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Some homotopy groups of the rotation group R_n

Dedicated to Professor Teiich Kobayashi on his 60th birthday

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Abstract. We determine the group strucures of the 2-primary components of the homotopy groups of the rotation group $\pi_k(R_n)$ for $k = 17$ and 18 by use of the fibration $R_{n+1}/R_n = S^n$.

Introduction

We denote by R_n the n-th rotation group. We know the homotopy groups $\pi_k(R_n)$ for $k \leq 15$ by [7]. According to [9] and [8], the group structures of $\pi_k(R_n)$ for $k \leq 22$ and $n \leq 9$ are known. For $k = 15$ and 16, we know the 2-primary components of $\pi_k(R_n)$ ([5]). We denote by $\pi_k(X : 2)$ a suitablly chosen subgroup of the homotopy group $\pi_k(X)$ which consists of the 2-primary component and a free part such that the index $[\pi_k(X) : \pi_k(X : 2)]$ is odd. The purpose of the present note is to determine $\pi_k(R_n : 2)$ for $k = 17$ and 18.

Our method is the composition methods developed by Toda [17]. We freely use generators and relations in the homotopy groups of spheres $\pi_{n+k}(S^n)$ for $k \leq 18$. In determining $\pi_{18}(R_n : 2)$, the precise informations of the generators of $\pi_{n+18}(S^n)$ for $10 \leq n \leq 12$ ([14]) are essentially used. Our main tool is to use the following exact sequence induced from the fibration $R_{n+1}/R_n = S^n$:

$$(k)_n \qquad \pi_{k+1}(S^n) \xrightarrow{\Delta} \pi_k(R_n) \xrightarrow{i_*} \pi_k(R_{n+1}) \xrightarrow{p_*} \pi_k(S^n) \xrightarrow{\Delta} \pi_{k-1}(R_n),$$

where $i : R_n \hookrightarrow R_{n+1}$ is the inclusion, $p : R_{n+1} \to S^n$ is the projection and Δ is the connecting map.

The metastable range is obtained from the splitting ([2]):

$$\pi_k(R_n) \cong \pi_k(R_\infty) \oplus \pi_{k+1}(V_{2n,n}) \qquad \text{for } k \leq 2n-1 \quad \text{and} \quad n \geq 13,$$

where $V_{m,r} = R_m/R_{m-r}$ for $m \geq r$ is the Stiefel manifold.

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By use of this splitting and [13], we have $\pi_k(R_n:2)$ for $k=17$ and 18 with $n\geq 13$. So our main task is to determine the unstably metastable range, that is to say, to determine $\pi_k(R_n)$ for $k=17$ and 18 in the case $10\leq n\leq 12$. Especially determinations of $\Delta(\pi_{18}(S^9))$ and $\mathrm{Ker}\{\Delta:\pi_{18}(S^{10})\to\pi_{17}(R_{10})\}$ play an essential role to get our result.

We use the notations and results of [5] and [17] freely. For an element $\alpha\in\pi_k(S^n)$, we denote by $[\alpha]\in\pi_k(R_{n+1})$ an element satisfying $p_*[\alpha]=\alpha$. Though $[\alpha]$ is only determined modulo $\mathrm{Im}\,i_*=i_*(\pi_k(R_n))$, we will sometimes give restrictions on $[\alpha]$ to fix it more concretely. We set $[\alpha]_m=i_*[\alpha]\in\pi_k(R_m)$, where $i:R_{n+1}\hookrightarrow R_m$ for $n+1\leq m$ is the inclusion. We state our result.

THEOREM 1. (i) $\pi_{17}(R_3:2)=\mathbf{Z}_2\{[\eta_2]\varepsilon_3\nu_{11}^2\}$;
$\pi_{17}(R_4:2)=\mathbf{Z}_2\{[\eta_2]_4\varepsilon_3\nu_{11}^2\}\oplus\mathbf{Z}_2\{[\iota_3]\varepsilon_3\nu_{11}^2\}$;
$\pi_{17}(R_5:2)=\mathbf{Z}_8\{[\nu_4^2]\sigma_{10}\}$;
$\pi_{17}(R_6:2)=\mathbf{Z}_8\{[\nu_4^2]_6\sigma_{10}\}\oplus\mathbf{Z}_2\{[\nu_5]\mu_8\}\oplus\mathbf{Z}_2\{[\nu_5]\nu_8^3\}\oplus\mathbf{Z}_2\{[\nu_5]\eta_8\varepsilon_9\}$.

(ii) $\pi_{17}(R_7:2)=\mathbf{Z}_8\{[\bar\nu_6+\varepsilon_6]\nu_{14}\}\oplus\mathbf{Z}_8\{[\nu_4^2]_7\sigma_{10}\}\oplus\mathbf{Z}_2\{[\nu_5]_7\mu_8\}\oplus\mathbf{Z}_2\{[\eta_6]\eta_7\mu_8\}$;
$\pi_{17}(R_8:2)=\mathbf{Z}_8\{[\bar\nu_6+\varepsilon_6]_8\nu_{14}\}\oplus\mathbf{Z}_8\{[\nu_4^2]_8\sigma_{10}\}\oplus\mathbf{Z}_2\{[\nu_5]_8\mu_8\}$
$\oplus\mathbf{Z}_2\{[\eta_6]_8\eta_7\mu_8\}\oplus\mathbf{Z}_8\{[\iota_7]\nu_7\sigma_{10}\}\oplus\mathbf{Z}_2\{[\iota_7]\eta_7\mu_8\}$;
$\pi_{17}(R_9:2)=\mathbf{Z}_8\{[\iota_7]_9\nu_7\sigma_{10}\}\oplus\mathbf{Z}_2\{[\nu_5]_9\mu_8\}\oplus\mathbf{Z}_2\{[\iota_7]_9\eta_7\mu_8\}$.

(iii) $\pi_{17}(R_{10}:2)=\mathbf{Z}_4\{[\iota_7]_{10}\nu_7\sigma_{10}\}\oplus\mathbf{Z}_2\{[\iota_7]_{10}\eta_7\mu_8\}$;
$\pi_{17}(R_{11}:2)=\mathbf{Z}_2\{[\iota_7]_{11}\nu_7\sigma_{10}\}\oplus\mathbf{Z}_2\{[\iota_7]_{11}\eta_7\mu_8\}$;
$\pi_{17}(R_n:2)=\mathbf{Z}_2\{[\iota_7]_n\eta_7\mu_8\}$ *for* $n=12,13$ *and* 14.

(iv) $\pi_{17}(R_{15}:2)=\mathbf{Z}_2\{[\iota_7]_{15}\eta_7\mu_8\}\oplus\mathbf{Z}_2\{[\eta_{14}^2]\eta_{16}\}$;
$\pi_{17}(R_{16}:2)=\mathbf{Z}_2\{[\iota_7]_{16}\eta_7\mu_8\}\oplus\mathbf{Z}_2\{[\eta_{14}^2]_{16}\eta_{16}\}\oplus\mathbf{Z}_2\{[\eta_{15}]\eta_{16}\}$;
$\pi_{17}(R_{17}:2)=\mathbf{Z}_2\{[\iota_7]_{17}\eta_7\mu_8\}\oplus\mathbf{Z}_2\{[\eta_{15}]_{17}\eta_{16}\}$;
$\pi_{17}(R_{18}:2)=\mathbf{Z}_2\{[\iota_7]_{18}\eta_7\mu_8\}\oplus\mathbf{Z}\{[2\iota_{17}]\}$;
$\pi_{17}(R_n:2)=\mathbf{Z}_2\{[\iota_7]_n\eta_7\mu_8\}$ *for* $n\geq 19$.

THEOREM 2. (i) $\pi_{18}(R_3:2)=\mathbf{Z}_2\{[\eta_2]\bar\varepsilon_3\}$;
$\pi_{18}(R_4:2)=\mathbf{Z}_2\{[\eta_2]_4\bar\varepsilon_3\}\oplus\mathbf{Z}_2\{[\iota_3]\bar\varepsilon_3\}$;
$\pi_{18}(R_5:2)=\mathbf{Z}_8\{[\nu_4\zeta_7]\}\oplus\mathbf{Z}_2\{[\iota_3]_5\bar\varepsilon_3\}$;
$\pi_{18}(R_6:2)=\mathbf{Z}_8\{[\nu_4\zeta_7]_6\}\oplus\mathbf{Z}_4\{[\nu_5]\sigma_8\nu_{15}\}\oplus\mathbf{Z}_2\{[\nu_5]\eta_8\mu_9\}$.

(ii) $\pi_{18}(R_7:2)=\mathbf{Z}_{16}\{[2[\iota_6,\iota_6]]\sigma_{11}\}\oplus\mathbf{Z}_8\{[\nu_4\zeta_7]_7\}\oplus\mathbf{Z}_2\{[\nu_5]_7\sigma_8\nu_{15}\}$;
$\pi_{18}(R_8:2)=\mathbf{Z}_{16}\{[2[\iota_6,\iota_6]]_8\sigma_{11}\}\oplus\mathbf{Z}_8\{[\nu_4\zeta_7]_8\}\oplus\mathbf{Z}_2\{[\nu_5]_8\sigma_8\nu_{15}\}$
$\oplus\mathbf{Z}_8\{[\iota_7]\zeta_7\}\oplus\mathbf{Z}_2\{[\iota_7]\bar\nu_7\nu_{15}\}$;
$\pi_{18}(R_9:2)=\mathbf{Z}_{16}\{[2[\iota_6,\iota_6]]_9\sigma_{11}\}\oplus\mathbf{Z}_8\{[\iota_7]_9\zeta_7\}\oplus\mathbf{Z}_2\{[\iota_7]_9\bar\nu_7\nu_{15}\}$.

(iii) $\pi_{18}(R_{10}:2)=\mathbf{Z}_{32}\{[[\iota_9,\iota_9]\eta_{17}]\}\oplus\mathbf{Z}_8\{[\eta_9\varepsilon_{10}]\}$;
$\pi_{18}(R_{11}:2)=\mathbf{Z}_8\{[\varepsilon_{10}]\}$;
$\pi_{18}(R_{12}:2)=\mathbf{Z}_{16}\{[2\sigma_{11}]\}\oplus\mathbf{Z}_4\{[\varepsilon_{10}]_{12}-2[2\sigma_{11}]\}$.

