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Certain bilateral basic hypergeometric transformations and mock theta functions

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ABSTRACT. In this paper we show that the bilateral mock theta functions corresponding to the mock theta functions of order five of the first kind can be related to the ones of the second kind. We give also some identities for the bilateral mock theta functions.

1. Introduction

Bilateral forms of mock theta functions have been defined by Watson [4, 5], Agarwal [1], A. Gupta [2] from time to time. They have studied their relationship with mock theta functions using bilateral basic hypergeometric transformations. It is interesting to note that in a number of cases the bilateral mock theta functions yield two different forms taking into account the positive and the negative series. This reveals some relations among ten bilateral mock theta functions. In §5, we present still other forms of the bilateral mock theta functions.

Ramanujan stated without proof that the several identities holding between mock theta functions belonging to the first kind $f_0(q)$, $\Phi_0(q)$, $\Psi_0(q)$, $F_0(q)$, $\chi_0(q)$ and the five mock theta functions of the second kind $f_1(q)$, $\Phi_1(q)$, $\Psi_1(q)$, $F_1(q)$, $\chi_1(q)$ are interrelated amongst themselves. In §6, we show that the bilateral mock theta functions corresponding to the mock theta functions of order five of the first kind can be related to the ones corresponding to the second kind.

In §7, we give identities for the bilateral mock theta functions that are analogous to the identities given by Ramanujan [Watson 2, p. 277].

Notations and symbols

We use the following usual basic hypergeometric notation: For $|q| < 1$, $(a)_n = (a;q)_n = (1-a)(1-aq)\ldots\ldots\ldots(1-aq^{n-1})$ for $1 \le n \le \infty$, $(a)_0 =$

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$(a;q)_0 = 1$ and $(a)_n = (a;q)_n = \prod_{i=1}^{-n}(1-q^{-i}a)^{-1}$ for $n<0$, where a does not equal to q^i. Moreover,

$$ {}_r\psi_r\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q; z\right] = \sum_{n=-\infty}^{\infty} (a_1, a_2, \dots, a_r; q)_n z^n \Big/ (b_1, b_2, \dots, b_r; q)_n, $$

where $|b_1 b_2 \dots b_r / a_1 a_2 \dots a_r| < |z| < 1$ and $(a_1, a_2, \dots, a_r; q)_n = (a_1;q)_n \dots (a_r;q)_n$.

The symbol idem $(c_1; c_2)$, after the expression, means that the preceding expression is repeated with c_1 and c_2 interchanged.

2. Definitions of the mock theta functions of order five

Ramanujan defined ten mock theta functions of order five. These ten functions fall into two groups and are defined as:

$$ f_0(q) = \sum_0^\infty q^{n^2} \Big/ (-q)_n, \qquad \Phi_0(q) = \sum_0^\infty q^{n^2}(-q;q^2)_n, $$

$$ \psi_0(q) = \sum_0^\infty q^{(n+1)(n+2)/2}(-q)_n, \qquad F_0(q) = \sum_0^\infty q^{2n^2} \Big/ (q;q^2)_n, $$

$$ \chi_0(q) = \sum_0^\infty q^n \Big/ (q^{n+1})_n, $$

and

$$ f_1(q) = \sum_0^\infty q^{n(n+1)} \Big/ (-q)_n, \qquad \Phi_1(q) = \sum_0^\infty q^{(n+1)^2}(-q;q^2)_n, $$

$$ \psi_1(q) = \sum_0^\infty q^{n(n+1)/2}(-q)_n, \quad F_1(q) = \sum_0^\infty q^{2n(n+1)}/(q;q^2)_{n+1}, $$

$$ \chi_1(q) = \sum_0^\infty q^n \Big/ (q^{n+1})_{n+1}. $$

3. Definition and transformation of series for bilateral mock theta functions of order five

We shall denote by $f_{0c}(q)$ the bilateral form of $f_0(q)$ with similar notation for other functions. Using the following transformation of Bailey [3, p. 137, (5.20)]:

