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The distribution of zeros of solutions of neutral differential equations

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ABSTRACT. In this paper we establish an estimate for the distance between adjacent zeros of the oscillatory solutions of the neutral delay differential equation $[x(t) + P(t)x(t-\tau)]' + Q(t)x(t-\sigma) = 0$, where $P, Q \in C([t_0, \infty), \mathbf{R}^+)$ and $\tau, \sigma \in \mathbf{R}^+$.

1. Introduction

Consider the first order neutral delay differential equation

$$[x(t) + P(t)x(t-\tau)]' + Q(t)x(t-\sigma) = 0 \tag{1}$$

where

$$P \in C([t_0, \infty), [0, \infty)), \qquad Q \in C([t_0, \infty), (0, \infty)), \qquad \sigma > \tau > 0. \tag{2}$$

When $P(t) \equiv 0$, Eq.(1) reduces to

$$x'(t) + Q(t)x(t-\sigma) = 0. \tag{3}$$

The oscillation theory of neutral differential equations has been extensively developed during the past several years. We refer to the monographs by Györi and Ladas [1], Bainov and Mishev [2], Erbe, Kong and Zhang [3], and the references cited therein. But the results dealing with the distribution of zeros of the oscillatory solution of neutral differential equation are relatively scarce. Recently, Erbe et al. [3] and Liang [4] established estimates for the distance between adjacent zeroes of the solutions of Eq.(3). Zhou and Wang [5] extend the results in [3]. In this paper, by using a new technique, we establish an estimate for the distance between adjacent zeroes of the solutions of Eq.(1). Our results improve the known results in [3–5].

Let $m = \max\{\tau, \sigma\}$. By a solution of Eq.(1) we mean a function $x \in C([t_x - m, \infty), \mathbf{R})$, for some $t_x \geq t_0$, such that $x(t) + P(t)x(t-\tau)$ is continuously differentiable on $[t_x, \infty)$ and such that Eq.(1) is satisfied for $t \geq t_x$.

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Assume that (2) holds and let $\phi \in C([t_0 - m, t_0], \mathbf{R})$ be a given initial function. Then one can easily see by the method of steps that Eq.(1) has a unique solution $x \in C([t_0 - m, \infty), \mathbf{R})$ such that $x(t) = \phi(t)$ for $t_0 - m \le t \le t_0$.

2. Main results

First we define a sequence $\{a_i\}$ by

$$a_1 = e^{\rho}, \qquad a_{i+1} = e^{\rho a_i}, \qquad i = 1, 2, \dots. \tag{4}$$

It is easily seen that for $\rho > 0$.

$$a_{i+1} > a_i, \qquad i = 1, 2, \dots.$$

Observe that when $\rho > \frac{1}{e}$ then

$$\lim_{i \to \infty} a_i = +\infty,$$

because otherwise the sequence $\{a_i\}$ would have a finite limit a, such that

$$a = e^{\rho a}.$$

Using the known ineqality

$$e^x \ge ex,$$

we have

$$a = e^{\rho a} \ge e\rho a > a$$

which is a contradiction.

When $\frac{1}{e} < \rho < 1$, we also define a sequence $\{b_j\}$ by

$$b_1 = \frac{2(1-\rho)}{\rho^2}, \qquad b_{j+1} = \frac{2(1-\rho)}{\rho^2 + \frac{2}{b_j^2}}, \qquad j = 1, 2, \dots. \tag{5}$$

Observe that for $\frac{1}{e} < \rho < 1$

$$b_{j+1} < b_j, \qquad j = 1, 2, \dots.$$

In the following, $D(x)$ denotes distance between adjacent zeros of the solution $x(t)$ of Eq.(1).

Our main result is the following theorem.

