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Induction of nilpotent orbits for real reductive groups and associated varieties of standard representations

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ABSTRACT. In [LS], Lusztig and Spaltenstein introduced the notion of induction of nilpotent orbits for complex reductive groups. It is known that the induction of representations and that of nilpotent orbits are compatible with respect to the operation taking associated variety for complex reductive groups (cf. [BV]). In this paper, we give a definition of induction of nilpotent orbits by θ-stable paraboric subalgebras and that by real paraboric subalgebras for real reductive groups, and show that the generic K-orbits in the associated varieties of certain standard $(\mathfrak{g}, K)$-modules can be described by using these inductions.

0. Introduction

Let G be a complex connected reductive algebraic group and $\tau : G \to G$ a complex conjugation which defines a real form $G(\mathbf{R})$ of G. Let $\theta : G \to G$ be a (complexified) Cartan involution of G which commutes with τ. Write $K = \{g \in G; \theta(g) = g\}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ the Cartan decomposition with respect to θ. For a closed subgroup H of G, we denote its Lie algebra by the corresponding small German letter $\mathfrak{h}$.

In § 1, to describe the $\mathfrak{g}$-principal (i.e. regular in $\mathfrak{g}$) K-orbits in the associated varieties of certain standard $(\mathfrak{g}, K)$-modules, we give a parametrization of $\mathfrak{g}$-principal nilpotent K-orbits in $\mathfrak{s}$.

In § 2, we discuss the induction of nilpotent K-orbits. For a θ-stable (resp. τ-stable) parabolic subgroup $Q = LU$ (resp. $P = MN$) with θ-stable and τ-stable Levi factor L (resp. M), we define

$$Ind^{\theta}((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g}) : 2^{\mathcal{N}_{\mathfrak{l} \cap \mathfrak{s}}/L \cap K} \to 2^{\mathcal{N}_{\mathfrak{s}}/K}$$

$$(\text{resp.} \quad Ind^{\mathbf{R}}((\mathfrak{m}, \mathfrak{p}) \uparrow \mathfrak{g}) : 2^{\mathcal{N}_{\mathfrak{m} \cap \mathfrak{s}}/M \cap K} \to 2^{\mathcal{N}_{\mathfrak{s}}/K})$$

as a generalization of induction of nilpotent orbits in the complex cases, where we write $2^{\mathcal{N}_{\mathfrak{s}}/K}$ for the set of subsets of nilpotent K-orbits in $\mathfrak{s}$. We describe

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the nilpotent K-orbits induced by $Ind^{\theta}((\mathfrak{l},\mathfrak{q})\uparrow\mathfrak{g})$ from $\mathfrak{l}$-principal nilpotent K-orbits. We also describe the nilpotent K-orbits induced by $Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g})$ from the zero orbit when $P=MN$ is a τ-stable Borel subgroup of G.

In §3, we recall the descriptions of the associated varieties of certain standard $(\mathfrak{g},K)$-modules and show that the generic K-orbits in their associated varieties can be described by using $Ind^{\theta}((\mathfrak{l},\mathfrak{q})\uparrow\mathfrak{g})$ and $Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g})$. We also show that some finite group F_L^G (cf. (3.3)) acts on the associated variety of the standard $(\mathfrak{g},K)$-module corresponding to a set of θ-stable data for $G(\mathbf{R})$.

1. Parametrization of $\mathfrak{g}$-principal nilpotent K-orbits

Let G be a complex reductive algebraic group defined over $\mathbf{R}$ and $\tau: G\to G$ a complex conjugation which defines the real form $G(\mathbf{R})=\{g\in G;\tau(g)=g\}$ of G. Let $\theta: G\to G$ be a (complexified) Cartan involution of G which commutes with τ. Throughout this paper, we use the following notations. For a closed subgroup of G, its Lie algebra is denoted by the corresponding small German letter. The involution of $\mathfrak{g}$, which is induced from τ (resp. θ), is also denoted by τ (resp. θ). Write $K:=\{g\in G;\theta(g)=g\}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$ the Cartan decomposition with respect to θ. The action of G on $\mathfrak{g}$, which we always consider, is the adjoint action. For a τ-stable subset A of G (resp. $\mathfrak{g}$), we write $A(\mathbf{R})=\{x\in A;\tau(x)=x\}$. For a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we denote by $R(\mathfrak{g},\mathfrak{h})$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and by $\mathfrak{g}_\alpha$ the root space corresponding to a root $\alpha\in R(\mathfrak{g},\mathfrak{h})$. For a $\mathfrak{h}$-stable subspace $V\subset\mathfrak{g}$, we write $R(V,\mathfrak{h}):=\{\alpha\in R(\mathfrak{g},\mathfrak{h});\mathfrak{g}_\alpha\subset V\}$. If $\mathfrak{h}$ is θ-stable, we write $R(V,\mathfrak{h})_{i\mathbf{R}}$ (resp. $R(V,\mathfrak{h})_{\mathbf{R}}$) the set of imaginary (resp. real) roots in $R(V,\mathfrak{h})$, i.e.

$$R(V,\mathfrak{h})_{i\mathbf{R}}=\{\alpha\in R(V,\mathfrak{h});\theta(\alpha)=\alpha\},\qquad R(V,\mathfrak{h})_{\mathbf{R}}=\{\alpha\in R(V,\mathfrak{h});\theta(\alpha)=-\alpha\}.$$

The set of all nilpotent elements in $\mathfrak{g}$ (resp. $\mathfrak{s},\mathfrak{g}(\mathbf{R})$) is denoted by $\mathcal{N}_{\mathfrak{g}}$ (resp. $\mathcal{N}_{\mathfrak{s}},\mathcal{N}_{\mathfrak{g}(\mathbf{R})}$). The set of orbits in $\mathcal{N}_{\mathfrak{g}}$ (resp. $\mathcal{N}_{\mathfrak{s}},\mathcal{N}_{\mathfrak{g}(\mathbf{R})}$) under the action of G (resp. $K,G(\mathbf{R})$) is denoted by $\mathcal{N}_{\mathfrak{g}}/G$ (resp. $\mathcal{N}_{\mathfrak{s}}/K,\mathcal{N}_{\mathfrak{g}(\mathbf{R})}/G(\mathbf{R})$).

DEFINITION 1.1 ([AV]). (i) Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. A positive system Σ of $R(\mathfrak{g},\mathfrak{h})_{i\mathbf{R}}$ is called of large type if every simple roots of Σ is non-compact (i.e. $\mathfrak{g}_\alpha\subset\mathfrak{s}$).

(ii) A θ-stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is called of large type if every simple root of $R(\mathfrak{b},\mathfrak{h})$ is complex (i.e. $\theta(\alpha)\neq\pm\alpha$) or non-compact imaginary for any θ-stable Cartan subalgebra $\mathfrak{h}\subset\mathfrak{b}$. We write $\mathscr{B}_{\mathfrak{g}}^L$ the set of θ-stable Borel subalgebras of $\mathfrak{g}$ of large type.

REMARK 1.2. For a θ-stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, since any θ-stable Cartan subalgebras in $\mathfrak{b}$ are conjugate under the action of $B\cap K,\mathfrak{b}$ is of large type if and only if there exists a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{b}$ such that every simple root of $R(\mathfrak{b},\mathfrak{h})$ is complex or non-compact imaginary.

