_i\| < \|\boldsymbol{v}\|$ and suppose that $f_i \neq 0$. Then, by the definition of $\boldsymbol{F}_0$, we have

$$\|\boldsymbol{v}\| = \langle v, f \rangle = \sum_{i=1}^{n} \langle v_i, f_i \rangle \leq \sum_{i=1}^{n} \|v_i\| \, \|f_i\| < \|\boldsymbol{v}\|.$$

This is a contradiction, and we must have $f_i = 0$. □

REMARK 5.1.4. (i) Suppose that $\|v_i\| = \|\boldsymbol{v}\|$ and $\|v_j\| < \|\boldsymbol{v}\|$ for $j \neq i$. Then, by Proposition 5.1.3, $f_j = 0$ for $j \neq i$. Hence, $\|\boldsymbol{v}\| = \|v_i\| = \langle \boldsymbol{v}, \boldsymbol{f} \rangle = \sum_{i=i}^{n} \langle v_i, f_i \rangle = \langle v_i, f_i \rangle$, which implies $\|v_i\| = \langle v_i, f_i \rangle$ and $\|f_i\| = 1$. This means that $f_i \in F_0(v_i)$. Hence $\boldsymbol{F}_0(v) = (0, \ldots, F_0(v_i), \ldots, 0)$ if $\|\boldsymbol{v}\| > \|v_j\|$ for $j \neq i$.

(ii) It is also clear from Proposition 5.1.3 that if $f_i \neq 0$ then $\|v_i\| = \|\boldsymbol{v}\|$.

Proposition 5.1.3 and its Remark 5.1.4 leads to following

LEMMA 5.1.5. *Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ and $\boldsymbol{f} = (f_1, f_2, , f_n) \in \boldsymbol{F}_0(\boldsymbol{v})$.*
(i) If $\|v_j\| < \|\boldsymbol{v}\|$ for $j \neq i$ then $\boldsymbol{F}_0(\boldsymbol{v}) = (0, \ldots, F_0(v_i), \ldots, 0)$.
(ii) If $\|\boldsymbol{v}\| = \|v_i\|$ for all i then $\boldsymbol{F}_0(\boldsymbol{v}) = \bigcup_{\substack{\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0 \\ \sum_{i=1}^{n} \alpha_i = 1}} (\alpha_1 F_0(v_1), \alpha_2 F_0(v_2), \ldots, \alpha_n F_0(v_n))$.

PROOF. Assertion (i) is already verified in Remark 5.1.4. We then prove the second assertion (ii) by induction. First we show the result for $n = 2$. Let $\boldsymbol{f} = (f_1, f_2)$, $\boldsymbol{v} = (v_1, v_2)$ and $(f_1, f_2) \in \boldsymbol{F}_0(v_1, v_2)$. Then by the definition of $\boldsymbol{F}_0$ we have $\|v_1\| = \|v_2\| = \|(v_1, v_2)\| = \langle (v_1, v_2), (f_1, f_2) \rangle = \langle v_1, f_1 \rangle + \langle v_2, f_2 \rangle \leq \|v_1\| \, \|f_1\| + \|v_2\| \, \|f_2\| \leq \|v_1\| = \|v_2\|$. From this we have $\langle v_1, f_1 \rangle = \|v_1\| \, \|f_1\|$ and $\langle v_2, f_2 \rangle = \|v_2\| \, \|f_2\|$. Let $f_1' = f_1/\|f_1\|$ and $f_2' = f_2/\|f_2\|$. Then $f_1' \in$

$F_0(v_1)$ and $f_2' \in F_0(v_2)$. Set $\alpha = \|f_1\|$ and $\beta = \|f_2\|$, then $\alpha + \beta = 1$, and so $(\alpha f_1', \beta f_2') \in \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} (\alpha F_0(v_1), \beta F_0(v_2))$. This means that $(f_1, f_2) \in \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} (\alpha F_0(v_1), \beta F_0(v_2))$. Hence

$$(5.1.1) \qquad F_0(v_1, v_2) \subset \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} (F_0(v_1), F_0(v_2)).$$