(iv) $\pi_{18}(R_{13}:2) = \mathbf{Z}_8\{[\nu_{12}^2]\}$;
$\pi_{18}(R_n:2) = \mathbf{Z}_8\{[\nu_{12}^2]_n\}$ *for* $n = 14$ *and* 15;
$\pi_{18}(R_{16}:2) = \mathbf{Z}_8\{[\nu_{12}^2]_{16}\} \oplus \mathbf{Z}_8\{[\nu_{15}]\}$;
$\pi_{18}(R_{17}:2) = \mathbf{Z}_8\{[\nu_{15}]_{17}\}$;
$\pi_{18}(R_{18}:2) = \mathbf{Z}_4\{[\nu_{15}]_{18}\}$;
$\pi_{18}(R_{19}:2) = \mathbf{Z}_2\{[\nu_{15}]_{19}\}$;
$\pi_{18}(R_n:2) = 0$ *for* $n \geq 20$.

1. Some relations among elements of $\pi_k(R_n)$

First of all, since R_n is a Hopf space, we have the following formula;

$$(\alpha+\beta)\circ\gamma = \alpha\circ\gamma + \beta\circ\gamma$$

for $\alpha, \beta \in \pi_k(R_n)$ and $\gamma \in \pi_m(S^k)$.

We recall a formula

$$\Delta(\alpha\circ\Sigma\beta) = \Delta(\alpha)\circ\beta$$

for $\alpha \in \pi_{j+1}(R_n)$ and $\beta \in \pi_k(S^j)$.

Let $J:\pi_k(R_{n+1}) \to \pi_{k+n+1}(S^{n+1})$ be the J homomorphism ([18]). We have a formula

$$J(\alpha\circ\beta) = J(\alpha)\circ\Sigma^{n+1}\beta$$

for $\alpha \in \pi_j(R_{n+1})$ and $\beta \in \pi_k(S^j)$.

Concerning the exact sequence $(k)_n$, the following formulae hold:

$$J(i_*\beta) = \Sigma(J(\beta));$$

$$H(J[\alpha]) = \Sigma^{n+1}\alpha;$$

$$J(\Delta\alpha) = [\alpha, \iota_n],$$

where $H:\pi_{k+n+1}(S^{n+1}) \to \pi_{k+n+1}(S^{2n+1})$ is the Hopf homomorphism and $[\ ,\iota_n]$ is the Whitehead product with the identity class ι_n of S^n.

Hereafter we only deal with the homotopy group $\pi_k(X:2)$ and it is denoted by $\pi_k(X)$ for simplicity.

We recall the following elements given in [5]; $[\eta_2] \in \pi_3(R_3)$, $[\iota_3] \in \pi_3(R_4)$, $[\nu_5] \in \pi_8(R_6)$, $[\eta_6] \in \pi_7(R_7)$, $[\iota_7] \in \pi_7(R_8)$, $[\nu_4^2] \in \pi_{10}(R_5)$, $[\eta_5\varepsilon_6] \in \pi_{14}(R_6)$ and $[\bar{\nu}_6 + \varepsilon_6] \in \pi_{14}(R_7)$.

The elements $[\iota_3]$ and $[\iota_7]$ are represented by the multiplications of the quaternions and Cayley numbers, respectively. The relations of $\Delta\iota_4$ and $\Delta\iota_8$ in (i) of the following lemma determine $[\eta_2]$ and $[\eta_6]$, respectively. $[\nu_5]$ and $[\nu_4^2]$ are unique. $[\eta_5\varepsilon_6]$ and $[\bar{\nu}_6 + \varepsilon_6]$ are fixed in (ii) of the lemma. The following result is an improvement of that of [5].

LEMMA 1.1. (i) $\Delta\iota_4 = 2[\iota_3] - [\eta_2]_4, \Delta\nu_4 = ([\iota_3] + a[\eta_2]_4)\nu'$ *for* $0 \le a \le 3$, $\Delta\nu_5 = 0$, $\Delta\eta_6 = 0$, $\Delta\nu_6 = 2[\nu_5]$, $\Delta\iota_8 = 2[\iota_7] - [\eta_6]_8$, $\Delta\iota_9 = [\nu_5]_9 + [\iota_7]_9\eta_7$ *and* $\Delta\iota_{11} = [\iota_7]_{11}\nu_7$.

(ii) $J[\iota_7] = \sigma_8$, $J[\eta_6] = \sigma'$, $J[\nu_5] = \bar{\nu}_6 + \varepsilon_6$, $J[\nu_4^2] = \nu_5\sigma_8$, $J[\eta_5\varepsilon_6] = -\sigma''\sigma_{13} + \bar{\nu}_6\nu_{14}^2$ *and* $J[\bar{\nu}_6 + \varepsilon_6] = \sigma'\sigma_{14}$.

(iii) $\Delta([\iota_6, \iota_6]) = [\nu_5]\eta_8^2 + 4[\nu_4^2]_6$.

(iv) $[\eta_6]\nu_7 = b[\nu_4^2]_7$ *for an odd integer* b.

(v) $[\eta_6]\sigma' = 4[\bar{\nu}_6 + \varepsilon_6] + [\nu_5]_7\nu_8^2 + [\eta_5\varepsilon_6]_7$.

PROOF. (i) is obtained from Table 3 of [5].
$J[\iota_7] = \sigma_8$ is the Hopf class. So, we have

$$\Sigma^2 J[\eta_6] = \Sigma(J[\eta_6]_8) = \Sigma 2J[\iota_7] - J(i_*\Delta\iota_8) = 2\sigma_9 = \Sigma^2\sigma'.$$

It follows $J[\eta_6] = \sigma'$ since $\Sigma^2 : \pi_{14}(S^7) \to \pi_{16}(S^9)$ is monic.

From the relations $H(J[\nu_5]) = \nu_{11}$, $H(\bar{\nu}_6) = \nu_{11}$ and $H(\varepsilon_6) = 0$, we have $J[\nu_5] = \bar{\nu}_6$ or $\bar{\nu}_6 + \varepsilon_6$. Since the stable J-image of the 8-stem group is generated by $\bar{\nu} + \varepsilon$, we have $J[\nu_5] = \bar{\nu}_6 + \varepsilon_6$.

Since $H : \pi_{15}(S^5) \to \pi_{15}(S^9)$ has the kernel $\{\eta_5\mu_6\}$ and since in the stable 10-stem group the J-image is trivial and $\eta\mu \neq 0$, the third relation is obtained from the fact $H(\nu_5\sigma_8) = \Sigma(\nu_4 \wedge \nu_4) = \nu_9^2$.

By [17], we have

$$H(J[\eta_5\varepsilon_6]) = \eta_{11}\varepsilon_{12} = \eta_{11}^2\sigma_{13} + \nu_{11}^3 = H(\sigma'')\sigma_{13} + H(\bar{\nu}_6)\nu_{14}^2 = H(\sigma''\sigma_{13} + \bar{\nu}_6\nu_{14}^2).$$

So, $J[\eta_5\varepsilon_6] \equiv \sigma''\sigma_{13} + \bar{\nu}_6\nu_{14}^2 \bmod \operatorname{Im}\Sigma = \{2\sigma''\sigma_{13}\}$. By Table 2 of [5], $2[\eta_5\varepsilon_6] \in i_*\pi_{14}(R_5)$. Then the fifth relation follows by choosing a representative of $[\eta_5\varepsilon_6]$.

Similarly the last relation is obtained from the relations

$$H(J[\bar{\nu}_6 + \varepsilon_6]) = \bar{\nu}_{13} + \varepsilon_{13} = \eta_{13}\sigma_{14} = H(\sigma'\sigma_{14})$$

and that $\operatorname{Im}\Sigma = \{2\sigma'\sigma_{14}, \bar{\nu}_7\nu_{15}^2\} = J(\operatorname{Im} i_*)$, completing the proof of (ii).

In the exact sequence $(10)_6$, we know that $\pi_{10}(R_6) = \mathbf{Z}_8\{[\nu_4^2]_6\} \oplus \mathbf{Z}_2\{[\nu_5]\eta_8^2\}$ and $\pi_{10}(R_7) = \mathbf{Z}_8\{[\nu_4^2]_7\}$ by Table 2 of [5]. So we have $\Delta[\iota_6, \iota_6] = [\nu_5]\eta_8^2 + 4x[\nu_4^2]_6$, where $x = 0$ or 1. By [12], we have $J\Delta([\iota_6, \iota_6]) = [\iota_6, [\iota_6, \iota_6]] = 0$. By Lemma 6.3, (7.10) and Theorem 7.3 of [17] and by (ii), we have

$$J([\nu_5]\eta_8^2) = (\bar{\nu}_6 + \varepsilon_6) \circ \eta_{14}^2 = \eta_6^2\varepsilon_8 = 4\nu_6\sigma_9 \neq 0$$

and $J[\nu_4^2]_6 = \nu_6\sigma_9$. So we have $x = 1$. This leads us to (iii).

Applying the J-homorphism $J : \pi_{10}(R_7) \to \pi_{17}(S^7)$, (iv) is obtained from the relation $\sigma'\nu_{14} = x\nu_7\sigma_{10}$ for an odd integer x by (7.19) of [17].

Since $p_*([\eta_6]\sigma' - 4[\bar{\nu}_6 + \varepsilon_6]) = \eta_6\sigma' - 4\bar{\nu}_6 = 0$ (p. 64 of [17]), we have

$$[\eta_6]\sigma' - 4[\bar{\nu}_6 + \varepsilon_6] \in i_*\pi_{14}(R_6) = \mathbf{Z}_8\{[\eta_5\varepsilon_6]_7\} \oplus \mathbf{Z}_2\{[\nu_5]_7\nu_8^2\}$$

by Table 2 of [5]. So we have

$$[\eta_6]\sigma' = 4[\bar{\nu}_6 + \varepsilon_6] + x[\nu_5]_7\nu_8^2 + y[\eta_5\varepsilon_6]_7 \qquad \text{for integers } x \text{ and } y.$$

We know ([17]) that $\varepsilon_5\nu_{13} = 0$, $2\kappa_7 \equiv \bar{\nu}_7\nu_{15}^2 \bmod 4\sigma'\sigma_{14}$ and $\pi_{21}(S^7) = \mathbf{Z}_8\{\sigma'\sigma_{14}\} \oplus \mathbf{Z}_4\{\kappa_7\}$. So, by (ii) and (iv), we have

$$2\sigma'\sigma_{14} = 4\sigma'\sigma_{14} + x\bar{\nu}_7\nu_{15}^2 \pm 2y\sigma'\sigma_{14} + y\bar{\nu}_7\nu_{15}^2.$$

Therefore $y \equiv 1 \bmod 4$ and x is odd. By a choice of $[\eta_5\varepsilon_6]_7$ modulo 4, we can put $y = 1$ and $x = 1$ with $J[\eta_5\varepsilon_6]$ in (ii) unchanged. This completes the proof. □

LEMMA 1.2. (i) $\Delta(\Sigma\sigma') = 2[\iota_7]\sigma' + 4[\bar{\nu}_6 + \varepsilon_6]_8 + [\nu_5]_8\nu_8^2 - [\eta_5\varepsilon_6]_8$.