PROPOSTION 1.3 ([AV, Proposition 6.25]). *Let* $\mathfrak{b}$ *be a* θ*-stable Borel subalgebra of* $\mathfrak{g}$ *and* $\mathfrak{h}\subset\mathfrak{b}$ *a* θ*-stable Cartan subalgebra. Then the positive system* $R(\mathfrak{b},\mathfrak{h})_{i\mathbf{R}}$ *is of large type if and only if* $\mathfrak{b}$ *is of large type.*

To describe the $\mathfrak{g}$-principal K-orbits in the associated variety of $X = X_{G(\mathbf{R})}(\mathfrak{q},H(\mathbf{R}),\delta,\nu)$ (cf. §3), we consider the sets $\mathscr{B}^L_{\mathfrak{q}}$ and $\mathscr{P}^L_{\mathfrak{q}}$. Let $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}\subset\mathfrak{g}$ be a θ-stable parabolic subalgebra with θ-stable Levi subalgebra $\mathfrak{l}$ and nilpotent radical $\mathfrak{u}$. We put

$$\mathscr{B}^L_{\mathfrak{q}} := \{\mathfrak{b}\in\mathscr{B}^L_{\mathfrak{g}}; k\mathfrak{b}\subset\mathfrak{q}\quad\text{for some } k\in K\}.$$

We write $\mathscr{P}^L_{\mathfrak{q}}$ the set of pairs $(\mathfrak{t},\Sigma^c)$ with the following properties:

(a) $\mathfrak{t}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$ (i.e. $\mathfrak{t}$ contains a Cartan subalgebra of $\mathfrak{k}$).

(b) Σ^c is a positive system of $R(\mathfrak{g},\mathfrak{t})_{i\mathbf{R}}$ of large type.

(c) There exists $k\in K$ such that $k\mathfrak{t}\subset\mathfrak{q}$ and that $k\Sigma^c\subset R(\mathfrak{q},k\mathfrak{t})$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{s}\cap[\mathfrak{g},\mathfrak{g}]$ and define a finite group F_G by

$$F_G := \{a\in exp(\mathfrak{a}); Ad(a^2)=id\}.$$

REMARK 1.4. (i) For $a\in F_G$ and $k\in K$, since a^2 is contained in the center of G, we have $\theta(aka^{-1})=a^{-1}ka=a^{-2}(aka^{-1})a^2=aka^{-1}$ and hence F_G normalizes K.

(ii) By [KR, Proposition 1], we have

$$Ad(K^0F_G)=[Ad(G)]^\theta := \{Ad(g); g\in G, \theta\circ Ad(g)\circ\theta^{-1}=Ad(g)\},$$

where we write K^0 the identity component of K. It is easily verified that $[Ad(G)]^\theta = Ad(N_G(\mathfrak{k}))$ and hence $Ad(K^0F_G)=Ad(N_G(\mathfrak{k}))=Ad(N_G(\mathfrak{s}))$. Since $N_G(\mathfrak{k})\supset K$, we have $Ad(KF_G)=Ad(N_G(\mathfrak{k}))=Ad(N_G(\mathfrak{s}))$.

A nilpotent element $X\in\mathfrak{g}$ is called $\mathfrak{g}$-principal if X is regular in $\mathfrak{g}$. We write $\mathscr{N}^{\mathfrak{g}-pr}_{\mathfrak{g}}$ (resp. $\mathscr{N}^{\mathfrak{g}-pr}_{\mathfrak{s}},\mathscr{N}^{\mathfrak{g}-pr}_{\mathfrak{g}(\mathbf{R})}$) the set of $\mathfrak{g}$-principal elements in $N_{\mathfrak{g}}$ (resp. $\mathscr{N}_{\mathfrak{s}},\mathscr{N}_{\mathfrak{g}(\mathbf{R})}$).

PROPOSITION 1.5 ([AV, Proposition 6.24]). *The following conditions on* θ *are equivalent.*

(a) $\mathfrak{g}$ *is quasisplit* (*i.e. there exists a Borel subalgebra of* $\mathfrak{g}$ *defined over* $\mathbf{R}$).

(b) $\mathscr{N}^{\mathfrak{g}-pr}_{\mathfrak{s}}\neq\varnothing$.

(c) $\mathscr{B}^L_{\mathfrak{g}}\neq\varnothing$.

(d) *For any θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}, R(\mathfrak{g},\mathfrak{h})_{i\mathbf{R}}$ has a positive system of large type.*

(d′) *There exists a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $R(\mathfrak{g},\mathfrak{h})_{i\mathbf{R}}$ has a positive system of large type.*

By using $\mathscr{B}_{\mathfrak{g}}^L/K$ and $\mathscr{P}_{\mathfrak{g}}^L/K$, the set $\mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K$ of K-orbits in $\mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}$ is parametrized as follows. For $x \in \mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}$, we can take a normal S-triple (h,x,y) $(h \in \mathfrak{k}, y \in \mathfrak{s})$ (cf. [KR]). Since h is a reguler semisimple element of $\mathfrak{g}, \mathfrak{t} := \mathfrak{z}_{\mathfrak{g}}(h)$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. Define a Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$ of $\mathfrak{g}$ by $R(\mathfrak{b},\mathfrak{t}) = \{\alpha \in R(\mathfrak{g},\mathfrak{t}); \alpha(h) > 0\}$ and write Δ the set of simple roots in $R(\mathfrak{b},\mathfrak{t})$. Then we have $\Delta = \{\alpha \in R(\mathfrak{g},\mathfrak{t}); \alpha(h) = 2\}$. Since $[h,x] = 2x, x$ can be written as a sum

$$x = \sum\nolimits_{\alpha \in \Delta} X_\alpha$$

for some root vectors $X_\alpha \in \mathfrak{g}_\alpha \backslash (0)$ and it holds that $\theta(X_\alpha) = -X_{\theta(\alpha)}$ $(\alpha \in \Delta)$. Hence any roots in Δ are complex or non-compact imaginary, and thus $\mathfrak{b}$ is of large type. Since $x \in \mathfrak{b}$ and x is $\mathfrak{g}$-principal, $\mathfrak{b}$ is the unique Borel subalgebra containing x. Then the correspondence $x \mapsto \mathfrak{b}$ defines a map

$$\varphi : \mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K \to \mathscr{B}_{\mathfrak{g}}^L/K.$$

For $\mathfrak{b} \in \mathscr{B}_{\mathfrak{g}}^L$, take a θ-stable Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{b}$. Since $\mathfrak{b}$ is θ-stable, $R(\mathfrak{b},\mathfrak{t})$ dose not have any real root and $\mathfrak{t}$ is fundamental. By Proposition 1.3, $\Sigma^c := R(\mathfrak{b},\mathfrak{t})_{i\mathbf{R}}$ is of large type and hence $(\mathfrak{t},\Sigma^c) \in \mathscr{P}_{\mathfrak{g}}^L$. Then the correspondence $\mathfrak{b} \mapsto (\mathfrak{t},\Sigma^c)$ defines a map

$$\psi : \mathscr{B}_{\mathfrak{g}}^L/K \to \mathscr{P}_{\mathfrak{g}}^L/K.$$

PROPOSITION 1.6 ([AV, Proposition A.7]). *The maps $\varphi : \mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K \to \mathscr{B}_{\mathfrak{g}}^L/K$ and $\varphi : \mathscr{B}_{\mathfrak{g}}^L/K \to \mathscr{P}_{\mathfrak{g}}^L/K$ are bijections. Furthermore the finite group F_G acts naturaly and transitively on the sets $\mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K, \mathscr{B}_{\mathfrak{g}}^L/K$, $\mathscr{P}_{\mathfrak{g}}^L/K$, and the maps φ, ψ are F_G-equivariant.*

We write $\mathscr{O}_{(\mathfrak{t},\Sigma^c)} \in \mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K$ the K-orbits corresponding to $(\mathfrak{t},\Sigma^c) \in \mathscr{P}_{\mathfrak{g}}^L$ by Proposition 1.6.

Let Q be a θ-stable parabolic subgroup of G, L a θ-stable Levi subgroup of Q and U the unipotent radical of Q. Then $Q = LU$. For a maximal abelian subspace $\mathfrak{a}_L$ of $[\mathfrak{l},\mathfrak{l}] \cap \mathfrak{s}$, we write

$$F_L = \{a \in exp(\mathfrak{a}_L); Ad(a^2)|_{\mathfrak{l}} = id_{\mathfrak{l}}\}, \quad F_L^G = \{a \in F_L; Ad(a^2) = id_{\mathfrak{g}}\}.$$

Then the set

$$[\mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K]_{\mathfrak{q}} := \{\mathscr{O} \in \mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K; \mathscr{O} \cap \mathfrak{q} \neq \varnothing\}$$

of $\mathfrak{g}$-principal K-orbits which intersect $\mathfrak{q}$ is parametrized as follows.

