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Positive solutions for a class of nonlinear elliptic problems

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ABSTRACT. This paper deals with multiplicity results of the boundary value problem

$$(p(t)\phi(u'))' = -p(t)(\lambda f(u) + g(u)), a < t < b$$

$$u(a) = 0 = u(b),$$

where f is ϕ-sublinear (superlinear) at 0, g is ϕ-superlinear (sublinear) at 0 and ∞, and λ is a positive parameter. Analogous results for systems will also be established.

1. Introduction

Consider the quasilinear elliptic boundary value problem

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) = -(\lambda|u|^{q-2}u + |u|^{\alpha-2}u), & x \in \Omega \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $1 < q < p < \alpha \le p^*$, with $p^* = \dfrac{Np}{N-p}$ if $p < N$ and $p^* = \infty$ if $p = N$, and λ is a positive parameter.

Problem (1.1) with $p = 2$ was considered in [1, 3, 5]. The general case $p > 1$ has been studied in [2, 4, 7]. It was shown in [4, 7] that (1.1) has at least two positive solutions for $\lambda > 0$ sufficiently small. These results were extended in [2], in which, assuming Ω to be a ball, the authors proved the existence of two positive radial solutions to (1.1) for $\lambda \in (0, \Lambda)$, where $\Lambda = \sup\{\lambda > 0 : (1.1)$ has a positive radial solution$\}$. In this paper, we shall extend the multiplicity result in [2] to positive radial solutions of the general quasilinear elliptic problem

$$\begin{cases} div(\alpha(|\nabla u|^2)\nabla u) + \lambda f(u) + g(u) = 0, & a < |x| < b \\ u = 0, \quad |x| \in \{a, b\} \end{cases}$$

on an annulus, where α, f, $g : R^+ \to R^+$. Since we look for radial solutions, we shall consider the following boundary value problem

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$$(1.2) \qquad \begin{cases} (p(t)\phi(u'))' = -p(t)(\lambda f(u) + g(u)), & a < t < b \\ u(a) = u(b) = 0, \end{cases}$$

where ϕ is an odd increasing homeomorphism on R, f is ϕ-sublinear (superlinear) at 0 and g is ϕ-superlinear (sublinear) at 0 and ∞. Similar results for systems will also be established. Note that the proof in [2] depends on scaling arguments and therefore does not apply to general quasilinear term and nonlinearities. We overcome this by first establishing a lower bound for the sup-norm of possible solutions of (1.2) and then define a suitable operator whose fixed points are positive solutions of (1.2). Our approach is based on degree theoretic argument and sub-supersolutions method.

2. Existence results

We first consider the case when f is ϕ-sublinear at 0 and g is ϕ-superlinear at 0 and ∞. We shall impose the following assumptions:

(A.1) $p : [a, b] \to (0, \infty)$ is continuous

(A.2) ϕ is an odd, increasing homeomorphism on R, and for each $c > 0$, there exists a positive number $A_c > 0$ such that

$$\phi(cx) \geq A_c \phi(x)$$

for every $x > 0$.

(A.3) f, g are increasing, continuous functions on R^+ such that

$$\lim_{u \to 0} \frac{f(u)}{\phi(u)} = \infty, \quad \lim_{u \to \infty} \frac{g(u)}{\phi(u)} = \infty$$

and

$$\lim_{u \to 0} \frac{g(u)}{\phi(u)} = 0.$$

Then we have

THEOREM 2.1. *Let* (A.1)–(A.3) *hold. Then there exists a positive number* $\lambda^* > 0$ *such that* (1.2) *has at least two positive solutions for* $\lambda < \lambda^*$, *at least one for* $\lambda = \lambda^*$ *and none for* $\lambda > \lambda^*$.

We first recall the following

LEMMA 2.2. [6] *Let* u *satisfy*

$$\begin{cases} (p(t)\phi(u'))' \leq 0, & a < t < b \\ u(a) = u(b) = 0. \end{cases}$$

Then

$$u(t) \geq K|u|_0 r(t),$$

where $r(t) = \dfrac{1}{b-a}\min(t-a, b-t)$ and K is a positive constant. Here $|\,.\,|_0$ denotes the sup-norm.

The next lemma gives a priori estimates for solutions of (1.2).