(ii) $\Delta\sigma_8 \equiv [\iota_7]\sigma' + c[\bar{\nu}_6 + \varepsilon_6]_8 \bmod \{[\nu_5]_8\nu_8^2, [\eta_5\varepsilon_6]_8\}$ *for an odd integer* c.

(iii) $[\eta_5\varepsilon_6]\eta_{14} = 4[\nu_5]\sigma_8 + [\nu_4\sigma'\eta_{14}]_6$ *and* $[\eta_5\varepsilon_6]_7\eta_{14} = [\nu_4\sigma'\eta_{14}]_7$.

(iv) $[\bar{\nu}_6 + \varepsilon_6]\eta_{14} \equiv [\eta_6](\bar{\nu}_7 + \varepsilon_7) \bmod \{[\nu_5]_7\sigma_8, [\nu_4\sigma'\eta_{14}]_7\}$.

(v) $\Delta\sigma_9 = [\nu_5]_9\sigma_8 + [\iota_7]_9\bar{\nu}_7 + [\iota_7]_9\varepsilon_7 + [\iota_7]_9\sigma'\eta_{14}$.

PROOF. By Lemma 1.1,

$$\Delta(\Sigma\sigma') = (2[\iota_7] - [\eta_6]_8) \circ \sigma' = 2[\iota_7]\sigma' - 4[\bar{\nu}_6 + \varepsilon_6]_8 - [\nu_5]_8\nu_8^2 - [\eta_5\varepsilon_6]_8.$$

This leads us to (i).

By Sugawara's theorem ([16]), we have $p_*\Delta\sigma_8 = \sigma'$, and so

$$\Delta\sigma_8 - [\iota_7]\sigma' \in i_*\pi_{14}(R_7) = \mathbf{Z}_8\{[\eta_5\varepsilon_6]_8\} \oplus \mathbf{Z}_8\{[\bar{\nu}_6 + \varepsilon_6]_8\} \oplus \mathbf{Z}_2\{[\nu_5]_8\nu_8^2\}$$

by Table 2 of [5]. We have

$$\Delta\sigma_8 = [\iota_7]\sigma' + x[\bar{\nu}_6 + \varepsilon_6]_8 + y[\nu_5]_8\nu_8^2 + z[\eta_5\varepsilon_6]_8 \qquad \text{for integers } x, y \text{ and } z.$$

By [1] and [12], we have

$$J\Delta(\sigma_8) = [\sigma_8, \iota_8] = [\iota_8, \iota_8] \circ \sigma_{15} = (2\sigma_8 - \Sigma\sigma') \circ \sigma_{15}$$

in $\pi_{22}(S^8) = \mathbf{Z}_{16}\{\sigma_8^2\} \oplus \mathbf{Z}_8\{\Sigma\sigma' \circ \sigma_{15}\} \oplus \mathbf{Z}_4\{\kappa_8\}$, where $2\kappa_8 \equiv \bar{\nu}_8\nu_{16}^2 \bmod 4\Sigma\sigma' \circ \sigma_{15}$ ([17]). Hence, by Lemma 1.1.(ii) and the fact $\varepsilon_8\nu_{16}^2 = 0$, we have

$$(2\sigma_8 - \Sigma\sigma') \circ \sigma_{15} = J\Delta(\sigma_8) = 2\sigma_8^2 + x\Sigma\sigma' \circ \sigma_{15} - 2z\Sigma\sigma' \circ \sigma_{15} + (y+z)\bar{\nu}_8\nu_{16}^2.$$

So x is odd. This leads us to (ii).

Since $\eta_5^2\varepsilon_7 = 4\nu_5\sigma_8$, we have

$$[\eta_5\varepsilon_6]\eta_{14} - 4[\nu_5]\sigma_8 \in i_*\pi_{15}(R_5) = \mathbf{Z}_2\{[\nu_4\sigma'\eta_{14}]_6\}$$

by Table 2 of [5]. So, by Proposition 2.1 of [5], we have

$$[\eta_5\varepsilon_6]\eta_{14} = 4[\nu_5]\sigma_8 + x[\nu_4\sigma'\eta_{14}]_6 \quad \textit{and}$$
$$[\eta_5\varepsilon_6]_7\eta_{14} = x[\nu_4\sigma'\eta_{14}]_7 \qquad \text{for } x = 0 \text{ or } 1.$$

By Proposition 2.1 of [5], the exact sequence $(15)_8$ is the form:

$$\pi_{16}(S^8) \cong (\mathbf{Z}_2)^4 \to (\mathbf{Z}_2)^7 \to \mathbf{Z} \oplus (\mathbf{Z}_2)^3.$$

This implies that $\Delta : \pi_{16}(S^8) \to \pi_{15}(R_8)$ is monic. By Lemma 1.2.(i), we have $\Delta(\Sigma\sigma'\eta_{15}) = \Delta(\Sigma\sigma')\eta_{14} = [\eta_5\varepsilon_6]_8\eta_{14}$ for one of the generators $\Sigma\sigma'\eta_{15}$ of $\pi_{16}(S^8)$. Then $[\eta_5\varepsilon_6]_8\eta_{14} \neq 0$, and hence we have $x = 1$. This leads us to (iii).

Since $p_*([\bar{\nu}_6 + \varepsilon_6]\eta_{14} - [\eta_6](\bar{\nu}_7 + \varepsilon_7)) = \bar{\nu}_6\eta_{14} + \varepsilon_6\eta_{14} - \eta_6\bar{\nu}_7 - \eta_6\varepsilon_7 = 0$, we have

$$[\bar{\nu}_6 + \varepsilon_6]\eta_{14} - [\eta_6](\bar{\nu}_7 + \varepsilon_7) \in i_*\pi_{15}(R_6) = \{[\nu_4\sigma'\eta_{14}]_7, [\nu_5]_7\sigma_8\}$$

by Proposition 2.1 of [5]. This leads us to (iv).

By (7.4) of [17] and Lemma 1.1.(i), we have

$$\Delta\sigma_9 = [\nu_5]_9\sigma_8 + [\iota_7]_9\eta_7\sigma_8 = [\nu_5]_9\sigma_8 + [\iota_7]_9\bar{\nu}_7 + [\iota_7]_9\varepsilon_7 + [\iota_7]_9\sigma'\eta_{14}.$$

This completes the proof. □

Finally we show

LEMMA 1.3. $[\eta_5\varepsilon_6]_9\eta_{14} = [\nu_4\sigma'\eta_{14}]_9 = 0$, $[\eta_6]_9\bar{\nu}_7 = 0$, $[\eta_6]_9\varepsilon_7 = 0$, $[\eta_6]_9\mu_7 = 0$, $[\nu_4^2]_9\nu_{10}^2 = 0$ *and* $[\iota_7]_9\sigma'\eta_{14} = [\bar{\nu}_6 + \varepsilon_6]_9\eta_{14}$.

PROOF. By the proof of Lemma 1.2.(iii), we have the first relations.

By Lemma 1.1.(i), we have $\Delta\bar{\nu}_8 = [\eta_6]_8\bar{\nu}_7$, $\Delta\varepsilon_8 = [\eta_6]_8\varepsilon_7$ and $\Delta\mu_8 = [\eta_6]_8\mu_7$. So we have the second, third and fourth relations.

By Lemma 1.1.(i) and (iv), $\Delta\nu_8^3 = [\eta_6]_8\nu_7^3 = [\nu_4^2]_8\nu_{10}^2$. So we have the fifth relation.

By Lemma 1.2.(ii),

$$\Delta(\sigma_8\eta_{15}) = (\Delta\sigma_8)\eta_{14} = [\iota_7]\sigma'\eta_{14} + [\bar{\nu}_6 + \varepsilon_6]_8\eta_{14} + d[\eta_5\varepsilon_6]_8\eta_{14} \qquad \text{for } d = 0 \text{ or } 1.$$

So we have the last relation by the first. This completes the proof. □

2. Determination of $\pi_{17}(R_n:2)$

Since $\pi_{17}(R_3) \cong \pi_{17}(S^3)$, $\pi_{17}(R_4) \cong \pi_{17}(R_3) \oplus \pi_{17}(S^3)$ and $\pi_{17}(S^3) = \mathbf{Z}_2\{\varepsilon_3\nu_{11}^2\}$, we have the first two of Theorem 1.(i).

For $n \geq 4$, we will determine the group $\pi_{17}(R_{n+1})$ by applying the exact sequence

$$(17)_n \qquad \pi_{18}(S^n) \xrightarrow{\Delta} \pi_{17}(R_n) \xrightarrow{i_*} \pi_{17}(R_{n+1}) \xrightarrow{p_*} \operatorname{Ker}\Delta \to 0,$$

where $\operatorname{Im} p_* = \operatorname{Ker}\Delta$ for $\Delta : \pi_{17}(S^n) \to \pi_{16}(R_n)$ and the results on this $\operatorname{Ker}\Delta$ will be referred to Proposition 4.1 of [5].