PROPOSITION 1.7. *The maps* φ, ψ *in* Proposition 1.6 *induce* F_L^G*-equivariant bijections*

$$[\mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K]_{\mathfrak{q}} \simeq \mathscr{B}_{\mathfrak{q}}^L/K \simeq \mathscr{P}_{\mathfrak{q}}^L/K.$$

2. Induction of nilpotent orbits for real reductive groups

2.1 Induction of nilpotent orbits by θ-stable parabolic subalgebras

Let $Q = LU$ be a θ-stable paraboric subgroup of G with θ-stable Levi factor L and unipotent radical U, and write $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ its Lie algebra. We put

$$K_L := L \cap K, \qquad \mathfrak{s}_L = \mathfrak{l} \cap \mathfrak{s}.$$

Let $\mathcal{O}$ be a K_L-orbit in $\mathscr{N}_{\mathfrak{s}_L}$ and $\mathcal{O}_0$ a connected component of $\mathcal{O}$. Since $\mathcal{O}_0 + \mathfrak{u} \cap \mathfrak{s} \subset \mathscr{N}_{\mathfrak{s}}$ is irreducible and $\mathscr{N}_{\mathfrak{s}}$ is a finite union of K-orbits, there exists a unique K-orbit $\tilde{\mathcal{O}} \in \mathscr{N}_{\mathfrak{s}}/K$ such that $(\mathcal{O}_0 + \mathfrak{u} \cap \mathfrak{s}) \cap \tilde{\mathcal{O}}$ is open and dense in $\mathcal{O}_0 + \mathfrak{u} \cap \mathfrak{s}$. Here we consider the Zariski topology. Any connected component of $\mathcal{O} + \mathfrak{u} \cap \mathfrak{s}$ can be written as $k\mathcal{O}_0 + \mathfrak{u} \cap \mathfrak{s}$ for some $k \in K_L$. Then $(k\mathcal{O}_0 + \mathfrak{u} \cap \mathfrak{s}) \cap \tilde{\mathcal{O}} = k\{(\mathcal{O}_0 + \mathfrak{u} \cap \mathfrak{s}) \cap \tilde{\mathcal{O}}\}$ is open and dense in $k\mathcal{O}_0 + \mathfrak{u} \cap \mathfrak{s} = k(\mathcal{O}_0 + \mathfrak{u} \cap \mathfrak{s})$. Therefore $\tilde{\mathcal{O}}$ is the unique nilpotent K-orbit in $\mathscr{N}_{\mathfrak{s}}$ such that $(\mathcal{O} + \mathfrak{u} \cap \mathfrak{s}) \cap \tilde{\mathcal{O}}$ is open and dense in $\mathcal{O} + \mathfrak{u} \cap \mathfrak{s}$.

DEFINITION 2.1. For a nilpotent K_L-orbit $\mathcal{O} \in \mathscr{N}_{\mathfrak{s}_L}/K_L$, we write

$$\tilde{\mathcal{O}} = \mathit{Ind}^{\theta}((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{O}) \in \mathscr{N}_{\mathfrak{s}}/K$$

the unique nilpotent K-orbit in $\mathscr{N}_{\mathfrak{s}}$ such that $(\mathcal{O} + \mathfrak{u} \cap \mathfrak{s}) \cap \tilde{\mathcal{O}}$ is open and dense in $\mathcal{O} + \mathfrak{u} \cap \mathfrak{s}$.

REMARK 2.2. (i) Suppose that $G(\mathbf{R})$ itself is a complex connected reductive group. We can see that

$$G = G(\mathbf{R}) \times G(\mathbf{R}), \quad \theta(g_1, g_2) = (g_2, g_1), \quad \tau(g_1, g_2) = (\bar{g}_2, \bar{g}_1), (g_1, g_2 \in G(\mathbf{R})),$$

where $g \mapsto \bar{g}$ is the complex conjugation of $G(\mathbf{R})$ corresponding to a compact real form of G. Then we have

$$K = \{(g, g); g \in G(\mathbf{R})\} \simeq G(\mathbf{R}), \qquad \mathfrak{s} = \{(X, -X); X \in \mathfrak{g}(\mathbf{R})\} \simeq \mathfrak{g}(\mathbf{R}).$$

Via the map $\mathfrak{g}(\mathbf{R}) \xrightarrow{\sim} \mathfrak{s}, X \mapsto (X, -X)$, we have a natural identification

$$\mathscr{N}_{\mathfrak{g}(\mathbf{R})}/G(\mathbf{R}) \xrightarrow{\sim} \mathscr{N}_{\mathfrak{s}}/K. \tag{2.1}$$

Let $Q(\mathbf{R}) = L(\mathbf{R})U(\mathbf{R})$ be a complex parabolic subgroup of $G(\mathbf{R})$ and write $Q = Q(\mathbf{R}) \times Q(\mathbf{R}), L = L(\mathbf{R}) \times L(\mathbf{R})$. In Lusztig-Spaltenstein [LS], the induction of nilpotent orbits

$$Ind^{\mathfrak{g}(\mathbf{R})}_{(\mathfrak{l}(\mathbf{R}),\mathfrak{q}(\mathbf{R}))} : \mathcal{N}_{\mathfrak{l}(\mathbf{R})}/L(\mathbf{R}) \to \mathcal{N}_{\mathfrak{g}(\mathbf{R})}/G(\mathbf{R})$$

for complex Lie algebras is defined. Then it is easily verified that, via the identification (2.1), $Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})$ coincides with $Ind^{\mathfrak{g}(\mathbf{R})}_{(\mathfrak{l}(\mathbf{R}),\mathfrak{q}(\mathbf{R}))}$. Therefore Definition 2.1 is a generalization of the induction in [LS].

(ii) It is known that the induction $Ind^{\mathfrak{g}(\mathbf{R})}_{(\mathfrak{l}(\mathbf{R}),\mathfrak{q}(\mathbf{R}))}$ of [LS] depends only on $\mathfrak{l}(\mathbf{R})$, not on $\mathfrak{q}(\mathbf{R})$. But the induction of Definition 2.1 depends on the choice of $\mathfrak{q}$.

$Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{O})(\mathcal{O} \in \mathcal{N}_{\mathfrak{s}_L}/K_L)$ defines a map

$$Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g}) : \mathcal{N}_{\mathfrak{s}_L}/K_L \to \mathcal{N}_{\mathfrak{s}}/K.$$

We extend this to a correspondence between the set $2^{\mathcal{N}_{\mathfrak{s}_L}/K_L}$ of subsets of $\mathcal{N}_{\mathfrak{s}_L}/K_L$ and $2^{\mathcal{N}_{\mathfrak{s}}/K}$ as follows.