We next prove the reverse inclusion (5.1.1) as follows. Let $(f_1', f_2') \in \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} (\alpha F_0(v_1), \beta F_0(v_2))$. Then there exists α, β such that $f_1' = \alpha f_1$, $f_1 \in F_0(v_1)$, $f_2' = \beta f_2$, $f_2 \in F_0(v_2)$. Then $\langle (v_1, v_2), (f_1', f_2') \rangle = \langle (v_1, v_2), (\alpha f_1', \beta f_2') \rangle = \langle v_1, \alpha f_1' \rangle + \langle v_2, \beta f_2' \rangle = \alpha \langle v_1, f_1' \rangle + \beta \langle v_2, f_2' \rangle = \|v_1\| \ (= \|v_2\|)$ which gives $\langle (v_1, v_2), (f_1', f_2') \rangle = \|(v_1, v_2)\|, \|f_1'\| + \|f_2'\| = 1$. Therefore we have $(f_1', f_2') \in F_0(v_1, v_2)$, which shows that $\bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} (\alpha F_0(v_1), \beta F_0(v_2)) \subset F_0(v_1, v_2)$. Combining this with (5.1.1), we have $F_0(v_1, v_2) = \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} (\alpha F_0(v_1), \beta F_0(v_2))$. Hence the proof is complete for $n = 2$.

Now we assume that the assertion (ii) for $n = p$. To prove that the Assertion (ii) is also true for $n = p + 1$, we employ the following notation. Let $\mathbf{X}' = X_1 \times X_2 \times \cdots \times X_p$, and write an element in X' as $\boldsymbol{v}'$. Then we can write

$$\boldsymbol{v} = (\boldsymbol{v}', v_{p+1}) \quad \text{and} \quad \boldsymbol{F}_0(\boldsymbol{v}) = \boldsymbol{F}_0(\boldsymbol{v}', v_{p+1}).$$

Applying the result for the case of $n = 2$, we have

$$\begin{aligned}
\boldsymbol{F}_0(\boldsymbol{v}) &= \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} (\alpha \boldsymbol{F}_0(\boldsymbol{v}'), \beta F_0(v_{p+1})) \\
&= \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} \alpha\Big(\bigcup_{\substack{\sum_{i=1}^p \alpha_i = 1 \\ \alpha_i \geq 0}} (\alpha_1 F_0(v_1), \alpha_2 F_0(v_2), \ldots, \alpha_p F_0(v_i)), \beta F_0(v_{p+1})\Big) \\
&= \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} \Big(\bigcup_{\substack{\sum_{i=1}^p \alpha_i = 1 \\ \alpha_i \geq 0}} (\alpha\alpha_1 F_0(v_1), \alpha\alpha_2 F_0(v_2), \ldots, \alpha\alpha_p F_0(v_i)), \beta F_0(v_{p+1})\Big) \\
&= \bigcup_{\alpha,\beta \geq 0, \alpha+\beta=1} \bigcup_{\substack{\sum_{i=1}^p \alpha_i = 1 \\ \alpha_i \geq 0}} ((\alpha\alpha_1 F_0(v_1), \alpha\alpha_2 F_0(v_2), \ldots, \alpha\alpha_p F_0(v_i)), \beta F_0(v_{p+1})) \\
&= \bigcup_{\substack{\sum_{i=1}^{p+1} \alpha_i' = 1 \\ \alpha_i' \geq 0}} ((\alpha_1' F_0(v_1), \alpha_2' F_0(v_2), \ldots, \alpha_p' F_0(v_i)), \alpha_{p+1}' F_0(v_{p+1})),
\end{aligned}$$

where $\alpha_i' = \alpha\alpha_i$ for $i = 1, \ldots, p$ and $\alpha_{p+1}' = \beta$. Hence the result is true for $n = p + 1$. By the induction principal we obtain the desired result. □

We can now state the above theorem in a more general setting:

THEOREM 5.1.6.

$$\boldsymbol{F}_0(\boldsymbol{v}) = \bigcup_{\substack{\sum_{i=1}^{n} \alpha_i = 1 \\ \alpha_i \geq 0, i \in I \\ \alpha_i = 0, i \notin I}} (\alpha_1 F_0(v_1), \ldots, \alpha_n F_0(v_n)),$$

where $I = \{i : \|\boldsymbol{v}\| = \|v_i\|\} \subset \{1, 2, \ldots, n\}$.