In the exact sequence $(17)_4$, $\operatorname{Ker}\Delta = \mathbf{Z}_8\{\nu_4^2\sigma_{10}\}$ for $\Delta : \pi_{17}(S^4) \to \pi_{16}(R_4)$ and $\pi_{18}(S^4) = \mathbf{Z}_8\{\nu_4\zeta_7\} \oplus \mathbf{Z}_2\{\nu_4\bar{\nu}_7\nu_{15}\} \oplus \mathbf{Z}_2\{\varepsilon_4\nu_{12}^2\}$. By Lemma 1.1.(i), we have

$$\Delta(\nu_4\zeta_7) = \Delta(\nu_4)\zeta_6 = ([\iota_3] + a[\eta_2]_4)\nu'\zeta_6 = 0,$$

because $\nu'\zeta_6 \in \nu'\{\nu_6, 8\iota_9, 2\sigma_9\} = -\{\nu', \nu_6, 8\iota_9\} \circ 2\sigma_{10} \subset 2\pi_{10}(S^3) \circ \sigma_{10} = 0$.

We also have

$$\Delta(\nu_4\bar{\nu}_7\nu_{15}) = ([\iota_3] + a[\eta_2]_4)\nu'\bar{\nu}_6\nu_{14} = ([\iota_3] + a[\eta_2]_4)\varepsilon_3\nu_{11}^2$$

and

$$\Delta(\varepsilon_4\nu_{12}^2) = (2[\iota_3] - [\eta_2]_4)\varepsilon_3\nu_{11}^2 = [\eta_2]_4\varepsilon_3\nu_{11}^2,$$

because $\nu'\bar{\nu}_6 = \varepsilon_3\nu_{11}$ by (7.12) of [17]. So we have $\pi_{17}(R_5) = \mathbf{Z}_8\{[\nu_4^2]\sigma_{10}\}$.

In the exact sequence $(17)_5$, $\pi_{18}(S^5) = \mathbf{Z}_2\{\nu_5\sigma_8\nu_{15}\} \oplus \mathbf{Z}_2\{\nu_5\eta_8\mu_9\}$ and so we have $\Delta(\pi_{18}(S^5)) = 0$ since $\Delta\nu_5 = 0$ by Lemma 1.1.(i). Also we have that $\operatorname{Ker}\Delta$ has a $\mathbf{Z}_2$-basis $\{\nu_5\alpha = p_*[\nu_5]\alpha\}$ for $\alpha = \nu_8^3, \mu_8$ and $\eta_8\varepsilon_9$. Then the sequence $(17)_5$ splits, and $\pi_{17}(R_6)$ is determined.

In the exact sequence $(17)_6$, $\pi_{18}(S^6) = \mathbf{Z}_{16}\{[\iota_6, \iota_6]\sigma_{11}\}$ and $\operatorname{Ker}\Delta = \mathbf{Z}_2\{\eta_6^2\mu_8\} \oplus \mathbf{Z}_4\{(\bar{\nu}_6 + \varepsilon_6)\nu_{14}\}$ for $\Delta : \pi_{17}(S^6) \to \pi_{16}(R_6)$. $\Delta(\pi_{18}(S^6))$ is generated by $\Delta([\iota_6, \iota_6]\sigma_{11}) = \Delta([\iota_6, \iota_6]) \circ \sigma_{10}$. By Lemma 1.1.(iii),

$$\Delta([\iota_6, \iota_6]) \circ \sigma_{10} = [\nu_5]\eta_8^2\sigma_{10} + 4[\nu_4^2]_6\sigma_{10} = [\nu_5]\nu_8^3 + [\nu_5]\eta_8\varepsilon_9 + 4[\nu_4^2]_6\sigma_{10},$$

because $\eta_8^2\sigma_{10} = \nu_8^3 + \eta_8\varepsilon_9$ by (7.3) and (7.4) of [17]. Therefore we have $\operatorname{Ker}\Delta = \mathbf{Z}_8\{2[\iota_6, \iota_6]\sigma_{11}\}$,

$$i_*\pi_{17}(R_6) = \mathbf{Z}_8\{[\nu_4^2]_7\sigma_{10}]\} \oplus \mathbf{Z}_2\{[\nu_5]_7\mu_8\} \oplus \mathbf{Z}_2\{[\nu_5]_7\nu_8^3\}$$

and the following exact sequence $(17)_6$:

$$0 \to i_*\pi_{17}(R_6) \to \pi_{17}(R_7) \to \mathbf{Z}_4\{p_*[\bar{\nu}_6 + \varepsilon_6]\nu_{14}\} \oplus \mathbf{Z}_2\{p_*[\eta_6]\eta_7\mu_8\} \to 0.$$

In order to determine this extension, we recall the following result on the fibering $G_2/SU(3) = S^6$ from p. 166 of [8].

LEMMA 2.1 (Mimura [8]). $4\langle\bar{\nu}_6\nu_{14}\rangle = i_*[\nu_5^2]'\nu_{11}^2$ *in* $\pi_{17}(G_2:2) = \mathbf{Z}_8\{\langle\bar{\nu}_6\nu_{14}\rangle\} \oplus \mathbf{Z}_2\{\langle\eta_6^2\rangle\mu_8\}$. *Here* $\langle\alpha\rangle$ *for* $\alpha \in \pi_k(S^6)$ *is an element satisfying* $p'_*\langle\alpha\rangle = \alpha$ *for the projection* $p': G_2 \to S^6$ *and* $[\beta]'$ *for* $\beta \in \pi_k(S^5)$ *is an element satisfying* $p''_*[\beta]' = \beta$ *for the projection* $p'': SU(3) \to S^5$.

Now we show

LEMMA 2.2. (i) $\nu_5\sigma_8\eta_{15} = \nu_5\varepsilon_8$.

(ii) $[\nu_5]\sigma_8\eta_{15} \equiv [\nu_5]\varepsilon_8 \bmod \{[\nu_4\sigma'\eta_{14}]_6\eta_{15}, [\nu_4^2]_6\nu_{10}^2\}$ *and*
$[\nu_5]\sigma_8\eta_{15}^2 \equiv [\nu_5]\eta_8\varepsilon_9 \bmod [\nu_4\sigma'\eta_{14}]_6\eta_{15}^2$.

(iii) $4[\bar{\nu}_6+\varepsilon_6]\nu_{14} \equiv [\nu_5]_7\nu_8^3 \bmod 4[\nu_4^2]_7\sigma_{10}$ *and* $[\bar{\nu}_6+\varepsilon_6]\nu_{14}$ *is of order* 8.

PROOF. We know the relation of (i) (p. 152 of [17]). Here we show a proof given by Oda. We know $\nu_5\sigma_8 \in \{\nu_5, 2\nu_8, \nu_{11}\}_1 \bmod 2(\nu_5\sigma_8)$ and $\varepsilon_5 \in \{2\nu_5, \nu_8, \eta_{11}\} \bmod 0$. So we have $\nu_5\sigma_8\eta_{15} \in \{\nu_5, 2\nu_8, \nu_{11}\}_1 \circ \eta_{15} = \nu_5\Sigma\{2\nu_7, \nu_{10}, \eta_{13}\} \ni \nu_5\varepsilon_8 \bmod \nu_5\Sigma\sigma'\eta_{15} = 2(\nu_5\sigma_8)\eta_{15} = 0$. This leads us to (i).

We have $[\nu_5]\sigma_8\eta_{15} - [\nu_5]\varepsilon_8 \in i_*\pi_{16}(R_5) = \{[\nu_4\sigma'\eta_{14}]_6\eta_{15}, [\nu_4^2]_6\nu_{10}^2\}$ by Proposition 4.1 of [5]. So we have (ii).

The double covering induces isomorphisms of $\pi_k(Spin(n+1))$ onto $\pi_k(R_{n+1})$ compatible with the projection homomorphisms to $\pi_k(S^n)$. So we use the same notation $[\alpha] \in \pi_k(Spin(n+1))$ as in $\pi_k(R_{n+1})$. Since $\varepsilon_6\nu_{14} = 0$ by (7.13) of [17], we have $p_*([\bar{\nu}_6+\varepsilon_6]\nu_{14}) = \bar{\nu}_6\nu_{14} = p'_*\langle\bar{\nu}_6\nu_{14}\rangle$. Let $i': G_2 \hookrightarrow Spin(7)$ and $i'': SU(3) \hookrightarrow Spin(7)$ be the inclusions. Then, by the naturality, we have $i'_*\langle\bar{\nu}_6\nu_{14}\rangle \equiv [\bar{\nu}_6+\varepsilon_6]\nu_{14} \bmod i_*\pi_{17}(R_6)$ and $i''_*([\nu_5^2]'\nu_{11}^2) = [\nu_5^2]\nu_{11}^2 \equiv [\nu_5]\nu_8^3 \bmod i'_*(i_*\pi_{11}(R_5)) \circ \nu_{11}^2 = 0$, because $\pi_{11}(R_5) = \mathbf{Z}_2\{[\iota_3]_5\varepsilon_3\}$ and $[\iota_3]_5\varepsilon_3\nu_{11} \in \pi_{14}(R_5) = \mathbf{Z}_{16}\{[2\nu_4\sigma']\}$ by Table 2 of [5]. Then Lemma 2.1 implies that $4[\bar{\nu}_6+\varepsilon_6]\nu_{14} \equiv [\nu_5]_7\nu_8^3 \bmod 4i_*\pi_{17}(R_6) = \{4[\nu_4^2]_7\sigma_{10}\}$. From the exact sequence $(17)_6$, we have the second half of (iii) and completes the proof. □

By the exact sequence $(17)_6$ and Lemma 2.2, we have the group $\pi_{17}(R_7)$. Since $\pi_k(R_8) \cong \pi_k(R_7) \oplus \pi_k(S^7)$, we have the group $\pi_{17}(R_8)$.

We consider the exact sequence $(17)_8$. First consider the image of Δ for generators of $\pi_{17}(S^8)$. By Lemma 1.2, we have $\Delta(\Sigma\sigma'\eta_{15}^2) = [\eta_5\varepsilon_6]_8\eta_{14}^2 = [\nu_4\sigma'\eta_{14}]_8\eta_{15}$. By Lemma 1.1.(i) and (iv), we have

$$\Delta(\nu_8^3) = \Delta\iota_8 \circ \nu_7^3 = [\eta_6]_8\nu_7^3 = [\nu_4^2]_8\nu_{10}^2, \quad \Delta(\eta_8\varepsilon_9) = [\eta_6]_8\eta_7\varepsilon_8 \ \text{ and } \ \Delta\mu_8 = [\eta_6]_8\mu_7.$$

By Lemmas 1.2 and 1.1.(iv), we have

$$\Delta(\sigma_8\eta_{15}^2) = \Delta(\sigma_8)\eta_{14}^2 \equiv [\iota_7]\sigma'\eta_{14}^2 + [\eta_6]_8(\bar{\nu}_7+\varepsilon_7)\eta_{15}$$
$$\equiv [\iota_7]\sigma'\eta_{14}^2 + [\eta_6]_8\eta_7\varepsilon_8 + [\nu_4^2]_8\nu_{10}^2 \bmod \{[\nu_5]_8\sigma_8\eta_{15}, [\nu_4\sigma'\eta_{14}]_8\eta_{14}\}.$$

We see in Proposition 4.1 of [5], that these Δ-images are independent in $\pi_{16}(R_8)$. Thus $\mathrm{Ker}\,\Delta = 0$, and $i_*: \pi_{17}(R_8) \to \pi_{17}(R_9)$ is an epimorphism.