DEFINITION 2.3. For a subset $S \in 2^{\mathcal{N}_{\mathfrak{s}_L}/K_L}$ of $\mathcal{N}_{\mathfrak{s}_L}/K_L$, we write $Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})(S)$ the set of orbits in $\{Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})(\mathscr{C}); \mathscr{C} \in S\}$ which are maximal with respect to the closure relation. This defines a map

$$Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g}) : 2^{\mathcal{N}_{\mathfrak{s}_L}/K_L} \to 2^{\mathcal{N}_{\mathfrak{s}}/K}.$$

PROPOSITION 2.4. *Let $Q = LU$ be a θ-stable parabolic subgroup of G with θ-stable Levi subgroup L and unipotent radical U. Suppose that L is quasi-split. Then the set $[Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}^{\mathfrak{l}-pr}_{\mathfrak{s}_L}/K_L)]^{\mathfrak{g}-pr}$ of $\mathfrak{g}$-principal K-orbits in $Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}^{\mathfrak{l}-pr}_{\mathfrak{s}_L}/K_L)$ can be written as*

$$[Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}^{\mathfrak{l}-pr}_{\mathfrak{s}_L}/K_L)]^{\mathfrak{g}-pr} = [\mathcal{N}^{\mathfrak{g}-pr}_{\mathfrak{s}}/K]_{\mathfrak{q}}.$$

PROOF. It is clear that

$$[Ind^{\theta}((\mathfrak{l},\mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}^{\mathfrak{l}-pr}_{\mathfrak{s}_L}/K_L)]^{\mathfrak{g}-pr} \subset [\mathcal{N}^{\mathfrak{g}-pr}_{\mathfrak{s}}/K]_{\mathfrak{q}}.$$

Suppose that $\mathcal{O} \in [\mathcal{N}^{\mathfrak{g}-pr}_{\mathfrak{s}}/K]_{\mathfrak{q}}$ and $x \in \mathfrak{q} \cap \mathcal{O}$. Choose a normal S-triple (h,x,y) $(h \in \mathfrak{k}, x, y \in \mathfrak{s})$ (cf. [KR]) and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $x \in \mathfrak{b} \subset \mathfrak{q}$. We can also choose a Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{g}$ such that $x, h \in \mathfrak{b}'$. Since x is $\mathfrak{g}$-principal, a Borel subalgebra which contains x is unique. Therefore

$$x, h \in \mathfrak{b} = \mathfrak{b}' \subset \mathfrak{q}.$$

Since h is regular in $\mathfrak{g}$ and $h \in \mathfrak{k}, \mathfrak{t} := \mathfrak{z}_{\mathfrak{g}}(h) \subset \mathfrak{b}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$.

Let $\mathfrak{t}'$ be a fundamental Cartan subalgebra of $\mathfrak{l}$. Then $R(\mathfrak{l},\mathfrak{t}')$ does not have any real root. Since $\mathfrak{u}$ is θ-stable, $R(\mathfrak{u},\mathfrak{t}')$ does not have any real root.

Hence $\mathfrak{t}'$ is also fundamental in $\mathfrak{g}$. Since $\mathfrak{t}_c = \mathfrak{t} \cap \mathfrak{k}$ and $\mathfrak{t}'_c = \mathfrak{t}' \cap \mathfrak{k}$ are both Cartan subalgebras of $\mathfrak{q} \cap \mathfrak{k}$, there exists $k \in Q \cap K$ such that $\mathfrak{t}'_c = k\mathfrak{t}_c$. Since

$$\mathfrak{l} \supset \mathfrak{t}' = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}'_c) = \mathfrak{z}_{\mathfrak{g}}(k\mathfrak{t}_c) = k\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_c) = k\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(kh),$$

by taking $kx \in \mathfrak{q} \cap \mathcal{O}$ instead of x, we may assume that $\mathfrak{t} \subset \mathfrak{l}$.

Let Δ be the base of $R(\mathfrak{b}, \mathfrak{t})$ and write

$$\Delta_{\mathfrak{l}} := \Delta \cap R(\mathfrak{l}, \mathfrak{t}), \qquad \Delta_{\mathfrak{u}} := \Delta \cap R(\mathfrak{u}, \mathfrak{t}).$$

Since $[h, x] = 2x, x$ can be written as $x = \Sigma_{\alpha \in \Delta} X_\alpha$ for suitable root vectors

$$X_\alpha \in \mathfrak{g}_\alpha \backslash \{0\} (\alpha \in \Delta).$$

If we write

$$x_{\mathfrak{l}} := \sum\nolimits_{\alpha \in \Delta_{\mathfrak{l}}} X_\alpha \in \mathfrak{l}, \qquad x_{\mathfrak{u}} := \sum\nolimits_{\beta \in \Delta_{\mathfrak{u}}} X_\beta \in \mathfrak{u},$$

then $x = x_{\mathfrak{l}} + x_{\mathfrak{u}}$. Since $\mathfrak{l}, \mathfrak{u}$ are θ-stable and $\theta(x) = -x$, we have $x_{\mathfrak{l}} \in \mathfrak{s}_L$ and $x_{\mathfrak{u}} \in \mathfrak{u} \cap \mathfrak{s}$. Since $\mathcal{O}$ is $\mathfrak{g}$-principal, $\mathcal{O} \cap (K_L x_{\mathfrak{l}} + \mathfrak{u} \cap \mathfrak{s})(\ni x)$ is open dense in $K_L x_{\mathfrak{l}} + \mathfrak{u} \cap \mathfrak{s}$. Therefore we have

$$\mathcal{O} = Kx = Ind^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(K_L x_{\mathfrak{l}}) \in Ind^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-pr}/K_L)$$

by noticing that $K_L x_{\mathfrak{l}} \in \mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-pr}/K_L$.

Remark 2.5. In the setting of Proposition 2.4, for two orbits $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}_{\mathfrak{s}_L}^{\mathfrak{l}-pr}/K_L$, it can happen that $Ind^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{O}_1)$ and $Ind^\theta((\mathfrak{l}, \mathfrak{q}) \uparrow \mathfrak{g})(\mathcal{O}_2)$ have different dimensions. We will exhibit such an example in the succeeding paper.

2.2 Induction of nilpotent orbits by real parabolic subalgebras

Let $P = MN$ be a τ-stable parabolic subgroup of G with τ-stable Levi factor M and unipotent radical N.

Definition 2.6. For a nilpotent orbit $\mathcal{O} \in \mathcal{N}_{\mathfrak{m}(\mathbf{R})}/M(\mathbf{R})$, we write $Ind^{\mathbf{R}}((\mathfrak{m}, \mathfrak{p}) \uparrow \mathfrak{g})(\mathcal{O})$ the set of orbits in $\{\mathcal{C} \in \mathcal{N}_{\mathfrak{g}(\mathbf{R})}/G(\mathbf{R}); (\mathcal{O} + \mathfrak{n}(\mathbf{R})) \cap \mathcal{C} \neq \varnothing\}$ which are maximal with respect to the closure relation.

Remark 2.7. In the setting of Remark 2.2(i), let $P(\mathbf{R}) = M(\mathbf{R})N(\mathbf{R})$ be a complex parabolic subgroup of $G(\mathbf{R})$ and write $P = P(\mathbf{R}) \times \overline{P(\mathbf{R})}, M = M(\mathbf{R}) \times \overline{M(\mathbf{R})}$. We can see that P and M are complexifications of $P(\mathbf{R})$ and $M(\mathbf{R})$ respectively. Then for $\mathcal{O} \in \mathcal{N}_{\mathfrak{m}(\mathbf{R})}/M(\mathbf{R})$ and $\tilde{\mathcal{O}} \in \mathcal{N}_{\mathfrak{g}(\mathbf{R})}/G(\mathbf{R}), \tilde{\mathcal{O}}$ is maximal in $\{\mathcal{C} \in \mathcal{N}_{\mathfrak{g}(\mathbf{R})}/G(\mathbf{R}); (\mathcal{O} + \mathfrak{n}(\mathbf{R})) \cap \mathcal{C} \neq \varnothing\}$ with respect to the closure relation if and only if $(\mathcal{O} + \mathfrak{n}(\mathbf{R})) \cap \tilde{\mathcal{O}}$ is open dense in $\mathcal{O} + \mathfrak{n}(\mathcal{R})$ (with respect to the

Zariski topology). Therefore we have

$$Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g})(\mathcal{O})=\{Ind^{\mathfrak{g}(\mathbf{R})}_{(\mathfrak{m}(\mathbf{R}),\mathfrak{p}(\mathbf{R}))}(\mathcal{O})\}.$$

Thus we can interpret $Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g})$ as a generalization of the Lusztig-Spaltenstein induction of nilpotent orbits for complex Lie algebras.

From now on, we suppose that M is θ-stable. Since $\mathfrak{p}(\mathbf{R})=\mathfrak{m}(\mathbf{R})+\mathfrak{n}(\mathbf{R})$ is a real parabolic subalgebra of $\mathfrak{g}(\mathbf{R}),\mathfrak{p}(\mathbf{R})$ contains a minimal parabolic subalgebra of $\mathfrak{g}(\mathbf{R})$. Hence there exists a τ-stable maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{s}\cap[\mathfrak{g},\mathfrak{g}]$ such that $\mathfrak{a}\subset\mathfrak{m}$. As before, we write

$$F_G=\{a\in exp(\mathfrak{a});Ad(a^2)=id\}.$$

Then we have the following.

REMARK 2.8. (i) Any element $a\in F_G$ can be written as $a=exp(iA)$ for some $A\in\mathfrak{a}(\mathbf{R})$ and hence $\tau(a)=a^{-1}$. By an argument similar to the one in Remark 1.4 (i), F_G normalizes both $M(\mathbf{R})$ and $G(\mathbf{R})$.

(ii) By [KR, Proposition 2], we have $Ad(N_G(\mathfrak{g}(\mathbf{R})))=Ad(F_GG(\mathbf{R}))$.

(iii) For $\mathcal{O}\in\mathcal{N}_{\mathfrak{m}(\mathbf{R})}/M(\mathbf{R})$, write $F_G(\mathcal{O}):=\{a\in F_G;a\mathcal{O}=\mathcal{O}\}$. Then for $a\in F_G(\mathcal{O})$ and $\tilde{\mathcal{O}}\in\mathcal{N}_{\mathfrak{g}(\mathbf{R})}/G(\mathbf{R})$, we have $(\mathcal{O}+\mathfrak{n}(\mathbf{R}))\cap\tilde{\mathcal{O}}\neq\varnothing$ if and only if $(\mathcal{O}+\mathfrak{n}(\mathbf{R}))\cap(a\tilde{\mathcal{O}})\neq\varnothing$. Hence $Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g})(\mathcal{O})$ is $F_G(\mathcal{O})$-stable.

For a subset $S\in 2^{\mathcal{N}_{\mathfrak{m}(\mathbf{R})}/M(\mathbf{R})}$ of $\mathcal{N}_{\mathfrak{m}(\mathbf{R})}/M(\mathbf{R})$, we write $Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g})(S)$ the set of $G(\mathbf{R})$-orbits in $\bigcup_{\mathcal{O}\in S}Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g})(\mathcal{O})$ which are maximal with respect to the closure relation. This defines a map

$$Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g}):2^{\mathcal{N}_{\mathfrak{m}(\mathbf{R})}/M(\mathbf{R})}\to 2^{\mathcal{N}_{\mathfrak{g}(\mathbf{R})}}/G(\mathbf{R}).$$

It is known by [S] that there exists a natural bijection

$$S_G:\mathcal{N}_{\mathfrak{s}}/K\xrightarrow{\sim}\mathcal{N}_{\mathfrak{g}(\mathbf{R})}/G(\mathbf{R})$$

which is called the Sekiguchi correspondence (for the details of the definition of S_G, see [O, Theorem 1]). It is easy to see that S_G is F_G-equivariant. Via the Sekiguchi correspondence, we regard $Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g})$ as a map

$$Ind^{\mathbf{R}}((\mathfrak{m},\mathfrak{p})\uparrow\mathfrak{g}):2^{\mathcal{N}_{\mathfrak{s}_M}/K_M}\to 2^{\mathcal{N}_{\mathfrak{s}}/K},$$

where we write $K_M:=M\cap K,\mathfrak{s}_M=\mathfrak{m}\cap\mathfrak{s}$.

PROPOSITION 2.9. *Let H be a maximal torus of G which is both τ-stable and θ-stable, and $P=HN$ a τ-stable Borel subgroup with Levi factor H and unipotent radical N (hence G is quasisplit). Then we have*

$$Ind^{\mathbf{R}}((\mathfrak{h},\mathfrak{p})\uparrow\mathfrak{g})(\{(0)_{\mathfrak{s}_H}\})=\mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K,$$

where $(0)_{\mathfrak{s}_H}$ is the K_H-orbit in $\mathfrak{s}_H$ consisting of 0.

PROOF. Let Δ be the base of the positive system $R(\mathfrak{p},\mathfrak{h})$ and $\Delta_{\mathbf{R}}$ (resp. Δ_c) the set of real (resp. complex) roots in Δ:

$$\Delta_{\mathbf{R}} := \{\alpha \in \Delta; \tau(\alpha) = \alpha\}, \qquad \Delta_c = \{\alpha \in \Delta; \tau(\alpha) \neq \alpha\},$$

where the root $\tau(\alpha)$ is defined by $\tau(\alpha)(h) := \overline{\alpha(\tau(h))}$ $(h \in \mathfrak{h})$. We notice that $\tau(\alpha) = -\theta(\alpha)$ for a root $\alpha \in R(\mathfrak{g},\mathfrak{h})$. Since $R(\mathfrak{p},\mathfrak{h})$ is τ-stable, Δ does not have any imaginary root: $\Delta = \Delta_{\mathbf{R}} \cup \Delta_c$. Choose root vectors $X_\alpha \in \mathfrak{g}_\alpha \backslash (0) (\alpha \in \Delta)$ such that $\tau(X_\alpha) = X_\alpha$ for $\alpha \in \Delta_{\mathbf{R}}$ and $\tau(X_\alpha) = X_{\tau(\alpha)}$ for $\alpha \in \Delta_c$. Write $x = \sum_{\alpha \in \Delta} X_\alpha$. Since $x \in \mathfrak{n}(\mathbf{R}) = (0)_{\mathfrak{h}(\mathbf{R})} + \mathfrak{n}(\mathbf{R})$ and x is $\mathfrak{g}$-principal, we have

$$G(\mathbf{R})x \in Ind^{\mathbf{R}}((\mathfrak{h},\mathfrak{p}) \uparrow \mathfrak{g})(\{(0)_{\mathfrak{h}(\mathbf{R})}\})$$

on the level of induction by real Lie algebra, where we write $(0)_{\mathfrak{h}(\mathbf{R})}$ the $H(\mathbf{R})$-orbit in $\mathfrak{h}(\mathbf{R})$ consisting of 0. Since $F_G = F_G((0)_{\mathfrak{h}(\mathbf{R})})$ acts on $Ind^{\mathbf{R}}((\mathfrak{h},\mathfrak{p}) \uparrow \mathfrak{g})(\{(0)_{\mathfrak{h}(\mathbf{R})}\})$ by Remark 2.8(iii) and acts transitively on $\mathcal{N}_{\mathfrak{g}(\mathbf{R})}^{\mathfrak{g}-pr}/G(\mathbf{R})$, we have

$$Ind^{\mathbf{R}}((\mathfrak{h},\mathfrak{p}) \uparrow \mathfrak{g})(\{(0)_{\mathfrak{h}(\mathbf{R})}\}) \supset \mathcal{N}_{\mathfrak{g}(\mathbf{R})}^{\mathfrak{g}-pr}/G(\mathbf{R}).$$

On the other hand, we notice that $\mathcal{N}_{\mathfrak{g}}^{\mathfrak{g}-pr}$ and $\mathcal{N}_{\mathfrak{g}}$ are defined over $\mathbf{R}, \mathcal{N}_{\mathfrak{g}}^{\mathfrak{g}-pr}$ is open dense in $\mathcal{N}_{\mathfrak{g}}$, and $\mathcal{N}_{\mathfrak{g}(\mathbf{R})}^{\mathfrak{g}-pr} = \mathcal{N}_{\mathfrak{g}}^{\mathfrak{g}-pr} \cap \mathfrak{g}(\mathbf{R}) \neq \emptyset$. Hence $\mathcal{N}_{\mathfrak{g}(\mathbf{R})}^{\mathfrak{g}-pr}$ is open dense in $\mathcal{N}_{\mathfrak{g}(\mathbf{R})} = \mathcal{N}_{\mathfrak{g}} \cap \mathfrak{g}(\mathbf{R})$. Therefore we have

$$Ind^{\mathbf{R}}((\mathfrak{h},\mathfrak{p}) \uparrow \mathfrak{g})(\{(0)_{\mathfrak{h}(\mathbf{R})}\}) = \mathcal{N}_{\mathfrak{g}(\mathbf{R})}^{\mathfrak{g}-pr}/G(\mathbf{R}).$$