Lemma 2.3. *There exist positive numbers C_λ and C, with $C_\lambda \to \infty$ as $\lambda \to \infty$, such that any nontrivial solution of*

$$\begin{cases} (p(t)\phi(u'))' \leq -p(t)(\lambda f(u) + g(u)), & a < t < b \\ u(a) = u(b) = 0 \end{cases} \tag{2.1}$$

satisfies

$$C_\lambda \leq |u|_0 \leq C.$$

In the rest of the paper, we assume that $0 < p_0 \leq p(t) \leq p_1$ for every $t \in [a,b]$, $f(u) = f(0)$ and $g(u) = g(0)$ for $u \leq 0$. We shall denote by $C_k, k = 1, 2, \ldots$ various constants.

Proof of lemma 2.3. Let u satisfy (2.1). A comparison argument shows that $u \geq v$, where v is the solution of

$$\begin{cases} (p(t)\phi(v'))' = -p(t)(\lambda f(u) + g(u)), & a < t < b \\ v(a) = v(b) = 0. \end{cases}$$

Note that

$$v(t) = \int_a^t \phi^{-1}\left\{\frac{M - \int_a^s p(\tau)(\lambda f(u) + g(u))\,d\tau}{p(s)}\right\} ds$$

where M is such that $v(b) = 0$.

Let $|v|_0 = v(t_0)$ for some $t_0 \in (a,b)$. Then $v'(t_0) = 0$ and we have

$$u(t) \geq \int_a^t \phi^{-1}\left\{\frac{\int_s^{t_0} p(\tau)(\lambda f(u) + g(u))\,d\tau}{p(s)}\right\} ds.$$

Let $[a_1, b_1] \subset (a,b)$. If $t_0 \geq \dfrac{a_1 + b_1}{2}$ then it follows from Lemma 2.2 that

$$|u|_0 \geq u(a_1) \geq (a_1 - a)\phi^{-1}\left\{\frac{\lambda p_0(b_1 - a_1)}{2p_1} f(|u|_0\delta)\right\},$$

where $\delta = K \min_{a_1 \le t \le b_1} r(t)$, or

$$\frac{\phi\left(\frac{|u|_0}{a_1 - a}\right)}{f(|u|_0 \delta)} \ge \frac{\lambda p_0 (b_1 - a_1)}{2 p_1}. \tag{2.2}$$

If $t_0 \le \frac{a_1 + b_1}{2}$ then by rewriting u as

$$u(t) \ge \int_t^b \phi^{-1}\left\{\frac{\int_{t_0}^s p(\tau)(\lambda f(u) + g(u))\,d\tau}{p(s)}\right\} ds,$$

we obtain

$$\frac{\phi\left(\frac{|u|_0}{b - b_1}\right)}{f(|u|_0 \delta)} \ge \frac{\lambda p_0 (b_1 - a_1)}{2 p_1}. \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$\frac{\phi(|u|_0 \gamma)}{f(|u|_0 \delta)} \ge \frac{\lambda p_0 (b_1 - a_1)}{2 p_1} = \lambda C_1,$$

where $\gamma = \max\left(\frac{1}{a_1 - a}, \frac{1}{b - b_1}\right)$, and hence

$$\frac{\phi(|u|_0 \delta)}{f(|u|_0 \delta)} \ge \lambda C_2$$

by (A.2). Since $\lim_{x \to 0} \frac{\phi(x)}{f(x)} = 0$, it follows that there exists $C_\lambda > 0$ with $C_\lambda \to \infty$ as $\lambda \to \infty$ such that $|u|_0 \ge C_\lambda$. Similarly, we have

$$\frac{\phi(|u|_0 \delta)}{g(|u|_0 \delta)} \ge C_3, \tag{2.4}$$

and therefore $|u|_0 \le C$ for some $C > 0$ independent of λ. □

From Lemmas 2.2 and 2.3, we see that u is a positive solution of (1.2) iff u satisfies

$$\begin{cases} (p(t)\phi(u'))' = -p(t)(\tilde{f}(t, u, \lambda) + g(u)), & a < t < b \\ u(a) = u(b) = 0, \end{cases} \tag{2.5$_\lambda$}$$

where $\tilde{f}(t, u, \lambda) = \lambda f(\max(u, \tilde{C}_\lambda r(t)))$ and $\tilde{C}_\lambda = K C_\lambda$. Without loss of generality, we assume that C_λ is nondecreasing with respect to λ. For each $v \in C[a, b]$, we define $u = A(\lambda, v)$ to be the solution of

$$\begin{cases} (p(t)\phi(u'))' = -p(t)(\tilde{f}(t,v,\lambda) + g(v)), & a < t < b \\ u(a) = u(b) = 0. \end{cases}$$

Then is can be verified that $A(\lambda, .) : C[a,b] \to C[a,b]$ is completely continuous and fixed points of $A(\lambda, .)$ are solutions of $(2.5)_\lambda$.

Now we show the existence of a solution to $(2.5)_\lambda$ of $\lambda > 0$ small.

LEMMA 2.4. *There exists a positive number $\bar{\lambda} > 0$ such that $(2.5)_\lambda$ has a solution for $\lambda < \bar{\lambda}$.*

PROOF. Let u satisfy $u = \theta A(\lambda, u)$ for some $\theta \in [0,1]$ and let $t_0 \in (a,b)$ be such that $u'(t_0) = 0$. By integrating, we obtain

$$\begin{aligned} u(t) &= \theta \int_a^t \phi^{-1}\left\{\frac{\int_s^{t_0} p(\tau)(\tilde{f}(\tau,u,\lambda) + g(u))\,d\tau}{p(s)}\right\} ds \\ &\le \int_a^t \phi^{-1}\left\{\frac{\int_s^{b_0} p(\tau)(\tilde{f}(\tau,u,\lambda) + g(u))\,d\tau}{p(s)}\right\} ds \end{aligned}$$

and so

$$|u|_0 \le (b-a)\phi^{-1}\{\lambda\bar{p}f(\max(|u|_0, \tilde{C}_\lambda) + \bar{p}g(|u|_0)\}, \tag{2.5}$$

where $\bar{p} = \dfrac{p_1}{p_0}(b-a)$.

From (A.2) and the fact that $\lim_{x\to 0} \dfrac{\phi(x)}{g(x)} = \infty$, is follows that there exists a positive number r such that

$$\phi\left(\frac{r}{b-a}\right) > 2\bar{p}g(r). \tag{2.6}$$

Now, let $\bar{\lambda} \in (0,1)$ be such that

$$\phi\left(\frac{r}{b-a}\right) > 2\bar{\lambda}\bar{p}f(\max(r, \tilde{C}_1)). \tag{2.7}$$

Adding (2.6) and (2.7), we obtain

$$\phi\left(\frac{r}{b-a}\right) > \bar{\lambda}\bar{p}f(\max(r, \tilde{C}_1)) + \bar{p}g(r),$$

which implies that

$$r > (b-a)\phi^{-1}\{\lambda\bar{p}f(\max(r, \tilde{C}_\lambda)) + \bar{p}g(r)\} \tag{2.8}$$

for $\lambda \in (0, \bar{\lambda})$.

Combining (2.5) and (2.8), we deduce that $|u|_0 \neq r$ and the existence of a fixed point of $A(\lambda, \cdot)$ follows from the Leray-Schauder fixed point Theorem. □

The following nonexistence result is an immediate consequence of Lemma 2.3.

LEMMA 2.5. *There is no positive solution to* $(2.5)_\lambda$ *for* $\lambda > 0$ *large enough.*

Let us define $\Lambda = \{\lambda > 0 : (2.5)_\lambda \text{ has a solution}\}$ and let $\lambda^* = \sup \Lambda$. By Lemmas 2.4 and 2.5, $0 < \lambda^* < \infty$. By standard limiting processes, it follows that $(2.5)_{\lambda^*}$ has a solution u_{λ^*}.

We are now ready to give the

PROOF OF THEOREM 2.1. Let $0 < \lambda < \lambda^*$. Since u_{λ^*} is a supersolution and 0 is a subsolution for $(2.5)_\lambda$, there exists a solution u_λ of $(2.5)_\lambda$ with $0 \leq u_\lambda \leq u_{\lambda^*}$. We next establish the existence of a second solution. Define

$$\Theta = \{u \in C^1[a,b] : 0 < u < u_{\lambda^*} \text{ on}(a,b), u'(b) > u'_{\lambda^*}(b)$$
$$u'(a) < u'_{\lambda^*}(a), u'(a) > 0, u'(b) < 0\},$$

and

$$\Delta = \{u \in C[a,b] : 0 \leq u \leq u_{\lambda^*}\}.$$

We claim that $A(\lambda, .) : \Delta \to \Theta$. Indeed, let $u = A(\lambda, v)$ with $v \in \Delta$. Then

$$\begin{aligned} (p(t)\phi(u'))' &= -p(t)(\lambda f(\max(v, \tilde{C}_\lambda r(t)) + g(v)) \\ &\geq -p(t)(\lambda f(\max(u_{\lambda^*}, \tilde{C}_{\lambda^*} r(t)) + g(u_{\lambda^*})) \\ &= -p(t)(\lambda f(u_{\lambda^*}) + g(u_{\lambda^*})) \\ &= (p(t)\phi(u'_{\lambda^*}))' + p(t)(\lambda^* - \lambda) f(u_{\lambda^*}). \end{aligned} \tag{2.9}$$