Next we consider $\mathrm{Ker}\, i_* = \Delta(\pi_{18}(S^8))$, where $\pi_{18}(S^8) = \mathbf{Z}_8\{\sigma_8\nu_{15}\} \oplus \mathbf{Z}_8\{\nu_8\sigma_{11}\} \oplus \mathbf{Z}_2\{\eta_8\mu_9\}$. By Lemma 1.2.(ii), we have

$$\Delta(\sigma_8\nu_{15}) = [\iota_7]\sigma'\nu_{14} + c[\bar{\nu}_6 + \varepsilon_6]_8\nu_{14} + y[\nu_5]_8\nu_8^3 + z[\eta_5\varepsilon_6]_8\nu_{14}$$

$$= x[\iota_7]\nu_7\sigma_{10} + c[\bar{\nu}_6 + \varepsilon_6]_8\nu_{14} + y[\nu_5]_8\nu_8^3 + z'[\nu_4^2]_8\sigma_{10},$$

because $\sigma'\nu_{14} = x\nu_7\sigma_{10}$ for an odd integer x by (7.19) of [17] and $[\eta_5\varepsilon_6]\nu_{14} \in i_*\pi_{17}(R_5) = \mathbf{Z}_8\{[\nu_4^2]_6\sigma_{10}\}$.

By Lemma 1.1.(i) and (iv), we have

$$\Delta(\nu_8\sigma_{11}) = (2[\iota_7] - [\eta_6]_8)\nu_7\sigma_{10} = 2[\iota_7]\nu_7\sigma_{10} - [\nu_4^2]_8\sigma_{10}$$

and $\Delta(\eta_8\mu_9) = [\eta_6]_8\eta_7\mu_8$. This determines the group $\pi_{17}(R_9)$. Next we show

LEMMA 2.3. (i) $[\nu_4^2]_9\sigma_{10} = 2[\iota_7]_9\nu_7\sigma_{10}$ *and* $4[\bar{\nu}_6 + \varepsilon_6]_9\nu_{14} = 4[\iota_7]_9\nu_7\sigma_{10}$.

(ii) $[\bar{\nu}_6 + \varepsilon_6]\eta_{14} \equiv [\eta_6](\bar{\nu}_7 + \varepsilon_7) + [\nu_5]_7\sigma_8 \bmod [\nu_4\sigma'\eta_{14}]_7$, $[\bar{\nu}_6 + \varepsilon_6]_9\eta_{14} = [\nu_5]_9\sigma_8$ *and* $[\nu_5]_9\sigma_8\eta_{15}^2 = 4[\iota_7]_9\nu_7\sigma_{10}$.

(iii) $[\iota_7]_9\sigma'\eta_{14}^2 = [\bar{\nu}_6 + \varepsilon_6]_9\eta_{14}^2 = [\nu_5]_9\sigma_8\eta_{15}$.

(iv) $[\nu_5]_9\sigma_8\eta_{15} = [\nu_5]_9\varepsilon_8$ *and* $[\nu_5]_9\eta_8\varepsilon_9 = [\nu_5]_9\nu_8^3 = 4[\iota_7]_9\nu_7\sigma_{10}$.

(v) $[\bar{\nu}_6 + \varepsilon_6]_9\nu_{14} = d[\iota_7]_9\nu_7\sigma_{10}$ *for an odd integer d.*

PROOF. Since $\Delta(\nu_8\sigma_{11}) = 2[\iota_7]\nu_7\sigma_{10} - [\nu_4^2]_8\sigma_{10}$, we have the first half of (i). So we have $4[\nu_4^2]_9\sigma_{10} = 0$. Therefore, by the above calculation of $\Delta(\sigma_8\nu_{15})$, we have the second half of (i).

By Lemma 1.2.(iv), we can set $[\bar{\nu}_6 + \varepsilon_6]\eta_{14} = [\eta_6](\bar{\nu}_7 + \varepsilon_7) + x[\nu_5]_7\sigma_8 + y[\nu_4\sigma'\eta_{14}]_7$ for $x, y = 0, 1$. By Lemma 1.3, we have $[\bar{\nu}_6 + \varepsilon_6]_9\eta_{14} = [\eta_6]_9(\bar{\nu}_7 + \varepsilon_7) + x[\nu_5]_9\sigma_8$. So we have

$$4[\bar{\nu}_6 + \varepsilon_6]_9\nu_{14} = [\bar{\nu}_6 + \varepsilon_6]_9\eta_{14}^3 = [\eta_6]_9(\bar{\nu}_7 + \varepsilon_7)\eta_{15}^2 + x[\nu_5]_9\sigma_8\eta_{15}^2.$$

Since $[\eta_6]_9(\bar{\nu}_7 + \varepsilon_7)\eta_{15}^2 = [\eta_6]_9(\nu_7^3 + 4\nu_6\sigma_{10}) = [\nu_4^2]_9\nu_{10}^2 + 4[\nu_4^2]_9\sigma_{10} = 0$ by (i) and Lemma 1.3, we have $4[\bar{\nu}_6 + \varepsilon_6]_9\nu_{14} = x[\nu_5]_9\sigma_8\eta_{15}^2$. By (i), we have $x[\nu_5]_9\sigma_8\eta_{15}^2 = 4[\iota_7]_9\nu_7\sigma_{10}$. By the group structure of $\pi_{17}(R_9)$, we have $4[\iota_7]_9\nu_7\sigma_{10} \neq 0$. Hence we have $x = 1$ and this implies the first and third assertions of (ii). The second relation follows from the first and Lemma 1.3. This leads us to (ii).

By the last relation of Lemma 1.3, we have the first equality of (iii). The second relation of (ii) implies the second equality of (iii).

The first relation of (iv) follows from Lemmas 1.3 and 2.2.(ii). Then, by (ii), we have

$$[\nu_5]_9\eta_8\varepsilon_9 = [\nu_5]_9\sigma_8\eta_{15}^2 = 4[\iota_7]_9\nu_7\sigma_{10}.$$

By (i) and Lemma 2.2.(iii), $[\nu_5]_9\nu_8^3 = 4[\iota_7]_9\nu_7\sigma_{10}$. This leads us to (iv).

We recall $\Delta(\sigma_8\nu_{15}) = x[\iota_7]\nu_7\sigma_{10} + c[\bar{\nu}_6 + \varepsilon_6]_8\nu_{14} + y[\nu_5]_8\nu_8^3 + z'[\nu_4^2]_8\sigma_{10}$, where x, c are odd integers and y, z' are integers. So, by (i) and (iv), we have the assertion of (v). This completes the proof. □

We consider the exact sequence $(17)_9$. By Lemmas 1.1.(v) and 2.3.(iii), we have

$$\Delta(\sigma_9\eta_{16}) = \Delta\sigma_9 \circ \eta_{15} = [\nu_5]_9\sigma_8\eta_{15} + [\iota_7]_9\sigma'\eta_{14}^2 + [\iota_7]_9\nu_7^3 + [\iota_7]_9\eta_7\varepsilon_8 = [\iota_7]_9\nu_7^3 + [\iota_7]_9\eta_7\varepsilon_8.$$

We also have

$$\Delta\bar{\nu}_9 = [\nu_5]_9\bar{\nu}_8 + [\iota_7]_9\nu_7^3 \quad \text{and} \quad \Delta\varepsilon_9 = [\nu_5]_9\varepsilon_8 + [\iota_7]_9\eta_7\varepsilon_8.$$

So, by Proposition 4.1 of [5], Δ is a monomorphism and i_* is an epimorphism.

By Lemma 2.3.(iv), we have

$$\Delta(\sigma_9\eta_{16}^2) = \Delta(\sigma_9\eta_{16})\eta_{16} = [\iota_7]_9\eta_7^2\varepsilon_9 = 4[\iota_7]_9\nu_7\sigma_{10} = [\nu_5]_9\nu_8^3,$$

$$\Delta(\nu_9^3) = \Delta(\bar{\nu}_9)\eta_{16} = [\nu_5]_9\nu_8^3$$

and

$$\Delta(\eta_9\varepsilon_{10}) = \Delta(\varepsilon_9)\eta_{16} = [\nu_5]_9\eta_8\varepsilon_9 + 4[\iota_7]_9\nu_7\sigma_{10} = 0.$$

By Lemma 1.1.(i), $\Delta\mu_9 = [\nu_5]_9\mu_8 + [\iota_7]_9\eta_7\mu_8$. Since $[\iota_9, \iota_9]\eta_{17} = \sigma_9\eta_{16}^2 + \nu_9^3 + \eta_9\varepsilon_{10}$, we have

$$\operatorname{Ker}\Delta = \mathbf{Z}_2\{\eta_9\varepsilon_{10}\} \oplus \mathbf{Z}_2\{[\iota_9, \iota_9]\eta_{17}\}$$

for $\Delta : \pi_{18}(S^9) \to \pi_{17}(R_9)$ and we obtain the group $\pi_{17}(R_{10})$.

Next we consider the exact sequence $(17)_{10}$. By (4.7) of [5], we know $\Delta\sigma_{10} = [2\sigma_9]$. So i_* is an epimorphism. In the exact sequence $(10)_{10}$, $\pi_{10}(R_{10}) = \mathbf{Z}_4\{[\iota_7]_{10}\nu_7\}$ and $\pi_{10}(R_{11}) = \mathbf{Z}_2\{[\iota_7]_{11}\nu_7\}$ by Table 2 of [5]. So we have $\Delta\eta_{10} = 2[\iota_7]_{10}\nu_7$. By this result, we have $\Delta(\eta_{10}\sigma_{11}) = 2[\iota_7]_{10}\nu_7\sigma_{10}$. Since $J([\iota_7]_n\eta_7\mu_8) = \sigma_n\eta_{n+7}\mu_{n+8} \neq 0$ for $n \geq 8$, we have $\Delta\bar{\nu}_{10}$ or $\Delta\varepsilon_{10} = 0$. This determines the group $\pi_{17}(R_{11})$.