(i) $${}_2\psi_2\left[\begin{matrix} a,\ b \\ c,\ d \end{matrix};q;z\right] = \frac{(az,d/a,c/b,dq/abz;q)_\infty}{(z,d,q/b,cd/abz;q)_\infty}\,{}_2\psi_2\left[\begin{matrix} a,\ abz/d \\ az,\ c \end{matrix};q;d/a\right],$$

(ii) $${}_2\psi_2\left[\begin{matrix} a,\ b \\ c,\ d \end{matrix};q;z\right] = \frac{(az,bz,cq/abz,dq/abz;q)_\infty}{(q/a,q/b,c,d;q)_\infty}\,{}_2\psi_2\left[\begin{matrix} abz/c, abz/d \\ az, bz \end{matrix};q;cd/abz\right],$$

we can get an alternative from for the bilateral mock theta function $f_{0c}(q)$.

Making $a, b \to \infty$ and taking $c = -q$, $d = 0$, $z = q/ab$ in (ii), we have

$$\sum_{-\infty}^{\infty} q^{n^2}\Big/(-q)_n = \sum_{-\infty}^{\infty} q^{n(n+1)/2}(-1)_n.$$

Put

$$f_{0c}(q) = \sum_{-\infty}^{\infty} q^{n^2}\Big/(-q)_n \tag{1}$$

$$= \sum_{-\infty}^{\infty} q^{n(n+1)/2}(-1)_n \tag{2}$$

where the first expression is the definition of the bilateral mock theta function and the second expression is its alternative form.

Similarly we have the following definition of the other bilateral mock theta functions. We have given, in brackets, the value of the parameters taken in each case.

$$\Phi_{0c}(q) = \sum_{-\infty}^{\infty} q^{n^2}(-q;q^2)_n \tag{3}$$

$$= 1/2\sum_{-\infty}^{\infty} q^{2n^2}\Big/(-q;q^2)_n \tag{4}$$

($a \to \infty, b = -q, c = d = 0, z = -q/a$ and base changed to q^2)

$$\psi_{0c}(q) = \sum_{-\infty}^{\infty} q^{(n+1)(n+2)/2}(-q)_n \tag{5}$$

$$= (1/2)(1+q)\sum_{-\infty}^{\infty} q^{(n+1)^2}\Big/(-q^2)_n \tag{6}$$

($a \to \infty, b = -q, c = d = 0, z = -q^2/a$)

$$F_{0c}(q) = \sum_{-\infty}^{\infty} q^{2n^2} \Big/ (q;q^2)_n \tag{7}$$

$$= \sum_{-\infty}^{\infty} (-1)^n q^{n^2} (q;q^2)_n \tag{8}$$

$(a, b \to \infty, c = q, d = 0, z = q^2/ab,$ base changed to $q^2)$

$$f_{1c}(q) = \sum_{-\infty}^{\infty} q^{n(n+1)} \Big/ (-q)_n \tag{9}$$

$$= 2\sum_{-\infty}^{\infty} q^{n(n+1)/2} (-q)_n \tag{10}$$

$(a, b \to \infty, c = -q, d = 0, z = q^2/ab)$

$$\Phi_{1c}(q) = \sum_{-\infty}^{\infty} q^{(n+1)^2} (-q;q^2)_n \tag{11}$$

$$= 1/(1+q)\sum_{-\infty}^{\infty} q^{2n(n+1)} \Big/ (-q^3;q^2)_n \tag{12}$$

$(a \to \infty, b = q, c = d = 0, z = -q^3/a,$ base changed to $q^2)$

$$\psi_{1c}(q) = \sum_{-\infty}^{\infty} q^{n(n+1)/2} (-q)_n \tag{13}$$

$$= 1/2\sum_{-\infty}^{\infty} q^{n(n+1)} \Big/ (-q)_n \tag{14}$$

$(a \to \infty, b = -q, c = d = 0, z = -q/a)$

$$F_{1c}(q) = \sum_{-\infty}^{\infty} q^{2n(n+1)} \Big/ (q;q^2)_{n+1} \tag{15}$$

$$= \sum_{-\infty}^{\infty} (-1)^n q^{n(n+2)} (q;q^2)_n \tag{16}$$

$(a, b \to \infty, c = q^3, d = 0, z = q^4/ab,$ base changed to $q^2)$

4. Relations between bilateral mock theta functions of order five

Using the definitions and the alternative definitions of bilateral mock theta functions as given in §3, we have the following relations:

$$f_{0c}(q) = \sum_{-\infty}^{\infty} q^{n(n+1)/2}(-1)_n = 2\sum_{-\infty}^{\infty} q^{(n+1)(n+2)/2}(-q)_n$$

$$= 2\psi_{0c}(q) \tag{1}$$

Similarly we have

$$F_{0c}(q) = \Phi_{0c}(-q) \tag{2}$$

$$f_{1c}(q) = 2\psi_{1c}(q) \tag{3}$$

$$F_{1c}(q) = -1/q\Phi_{1c}(-q) \tag{4}$$

5. Still another definition of bilateral mock theta functions of order five

Using the general transformation of Slater [3, p 129 (5.4.3)] between ${}_2\psi_2$ series, we have still other alternative forms for the bilateral mock theta functions as given below. The advantage of using this transformation is that the c's are absent on the left-hand side, and we can choose c's in any convenient way. For $r = 2$, we have the transformation

$$\frac{(b_1, b_2, q/a_1, q/a_2, dz, q/dz; q)_\infty}{(c_1, c_2, q/c_1, q/c_2; q)_\infty} {}_2\psi_2\left[\begin{matrix} a_1, & a_2 \\ & & ; q; z \\ b_1, & b_2 \end{matrix}\right]$$

$$= \frac{(c_1/a_1, c_1/a_2, qb_1/c_1, qb_2/c_1, dc_1z/q, q^2/dc_1z; q)_\infty}{c_1/q(c_1, q/c_1, c_1/c_2, qc_2/c_1; q)_\infty}$$

$$\times {}_2\psi_2\left[\begin{matrix} qa_1/c_1, & qa_2/c_1 \\ & & ; q; z \\ qb_1/c_1, & qb_2/c_1 \end{matrix}\right],$$

where $d = a_1a_2/c_1c_2$ and $|b_1b_2/a_1a_2| < |z| < 1$.

Making $a_1, a_2 \to \infty$, and taking $b_1 = -q, b_2 = 0, z = q/a_1a_2$, we have another form for $f_{0c}(q)$.

$$\frac{(-q, q/c_1c_2, c_1c_2; q)_\infty}{(c_1c_2, q/c_1, q/c_2; q)_\infty} f_{0c}(q)$$

$$= \frac{q/c_1(-q^2/c_1, 1/c_2, c_2q; q)_\infty}{(c_1, q/c_1, c_2q/c_1, c_1/c_2; q)_\infty} \sum_{-\infty}^{\infty} q^{n(n+2)}/c_1^{2n}(-q^2/c_1; q)_n$$

$$+ \operatorname{idem}(c_1; c_2) \tag{1}$$

Similarly we obtain the following relations:

$$\frac{(-q,q^2/c_1c_2,c_1c_2/q;q)}{(c_1,c_2,q/c_1,q/c_2;q)_\infty} f_{1c}(q)$$

$$= \frac{q/c_1(-q^2/c_1,q/c_2,c_2;q)_\infty}{(c_1,q/c_1,c_2q/c_1,c_1/c_2;q)_\infty} \sum_{-\infty}^{\infty} q^{n(n+3)}/c_1^{2n}(-q^2/c_1;q)_n$$

$$+ \text{idem}(c_1;c_2) \tag{2}$$

$$\frac{(q,q^2/c_1c_2,c_1c_2;q^2)_\infty}{(c_1,c_2,q^2/c_1,q^2/c_2;q^2)_\infty} F_{0c}(q)$$

$$= \frac{q^2/c_1(q^3/c_1,1/c_2,c_2q^2;q^2)_\infty}{(c_1,q^2/c_1,c_2q^2/c_1,c_1/c_2;q^2)_\infty} \sum_{-\infty}^{\infty} q^{2n(n+2)}/c_1^{2n}(q^3/c_1;q^2)_n$$

$$+ \text{idem}(c_1;c_2) \tag{3}$$

$$\frac{(q^2,q^8/c_1c_2,c_1c_2/q^4;q^4)}{(c_1,c_2,q^4/c_1,q^4/c_2;q^4)_\infty} F_{1c}(q^2)$$

$$= \frac{q^4/c_1(q^{10}/c_1,q^4/c_2,c_2;q^4)_\infty}{(c_1,q^4/c_1,c_2q^4/c_1,c_1/c_2;q^4)_\infty} \sum_{-\infty}^{\infty} q^{4n(n+3)}/c_1^{2n}(q^{10}/c_1;q^4)_n$$

$$+ \text{idem}(c_1;c_2) \tag{4}$$

As $\varphi_{0c}(q), \psi_{0c}(q), \varphi_{1c}(q), \psi_{1c}(q)$ are related to the above functions (§4), we only give the transformations for the above four fifth order bilateral mock theta functions.

6. Some relations between bilateral mock theta functions of the first kind and the second kind

6.1. A relation between $f_{0c}(q)$ and $F_{1c}(q)$

If we put $c_1 = -q^6$ in (5.4) and use (3.1), we have

$$\frac{(q^2,-q^2/c_2,-c_2q^2;q^4)_\infty}{(-q^6,c_2,-q^{-2},q^4/c_2;q^4)_\infty} F_{1c}(q^2)$$

$$= \frac{-q^{-2}(-q^4,q^4/c_2,c_2;q4)_\infty}{(-q^6,-q^{-2},-q^6/c_2,-c_2q^{-2};q^4)_\infty} f_{0c}(q^4)$$

$$+ \frac{q^4/c_2(q^{10}/c_2,-q^{-2},-q^6;q^4)_\infty}{(c_2,q^4/c_2,-c_2q^{-6},-q^{10}/c_2;q^4)_\infty} \sum_{-\infty}^{\infty} q^{4n(n+3)}/c_2^{2n}(q^{10}/c_2;q^4)_n \tag{1}$$

If we take $c_2 = q^6$, the above relation can be written in the form

$$\frac{4q^2(1+q^4)^2(-q^8,-q^8;q^4)_\infty}{(1+q^2)(-q^6,-q^2,q^2;q^4)}F_{1c}(q^2)$$
$$=\frac{-(1-q^2)^2(q^6,q^6;q^4)_\infty}{(1+q^2)^2(-q^6,-q^4,-q^6;q^4)_\infty}f_{0c}(q^4)$$
$$+\frac{(1+q^2)^2(q^4,-q^6,-q^6;q^4)_\infty}{(1-q^2)^2(-q^4,-q^4,q^6,q^6;q^4)_\infty}\sum_{-\infty}^{\infty}q^{4n}/(q^4;q^4)_n \tag{2}$$