Finally, we prepare the following lemma:

LEMMA 5.1.7. *Let* f, $g \in \boldsymbol{F}_0(v)$. *If* $f_i \neq 0$ *for any* i, *then* $\alpha f_i + g_i \neq 0$ *for any* $\alpha > 0$.

PROOF. By Proposition 5.1.3, $\langle v_i, f_i \rangle = \|v_i\| \, \|f_i\|$ and $\langle v_i, g_i \rangle = \|v_i\| \, \|g_i\|$. Hence $\langle v_i, \alpha f_i + g_i \rangle = \alpha \langle v_i, f_i \rangle + \langle v_i, f_i \rangle = \|v_i\|(\alpha\|f_i\| + \|g_i\|)$, which implies $\alpha f_i + g_i \neq 0$. This completes the proof of the Lemma. □

5.2. Extremal Points in $\boldsymbol{F}_0(\boldsymbol{f})$

In this subsection we make an attempt to give a characterization of the extremal points of $\boldsymbol{F}_0(v))$. First we prepare the following which plays a major roll in the following.

PROPOSITION 5.2.1. *For any* $\boldsymbol{f} \in \operatorname{ext} \boldsymbol{F}_0(v)$, $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)$ *has only one nonzero component.*

PROOF. Suppose $\boldsymbol{f}$ has two or more non-zero componenets. Then by Proposition 5.1.3, $\|v_i\| = \|\boldsymbol{v}\|$, for the corresponding v_i's. Now by Lemma 5.1.5 for these i's $(0, \ldots, f_i, \ldots, 0) \in F_0(v_i)$. Without lost of generality suppose $\|f\| \neq 0$. Then,

$$\boldsymbol{f} = \|f_1\|(f_1/\|f_1\|, 0, \ldots, 0) + (1 - \|f_1\|)(0, f_2/(1 - \|f_1\|), \\ f_3/(1 - \|f_1\|), \ldots, f_n/(1 - \|f_1\|)).$$

Since $(f_1/\|f_1\|, 0, \ldots, 0), (0, f_2/(1 - \|f_1\|), f_3/(1 - \|f_1\|), \ldots, f_n/(1 - \|f_1\|)) \in \boldsymbol{F}_0(\boldsymbol{v})$ by

Theorem 5.1.5 and $\boldsymbol{f}$ is an extremal point of $\boldsymbol{F}_0(\boldsymbol{v})$ we have

$$\boldsymbol{f} = (f_1/\|f_1\|, 0, \ldots, 0) = (0, f_2/(1 - \|f_1\|), f_3/(1 - \|f_1\|), \ldots, f_n/(1 - \|f_1\|)).$$

Thus $\boldsymbol{f} = 0$, and this gives a contradiction. Hence the proof is complete. □

PROPOSITION 5.2.2. *Let* $v = (v_1, v_2, \dots, v_n) \in \mathbf{X}$. *Then we have:*

(i) $\operatorname{ext}(0, \dots, F_0(v_i), \dots, 0) = (0, \dots, \operatorname{ext} F_0(v_i), \dots, 0)$,

(ii) $\operatorname{ext}\{\bigcup_{j=1}^n (\delta_{1j} F_0(v_1), \delta_{2j} F_0(v_2), \dots, \delta_{nj} F_0(v_n))\}$
$= \bigcup_{j=1}^n (\delta_{1j} \operatorname{ext} F_0(v_1), \delta_{2j} \operatorname{ext} F_0(v_2), \dots, \delta_{nj} \operatorname{ext} F_0(v_n))$