THEOREM 1. *Assume that* (2) *holds. Suppose that*
(A) *there exist a function* $H(t) \in C^1([t_0,\infty),[0,\infty))$ *such that*

$$P(t-\sigma)\frac{Q(t)}{Q(t-\tau)} \le H(t) \qquad \text{and} \qquad H'(t) \le 0;$$

(B) *there exist* t_1 $(t_1 \ge t_0)$ *and positive constant* ρ *such that*

$$\int_{t+\tau-\sigma}^{t} \frac{Q(s)}{1+H(s+\tau-\sigma)}ds \ge \rho > \frac{1}{e} \qquad \text{for } t \ge t_1.$$

Let $x(t)$ *be a solution of Eq.*(1) *on* $[t_x,\infty)$, *where* $t_x \ge t_1$. *Then* $x(t)$ *has arbitrarily large zeros and* $D(x) < 2\sigma + n_\rho(\sigma-\tau)$ *on* $[t_x,\infty)$, *where*

$$n_\rho = \begin{cases} 1, & \text{when } \rho \ge 1; \\ \min\limits_{i\ge 1, j\ge 1}\{i+j \mid a_i \ge b_j\}, & \text{when } 1/e < \rho < 1; \end{cases} \tag{6}$$

and a_i, b_j *are defined by* (4) *and* (5).

PROOF. It suffices to prove that for $T_0 \ge t_x$ the solution $x(t)$ of Eq.(1) has zeros on $[T_0, T_0+2\sigma+n_\rho(\sigma-\tau)]$. Otherwise, without loss of generality, we assume that $x(t)$ is positive on $[T_0, T_0+2\sigma+n_\rho(\sigma-\tau)]$.
Let

$$z(t) = x(t) + P(t)x(t-\tau) \qquad \text{for } t \ge T_0+\tau. \tag{7}$$

Then we get

$$z(t) > 0 \qquad \text{for } t \in [T_0+\tau, T_0+2\sigma+n_\rho(\sigma-\tau)] \tag{8}$$

and

$$z'(t) = -Q(t)x(t-\sigma) < 0 \qquad \text{for } t \in [T_0+\sigma, T_0+2\sigma+n_\rho(\sigma-\tau)]. \tag{9}$$

From (1) and (7), we have

$$\begin{aligned} z'(t) &= -Q(t)x(t-\sigma) \\ &= -Q(t)[z(t-\sigma) - P(t-\sigma)x(t-\tau-\sigma)] \\ &= -Q(t)z(t-\sigma) - P(t-\sigma)\frac{Q(t)}{Q(t-\tau)}z'(t-\tau) \qquad \text{for } t \ge T_0+\sigma+\tau. \end{aligned} \tag{10}$$

By condition (A) and (10), we get

$$z'(t) + H(t)z'(t-\tau) + Q(t)z(t-\sigma) \le 0 \qquad \text{for } t \ge T_0+\sigma+\tau. \tag{11}$$

Set

$$w(t) = z(t) + H(t)z(t-\tau) \qquad \text{for } t \ge T_0+2\tau. \tag{12}$$

From (8) and (12), we have

$$w(t) > 0 \qquad \text{for } t \in [T_0 + 2\tau, T_0 + 2\sigma + n_\rho(\sigma - \tau)] \tag{13}$$

and

$$w'(t) = z'(t) + H'(t)z(t - \tau) + H(t)z'(t - \tau) \qquad \text{for } t \geq T_0 + 2\tau. \tag{14}$$

By (11) and (14), we get

$$w'(t) \leq H'(t)z(t - \tau) - Q(t)z(t - \sigma) < 0, \qquad \text{for}$$
$$t \in [T_0 + \sigma + \tau, T_0 + 2\sigma + n_\rho(\sigma - \tau)]. \tag{15}$$

Since $z(t)$ is decreasing for $t \in [T_0 + \sigma, T_0 + 2\sigma + n_\rho(\sigma - \tau)]$, by (12) we have

$$w(t) < (1 + H(t))z(t - \tau) \qquad \text{for } t \in [T_0 + \tau + \sigma, T_0 + 2\sigma + n_\rho(\sigma - \tau)] \tag{16}$$

and so

$$z(t - \sigma) > \frac{w(t + \tau - \sigma)}{1 + H(t + \tau - \sigma)} \qquad \text{for } t \in [T_0 + 2\sigma, T_0 + 2\sigma + n_\rho(\sigma - \tau)]. \tag{17}$$