6.2. A relation between $f_1c(q)$ and $F_{0c}(q)$

If we put $c_1=-q$ in (5.3) and use (3.9), we have

$$\frac{(-q,-qc_2,-q/c_2;q^2)_\infty}{(-q,c_2,-q,q^2/c_2;q^2)_\infty}F_{0c}(q)$$
$$=\frac{-q(-q^2,1/c_2,c_2q^2;q_2)_\infty}{(-q,-q,-c_2q,-q/c_2;q^2)_\infty}f_{1c}(q^2)$$
$$+\frac{q^2/c_2(q^3/c_2,-1/q,-q^3;q^2)_\infty}{(c_2,q^2/c_2,-c_2/q,-q^3/c_2;q^2)_\infty}\sum_{-\infty}^{\infty}q^{2n(n+2)}/c_2^{2n}(q^3/c_2;q^2)_n \tag{3}$$

If we take $c_2=q$, the above relation can be put in the form

$$\frac{4(-q^2,-q^2;q^2)_\infty}{(q,-q,-q;q^2)_\infty}F_{0c}(q)=\frac{(q,q;q^2)_\infty}{(-q,-q,-q^2;q^2)_\infty}f_{1c}(q^2)$$
$$+\frac{(-q,-q,q^2;q^2)_\infty}{(-q^2,-q^2,q,q;q^2)_\infty}\sum_{-\infty}^{\infty}q^{2n(n+1)}/(q^2;q^2)_n \tag{4}$$

7. Bilateral equivalent forms of certain identities

The following identities for the bilateral mock theta functions are analogous to the identities of Ramanujan [Watson, 2, p. 277]

(Bo′) $$2F_{0c}(q)-\varphi_{0c}(-q)=\chi_0(q)+3F_0(q)-1$$

which can be written as

$$F_{0c}(q)=\chi_0(q)+3F_0(q)-1$$

(Co′) $$f_{0c}(-q)+2F_{0c}(q^2)=3(2\varphi_0(-q^2)-f_0(q))$$

$$\varphi_{0c}(-q^2)+\psi_{0c}(-q)=3\varphi_0(-q^2)-3/2f_0(q)$$

$$f_{0c}(-q)+2F_{0c}(q^2)=3\varphi_0(-q^2)+3\psi_0(-q)$$

(Bl′)
$$2F_{1c}(q) + 1/q\varphi_{1c}(-q) = -\chi_1(q) + 3F_1(q)$$
$$-2F_{1c}(q) + 1/q\varphi_{1c}(-q) = 3\chi_1(q) - 9F_1(q)$$
$$1/q\varphi_{1c}(-q) = \chi_1(q) - 3F_1(q)$$
$$F_{1c}(q) = -\chi_1(q) + 3F_1(q)$$

(Cl′)
$$\begin{aligned} f_{1c}(-q) - 2qF_{1c}(q^2) &= 3(1/q\varphi_1(-q^2) + \psi_1(-q)) \\ &= 6/q\varphi_1(-q^2) + 3f_1(q) \\ &= 3\vartheta_4(o,q)H(q) \end{aligned}$$

(Dl′)
$$2\psi_{1c}(q) - qF_{1c}(q^2) = 3\psi(q^2)G(q^4)$$

where

$$G(q) = \sum_0^\infty q^{n^2}/(q)_n, \quad H(q) = \sum_0^\infty q^{n(n+1)}/(q)_n, \quad \psi(q) = \sum_0^\infty q^{n(n+1)/2}.$$

We see that a study of bilateral mock theta functions is interesting in the sense that it gives various alternative forms and shows a close relationship among the mock theta functions, their bilateral forms and bilateral hypergeometric series.

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References

[1] Agarwal R. P., Mock theta functions-An analytical point of view, Proc. Nat. Acad. Sci. India **64** (1994), 95–107.

[2] Gupta Anju, On certain Ramanujan's mock theta functions, Proc. Indian Acad. Sci. **103** (1993), 257–267.

[3] Gasper G. and Rahman M., Basic Hypergeometric Series, Camb. Univ. Press, (1990).

[4] Watson G. N., The final problem: an account of the mock theta functions, J. London Math. Soc. **6** (1936), 55–80.

[5] Watson G. N., The mock theta functions (2), Proc. London Math. Soc. (2), **42** (1937), 274–304.

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