PROOF. (i): Without loss of generality we may assume that $i = 1$. Then we show that $\operatorname{ext}(F_0(v_1), 0, \dots, 0) = (\operatorname{ext} F_0(v_1), 0, \dots, 0)$. Let $(f, 0, \dots, 0) \in \operatorname{ext}(F_0(v_1), 0, \dots, 0)$. Assume that there exist $\alpha \in (0, 1)$ and $f_1, f_2 \in F_0(v_1)$ such that $(f, 0, \dots, 0) = \alpha(f_1, 0, \dots, 0) + (1 - \alpha)(f_2, 0, \dots, 0)$. Since $(f_1, 0, \dots, 0)$, $(f_2, 0, \dots, 0) \in (F_0(v_1), 0, \dots, 0)$ and $(f, 0, \dots, 0) \in \operatorname{ext}(F_0(v_1), 0, \dots, 0)$, we have $(f, 0, \dots, 0) = (f_1, 0, \dots, 0) = (f_2, 0, \dots, 0)$ which implies $f = f_1 = f_2$. This shows that $f \in \operatorname{ext} F_0(v_1)$, and as that $(f, 0, \dots, 0) \in (\operatorname{ext} F_0(v_1), 0, \dots, 0)$. Next let $(f, 0, \dots, 0) \in (\operatorname{ext} F_0(v_1), 0, \dots, 0)$, which means that $f \in \operatorname{ext} F_0(v_1)$. Suppose that there exist $\alpha(0, 1) \in (0, 1)$ and $(f_1, 0, \dots, 0)$, $(f_2, 0, \dots, 0) \in F_0(v_1)$ such that $(f, 0, \dots, 0) = \alpha(f_1, 0, \dots, 0) + (1 - \alpha)(f_2, 0, \dots, 0)$. This gives that $f = f_1 = f_2$ and also we have $f_1, f_2 \in F_0(v_1)$. Since $f \in \operatorname{ext} F_0(v_1)$, we have $f = f_1 = f_2$ and $(f, 0, \dots, 0) = (f_1, 0, \dots, 0) = (f_2, 0, \dots, 0)$. Thus we have $(f, 0, \dots, 0) \in \operatorname{ext}(F_0(v_1), 0, \dots, 0)$.

(ii): Since $(F_0(v_1), 0, \dots, 0), (0, F_0(v_2), 0, \dots, 0), \dots, (0, \dots, 0, F_0(v_n))$ are subsets of the union $\bigcup_{j=1}^n (\delta_{1j} F_0(v_1), \delta_{2j} F_0(v_2), \dots, \delta_{nj} F_0(v_n))$, for any $i \in \{1, \dots, n\}$, we have by the first result (i)

$$\operatorname{ext}\left\{\bigcup_{j=1}^n (\delta_{1j} F_0(v_1), \delta_{2j} F_0(v_2), \dots, \delta_{nj} F_0(v_n))\right\} \cap \{0, \dots, F_0(v_i), \dots, 0)$$

$$\subset \operatorname{ext}\{0, \dots, , F_0(v_i), \dots, 0)\} = (0, \dots, \operatorname{ext} F_0(v_i), \dots, 0).$$

Therefore

$$\operatorname{ext}\left\{\bigcup_{j=1}^n (\delta_{1j} F_0(v_i), \dots, \delta_{nj} F_0(v_n))\right\} \cap \bigcup_{j=1}^n (\delta_{1j} F_0(v_1), \delta_{2j} F_0(v_2), \dots, \delta_{nj} F_0(v_n))$$

$$\subset \bigcup_{j=1}^n (\delta_{1j} \operatorname{ext} F_0(v_1), \delta_{2j} \operatorname{ext} F_0(v_2), \dots, \delta_{nj} \operatorname{ext} F_0(v_n)).$$

From this it follows that

$$\operatorname{ext}\left\{\bigcup_{j=1}^n (\delta_{1j} F_0(v_1), \dots, \delta_{nj} F_0(v_n))\right\} \subset \bigcup_{j=1}^n (\delta_{1j} \operatorname{ext} F_0(v_1), \dots, \delta_{nj} \operatorname{ext} F_0(v_n)).$$