Substituting (17) into (15), we have

$$w'(t) + \frac{Q(t)}{1 + H(t + \tau - \sigma)} w(t + \tau - \sigma) < H'(t)z(t - \tau) \leq 0, \qquad \text{for}$$
$$t \in [T_0 + 2\sigma, T_0 + 2\sigma + n_\rho(\sigma - \tau)]. \tag{18}$$

Next, for convenience, we set

$$q(t) = \frac{Q(t)}{1 + H(t + \tau - \sigma)}.$$

Thus, (18) implies that

$$w'(t) + q(t)w(t + \tau - \sigma) < 0 \qquad \text{for } t \in [T_0 + 2\sigma, T_0 + 2\sigma + n_\rho(\sigma - \tau)]. \tag{19}$$

We consider the following two cases:

Case 1. $\rho \geq 1$.

From (13) and (15), we have

$$w(t) > 0 \qquad \text{for } t \in [T_0 + 2\tau, T_0 + 2\sigma + (\sigma - \tau)] \tag{20}$$

and

$$w'(t) < 0 \qquad \text{for } t \in [T_0 + \sigma + \tau, T_0 + 2\sigma + (\sigma - \tau)], \tag{21}$$

which implies that $w(t)$ is decreasing, and thus

$$w(t) > w(T_0 + 2\sigma) \qquad \text{for } t \in [T_0 + \sigma + \tau, T_0 + 2\sigma].$$

Integrating both sides of (19) from $T_0+2\sigma$ to $T_0+2\sigma+(\sigma-\tau)$, we obtain

$$\begin{aligned} w(T_0+2\sigma+(\sigma-\tau)) &< w(T_0+2\sigma) - \int_{T_0+2\sigma}^{T_0+2\sigma+(\sigma-\tau)} q(s)w(s+\tau-\sigma)ds \\ &< w(T_0+2\sigma)\left\{1-\int_{T_0+2\sigma}^{T_0+2\sigma+(\sigma-\tau)} q(s)ds\right\}. \end{aligned} \tag{22}$$

By (22) and condition (B), we have

$$w(T_0+2\sigma+(\sigma-\tau)) < w(T_0+2\sigma)(1-\rho) \le 0,$$

which contradicts (20).

Case 2. $1/e < \rho < 1$.

Setting $n_\rho = i^* + j^*$, under the condition (B), when $t \ge t_x$ (where $t_x \ge t_1$), we know that

$$\int_{t+\tau-\sigma}^{t} q(s)ds \ge \rho > \frac{1}{e} \qquad \text{and} \qquad \int_{t}^{t-\tau+\sigma} q(s)ds \ge \rho > \frac{1}{e}.$$

Observe that $f(\lambda) = \int_t^\lambda q(s)ds$ is a continuous function, $f(t) = 0$ and $f(t-\tau+\sigma) \ge \rho$ and there exists a λ_t such that $\int_t^{\lambda_t} q(s)ds = \rho$, where $t < \lambda_t \le t+(\sigma-\tau)$. In view of (6), we know that $n_\rho \ge 2$. When $T_0+2\sigma+(\sigma-\tau) \le t \le T_0+2\sigma+(i^*+j^*-1)(\sigma-\tau)$, integrating both sides of (19) from t to λ_t, we obtain

$$w(t) - w(\lambda_t) \ge \int_t^{\lambda_t} q(s)w(s+\tau-\sigma)ds. \tag{23}$$

Since $t \le s \le t+(\sigma-\tau)$, we easily see that $T_0+2\sigma \le t-(\sigma-\tau) \le s-(\sigma-\tau) \le t$. Integrating both side of (19) from $s-(\sigma-\tau)$ to t, we get