Let $t_0 \in (a,b)$ be such that $u'(t_0) = u'_{\lambda^*}(t_0)$. By (2.9),

$$p(t)(\phi(u') - \phi(u'_{\lambda^*})) > 0 \text{ on } (t_0, b],$$

which implies that $u < u_{\lambda^*}$ on (t_0, b) and $u'(b) > u'_{\lambda^*}(b)$. Similarly, $u < u_{\lambda^*}$ on $(a, t_0]$ and $u'(a) < u'_{\lambda^*}(a)$. By Lemma 2.2, $u'(a) > 0$, $u'(b) < 0$ and the claim is proved. Since Θ is open, convex and $u_\lambda \in \Theta$, we infer that

$$\deg(I - A(\lambda, .), \Theta, 0) = 1.$$

On the other hand, since solutions of $(2.5)_\lambda$ and bounded in the C^1-norm uniformly on bounded intervals,

$$\deg(I - A(\lambda, .), B(0, R), 0) = \text{constant},$$

when R is large enough. Here $B(0,R)$ denotes the open ball centered at 0 with radius R in $C^1[a,b]$. By Lemma 2.5, the constant is zero and therefore

$$\deg(I - A(\lambda, .), B(0,R)\backslash\overline{\Theta}, 0) = -1.$$

Hence $A(\lambda, .)$ has a fixed point $u \notin \overline{\Theta}$, completing the proof of Theorem 2.1. □

Next, we consider the case when f is ϕ-superlinear at 0 and g is ϕ-sublinear at 0 and ∞.

Assume

(A.3′) f, g are increasing continuous functions on R^+ such that

$$\lim_{u\to 0}\frac{f(u)}{\phi(u)} = 0, \quad \lim_{u\to 0}\frac{g(u)}{\phi(u)} = \infty$$

and

$$\lim_{u\to\infty}\frac{g(u)}{\phi(u)} = 0.$$

Then we have

THEOREM 2.6. *Let* (A.1), (A.2) *and* (A.3′) *hold. Then there exists a positive number* λ^* *such that* (1.2) *has at least two positive solutions for* $\lambda < \lambda^*$, *at least one for* $\lambda = \lambda^*$, *and none for* $\lambda > \lambda^*$.

The proof of Theorem 2.6 follows the same lines as that of Theorem 2.1, with Lemma 2.3 replaced by

LEMMA 2.7. *There exist positive numbers* C_λ *and* C, *with* $C_\lambda \to 0$ *as* $\lambda \to \infty$, *such that any nontrivial solution of*

$$\begin{cases} (p(t)\phi(u'))' = -p(t)(\lambda f(u) + g(u)) \\ u(a) = u(b) = 0 \end{cases}$$

satisfies

$$C \le |u|_0 \le C_\lambda.$$

Finally, we consider the following system

$$\begin{cases} (p(t)\phi(u'))' = -p(t)(\lambda f(v) + g(v)), & a < t < b \\ (q(t)\psi(v'))' = -q(t)(\lambda h(u) + k(u)), & a < t < b \\ u(a) = u(b) = 0, \quad v(a) = v(b) = 0. \end{cases} \tag{2.10}$$

It is assumed that

(A.4) $q : [a,b] \to (0,\infty)$ is continuous.

(A.5) ψ is an odd, increasing homeomorphism on R, and for each $c > 0$, there exists a positive number B_c such that

$$\psi(cx) \geq B_c \psi(x)$$

for every $x > 0$.

(A.6) h, k are increasing, continuous functions on R^+ such that

$$\lim_{u\to 0} \frac{h(u)}{\psi(u)} = \infty, \quad \lim_{u\to\infty} \frac{k(u)}{\psi(u)} = \infty$$

and

$$\lim_{u\to 0} \frac{k(u)}{\psi(u)} = 0.$$

THEOREM 2.8. *Let* (A.1)–(A.6) *hold. Then there exists a positive number* $\lambda^* > 0$ *such that* (2.10) *has at least two positive solutions for* $\lambda < \lambda^*$, *at least one for* $\lambda = \lambda^*$, *and none for* $\lambda > \lambda^*$.

We first establish a result analogous to Lemma 2.3 for the system (2.10).

LEMMA 2.9. *There exist positive numbers* C_λ *and* C, *with* $C_\lambda \to \infty$ *as* $\lambda \to \infty$, *such that any nontrivial solutions* (u, v) *of*

$$\begin{cases} (p(t)\phi(u'))' \leq -p(t)(\lambda f(v) + g(v)), & a < t < b \\ (q(t)\psi(v'))' \leq -q(t)(\lambda h(u) + k(u)), & a < t < b \\ u(a) = u(b) = 0, \quad v(a) = v(b) = 0 \end{cases}$$

satisfies

$$C_\lambda \leq |u|_0, |u|_0 \leq C.$$

PROOF. Let $K_1 > 0$ be such that $u(t) \geq K_1 |u|_0 r(t)$, $t \in [a,b]$, for every u satisfying $(q(t)\psi(u'))' \leq 0$, $u(a) = u(b) = 0$. As in the proof of Lemma 2.3, we have

$$\phi(\gamma|u|_0) \geq \lambda C_1 f(\delta|v|_0) + C_1 g(\delta|v|_0),$$

and

$$\psi(\gamma|v|_0) \geq \lambda C_4 h(\delta_1|u|_0) + C_4 k(\delta_1|u|_0),$$

where $\delta = K \min_{a_1 \leq t \leq b_1} r(t)$, $\delta_1 = K_1 \min_{a_1 \leq t \leq b_1} r(t)$, $[a_1, b_1] \subset (a,b)$, and $\gamma = \max\left(\frac{1}{a_1 - a}, \frac{1}{b - b_1}\right)$.

If $|u|_0 \geq |v|_0$ then $\psi(\gamma|v|_0) \geq \lambda C_4 h(\delta_1|v|_0)$, and hence

$$\frac{\psi(\delta_1|v|_0)}{h(\delta_1|v|_0)} \geq \lambda C_5,$$

which implies $|v|_0 \geq C_{1,\lambda} > 0$. Similarly, if $|u|_0 \leq |v|_0$ then $|u|_0 \geq C_{2,\lambda} > 0$. In either case, $|u|_0, |v|_0 \geq C_\lambda$ where $C_\lambda \to \infty$ as $\lambda \to \infty$. The uniform bounds for u, v can be derived in a similar manner. □

For $(\tilde{u}, \tilde{v}) \in C[a,b] \times C[a,b]$, Let $(u,v) = B(\lambda, \tilde{u}, \tilde{v})$ be the solution of

$$(2.11)_\lambda \qquad \begin{cases} (p(t)\phi(u'))' = -p(t)(f(t,\tilde{v},\lambda) + g(\tilde{v})), & a < t < b \\ (q(t)\psi(v'))' = -q(t)(h(t,\tilde{u},\lambda) + k(\tilde{u})), & a < t < b \\ u(a) = u(b) = 0, v(a) = v(b) = 0, \end{cases}$$

where $f(t,\tilde{v},\lambda) = \lambda f(\max(\tilde{v}, \tilde{C}_\lambda r(t))$, $h(t,\tilde{u},\lambda) = \lambda h(\max(\tilde{u}, \tilde{C}_\lambda r(t))$, $\tilde{C}_\lambda = \min(K, K_1)C_\lambda$ and C_λ is given by Lemma 2.9. Then (u,v) is a solution of $(2.11)_\lambda$ iff (u,v) is a positive solution of (2.10).

The next Lemma gives existence of solutions to $(2.11)_\lambda$ for $\lambda > 0$ small.

Lemma 2.10. *There exists* $\tilde{\lambda} > 0$ *such that* $(2.11)_\lambda$ *has a solution for* $\lambda < \tilde{\lambda}$.

Proof of Lemma 2.10. Let (u,v) be a solution of $(u,v) = \theta B(\lambda, u, v)$ for some $\theta \in (0,1)$. Suppose that $0 < q_0 \leq q(t) \leq q_1$ for every $t \in [a,b]$. Then we have

$$|u|_0 \leq (b-a)\phi^{-1}\{\lambda \bar{p} f(\max(|v|_0, \tilde{C}_\lambda)) + \bar{p} g(|v|_0)\}$$

and

$$|v|_0 \leq (b-a)\psi^{-1}\{\lambda \bar{q} h(\max(|u|_0, \tilde{C}_\lambda)) + \bar{q} k(|u|_0)\},$$

where $\bar{p} = \dfrac{p_1(b-a)}{p_0}$, $\bar{q} = \dfrac{q_1(b-a)}{q_0}$.

Let $|(u,v)|_0 = \max(|u|_0, |v|_0)$. If $|u|_0 \geq |v|_0$ then

$$\phi\left(\frac{|u|_0}{b-a}\right) \leq \lambda \bar{p} f(\max(|u|_0, \tilde{C}_\lambda)) + \bar{p} g(|u|_0),$$

while if $|u|_0 \leq |v|_0$, we have

$$\psi\left(\frac{|v|_0}{b-a}\right) \leq \lambda \bar{q} h(\max(|v|_0, \tilde{C}_\lambda)) + \bar{q} k(|v|_0),$$

Choose $r > 0$ so that

$$\phi\left(\frac{r}{b-a}\right) > 2\bar{p}g(r), \quad \psi\left(\frac{r}{b-a}\right) > 2\bar{q}k(r),$$

and let $\tilde{\lambda} \in (0,1)$ be such that

$$\phi\left(\frac{r}{b-a}\right) > 2\tilde{\lambda}\bar{p}f(\max(r, \tilde{C}_1)), \quad \psi\left(\frac{r}{b-a}\right) > 2\tilde{\lambda}\bar{q}h(\max(r, \tilde{C}_1)).$$

Then it is easy to see that $|(u,v)|_0 \neq r$ for $\lambda < \tilde{\lambda}$, and the Lemma follows from the Leray-Schauder fixed point Theorem. □

Proof of Theorem 2.8. We shall only give a sketch of proof since the details are similar to that of Theorem 2.1. Define $\Lambda = \{\lambda > 0 : (2.11)_\lambda \text{ has a solution}\}$ and let $\lambda^* = \sup \Lambda$. By Lemmas 2.9 and 2.10, $0 < \lambda^* < \infty$. By standard limiting processes $(2.11)_{\lambda^*}$ has a solution $(u_{\lambda^*}, v_{\lambda^*})$. Let $\lambda \in (0, \lambda^*)$, then there exists a solution (u_λ, v_λ) of $(2.11)_\lambda$ with $0 \leq u_\lambda \leq u_{\lambda^*}$ and $0 \leq v_\lambda \leq v_{\lambda^*}$.

Let

$$\Theta = \Big\{ (u,v) \in C^1[a,b] \times C^1[a,b] : 0 < u < u_{\lambda^*}, 0 < v < v_{\lambda^*},$$
$$\frac{\partial u}{\partial n} > 0, \frac{\partial(u_{\lambda^*} - u)}{\partial n} > 0, \frac{\partial v}{\partial n} > 0, \frac{\partial(v_{\lambda^*} - v)}{\partial n} > 0 \text{ at } a, b \Big\},$$

where n denotes the unit outer normal of (a,b). Then Θ is open, convex in $C^1[a,b] \times C^1[a,b]$ and $(u_\lambda, v_\lambda) \in \Theta$. As in the proof of Theorem 2.1, we obtain

$$\deg(I - B(\lambda, .), B(0,R) \backslash \overline{\Theta}, 0) = -1,$$

for large R, where $B(0,R)$ denotes the open ball centered at 0 with radius R in $C^1[a,b] \times C^1[a,b]$, and the existence of a second solution follows. □

References

[1] A. Ambrozetti, H. Brezis, and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), 519–543.

[2] A. Ambrozetti, J. Garcia Azorero and I. Peral, Multiplicity results for some nonlinear elliptic equations, *J. Funct. Anal* **137** (1996), 219–242.

[3] T. Bartsch and M. Willem, On an elliptic equation with concave and convex nonlinearities, *Proc. A. M. S.* **123** (1995), 3555–3561.

[4] L. Boccardo, M. Escobedo and I. Peral, A Dirichlet problem involving critical exponents, *Nonlinear Anal.* **24** (1995), 1639–1648.

[5] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), 437–477.

[6] H. Dang, R. Manasevich and K. Schmitt, Positive radial solutions for some nonlinear partial differential equations, *Math. Nachr.* **186** (1997), 101–113.

[7] J. Garcia Azorero and I. Peral Alonso, Some results about the existence of a second positive solution in a quasilinear critical problem, *Indiana Univ. Math, J.* **43** (1994), 941–957.

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