To show the inverse implication

$$\operatorname{ext}\left\{\bigcup_{j=1}^n (\delta_{1j} F_0(v_i), \dots, \delta_{nj} F_0(v_n))\right\} \supseteq \bigcup_{j=1}^n (\delta_{1j} \operatorname{ext} F_0(v_1), \dots, \delta_{nj} \operatorname{ext} F_0(v_n)),$$

let $i \in \{1,\dots,n\}$ and $(0,\dots,f_i,\dots,0) \in (0,\dots,\operatorname{ext} F_0(v_i),\dots,0)$. Suppose then that there exist $\alpha \in (0,1)$ and $\boldsymbol{h} = (h_1,\dots,h_n)$, $\boldsymbol{g} = (g_1,\dots,g_n) \in \bigcup_{j=1}^{n}(\delta_{1j}F_0(v_1),\dots,\delta_{nj}F_0(v_n))$ such that $(0,\dots,f_i,\dots,0) = \alpha h + (1-\alpha)g$. This implies that $h_j = g_j = 0$ for $i \neq j$ and $f_i = \alpha h_i + (1-\alpha)g_i$. Since $f_i \in \operatorname{ext} F_0(v_i)$ we have $f_i = h_i = g_i$. Therefore $(0,\dots,f_i,\dots,0) \in \operatorname{ext}\{\bigcup_{j=1}^{n}(\delta_{1j}F_0(v_i),\dots,\delta_{nj}F_0(v_n)\}$. Since i is arbitrary, we conclude that $\bigcup_{j=1}^{n}(\delta_{1j}\operatorname{ext} F_0(v_n),\dots,\delta_{nj}\operatorname{ext} F_0(v_n)) \subset \operatorname{ext}\{\bigcup_{j=1}^{n}(\delta_{1j}F_0(v_1),\dots,\delta_{nj}F_0(v_n))\}$. Thus we obtain the desired assertion (ii). $\square$

THEOREM 5.2.3. *Let $\boldsymbol{v} = (v_1,\dots,v_n) \in \mathbf{X}$. Then the set of extremal points of $\boldsymbol{F}_0(\boldsymbol{v})$ is characterized by*

$$\operatorname{ext} \boldsymbol{F}_0(v) = \bigcup_{\|v_j\|=\|\boldsymbol{v}\|} (\delta_{1j}\operatorname{ext} F_0(v_1),\dots,\delta_{nj}\operatorname{ext} F_0(v_n)).$$

PROOF. Using the Theorem 5.1.6 we can write

$$\boldsymbol{F}_0(v) = \bigcup_{\substack{\sum_{i=1}^{n}\alpha_i=1\\ \alpha_i\geq 0,\, i\in I\\ \alpha_i=0,\, i\notin I}} (\alpha_1 F_0(v_1),\dots,\alpha_n F_0(v_n))$$

where $I = \{i : \|\boldsymbol{v}\| = \|v_i\|\} \subset \{1,2,\dots,n\}$. For simplicity in the proof we assume here that

$$\boldsymbol{F}_0(v) = \bigcup_{\substack{\sum_{i=1}^{n}\alpha_i=1\\ 0\leq\alpha_i\leq 1}} (\alpha_1 F_0(v_1),\dots,\alpha_n F_0(v_n)).$$

Therefore $\bigcup_{j=1}^{n}(\delta_{1j}F_0(v_1),\delta_{2j}F_0(v_2),\dots,\delta_{nj}F_0(v_n)) \subset \boldsymbol{F}_0(v)$. Hence

$$\operatorname{ext}\boldsymbol{F}_0(v) \cap \left\{\bigcup_{j=1}^{n}(\delta_{1j}F_0(v_1),\dots,\delta_{nj}F_0(v_n))\right\} \subset \operatorname{ext}\left\{\bigcup_{j=1}^{n}(\delta_{1j}F_0(v_1),\dots,\delta_{nj}F_0(v_n))\right\}.$$

Combine this with Proposition 5.2.2 gives

$$\operatorname{ext}\boldsymbol{F}_0(v) \cap \left\{\bigcup_{j=1}^{n}(\delta_{1j}F_0(v_1),\dots,\delta_{nj}F_0(v_n)\right\}$$
$$\subset \bigcup_{j=1}^{n}(\delta_{1j}\operatorname{ext} F_0(v_1),\dots,\delta_{nj}\operatorname{ext} F_0(v_n)).$$