$$w(s+\tau-\sigma) - w(t) \ge \int_{s+\tau-\sigma}^{t} q(u)w(u+\tau-\sigma)du.$$

From (15), it is clear that $w(u+\tau-\sigma)$ is decreasing on $T_0+2\sigma \le s-(\sigma-\tau) \le u \le t$, thus, we have

$$\begin{aligned} w(s+\tau-\sigma) &\ge w(t) + w(t+\tau-\sigma)\int_{s+\tau-\sigma}^{t} q(u)du \\ &= w(t) + w(t+\tau-\sigma)\left\{\int_{s+\tau-\sigma}^{s} q(u)du - \int_t^s q(u)du\right\} \\ &\ge w(t) + w(t+\tau-\sigma)\left\{\rho - \int_t^s q(u)du\right\}. \end{aligned} \tag{24}$$

From (23) and (24), we have

$$\begin{aligned} w(t) &\geq w(\lambda_t) + \int_t^{\lambda_t} q(s)w(s+\tau-\sigma)ds \\ &\geq w(\lambda_t) + \int_t^{\lambda_t} q(s)\left\{w(t) + w(t+\tau-\sigma)\left(\rho - \int_t^s q(u)du\right)\right\}ds \\ &= w(\lambda_t) + \rho w(t) + \rho^2 w(t+\tau-\sigma) - w(t+\tau-\sigma)\int_t^{\lambda_t} ds \int_t^s q(s)q(u)du. \end{aligned} \tag{25}$$

As is well known, the identical relation

$$\int_t^{\lambda_t} ds \int_t^s q(s)q(u)du = \int_t^{\lambda_t} du \int_u^{\lambda_t} q(s)q(u)ds$$

holds. On the right hand, exchanging the variable notation of integration s and u, the above equality becomes

$$\int_t^{\lambda_t} ds \int_t^s q(s)q(u)du = \int_t^{\lambda_t} ds \int_s^{\lambda_t} q(u)q(s)du,$$

which implies

$$\begin{aligned} \int_t^{\lambda_t} ds \int_t^s q(s)q(u)du &= \frac{1}{2}\int_t^{\lambda_t} ds \int_t^{\lambda_t} q(u)q(s)du \\ &= \frac{1}{2}\left(\int_t^{\lambda_t} q(s)ds\right)^2 = \frac{\rho^2}{2}. \end{aligned}$$

Substituting this into (25), we have

$$w(t) > w(\lambda_t) + \rho w(t) + \frac{\rho^2}{2} w(t+\tau-\sigma). \tag{26}$$

Noting that

$$w(\lambda_t) > 0 \qquad \text{for } t \in [T_0 + 2\sigma + (\sigma-\tau), T_0 + 2\sigma + (i^* + j^* - 1)(\sigma - \tau)],$$

(26) implies

$$\begin{aligned} &\frac{w(t-(\sigma-\tau))}{w(t)} < \frac{2(1-\rho)}{\rho^2} = b_1, \\ &t \in [T_0 + 2\sigma + (\sigma-\tau), T_0 + 2\sigma + (i^* + j^* - 1)(\sigma-\tau)]. \end{aligned} \tag{27}$$

When $T_0 + 2\sigma + (\sigma - \tau) \leq t \leq T_0 + 2\sigma + (i^* + j^* - 2)(\sigma - \tau)$, we easily see that $T_0 + 2\sigma + (\sigma - \tau) \leq t \leq \lambda_t \leq t + (\sigma - \tau) \leq T_0 + 2\sigma + (i^* + j^* - 1)(\sigma - \tau)$. Thus, by (27), we have

$$w(\lambda_t) > \frac{1}{b_1} w(\lambda_t - (\sigma - \tau)). \tag{28}$$

Since $w(t)$ is decreasing on $[T_0 + \sigma + \tau, T_0 + 2\sigma + (i^* + j^*)(\sigma - \tau)]$ and $T_0 + 2\sigma \leq \lambda_t - (\sigma - \tau) < t < \lambda_t < T_0 + 2\sigma + (i^* + j^* - 1)(\sigma - \tau)$, we get

$$w(\lambda_t) > \frac{1}{b_1} w(\lambda_t - (\sigma - \tau)) > \frac{1}{b_1} w(t) > \frac{1}{b_1^2} w(t - (\sigma - \tau)).$$

Substituting this into (26), we have

$$w(t) > \frac{1}{b_1^2} w(t - (\sigma - \tau)) + \rho w(t) + \frac{\rho^2}{2} w(t - (\sigma - \tau)).$$

Therefore

$$\frac{w(t - (\sigma - \tau))}{w(t)} < \frac{2(1 - \rho)}{\rho^2 + \dfrac{2}{b_1^2}} = b_2,$$

$$t \in [T_0 + 2\sigma + (\sigma - \tau), T_0 + 2\sigma + (i^* + j^* - 2)(\sigma - \tau)].$$

Repeating the above procedure, we obtain

$$\frac{w(t - (\sigma - \tau))}{w(t)} < \frac{2(1 - \rho)}{\rho^2 + \dfrac{2}{b_{j^*-1}^2}} = b_{j^*}, \tag{29}$$

$$t \in [T_0 + 2\sigma + (\sigma - \tau), T_0 + 2\sigma + i^*(\sigma - \tau)].$$

Setting $t = T_0 + 2\sigma + i^*(\sigma - \tau)$ in (29), we get

$$\frac{w(T_0 + 2\sigma + (i^* - 1)(\sigma - \tau))}{w(T_0 + 2\sigma + i^*(\sigma - \tau))} < b_{j^*}. \tag{30}$$

On the other hand, from (15) we know that $w(t)$ is decreasing on $[T_0 + \sigma + \tau, T_0 + 2\sigma + (i^* + j^*)(\sigma - \tau)]$, hence

$$\frac{w(t - (\sigma - \tau))}{w(t)} > 1 \qquad \text{for } t \in [T_0 + 2\sigma, T_0 + 2\sigma + (i^* + j^*)(\sigma - \tau)]. \tag{31}$$

When $T_0 + 2\sigma + (\sigma - \tau) \leq t \leq T_0 + 2\sigma + (i^* + j^*)(\sigma - \tau)$, dividing (19) by $w(t)$, and integrating from $t - (\sigma - \tau)$ to t, we get

$$ln\left(\frac{w(t)}{w(t - (\sigma - \tau))}\right) + \int_{t-(\sigma-\tau)}^{t} q(s) \frac{w(s - (\sigma - \tau))}{w(s)} ds < 0.$$

By using (31) and (B), we have

$$ln\left(\frac{w(t-(\sigma-\tau))}{w(t)}\right) > \int_{t-(\sigma-\tau)}^{t} q(s)\frac{w(s-(\sigma-\tau))}{w(s)}ds > \rho.$$

If follows that

$$\frac{w(t-(\sigma-\tau))}{w(t)} > e^{\rho} = a_1 \quad \text{for} \quad t \in [T_0+2\sigma+(\sigma-\tau), T_0+2\sigma+(i^*+j^*)(\sigma-\tau)]. \tag{32}$$

Repeating the above procedure, we get

$$\frac{w(t-(\sigma-\tau))}{w(t)} > e^{\rho a_{i^*-1}} = a_{i^*} \quad t \in [T_0+2\sigma+i^*(\sigma-\tau), T_0+2\sigma+(i^*+j^*)(\sigma-\tau)]. \tag{33}$$

Seting $t = T_0+2\sigma+i^*(\sigma-\tau)$ in (33), we have

$$\frac{w(T_0+2\sigma+(i^*-1)(\sigma-\tau))}{w(T_0+2\sigma+i^*(\sigma-\tau))} > a_{i^*}. \tag{34}$$

From (30) and (34), we obtain

$$a_{i^*} < b_{j^*},$$

which contradicts (6) and completes the proof of the theorem.

Remake 1. If we choose $H(t) = P(t-\sigma)Q(t)/Q(t-\tau)$ or $H(t) = \alpha \in \mathbf{R}^+$, then conditions (B) becomes

$$\int_{t+\tau-\sigma}^{t} \frac{Q(s)Q(s-\tau)}{Q(s-\tau)+P(s-\sigma)Q(s)}ds \geq \rho > \frac{1}{e} \quad \text{for } t \geq t_1$$

or

$$\int_{t+\tau-\sigma}^{t} \frac{Q(s)}{1+\alpha}ds \geq \rho > \frac{1}{e} \quad \text{for } t \geq t_1.$$

Corollary 1. *Assume that*

(A_1) $P(t) = p \geq 0$, $Q(t) = q > 0$ *are constants*, $\sigma > \tau > 0$;

(B_1) $\dfrac{q(\sigma-\tau)}{1+p} = \rho > \dfrac{1}{e}$.

Let $x(t)$ *be a solution of Eq.*(1) *on* $[t_x, \infty)$. *Then* $x(t)$ *has arbitrarily large zeros and* $D(x) < 2\sigma + n_\rho(\sigma-\tau)$ *on* $[t_x, \infty)$, *where* n_ρ *is defined by* (6).

Corollary 2. *Assume that*

(A_2) $P(t) \equiv 0$, $Q(t) \geq 0$, $\sigma > 0$;

(B_2) *there exist* t_1 ($t_1 \geq t_0$) *and positive constant* ρ *such that*

$$\int_{t-\tau}^{t} Q(s)ds \geq \rho > \frac{1}{e} \qquad \text{for } t \geq t_1.$$

Let $x(t)$ *is a solution of Eq.* (3) *on* $[t_x, \infty)$, *where* $t_x \geq t_1$. *Then* $x(t)$ *has arbitrarily large zeros and* $D(x) < 2\sigma + n_\rho(\sigma - \tau)$ *on* $[t_x, \infty)$, *where* n_ρ *is defined by* (6).

Remark 2. Theorem 1 improve Theorem 1 in [5]. Corollary 2 improve the Theorem 2.2.1 and 2.2.2 in [3] and all theorems in [4].

Example 1. Consider the delay differential equation

$$x'(t) + x(t - 0.4) = 0$$

where $Q(t) = 1$. We have $\rho = \sigma = 0.4$ and $a_1 = 1.491\ldots$, $a_2 = 1.816\ldots, \ldots$, $a_{10} = 4.387\ldots$, $a_{11} = 5.784\ldots$, $a_{12} = 10.111\ldots, \ldots$; $b_1 = 7.500$, $b_2 = 6.136\ldots$, $b_3 = 5.631\ldots$, $b_4 = 5.379\ldots, \ldots$; Thus, we find

$$a_i < 5 < b_j, \quad 1 \leq i \leq 10, \quad j \geq 1; \qquad a_{11} > b_j, \quad j \geq 3; \qquad a_{12} > b_j, \quad j \geq 1;$$

Hence, by Corollary 2, we have $n_\rho = 12 + 1 = 13$ and $D(x) < 15 \times 0.4$. This improves the result in [3, 4]: $D(x) < 28 \times 0.4$.

Example 2. Consider the neutral differential equation

$$[x(t) + x(t - 0.45)]' + 2x(t - 1) = 0$$

where $p = 1$, $q = 2$, and $\tau = 0.45$, $\sigma = 1$. We have $\rho = \dfrac{2(1 - 0.45)}{1 + 1} = 0.55$ and $a_1 = 1.733\ldots$, $a_2 = 2.594\ldots$, $a_3 = 4.165\ldots$, $a_4 = 9.884\ldots, \ldots$; $b_1 = 2.975\ldots$, $b_2 = 1.703\ldots$, $b_3 = 0.907\ldots, \ldots$; Thus, we find

$$a_1 > b_j, \quad j \geq 2; \qquad a_2 > b_j, \quad j \geq 2; \qquad a_3 > b_j, \quad j \geq 1.$$

Hence, by Corollary 1, we have $n_\rho = 1 + 2 = 3$ and $D(x) < 2 \times 1 + 3 \times (1 - 0.45) = 3.65$.

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