In the exact sequence $(17)_{11}$, $\Delta(\sigma_{11}) = [\iota_7]_{11}\nu_7\sigma_{10}$ by Lemma 1.1.(i). By Proposition 4.1 of [5], $\Delta : \pi_{17}(S^{11}) \to \pi_{16}(R_{11})$ is a monomorphism. This determines the group $\pi_{17}(R_{12}) = \mathbf{Z}_2\{[\iota_7]_{12}\eta_7\mu_8\}$. Since $[\iota_7]_{12}\eta_7\mu_8$ survives stably, we have $\Delta(\pi_{18}(S^{12})) = 0$ and we obtain the groups $\pi_{17}(R_n)$ for $n = 13$ and 14.

In the exact sequence $(17)_{14}$:

$$0 \to \pi_{17}(R_{14}) \xrightarrow{i_*} \pi_{17}(R_{15}) \xrightarrow{p_*} \pi_{17}(S^{14}) \xrightarrow{\Delta} \pi_{16}(R_{14}),$$

we have $\Delta\nu_{14} = [2\nu_{13}]$ by (4.11) of [5]. So, by Proposition 4.1 of [5], we have $\mathrm{Ker}\,\Delta = \mathbf{Z}_2\{\eta_{14}^3\}$. Therefore we settle the group $\pi_{17}(R_{15})$.

By Table 2 of [5], we have $\Delta\iota_{15} = [\bar{\nu}_6 + \varepsilon_6]_{15}$. We know the element $[\eta_{15}] \in \pi_{16}(R_{16})$ of order 2 by Proposition 4.1 of [5]. In the exact sequence $(17)_{15}$, there exists an element $[\eta_{15}]\eta_{16} \in \pi_{17}(R_{16})$. Since $[\iota_7]_{12}\nu_7\sigma_{10} = 0$, we have

$$\Delta\nu_{15} = [\bar{\nu}_6 + \varepsilon_6]_{15}\nu_{14} = [\iota_7]_{15}\nu_7\sigma_{10} = 0$$

by Lemma 2.3.(v). This determines the group $\pi_{17}(R_{16})$.

By the results of $\pi_{16}(R_n)$, $\pi_{17}(R_n) = \mathbf{Z}_2\{[\iota_7]_n\eta_7\mu_8\}$ for $n \geq 19$ and by the exact sequences $(17)_n$ for $n = 16, 17$ and 18, we have the rest of Theorem 1. This completes the proof of Theorem 1.

In the above arguments determining $\pi_{17}(R_n)$ for $n = 11$ and 12, we have the following result.

LEMMA 2.4. $\Delta\eta_{10} = 2[\iota_7]_{10}\nu_7$ *and* $[\bar{\nu}_6 + \varepsilon_6]_{12}\nu_{14} = [\iota_7]_{12}\nu_7\sigma_{10} = 0$.

3. Determination of $\pi_{18}(R_n : 2)$

First we recall the following result obtained from (10.7), Lemma 12.12 of [17] and Propositions 2.13.(7), 2.17.(7) and (11) of [15].

LEMMA 3.1. (i) $\nu_6\rho' \equiv 0 \bmod 2\zeta_6\sigma_{17}$.

(ii) $\sigma''\zeta_{13} = \pm 2\zeta_6\sigma_{17}$ *and* $\sigma'\zeta_{14} = x\zeta_7\sigma_{18}$ *for an odd integer* x.

(iii) $\mu_3\varepsilon_{12} \equiv \eta_3\mu_4\sigma_{13} \bmod 2\bar{\varepsilon}'$.

Since $\pi_{18}(S^3) = \mathbf{Z}_2\{\bar{\varepsilon}_3\}$, we have the groups $\pi_{18}(R_n)$ for $n = 3$ and 4.

In the exact sequence $(18)_4$, we recall $\mathrm{Ker}\,\Delta = \mathbf{Z}_8\{\nu_4\zeta_7\}$ for $\Delta : \pi_{18}(S^4) \to \pi_{17}(R_4)$. We know $\pi_{19}(S^4) = \mathbf{Z}_2\{\bar{\varepsilon}_4\}$ and we have $\Delta\bar{\varepsilon}_4 = \Delta(\iota_4)\bar{\varepsilon}_3 = [\eta_2]_4\bar{\varepsilon}_3$ by Lemma 1.1.(i). Since $\nu_4\zeta_7 \in \nu_4\{\nu_7, 8\iota_{10}, 2\sigma_{10}\} \in \{\nu_4^2, 8\iota_{10}, 2\sigma_{10}\}$, we can take $[\nu_4\zeta_7]$ as an element of $\{[\nu_4^2], 8\iota_{10}, 2\sigma_{10}\}$. We have $8[\nu_4\zeta_7] \in 8\{[\nu_4^2], 8\iota_{10}, 2\sigma_{10}\} = -[\nu_4^2]\{8\iota_{10}, 2\sigma_{10}, 8\iota_{17}\} \ni 0 \bmod 0$. This shows that $\pi_{18}(R_5) = \mathbf{Z}_8\{[\nu_4\zeta_7]\} \oplus \mathbf{Z}_2\{[\iota_3]_5\varepsilon_3\}$. We show

LEMMA 3.2. $J[\nu_4\zeta_7] \equiv \zeta_5\sigma_{16} \bmod \nu_5\bar{\varepsilon}_8$ *and* $J[\nu_4\zeta_7]_6 = \zeta_6\sigma_{17}$.

PROOF. We know that $H(\zeta_5) = 8\sigma_9$ by Lemma 6.7 of [17] and that $HJ[\nu_4\zeta_7] = \nu_9\zeta_{20} = 8\sigma_9^2$ by (10.7) of [17]. So we have $J[\nu_4\zeta_7] - \zeta_5\sigma_{16} \in \Sigma\pi_{22}(S^4) = \{\nu_5\bar{\varepsilon}_8, \eta_5\bar{\mu}_6\}$. Since $\zeta\sigma = 0$ in the stable 18-stem and $\eta\bar{\mu}$ is not in the stable J-image, we have the first half. We know $\nu_6\bar{\varepsilon}_9 = 0$ (p. 148 of [17]). This leads us to the second half and completes the proof. □

We show

LEMMA 3.3. (i) $2[\nu_5]\nu_8 = [\iota_3]_6\varepsilon_3$ *and* $[\iota_3]_7\varepsilon_3 = 0$.

(ii) $J[2[\iota_6, \iota_6]] = \zeta_7$.

PROOF. By the exact sequence $(11)_n$ for $n = 5, 6$ and Table 2 of [5], we have (i).

Since $\pi_{11}(R_7) = \mathbf{Z}\{[2[\iota_6, \iota_6]]\} \oplus \mathbf{Z}_2\{[\nu_5]_7\nu_8\}$, $\pi_{18}(S^7) = \mathbf{Z}_8\{\zeta_7\} \oplus \mathbf{Z}_2\{\bar{\nu}_7\nu_{15}\}$, $J([\nu_5]\nu_8) = (\bar{\nu}_6 + \varepsilon_6)\nu_{14} = \bar{\nu}_6\nu_{14}$ and $\bar{\nu}_{10}\nu_{18} = 0$, we have (ii) by the choice of a representative $[2[\iota_6, \iota_6]]$. This completes the proof. □

Next we show

LEMMA 3.4. (i) $[\iota_3]_7\bar{\varepsilon}_3 = 0$.

(ii) $2[\nu_6]\sigma_8\nu_{15} = [\iota_3]_6\bar{\varepsilon}_3$.

(iii) $[\nu_4\zeta_7]_7 \equiv [\eta_6]\zeta_7 \bmod 8[2[\iota_6, \iota_6]]\sigma_{11}$.

PROOF. We know that $\bar{\varepsilon}_3 \in \{\varepsilon_3, 2\iota_{11}, \nu_{11}^2\} \in \pi_{17}(S^3)$ (p. 97 of [17]) and $\pi_{12}(R_7) = 0$ (Table 2 of [5]). So we have $[\iota_3]_7\bar{\varepsilon}_3 \in [\iota_3]_7\{\varepsilon_3, 2\iota_{11}, \nu_{11}^2\} = -\{[\iota_3]_7, \varepsilon_3, 2\iota_{11}\} \circ \nu_{12}^2 \subset \pi_{12}(R_7) \circ \nu_{12}^2 = 0$. This leads us to (i).

We know $\pi_{18}(S^5) = \mathbf{Z}_2\{\nu_5\sigma_8\nu_{15}\} \oplus \mathbf{Z}_2\{\nu_5\eta_8\mu_9\}$ and $\pi_{19}(S^5) = \mathbf{Z}_2\{\nu_5\zeta_8\} \oplus \mathbf{Z}_2\{\nu_5\bar{\nu}_8\nu_{16}\}$. By the fact $\Delta\nu_5 = 0$ of Lemma 1.1.(i), $\Delta(\pi_{18}(S^5)) = 0$ and $\Delta(\pi_{19}(S^5)) = 0$. So we have $[\iota_3]_6\bar{\varepsilon}_3 \neq 0$ and that $\pi_{18}(R_6)$ is generated by $[\nu_4\zeta_7]_6, [\nu_5]\sigma_8\nu_{15}, [\nu_5]\eta_8\mu_9$ and $[\iota_3]_6\bar{\varepsilon}_3$. Since $\pi_{19}(S^6) = \mathbf{Z}_2\{\nu_6\sigma_9\nu_{16}\}$ and $\Delta(\nu_6\sigma_9\nu_{16}) = 2[\nu_5]\sigma_8\nu_{15}$ by Lemma 1.1.(i), the kernel of $i_* : \pi_{18}(R_6) \to \pi_{18}(R_7)$ is 0 or $\mathbf{Z}_2$ generated by $2[\nu_5]\sigma_8\nu_{15}$. Since $[\iota_3]_6\bar{\varepsilon}_3 \neq 0$ and $[\iota_3]_7\bar{\varepsilon}_3 = 0$, $[\iota_3]_6\bar{\varepsilon}_3$ is a non-zero kernel of i_*. This leads us to (ii).

Since $[\eta_6]\nu_7 = b[\nu_4^2]_7$ by Lemma 1.1.(iv), we have

$$\begin{aligned} [\eta_6]\zeta_7 &\in [\eta_6]\{\nu_7, 8\iota_{10}, 2\sigma_{10}\} \\ &\subset \{b[\nu_4^2]_7, 8\iota_{10}, 2\sigma_{10}\} \\ &\ni b[\nu_4\zeta_7]_7 \bmod [\nu_4^2]_7\pi_{18}(S^{10}) + \pi_{11}(R_7) \circ 2\sigma_{11}. \end{aligned}$$

So we have $[\eta_6]\zeta_7 \equiv b[\nu_4\zeta_7]_7 \bmod 2[2[\iota_6, \iota_6]]\sigma_{11}$. For $[\nu_4^2]_7\varepsilon_{10} = [\eta_6]\nu_7\varepsilon_{10} = 0$, $[\nu_4^2]_7\bar{\nu}_{10} = [\eta_6]\nu_7\bar{\nu}_{10} = 0$ and $\pi_{11}(R_7) = \mathbf{Z}\{[2[\iota_6, \iota_6]]\}$ by Table 2 of [5]. Therefore we have $[\nu_4\zeta_7]_7 = [\eta_6]\zeta_7 + 2x[2[\iota_6, \iota_6]]\sigma_{11}$ for an integer x. We have $p_*[\nu_4\zeta_7]_7 = 0$ and $p_*[2[\iota_6, \iota_6]]\sigma_{11} = 2[\iota_6, \iota_6]\sigma_{11}$ is of order 8. Since

$$\eta_5\zeta_6 \in \eta_5\{\nu_6, 8\iota_9, \nu_9\} = -\{\eta_5, \nu_6, 8\iota_9\} \circ \nu_{10} \subset \pi_{10}(S^5) \circ \nu_{10} = 0,$$

we have $p_*[\eta_6]\zeta_7 = \eta_6\zeta_7 = 0$. This leads us to (iii) and completes the proof. □

By Lemma 3.4.(ii) and its proof, we obtain the group $\pi_{18}(R_6)$.

In the exact sequence $(18)_6$, we know $\pi_{18}(S^6)=\mathbf{Z}_{16}\{[\iota_6,\iota_6]\sigma_{11}\}$ and $\Delta([\iota_6,\iota_6]\sigma_{11})=([\nu_5]\eta_8^2+4[\nu_4^2]_6)\sigma_{10}=[\nu_5]\nu_8^3+[\nu_5]\eta_8\varepsilon_9+4[\nu_4^2]_6\sigma_{10}$.

By use of the fibering $G_2/SU(3)=S^6$, Mimura obtained the relation

$$8\langle 2[\iota_6,\iota_6]\rangle\sigma_{11}=i_*[\nu_5\eta_8\mu_9] \qquad \text{in } \pi_{18}(G_2:2)$$

(p. 166 of [8]). By a parallel argument to the proof of Lemma 2.2, we have

LEMMA 3.5. $8[2[\iota_6,\iota_6]]\sigma_{10}\equiv[\nu_5]_7\eta_8\mu_9 \bmod 4[\nu_4\zeta_7]_7$ *and* $[2[\iota_6,\iota_6]]\sigma_{11}$ *is of order* 16.

This lemma determines the group $\pi_{18}(R_7)$. By Theorem 7.4 of [17], we know that $\pi_{n+11}(S^n)=\mathbf{Z}_8\{\zeta_n\}\oplus\mathbf{Z}_2\{\bar{\nu}_n\nu_{n+8}\}$ for $n=7,8$ and 9. So we have the group $\pi_{18}(R_8)$.

In the exact sequence $(18)_8$, $\Delta:\pi_{18}(S^8)\to\pi_{17}(R_8)$ is a monomorphism by the argument determining $\pi_{17}(R_9)$. We show

LEMMA 3.6. (i) $\Delta\zeta_8\equiv 2[\iota_7]\zeta_7-[\nu_4\zeta_7]_8 \bmod 8[2[\iota_6,\iota_6]]_8\sigma_{11}$.

(ii) $[\eta_6]\bar{\nu}_7\nu_{15}=[\nu_5]_7\sigma_8\nu_{15}$ *and* $\Delta(\bar{\nu}_8\nu_{16})=[\eta_6]_8\bar{\nu}_7\nu_{15}=[\nu_5]_8\sigma_8\nu_{15}$.

PROOF. By Lemmas 1.1.(i) and 3.4.(iii), we have

$$\Delta\zeta_8=2[\iota_7]\zeta_7-[\eta_6]_8\zeta_7\equiv 2[\iota_7]\zeta_7-[\nu_4\zeta_7]_8 \bmod 8[2[\iota_6,\iota_6]]_8\sigma_{11}.$$

This leads us to (i).

By Lemmas 2.3.(ii) and 1.2.(iii), we have

$$0=[\bar{\nu}_6+\varepsilon_6]\eta_{14}\nu_{15}\equiv[\eta_6]\bar{\nu}_7\nu_{15}+[\nu_5]_7\sigma_8\nu_{15} \bmod [\nu_4\sigma'\eta_{14}]_7\nu_{15}=0.$$

This leads us to the first half of (ii). The first equality of the second relation of (ii) is directly obtained from Lemma 1.1.(i) and the first half implies the second equality. This leads us to (ii) and completes the proof. □

By Lemma 3.6, we have the group $\pi_{18}(R_9)$. Next we show

LEMMA 3.7. (i) $[\eta_6]_9\zeta_7$ *is of order* 4.

(ii) $[\nu_5]_8\eta_8\mu_9$ *is of order* 2 *and* $8[2[\iota_6,\iota_6]]\sigma_{10}=[\nu_5]_7\eta_8\mu_9+4[\nu_4\zeta_7]_7$.

PROOF. Since $[\eta_6]_9\zeta_7=2[\iota_7]_9\zeta_7$, we have $4[\eta_6]_9\zeta_7=0$. Since $J([\iota_7]_9\zeta_7)=\sigma_9\zeta_{16}$ is of order 8, we have (i).

Since $2\Sigma^2\bar{\varepsilon}'=2\nu_5\kappa_8$, Lemma 3.1.(iii) implies

$$\mu_5\varepsilon_{14}\equiv\eta_5\mu_6\sigma_{15} \bmod 2\nu_5\kappa_8.$$

By Lemma 1.1.(ii),

$$J([\nu_5]_8\eta_8\mu_9)=(\bar{\nu}_8+\varepsilon_8)\eta_{16}\mu_{17}=\nu_8^3\mu_{17}+\eta_8\varepsilon_9\mu_{17}=\eta_8^2\mu_{10}\sigma_{19}=4\zeta_8\sigma_{19}\neq 0.$$

This leads us to the first half of (ii).

By Lemma 3.1.(ii), we have $4\zeta_9\sigma_{20} = 8\sigma_9\zeta_{16} = 0$. By Lemma 3.2, $J[\nu_4\zeta_7]_7 = \pm\zeta_7\sigma_{18}$ and it is of order 8. By Lemma 3.5, $8[2[\iota_6,\iota_6]]\sigma_{10} = [\nu_5]_7\eta_8\mu_9 + 4x[\nu_4\zeta_7]_7$ for $x = 0, 1$. So, by (i), Lemmas 1.1.(i), 3.2 and 3.3.(ii), we have

$$0 = 8\zeta_7\sigma_{18} = (\bar{\nu}_7 + \varepsilon_7)\eta_{15}\mu_{16} + 4x\zeta_7\sigma_{18} = 4(1+x)\zeta_7\sigma_{18}.$$

This concludes $x = 1$ and completes the proof. □

In the exact sequence $(18)_9$, $\mathrm{Ker}\,\Delta = \mathbf{Z}_2\{\eta_9\varepsilon_{10}\} \oplus \mathbf{Z}_2\{[\iota_9,\iota_9]\eta_{17}\}$ for $\Delta : \pi_{18}(S^9) \to \pi_{17}(R_9)$.

We know $\pi_{19}(S^9) = \mathbf{Z}_8\{\sigma_9\nu_{16}\} \oplus \mathbf{Z}_2\{\eta_9\mu_{10}\}$. By Lemmas 1.1.(i), 2.3.(ii) and by the fact $\eta_7\sigma_8\nu_{15} = \bar{\nu}_7\nu_{15}$, we have

$$\Delta(\sigma_9\nu_{16}) = ([\nu_5]_9 + [\iota_7]_9\eta_7)\sigma_8\nu_{15} = [\nu_5]_9\sigma_8\nu_{15} + [\iota_7]_9\bar{\nu}_7\nu_{15} = [\iota_7]_9\bar{\nu}_7\nu_{15}.$$

We have $4[\nu_4\zeta_7]_9 = 4[\eta_6]_9\zeta_7 = 0$ by Lemmas 3.4.(iii) and 3.7.(i). So, by Lemmas 1.1.(i) and 3.7.(ii), we have

$$\Delta(\eta_9\mu_{10}) = [\nu_5]_9\eta_8\mu_9 + [\iota_7]_9\eta_7^2\mu_9 = 4([\iota_7]_9\zeta_7 + 2[2[\iota_6,\iota_6]]_9\sigma_{11}).$$

Therefore we have a short exact sequence

$$0 \to H \to \pi_{18}(R_{10}) \to G \to 0,$$

where

$$H = \mathbf{Z}_{16}\{[2[\iota_6,\iota_6]]_{10}\sigma_{11}\} \oplus \mathbf{Z}_4\{[\iota_7]_{10}\zeta_7 + 2[2[\iota_6,\iota_6]]_{10}\sigma_{11}\}$$

and

$$G = \mathbf{Z}_2\{p_*[\eta_9\varepsilon_{10}]\} \oplus \mathbf{Z}_2\{p_*[[\iota_9,\iota_9]\eta_{17}]\}.$$

Now we recall the homotopy groups $\pi_{n+18}(S^n)$ for $9 \le n \le 13$. According to Toda and Oda, we have the following

PROPOSITION 3.8 (Toda [17], Oda [14]).