Then by Proposition 5.2.1 we have $\operatorname{ext}\boldsymbol{F}_0(\boldsymbol{v}) \subset \bigcup_{j=1}^{n}(\delta_{1j}\operatorname{ext} F_0(v_1),\dots,\delta_{nj}\operatorname{ext} F_0(v_n))$. To get reverse inclusion, let $(f,0,\dots,0) \in (\operatorname{ext} F_0(v_1),0,\dots,0)$. Suppose that there exist $\alpha \in (0,1)$ and $(f_1,f_2,\dots,f_n),(g_1,g_2,\dots,g_n) \in \boldsymbol{F}_0(v)$ such that $(f,0,\dots,0) = \alpha(f_1,f_2,\dots,f_n) + (1-\alpha)(g_1,g_2,\dots,g_n)$. This gives $(f,0,\dots,0) = (\alpha f_1 + (1-\alpha)g_2, \alpha f_2 + (1-\alpha)g_2,\dots)$. From Lemma 5.1.7 it follows that $f_i, g_i = 0$ for $2 \leq i \leq n$. Hence $f = \alpha f_1 + (1-\alpha)g_1$. Since f is an extremal point of $F_0(v_1)$, $f = f_1 = g_1$. Therefore $(f, 0, \ldots, 0) \in \operatorname{ext} \boldsymbol{F}_0(v)$. In the same way as above we can prove if $(0, \ldots, f_i, \ldots, 0) \in (0, \ldots, \operatorname{ext} F_0(v_i), \ldots, 0)$ then $(0, \ldots, f_i, 0, \ldots, 0) \in \boldsymbol{F}_0(v)$. This shows that $\bigcup_{j=1}^{n}(\delta_{1j} \operatorname{ext} F_0(v_1), \ldots, \delta_{nj} \operatorname{ext} F_0(v_n)) \subset \operatorname{ext} \boldsymbol{F}_0(v)$, and the proof of the theorem is complete. □

5.3. A typical example in $L^\infty(\Omega)^3$

We here illustrate the use of Theorem 5.2.3 by considering a quasilinear diffusion operator in $L^\infty(\Omega)^3$. In order to formulate the operator we impose the following conditions:

(i) $\beta_i \in C(\overline{\Omega} \times \mathbf{R}^d)$ and there exists $c > 0$ such that $\beta_i \geq c$ on $\overline{\Omega} \times \mathbf{R}^d$ for $i = 1, 2, 3$.
(ii) $a_i \in C^1(\partial\Omega)$ and $a_i > 0$ on $\partial\Omega$ for $i = 1, 2, 3$.

We then define $D(\mathscr{A})$ to be the set

$$D(\mathscr{A}) = \left\{ \boldsymbol{v} = (v_i) \in L^\infty(\Omega)^3 : v_i \in W^{2,p}(\Omega) \quad \text{for } p > d,\ \beta_i(\cdot, \nabla v_i)\Delta v_i \in L^\infty(\Omega), \right.$$
$$\left. \frac{\partial v_i}{\partial \nu} + a_i v_i = 0 \quad \text{on } \partial\Omega, i = 1, 2, 3 \right\},$$

and formulate a quasilinear diffusion operator

$$\mathscr{A}\boldsymbol{v} = [\beta_1(\cdot, \nabla v_1)\Delta v_1, \beta_2(\cdot, \nabla v_2)\Delta v_2, \beta_3(\cdot, \nabla v_3)\Delta v_3].$$

Then we have following theorem.

THEOREM 5.3.1. *$\mathscr{A}$ is dissipative in $L^\infty(\Omega)^3$.*

PROOF. We must prove that for any pair of $\boldsymbol{u}, \boldsymbol{v} \in D(\mathscr{A})$, there exists $\boldsymbol{f} \in \boldsymbol{F}_0(\boldsymbol{v})$ such that

$$\langle \mathscr{A}(\boldsymbol{u}) - \mathscr{A}(\boldsymbol{v}), \boldsymbol{f} \rangle \leq 0.$$

Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. Without loss of generality we may assumme that $|u_1 - v_1| = \|\boldsymbol{u} - \boldsymbol{v}\|$. Now consider the first component of the left-hand side $\langle \beta_1(\cdot, \nabla u_1)\Delta u_1 - \beta_1(\cdot, \nabla v_1)\Delta v_1 \rangle$. Then by Lemma 4.3.2 there exist a 0-1 measure $f_1 \in F_0(u_1 - v_1)$ such that $\langle \beta_1(\cdot, \nabla u_1)\Delta u_1 - \beta_1(\cdot, \nabla v_1)\Delta v_1 \rangle \leq 0$. Then since this measure is an extremal point of $F_0(u_1 - v_1)$, Theorem 5.2.3 implies that $(f_1, 0, 0) \in \boldsymbol{F}_0(\boldsymbol{u} - \boldsymbol{v})$. This implies $\langle \mathscr{A}(\boldsymbol{u}) - \mathscr{A}(\boldsymbol{v}), (\omega_1, 0, 0) \rangle \leq 0$. Since $(f_1, 0, 0) \in \boldsymbol{F}_0(\boldsymbol{u} - \boldsymbol{v})$, it follows that $\mathscr{A}$ is dissipative.

REMARK 5.3.2. Finally, we would like to add the following comment. Theorem 3.1.5 remains valid in the case where Ω is unbounded and ω has its support at infinity.

References

[1] J. Diestel, Geometry of Banach Spaces, Lecture Notes in Math. 338, Springer, 1975.

[2] N. Dunford and J. Schawrtz, Linear Operators, Part 1: General Theory, Interscience Publishers, New York, 1958.

[3] Marián Fabian, A remark on the duality mapping on l^∞, Tokyo J. Math. **8-1** (1985), 215–217.

[4] D. H. Fremlin, Topological Riesz Spaces and Measure Theory, Lectures in Mathematics, University of Essex, 1974.

[5] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equation of Second Order, 2nd ed., Springer-Verlarg, Berlin, 1983.

[6] J. A. Goldstein, K. Hashimoto and S. Oharu, The duality mapping of $L^\infty[0,1]$ and its application to nonlinear degenerate diffusion operators, preprint.

[7] J. A. Goldstein and C. Y. Lin, Singular nonlinear parabolic boundary value problems in one space dimensions, J. Differential Equations **68** (1987), 429–443.

[8] J. A. Goldstein and C. Y. Lin, Highly degenerate parabolic boundary value problems, Differential Integral Equations **2** (1989), 216–227.

[9] Gottfried Köthe, Topological Vector Spaces, Springer-Verlarg, New York, 1969.

[10] I. Hada, K. Hashimoto and S. Oharu, On the duality mapping of l^∞, Tokyo J. Math. **2-1** (1979), 71–97.

[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics 840, Springer-Verlarg, Berlin-Heidelberg 1981.

[12] K. Kobayasi, Y. Kobayashi and S. Oharu, Nonlinear evolution operators in Banach spaces, Osaka J. Math. **21** (1984), 281–310.

[13] K. Kobayasi and S. Oharu, Nonlinear evolution operators associated with certain nonlinear equations of evolution, Mathematical Analysis on Structures in Nonlinear Phenomena, Lecture Notes in Num. Appl. Anal. **2** (1980), 139–210.

[14] C. Y. Lin, Degenerate nonlinear parabolic boundary value problems, Nonlinear Anal. **13** (1989), 1303–1315.

[15] M. Mimura and A. Nakaoka, On some degenerate diffusion system related with a certain reaction system, J. Math. Kyoto Univ. (JMKYAZ) **12-1** (1972), 95–121.

[16] T. Matsumoto, Time-dependent nonlinear perturbations of integrated semigroups, Nihonkai Math. J. **7** (1996), 1–28.

[17] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[18] J. Peiris and T. Matsumoto, Quasilinear reaction-diffusion systems in L^∞ spaces, Nihonkai Math. J. **9-2** (1998), to appear.

[19] H. H. Schaefer, Banach Lattices and Positive Operators, Springer-Verlag, 1974.

[20] K. Yosida and E. Hewitt, Finitely additive measures, Trans. Amer. Math. Soc. **72** (1952) 46–66.

[21] K. Yosida, Funtional Analysis, Second Edition, Springer-Verlarg, Berlin, 1966.

Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan