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The expression of functions as sums of finite differences on compact Abelian groups

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ABSTRACT. Let G be a Hausdorff compact Abelian group whose weight is not greater than the cardinality of the set of real numbers. We consider a special class, $\mathscr{H}(G)$, of functions in $L^2(G)$ whose Fourier series satisfy certain convergence conditions (stronger than absolute convergence). We show that G is topologically generated by not more than n elements if and only if, for each function f in $\mathscr{H}(G)$, there are $a_1, \ldots, a_n$ in G and functions $f_1, \ldots, f_n$ in $\mathscr{H}(G)$ such that $f = \sum_{j=1}^{n}(f_j - \delta_{a_j} * f_j)$, where $*$ is convolution defined in the usual sense, and δ_a denotes the Dirac measure at $a \in G$.

Introduction

Let G be a Hausdorff compact Abelian group. Let f be a Haar-measurable ction on G. Then a *difference of f* is a function of the form $f - \delta_y * f$ for ıe $y \in G$, where $\delta_y * f$ is the function given by $(\delta_y * f)(x) = f(x - y)$ for ıost all $x \in G$. Suppose V is a vector space of Haar-measurable functions G which is *translation invariant*; that is, $\{\delta_a * f : a \in G, f \in V\} \subseteq V$. Let f a given function in V. Suppose there are $(a_1, a_2, \ldots, a_n) \in G^n$ and functions $f_2, \ldots, f_n \in V$ such that

$$f = \sum_{j=1}^{n}(f_j - \delta_{a_j} * f_j).$$

Then f is said to be *represented as a sum of n differences* (in V).

The objective of this paper is to find out some relationships between the number of differences required to represent a function and the number of elements required to generate a compact Abelian group topologically and it is a continuation of the authors' work in [6]. In [6], it was shown that if G is a compact Abelian group and C is the component of the identity of G, and

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$\mathscr{H}(G)$ is a certain vector subspace of $L^2(G)$, then G/C can be topologically generated by n elements if and only if each function in $\mathscr{H}(G)$ can be represented as a sum of n differences. In this paper, we apply some results on the structure of Abelian groups to generalize the results in [6]. We will show that if G is a compact Abelian group whose *weight* is not greater than the cardinality of the set of real numbers (the weight of a topological space is the least cardinal number of an open basis of that space [8, p. 102]), then G can be topologically generated by n elements if and only if each function in $\mathscr{H}(G)$ can be represented as a sum of n differences. In particular, such result holds for all metrizable compact Abelian groups.

2. Preliminaries

Throughout this paper, all topological groups are assumed to be Hausdorff. If G is a locally compact Abelian group, then its dual group will be denoted by $\hat{G}$. Then $\boldsymbol{T}$ will denote the multiplicative circle group and $\boldsymbol{Q}$ will denote the additive group of rational numbers. The cardinality of the set of real number will be denoted by $\boldsymbol{c}$.

Let $\mathscr{J}$ be a non-void index set. Suppose for each $j \in \mathscr{J}$, G_j is an Abelian group with identity e_j. Then the *weak direct product* of G_j, denoted by $\prod_{j\in\mathscr{J}}^* G_j$, is the Abelian group consisting of $x = \prod_{j\in\mathscr{J}} x_j$ such that $x_j = e_j$ for all but a finite number of $j \in \mathscr{J}$. If $G_j = G$ for each $j \in \mathscr{J}$ and $\mathscr{A}$ is the cardinal number of $\mathscr{J}$, then $\prod_{j\in\mathscr{J}}^* G_j$ will also denoted by $G^{\mathscr{A}_*}$. For completeness, G^{0_*} is refered to as the trivial group.

Let G be an Abelian group. Then the *torsion free rank* of G will be denoted by $r_0(G)$ and for each prime number p, the *p-rank* of G will be denoted by $r_p(G)$. For details of the torsion free rank and the p-rank, please refer to [3, A.12, p. 444]. Note that $r_0(G)$ and $r_p(G)$ are cardinal numbers of certain subsets of G and they are not greater than that of G.

Let p be a prime number. Then $\boldsymbol{Z}(p^\infty)$ will be the countable subgroup of $\boldsymbol{T}$ whose elements are the solutions of $x^{p^n} = 1$ for some $n \in \boldsymbol{N}$. Thus, $\boldsymbol{Z}(p^\infty) = \bigcup_{n\in\boldsymbol{N}} \boldsymbol{Z}(p^n)$. As $\boldsymbol{Z}(p^\infty)$ is an increasing union of $\boldsymbol{Z}(p^n)$ for $n \in \boldsymbol{N}$, it follows that if H is a non-trivial proper subgroup of $\boldsymbol{Z}(p^\infty)$, then $H = \boldsymbol{Z}(p^n)$ for some $n \in \boldsymbol{N}$ [2, p. 16].

The following is a known result about the algebraic structure of the circle group and it can be derived from [3, 25.6–25.7, pp. 405–407].

Theorem 1. *Let $\mathscr{P}$ be the set of prime numbers. Then $\boldsymbol{T}$ is algebraically isomorphic to $\boldsymbol{Q}^{\boldsymbol{c}_*} \times \prod_{p\in\mathscr{P}}^* \boldsymbol{Z}(p^\infty)$.*

The following theorem is a consequence of [3, A.14–A.16, pp. 444–446].

THEOREM 2. *Let G be an Abelian group. Then G can be algebraically embedded into* $\boldsymbol{Q}^{r_0(G)_*} \times \left(\prod_{p\in\mathscr{P}}^* \boldsymbol{Z}(p^\infty)^{r_p(G)_*}\right)$, *where $\mathscr{P}$ is the set of prime numbers.*

By Theorem 1 and Theorem 2, one can determine whether or not a given Abelian group can be algebraically embedded into a product of a finite number of copies of the circle group. Later, these two theorems will be applied to obtain some of the main results of this paper.

3. Main results

Let G be a topological group and let X be a non-empty subset of G. Then G is said to be *topologically generated by X* if the smallest subgroup of G which contains X is dense in G. If G is topologically generated by X and X is a finite set with n elements, then G is said to be *n-thetic*. For the case when $n = 1$, G is usually called a *monothetic group*. If G is n-thetic but not $(n-1)$-thetic, then G is said to be *strictly n-thetic*. By definition, it is clear that an n-thetic group is strictly m-thetic for some $m \leq n$.

The structure of monothetic groups has been well studied. For example, it can be easily shown that a monothetic group is Abelian [5, Theorem 4.1 p. 267]. Also, a compact Abelian group is monothetic if and only if its dual group can be algebraically embedded into the circle group [3, 24.32, p. 390]. This result was generalized by Johnson [4]. In [4, Lemma 3.2], Johnson showed that a compact Abelian group is n-thetic if and only if its dual group can be algebraically embedded into $\boldsymbol{T}^n$. This, together with the previous theorems, means we can determine whether or not a given compact Abelian group is n-thetic.

The following is Lemma 2.2 of [1], in which the structure of strictly $(n+1)$-thetic connected non compact locally compact groups is discussed.

LEMMA 1. *Let G be an n-thetic topological group and let H be a continuous homomorphic image of G. Then H is n-thetic too.*

The following is a known result extracted from [4].

LEMMA 2. *Let G be a compact Abelian group and let C be the component of the identity. Then G/C is n-thetic if and only if $r_p(\hat{G}) \leq n$ for each prime number p.*

THEOREM 3. *Let G be a compact Abelian group and let C be the component of the identity. Let the weight of G be no greater than $\mathfrak{c}$ and let $n \in \boldsymbol{N}$ be given. Then G is n-thetic if and only if G/C is n-thetic.*

Proof. Suppose G is n-thetic. Then by Lemma 1, G/C is n-thetic too. Conversely, suppose G/C is n-thetic. Then for each prime number p, $r_p(\hat{G}) \leq n$ be Lemma 2. Meanwhile, the weight of $\hat{G}$ and the weight of G are equal each other [3, 24.14, p. 381] and, as $\hat{G}$ is discrete, it follows that the cardinality of $\hat{G}$ is at most $\boldsymbol{c}$. Thus, $r_0(\hat{G}) \leq \boldsymbol{c}$. Therefore, $\hat{G}$ can be algebraically embedded into $\boldsymbol{T}^n$ by Theorem 1 and Theorem 2. Consequently, G is n-thetic by [4, Lemma 3.2]. This completes the proof.

Theorem 4. *Let G be a compact Abelian group whose weight is not greater than $\boldsymbol{c}$. Let $n \in \boldsymbol{N}$ be given. Then G is strictly n-thetic if and only if G/C is strictly n-thetic.*

Proof. If $n = 1$, then the result holds trivially by Theorem 3. Thus, we only show the proof for the case when $n > 1$. Suppose $n > 1$. Let G be strictly n-thetic. Then G is n-thetic. Therefore, G/C is n-thetic by Theorem 3. Now we show that G/C is strictly n-thetic by a contrapositive argument. Assume that G/C is not strictly n-thetic. Then G/C is also m-thetic for some $m < n$. Thus, by Theorem 3, G is m-thetic with $m < n$. However, this would imply that G is not strictly n-thetic which contradicts the assumptions on G. Therefore, G/C is strictly n-thetic as well. Similarly, one can show that if G/C is strictly n-thetic, then G is strictly n-thetic too. Hence, the result follows.

Definition 1. Let G be a compact Abelian group and let μ_G denote a given Haar measure on G. Let

$$\mathscr{H}(G) = \left\{ f \in L^2(G) : \int_G f \, d\mu_G = 0 \quad \text{and} \quad \sum_{\gamma \in \hat{G}} |\hat{f}(\gamma)|^{1/2} < \infty \right\},$$

where $\hat{f}$ is the Fourier transform of f. Note that $\mathscr{H}(G)$ is a vector subspace of $L^2(G)$.

Definition 2. Let G be a compact Abelian group. Then $\mathscr{N}_{\#}(G) \in \boldsymbol{N} \cup \{\infty\}$ is defined as follows. If there is $n \in \boldsymbol{N}$ such that each function in $\mathscr{H}(G)$ can be represented as a sum of n differences but there is $f \in \mathscr{H}(G)$ such that f cannot be represented as a sum of $n-1$ differences, then $\mathscr{N}_{\#}(G)$ is defined to be n. Otherwise, $\mathscr{N}_{\#}(G)$ is taken to be ∞. Thus, $\mathscr{N}_{\#}(G)$ is the least positive integer n such that each function in $\mathscr{H}(G)$ can be represented as a sum of n differences, if such an integer exists; otherwise $\mathscr{N}_{\#}(G) = \infty$.

In [6, Theorem 8], it was shown that if G is a compact Abelian group, then $\mathscr{N}_{\#}(G) = n$ if and only if G/C is strictly n-thetic, where C is the component of the identity of G. This implies that a totally disconnected compact Abelian group is strictly n-thetic if and only if $\mathcal{N}_{\#}(G) = n$. The following result extends this observation to groups which are not necessarily totally disconnected.

THEOREM 5. *Let G be a compact Abelian group whose weight is not greater than $\mathbf{c}$. Then G is strictly n-thetic if and only if $\mathcal{N}_{\#}(G) = n$.*

PROOF. As the weight of G is not greater than $\mathbf{c}$, G is strictly n-thetic if and only if G/C is strictly n-thetic by Theorem 4. Hence, the result follows by [6, Theorem 8].

As an n-thetic topological group is strictly m-thetic for some $m \leq n$, by Theorem 5 and the definition of $\mathcal{N}_{\#}(G)$ for a given compact Abelian group G, one can deduce the following result.

THEOREM 6. *Let G be a compact Abelian group whose weight is not greater than $\mathbf{c}$. Then G is n-thetic if and only if for each f in $\mathscr{H}(G)$, there are $(a_1, a_2, \dots, a_n) \in G^n$ and functions $f_1, f_2, \dots, f_n$ in $\mathscr{H}(G)$ such that*

$$f = \sum_{j=1}^{n} (f_j - \delta_{a_j} * f_j).$$

COROLLARY. *Let G be a metrisable compact Abelian group. Then G is n-thetic if and only if for each f in $\mathscr{H}(G)$, there are $(a_1, a_2, \dots, a_n) \in G^n$ and functions $f_1, f_2, \dots, f_n$ in $\mathscr{H}(G)$ such that*

$$f = \sum_{j=1}^{n} (f_j - \delta_{a_j} * f_j).$$

PROOF. As G is metrisable, $\hat{G}$ is countable [8, p. 96]. Thus, the weight of G is less than $\mathbf{c}$. Hence, the result follows by Theorem 6.

REMARK 1. In Theorem 5 and Theorem 6, the condition that the group's weight is not greater than $\mathbf{c}$ cannot be dropped. For, let G be a compact Abelian group whose weight is greater than $\mathbf{c}$. Then $\hat{G}$ has the same weight as G and, as $\hat{G}$ is discrete, this weight is the cardinality of $\hat{G}$. Thus, by Theorem 1, $\hat{G}$ is not algebraically isomorphic to any subgroup of $\boldsymbol{T}^n$ for any $n \in \boldsymbol{N}$. Thus, by Lemma 3.2 of [4], such a compact Abelian group is not n-thetic for any $n \in \boldsymbol{N}$. In particular, such a group G is not monothetic. On the other hand, it is proved in [7] that when G is a compact connected abelian group, each function in $\mathscr{H}(G)$ can be represented as a single difference. Thus, Theorem 5 and Theorem 6 both fail when G is a connected compact Abelian group whose weight is greater than $\mathbf{c}$.

REMARK 2. For a compact Abelian group G, if $\mathscr{H}(G)$ is given by

$$\mathscr{H}(G) = \left\{ f \in L^2(G) : \int_G f \, d\mu_G = 0 \text{ and } \sum_{\gamma \in \hat{G}} |\hat{f}(\gamma)|^\varepsilon < \infty \right\},$$

where ε is some given number in $(0, 1/2]$, then the results of Theorem 5 and Theorem 6 are still valid.

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