Via the Sekiguchi correspondence, we have

$$Ind^{\mathbf{R}}((\mathfrak{h},\mathfrak{p}) \uparrow \mathfrak{g})(\{(0)_{\mathfrak{s}_H}\}) = \mathcal{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K. \qquad \text{q.e.d.}$$

3. Induction of nilpotent orbits and associated varieties of standard $(\mathfrak{g}, K)$-modules

3.1 Standard $(\mathfrak{g}, K)$-modules

In this section, we show that the generic K-orbits in the associated varieties of certain standard $(\mathfrak{g}, K)$-modules can be described by induction of nilpotent orbits. We first describe the standard $(\mathfrak{g}, K)$-modules $X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ according to Vogan [V1].

Let H be a θ-stable and τ-stable maximal torus of G and $H = H_c H_s$ the Cartan decomposition (i.e. $H_c = H \cap K, H_s = \{h \in H; \theta(h) = h^{-1}\}$).

Let $\langle , \rangle$ be a non-degenerate G-invariant symmetric bilinear form on $\mathfrak{g}$ which is real valued on $\mathfrak{g}(\mathbf{R})$ such that $\mathfrak{k}$ and $\mathfrak{s}$ are orthgonal with respect to $\langle , \rangle, \langle , \rangle|_{\mathfrak{k}(\mathbf{R})}$ is negative definite and $\langle , \rangle|_{\mathfrak{s}(\mathbf{R})}$ is positive definite. The bilinear form on $\mathfrak{g}^*$, which is induced from $\langle , \rangle$, is also denoted by $\langle , \rangle$. For a $\mathfrak{h}$-stable subspace $V \subset \mathfrak{g}$, we write $\rho_V := (\sum_{\alpha \in R(V,\mathfrak{h})} \alpha)/2$. Let us consider a set $(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ of θ-stable data for $G(\mathbf{R})$ which is a quadruple such that

(a) $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ with θ-stable and τ-stable Levi factor $\mathfrak{l}$.

(b) The connected subgroup L of G corresponding to $\mathfrak{l}$ is quasisplit and has a maximally split Cartan subgroup $H \subset L$.

(c) δ is a character of $H_c(\mathbf{R})$ which is fine with respect to $L(\mathbf{R})$ (cf. [V1, Definition 4.3.8]).

(d) ν is a character of $H_s(\mathbf{R})$.

(e) Write $\lambda^L := d\delta \in \mathfrak{h}_c^*$ for the differential of δ, and $\lambda^G := \lambda^L + \rho\mathfrak{u} \in \mathfrak{h}_c^* \subset \mathfrak{h}^*$. Then $\langle \alpha, \lambda^G \rangle > 0$ for $\alpha \in R(\mathfrak{u}, \mathfrak{h})$.

Choose a τ-stable Borel subgroup $P_L = HN_L$ of L such that ν is negative with respect to $\mathfrak{n}_L$ (i.e. $Re\langle \alpha, \nu \rangle \leq 0$ for $\alpha \in R(\mathfrak{n}_L, \mathfrak{h})$). The standard $(\mathfrak{g}, K)$-module corresponding to $(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ is

$$X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu) = X(\mathfrak{q}, \delta \otimes \nu) = (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^{\dim(\mathfrak{u} \cap \mathfrak{k})}(Ind_{P_L(\mathbf{R})}^{L(\mathbf{R})}(\delta \otimes \nu)).$$

Let $A_{L(\mathbf{R})}(\delta)$ be the set of fine $L(\mathbf{R}) \cap K(\mathbf{R})$-types μ such that δ occurs in $\mu|_{H_c(\mathbf{R})}$ ([V1, Definition 4.3.15]). It is known that there is a bijection between $A_{L(\mathbf{R})}(\delta)$ and the set $A_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta)$ of lambda-lowest $K(\mathbf{R})$-types in $X(\mathfrak{q}, \delta \otimes \nu)$. Any $\pi \in A_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta)$ occurs in $X(\mathfrak{q}, \delta \otimes \nu)$ with multiplicity one and defines an irreducible submodule $\bar{X}(\mathfrak{q}, \delta \otimes \nu)(\pi)$ which contains the lambda-lowest $K(\mathbf{R})$-type π. It is known that any irreducible $(\mathfrak{g}, K)$-module is isomorphic to some $\bar{X}(\mathfrak{q}, \delta \otimes \nu)(\pi)$.

Standard $(\mathfrak{g}, K)$-module is also described as follows. Let $\mathfrak{m} \supset \mathfrak{h}$ be the Levi subalgebra defined by $R(\mathfrak{m}, \mathfrak{h}) = R(\mathfrak{g}, \mathfrak{h})_{i\mathbf{R}}$ and M the connected group corresponding to $\mathfrak{m}$. Let $\mathfrak{q}_M = \mathfrak{h} + \mathfrak{u}_M$ be the θ-stable Borel subalgebra of $\mathfrak{m}$ defined by $R(\mathfrak{q}_M, \mathfrak{h}) = \{\alpha \in R(\mathfrak{m}, \mathfrak{h}); \langle \alpha, \lambda^G \rangle > 0\}$. Take a τ-stable parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{m}$ as in [V1, 6.6.14]. Then the standard $(\mathfrak{g}, K)$-module can be written as

$$X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu) \simeq Ind_{P(\mathbf{R})}^{G(\mathbf{R})}((\mathscr{R}_{\mathfrak{q}_M}^{\mathfrak{m}})^{\dim(\mathfrak{u}_M \cap \mathfrak{k})}(\delta \otimes \nu)).$$

Here we note that $R(\mathfrak{u}, \mathfrak{h})_{i\mathbf{R}} = R(\mathfrak{u}_M, \mathfrak{h})$ is a positive system of the root system $R(\mathfrak{g}, \mathfrak{h})_{i\mathbf{R}}$.

REMARK 3.1. Write $\mathfrak{n} := \mathfrak{n}_L + \mathfrak{u}, \mathfrak{b} := \mathfrak{p}_L + \mathfrak{u} = \mathfrak{h} + \mathfrak{n}, \mathbf{C}_{-2\rho\mathfrak{n}_L} := (\bigwedge^{top}\mathfrak{n}_L)^*$ (one dimensional $H(\mathbf{R})$-module) and $\mathbf{C}_{|\rho\mathfrak{n}_L|} :=$ (positive square root of $|2\rho\mathfrak{n}_L| : H(\mathbf{R}) \to \mathbf{R}^\times$). These one dimensional $H(\mathbf{R})$-modules can be seen as $(\mathfrak{b}, H_c)$-modules. We write $\mathbf{C}_{\delta \otimes \nu}$ the $H(\mathbf{R})$-module corresponding to $\delta \otimes \nu : H(\mathbf{R}) \to \mathbf{C}^\times$ and $E_{\rho(\mathfrak{b})}$ the genuine one dimensional $(\mathfrak{b}, (H_c)^{\rho(\mathfrak{b})})$-module induced from $(\mathfrak{b}, H_c)$-module $E_{2\rho(\mathfrak{b})} = \bigwedge^{top}(\mathfrak{g}/\mathfrak{b})^*$ (for the definition of $E_{\rho(\mathfrak{b})}$, see [AV, Definition 8.11]). $\mathbf{C}_\Lambda := (\mathbf{C}_{\delta\otimes\nu} \otimes \mathbf{C}_{-2\rho\mathfrak{n}_L} \otimes \mathbf{C}_{|\rho\mathfrak{n}_L|}) \otimes E_{\rho(\mathfrak{b})}$ can be seen as a genuine $(\mathfrak{b}, (H_c)^{\rho(\mathfrak{b})})$-module. Then $\mathfrak{b}$ and $\mathbf{C}_\Lambda$ are in good position ([AV, Definition 8.18]). Furthermore the stndard representation $I(B, \mathbf{C}_\Lambda)$ defined in [AV, Definition 8.18] coincides with $X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$.

3.2. Induction of nilpotent orbits and associated varieties of standard $(\mathfrak{g}, K)$-modules

For a finitely generated $(\mathfrak{g}, K)$-module X, we write $Ass(X) \subset \mathfrak{g}^*$ the associated variety of X (for the definition of $Ass(X)$, see [V2]). By the identification $\mathfrak{g} \simeq \mathfrak{g}^*(x \mapsto \langle x, \cdot \rangle)$ which is induced by the G-invariant bilinear form $\langle , \rangle$ on $\mathfrak{g}$, we see that $Ass(X)$ is a subset of $\mathfrak{g}$. Then $Ass(X)$ is a K-invariant subset of $\mathfrak{g}$. It is known that if X has a finite composition series, then $Ass(X) \subset \mathcal{N}_\mathfrak{s}$ (cf. [V2]). We write $Ass(X)^{\mathfrak{g}-pr}$ the set of $\mathfrak{g}$-principal elements in $Ass(X)$ and $Ass(X)^{\mathfrak{g}-pr}/K$ that of K-orbits in $Ass(X)^{\mathfrak{g}-pr}$.

Proposition 3.2 ([AV, Proposition A.9]). *Suppose that $\mathfrak{g}$ is quasisplit. Let $(\mathfrak{b}, T(\mathbf{R}), \delta, \nu)$ be a set of θ-stable data for $G(\mathbf{R})$ such that $\mathfrak{b}$ is a θ-stable Borel subalgebra of large type. Write $\Sigma^c := R(\mathfrak{b}, \mathfrak{t})_{i\mathbf{R}}$. Then $(\mathfrak{t}, -\Sigma^c) \in \mathscr{P}^L_\mathfrak{g}$ and we have*

$$Ass(X_{G(\mathbf{R})}(\mathfrak{b}, T(\mathbf{R}), \delta, \nu)) = \bar{\mathcal{O}}_{(\mathfrak{t}, -\Sigma^c)}.$$

Theorem 3.3 ([AV, Theorem A.10]). *Suppose that $\mathfrak{g}$ is quasisplit. Let H be a θ-stable and τ-stable maximal torus of G, $(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ $(\mathfrak{q} = \mathfrak{l} + \mathfrak{u}, \mathfrak{h} \subset \mathfrak{l})$ a set of θ-stable data for $G(\mathbf{R})$ and $X := X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ the corresponding standard $(\mathfrak{g}, K)$-module. Write*

$$\Sigma := R(\mathfrak{u}, \mathfrak{h})_{i\mathbf{R}} = \{\alpha \in R(\mathfrak{g}, \mathfrak{h})_{i\mathbf{R}}; \langle \alpha, \lambda^G \rangle > 0\}.$$

(i) *If Σ is not of large type, $Ass(X)^{\mathfrak{g}-pr} = \varnothing$.*

(ii) *Suppose that Σ is of large type. Let $Q' = L'U'$ be a θ-stable parabolic subgroup of G such that $L' \supset H$, and that Σ is contained in the set of roots of $\mathfrak{h}$ in $\mathfrak{u}'$ (hence L' is quasisplit). Then we have*

$$Ass(X)^{\mathfrak{g}-pr}/K = \{\mathcal{O}_{(\mathfrak{t}, -\Sigma^c)}; (\mathfrak{t}, \Sigma^c) \in \mathscr{P}^L_{\mathfrak{q}'}\}.$$

In particular, $Ass(X)^{\mathfrak{g}-pr}/K$ depends only on $(\mathfrak{h}, \Sigma)$.

Under the assumption of Theorem 3.3(ii), we have

$$Ass(X)^{\mathfrak{g}-pr}/K = [\mathcal{N}_\mathfrak{s}^{\mathfrak{g}-pr}/K]_{(\mathfrak{q}')^-}$$

by Theorem 3.3(ii) and Proposition 1.7 where we write $(\mathfrak{q}')^-$ the parabolic subalgebra corresponding to $-R(\mathfrak{q}', \mathfrak{h})$. For a subset $\mathscr{S} \subset \mathcal{N}_\mathfrak{s}/K$, we write

$\mathscr{S}^{\mathfrak{g}-pr}$ the set of $\mathfrak{g}$-principal K-orbits in $\mathscr{S}$. Then, by Proposition 2.4 and Proposition 2.9, we obtain the following:

THEOREM 3.4. *Under the assumption of* Theorem 3.3(ii), *we have*

$$(3.1)\qquad Ass(X)^{\mathfrak{g}-pr}/K = [\mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K]_{(\mathfrak{q}')^-} = [Ind^{\theta}((\mathfrak{l}', (\mathfrak{q}')^-) \uparrow \mathfrak{g})(\mathscr{N}_{\mathfrak{s}_{L'}}^{\mathfrak{l}'-pr}/K_{L'})]^{\mathfrak{g}-pr}$$

$$= [Ind^{\theta}((\mathfrak{l}', (\mathfrak{q}')^-) \uparrow \mathfrak{g}) \circ Ind^{\mathbf{R}}((\mathfrak{h}, \mathfrak{p}_{L'}) \uparrow \mathfrak{l}')(\{(0)_{\mathfrak{s}_H}\})]^{\mathfrak{g}-pr}$$

where we write $P_{L'} = HN_{L'}$ *a* τ*-stable Borel subgroup of* L'.

REMARK 3.5. (i) In the setting of Theorem 3.3(ii), if Σ is not of large type, it can be shown that

$$[\mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K]_{(\mathfrak{q}')^-} = [Ind^{\theta}((\mathfrak{l}', (\mathfrak{q}')^-) \uparrow \mathfrak{g})(\mathscr{N}_{\mathfrak{s}_{L'}}^{\mathfrak{l}'-pr}/K_{L'})]^{\mathfrak{g}-pr} = \varnothing.$$

Therefore (3.1) holds even when Σ is not of large type. We will show this in the succeeding paper.

(ii) For a standard $(\mathfrak{g}, K)$-module

$$X = X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu) = (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^{\dim(\mathfrak{u}\cap\mathfrak{k})}(Ind_{P_L(\mathbf{R})}^{L(\mathbf{R})}(\delta \otimes \nu))$$

in 3.1, write $Y := Ind_{P_L(\mathbf{R})}^{L(\mathbf{R})}(\delta \otimes \nu)$. We can see that Y itself is a standard $(\mathfrak{l}, L \cap K)$-module corresponding to the data $(\mathfrak{l}, H(\mathbf{R}), \delta, \nu)$ for $L(\mathbf{R})$. By Theorem 3.4, we have $Ass(Y) = \mathscr{N}_{\mathfrak{l}\cap\mathfrak{s}}$ and hence $Ass(Y)$ is stable under the action of F_L. On the other hand, if $\Sigma = R(\mathfrak{u}, \mathfrak{h})_{i\mathbf{R}}$ is of large type, the subgroup F_L^G of F_L acts on $Ass(X)^{\mathfrak{g}-pr}/K = [\mathscr{N}_{\mathfrak{s}}^{\mathfrak{g}-pr}/K]_{(\mathfrak{q})^-}$ by Proposition 1.7.

(iii) The dual group $\hat{R}_{\delta}$ of the R-group R_{δ} in [V1] is a quotient of F_L ([V1, Lemma 4.3.46]). Since $\hat{R}_{\delta}$ acts transitively on the set $A_{L(\mathbf{R})}(\delta)$ of lowest $L \cap K$-types of Y ([V1, Theorem 4.3.16]), F_L also acts on $A_{L(\mathbf{R})}(\delta)$ transitively. Since there is a natural 1-1 correspondence between $A_{L(\mathbf{R})}(\delta)$ and the set $A_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta)$ of lambda lowest K-types of X, F_L acts on $A_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta)$ transitively.

Suppose that $G(\mathbf{R})$ is connected, semisimple and has a compact Cartan subgroup $T(\mathbf{R})$. Let $\mathfrak{b} = \mathfrak{t} + \mathfrak{u}$ be a θ-stable Borel subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{t}$ and nilpotent radical $\mathfrak{u}$. Let $(\mathfrak{b}, T(\mathbf{R}), \delta)$ be a set of θ-stable data for $G(\mathbf{R})$ which contains $\mathfrak{b}$ and $T(\mathbf{R})$, and write

$$X = X_{G(\mathbf{R})}(\mathfrak{b}, T(\mathbf{R}), \delta) = (\mathscr{R}_{\mathfrak{b}}^{\mathfrak{g}})^{\dim(\mathfrak{u}\cap\mathfrak{k})}(\delta)$$

the corresponding standard $(\mathfrak{g}, K)$-module. Then $\lambda^G = d\delta + \rho\mathfrak{u}$ is regular, $R(\mathfrak{b}, \mathfrak{t}) = \{\alpha \in R(\mathfrak{g}, \mathfrak{t}); \langle \alpha, \lambda^G \rangle > 0\}$ and X is the $(\mathfrak{g}, K)$-module of the discrete series representation with Harish-Chandra parameter λ^G. Write $\mathfrak{u}^-$ the nilpotent radical of the opposite Borel subalgebra $\mathfrak{b}^-$ of $\mathfrak{b}$. Then the following fact seems to be well known to experts. A proof based on a work of Hotta-Parthasarathy is given in Yamashita [Y]. A brief sketch of a proof can be found in Binegar-Zierau [BZ].

THEOREM 3.6 (Yamashita [Y, Theorem 1]). *In the above setting, we have*

$$Ass(X) = K(\mathfrak{u}^- \cap \mathfrak{s}).$$

This is the closure of the unique K-orbit in $\mathcal{N}_{\mathfrak{s}}$, whose intersection with $\mathfrak{u}^- \cap \mathfrak{s}$ is open dense in $\mathfrak{u}^- \cap \mathfrak{s}$.

In the above setting, clearly the unique K-orbit in $\mathcal{N}_{\mathfrak{s}}$, whose intersection with $\mathfrak{u}^- \cap \mathfrak{s}$ is open dense in $\mathfrak{u}^- \cap \mathfrak{s}$, is $Ind^{\theta}((\mathfrak{l}, \mathfrak{b}^-) \uparrow \mathfrak{g})((0)_{\mathfrak{s}_H})$. Hence we have the following.

PROPOSITION 3.7. $Ass(X) = \overline{Ind^{\theta}((\mathfrak{l}, \mathfrak{b}^-) \uparrow \mathfrak{g})((0)_{\mathfrak{s}_H})}$.

3.3 The action of F_L^G on the associated varieties of the standard $(\mathfrak{g}, K)$-modules

Let $(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ and $P_L = HN_L$ be as in 3.1, and write

$$X = X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu) = (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^{\dim(\mathfrak{u} \cap \mathfrak{k})}(Ind_{P_L(\mathbf{R})}^{L(\mathbf{R})}(\delta \otimes \nu))$$

the corresponding standard $(\mathfrak{g}, K)$-module. Define finite groups F_L and F_L^G by

$$F_L := \{a \in exp(\mathfrak{h}_s \cap [\mathfrak{l}, \mathfrak{l}]); Ad(a^2)|_{\mathfrak{l}} = id_{\mathfrak{l}}\}, \qquad F_L^G = \{a \in F_L; Ad(a^2) = id\}.$$

We will show that the action of K on X can be extended to that of KF_L^G and that X has a structure of $(\mathfrak{g}, KF_L^G)$-module. Consequently $Ass(X)$ is stable under the action of KF_L^G.

For a subgroup S of G normalized by F_L^G, we write

$$[S]_L^G := F_L^G S = SF_L^G = \langle S \cup F_L^G \rangle.$$

Then S is a normal subgroup of $[S]_L^G$ of finite index. We notice that the subgroups $G(\mathbf{R}), K, K(\mathbf{R}), L(\mathbf{R}), L \cap K, L(\mathbf{R}) \cap K(\mathbf{R}), H(\mathbf{R}), H_c(\mathbf{R})$ and $P_L(\mathbf{R}) = H_c(\mathbf{R})H_s(\mathbf{R})N_L(\mathbf{R})$ are all normalized by F_L^G. $[G(\mathbf{R})]_L^G$ is a real reductive linear group in the sense of [V1, Chap. 0], $[K(\mathbf{R})]_L^G$ is a maximal compact subgroup of $[G(\mathbf{R})]_L^G$ and $N_{[G(\mathbf{R})]_L^G}(\mathfrak{q}) = [L(\mathbf{R})]_L^G$. For the character δ of $H_c(\mathbf{R})$, we can take a character $\tilde{\delta}$ of $[H_c(\mathbf{R})]_L^G$ such that $\tilde{\delta}|_{H_c(\mathbf{R})} = \delta$. Then $\tilde{\delta}$ is fine with respect to $[L(\mathbf{R})]_L^G$ and $(\mathfrak{q}, [H(\mathbf{R})]_L^G, \tilde{\delta}, \nu)$ is a set of θ-stable data for $[G(\mathbf{R})]_L^G$. On the other hand, $[P_L(\mathbf{R})]_L^G = (F_L^G H_c(\mathbf{R}))H_s(\mathbf{R})N_L(\mathbf{R})$ is a minimal parabolic subgroup of $[L(\mathbf{R})]_L^G$ and ν is negative with respect to $N_L(\mathbf{R})$. Hence we obtain the standard $(\mathfrak{g}, KF_L^G)$-module

$$\tilde{X} = (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^{\dim(\mathfrak{u} \cap \mathfrak{k})}(Ind_{[P_L(\mathbf{R})]_L^G}^{[L(\mathbf{R})]_L^G}(\tilde{\delta} \otimes \nu)).$$

By the construction of standard $(\mathfrak{g}, K)$-module, it is verified that $\tilde{X}$ is isomorphic to X as $(\mathfrak{g}, K)$-modules. The associated variety $Ass(X)$ coincides with $Ass(\tilde{X})$ and hence $Ass(X)$ is stable under the action of $KF_L^G = [K]_L^G$.

PROPOSITION 3.8. *For the standard* $(\mathfrak{g}, K)$*-module*

$$X = X_{G(\mathbf{R})}(\mathfrak{q}, H(\mathbf{R}), \delta, \nu) = (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^{\dim(\mathfrak{u}\cap\mathfrak{k})}(Ind_{P_L(\mathbf{R})}^{L(\mathbf{R})}(\delta \otimes \nu))$$

corresponding to a set $(\mathfrak{q}, H(\mathbf{R}), \delta, \nu)$ *of* θ*-stable data for* $G(\mathbf{R})$*, the associated variety* $Ass(X)$ *is stable under the action of* $KF_L^G = [K]_L^G$.

REMARK 3.9. In the above setting, if $\Sigma = R(\mathfrak{g}, \mathfrak{h})_{i\mathbf{R}}$ is of large type (hence $\mathfrak{g}$ is quasisplit) and θ is of inner type, we can show that the action of F_L^G on $Ass(X)^{\mathfrak{g}-pr}/K \neq \emptyset$ is transitive. We will prove this in the succeeding paper.

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