(i) $\pi_{27}(S^9) = \mathbf{Z}_8\{\sigma_9\zeta_{16}\} \oplus \mathbf{Z}_2\{\eta_9\bar{\mu}_{10}\}$ *and* $\Sigma\pi_{27}(S^9) = \mathbf{Z}_4\{\sigma_{10}\zeta_{17}\} \oplus \mathbf{Z}_2\{\eta_{10}\bar{\mu}_{11}\}$.

(ii) $\pi_{28}(S^{10}) = \mathbf{Z}_8\{\xi''\} \oplus \mathbf{Z}_2\{\xi'' \pm \lambda''\} \oplus \mathbf{Z}_2\{\eta_{10}\bar{\mu}_{11}\} = \mathbf{Z}_8\{\bar{\bar{\lambda}}''\} \oplus \mathbf{Z}_2\{\tilde{\lambda}''\} \oplus \mathbf{Z}_2\{\eta_{10}\bar{\mu}_{11}\}$, *where* $H(\bar{\bar{\lambda}}'') = H(\lambda'') = \eta_{19}\varepsilon_{20}$, $H(\tilde{\lambda}'') = \nu_{19}^3$, $\xi'' \equiv \bar{\bar{\lambda}}'' + \tilde{\lambda}''$ mod $\{\sigma_{10}\zeta_{17}, \eta_{10}\bar{\mu}_{11}\}$ *and* $\tilde{\lambda}'' \equiv \lambda'' + \xi''$ mod $\{\sigma_{10}\zeta_{17}, \eta_{10}\bar{\mu}_{11}\}$ *and* $2\xi'' \equiv 2\lambda'' \equiv \sigma_{10}\zeta_{17}$ mod $2\sigma_{10}\zeta_{17}$.

(iii) $\Sigma\pi_{28}(S^{10}) = \mathbf{Z}_4\{\Sigma\bar{\bar{\lambda}}''\} \oplus \mathbf{Z}_2\{\Sigma\tilde{\lambda}''\} \oplus \mathbf{Z}_2\{\eta_{11}\bar{\mu}_{12}\}$ *and* $\pi_{29}(S^{11}) = \mathbf{Z}_8\{\xi'\} \oplus \mathbf{Z}_4\{\xi' + \lambda'\} \oplus \mathbf{Z}_2\{\eta_{11}\bar{\mu}_{12}\}$, *where* $2\xi' = \Sigma\xi''$, $2\lambda' = \Sigma\lambda''$, $H(\lambda') = \varepsilon_{21}$, $H(\xi') = \bar{\nu}_{21} + \varepsilon_{21}$ *and* $4\lambda' = 4\xi' = \sigma_{11}\zeta_{18}$.

(iv) $\pi_{30}(S^{12}) = \mathbf{Z}_{32}\{\xi_{12}\} \oplus \mathbf{Z}_4\{\Sigma\xi' + 4\xi_{12}\} \oplus \mathbf{Z}_4\{\Sigma\xi' + \Sigma\lambda'\} \oplus \mathbf{Z}_2\{\eta_{12}\bar{\mu}_{13}\}$, *where* $H(\xi_{12}) \equiv \sigma_{23} \bmod 2\sigma_{23}$, $\Sigma\xi' - 2\xi_{12} = \pm [\iota_{12}, \sigma_{12}]$, $16\xi_{12} = \sigma_{12}\zeta_{19}$, $\Sigma^2\tilde{\lambda}'' \equiv \varepsilon^*_{12}\eta_{29} \bmod \eta_{12}\bar{\mu}_{13}$ *and* $\varepsilon^*_{12}\eta_{29} \equiv 2\Sigma\lambda' + 2\Sigma\xi' \bmod \sigma_{12}\zeta_{19}$.

(v) $\pi_{31}(S^{13}) = \mathbf{Z}_8\{\xi_{13}\} \oplus \mathbf{Z}_8\{\lambda\} \oplus \mathbf{Z}_2\{\eta_{13}\bar{\mu}_{14}\}$, *where* $H(\lambda) = \nu^2_{25}$, $2\lambda = \Sigma^2\lambda'$ *and* $\varepsilon^*_{13}\eta_{30} = 4\xi_{13} + 4\lambda$.

(vi) $\pi^S_{18}(S^0) = \mathbf{Z}_8\{\nu^*\} \oplus \mathbf{Z}_2\{\eta\bar{\mu}\}$, *where* $\nu^* = \Sigma^\infty\nu^*_{16}$, $H(\nu^*_{16}) \equiv \nu_{31} \bmod 2\nu_{31}$, $\Sigma^3\lambda - 2\nu^*_{16} = \pm [\iota_{33}, \nu_{33}]$, $\nu^* = -\xi$, $\varepsilon^*\eta = 4\nu*$ *and* $\Sigma^\infty\lambda = 2\nu^*$.

Since $HJ[\eta_9\varepsilon_{10}] = \eta_{19}\varepsilon_{20}$, we have

$$J[\eta_9\varepsilon_{10}] \equiv \lambda'' \bmod \Sigma\pi_{27}(S^9).$$

Since $HJ([[\iota_9, \iota_9]\eta_{17}]) = 0$ and $\eta\bar{\mu}$ is not in the stable J-image, we have

$$J[[\iota_9, \iota_9]\eta_{17}] = x\sigma_{10}\zeta_{17} \qquad \text{for an integer } x.$$

So the order of $[\eta_9\varepsilon_{10}]$ is a multiple of 8. Therefore, by the exact sequence preceding to Proposition 3.8, we have

LEMMA 3.9. *$2[\eta_9\varepsilon_{10}] \equiv y[\iota_7]_{10}\zeta_7 \bmod [2[\iota_6, \iota_6]]_{10}\sigma_{11}$ for an odd integer y.*

From Lemma 3.9, we have three possibilities of the group extension of $\pi_{18}(R_{10})$; $\pi_{18}(R_{10}) \cong \mathbf{Z}_{32} \oplus \mathbf{Z}_8$, $\cong \mathbf{Z}_{32} \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2$ or $\cong \mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$. To determine the group extension, we shall settle the group $\pi_{18}(R_n)$ in the metastable range by using of the splitting theorem of [2].

By [3], we know $\pi_{18}(R_n) = 0$ for $n \geq 20$. By [6], we also know $\pi_{18}(R_n)$ for $n \geq 14$. We denote by $\mathbf{RP}^n_k = \mathbf{RP}^n/\mathbf{RP}^{k-1}$ the stunted real projective space and by $V_{n,k}$ the Stiefel manifold of k-frames in $\mathbf{R}^n$. By [2] and [3], we have

$$\pi_{18}(R_{19}) \cong \pi_{19}(V_{21,2}) = \pi_{19}(\mathbf{RP}^{20}_{19}) \cong \mathbf{Z}_2.$$

By the James periodicity theorem ([4]), we have

$$\pi_{18}(R_{18}) \cong \pi_{19}(V_{21,3}) = \pi_{19}(\mathbf{RP}^{20}_{18}) = \pi_{19}(\Sigma^{16}\mathbf{RP}^4_2) = \pi^S_3(\mathbf{RP}^4_2) \cong \mathbf{Z}_4.$$

By [11], we have

$$\pi_{18}(R_{17}) \cong \pi_{19}(V_{21,4}) = \pi_{19}(\mathbf{RP}^{20}_{17}) = \pi^S_3(\mathbf{RP}^4) \cong \mathbf{Z}_8$$

and

$$\pi_{18}(R_{16}) \cong \pi_{19}(V_{21,5}) = \pi_{19}(\mathbf{RP}^{20}_{16}) = \pi^S_3(\mathbf{RP}^4_0) \cong \mathbf{Z}_8 \oplus \mathbf{Z}_8.$$

By use of the exact sequences $(18)_n$ for $n = 14$ and 15, we have

$$\pi_{18}(R_{14}) \cong \pi_{18}(R_{15}) \cong \mathbf{Z}_8.$$

In the exact sequence $(15)_{13}$, we have $\Delta(\nu_{13}) = [\nu_5]_{13}\sigma_8$ by Proposition 2.1 of [5] and by Theorem 3.(i) of [6]. So, in the exact sequence $(18)_{13}$, we have $\Delta(\nu_{13}^2) = [\nu_5]_{13}\sigma_8\nu_{15} = 0$ by Lemma 3.6. Therefore we have

$$\pi_{18}(R_{13}) \cong \mathbf{Z}_8.$$

In the exact sequence $(18)_{12}$, we have $\operatorname{Ker}\Delta = \mathbf{Z}_2\{\nu_{12}^2\}$ for $\Delta : \pi_{18}(S^{12}) \to \pi_{17}(R_{12})$.

Since $8J\Delta(\sigma_{12}) = 8[\iota_{12}, \sigma_{12}] = 16\xi_{12} = \sigma_{12}\zeta_{19} \neq 0$ by Proposition 3.8, we have (cf. [13])

$$\pi_{18}(R_{12}) \cong \mathbf{Z}_{16}\{\Delta(\sigma_{12})\} \oplus \mathbf{Z}_4.$$

In the exact sequence $(18)_{11}$, we have $\Delta\sigma_{11} = [\iota_7]_{11}\nu_7\sigma_{10}$, $\Delta\bar{\nu}_{11} = [\iota_7]_{11}\nu_7\bar{\nu}_{10} = 0$ and $\Delta\varepsilon_{11} = [\iota_7]_{11}\nu_7\varepsilon_{10} = 0$ by Lemma 1.1.(i). So we can take $[2\sigma_{11}] = \Delta\sigma_{12}$ by Lemma 3.10 of [5] and we have

$$\pi_{18}(R_{11}) \cong \mathbf{Z}_4?\mathbf{Z}_2 \qquad (= \mathbf{Z}_4 \oplus \mathbf{Z}_2 \text{ or } \mathbf{Z}_8).$$

By the argument determining $\pi_{17}(R_{11})$, we know $\operatorname{Ker}\Delta = \mathbf{Z}_2\{[\tau]\}$, where $\tau = \bar{\nu}_{10}$ or ε_{10}. We know that $\pi_{19}(S^{10}) = \mathbf{Z}\{[\iota_{10}, \iota_{10}]\} \oplus \mathbf{Z}_2\{\eta_{10}\bar{\nu}_{11}\} \oplus \mathbf{Z}_2\{\eta_{10}\varepsilon_{11}\} \oplus \mathbf{Z}_2\{\mu_{10}\}$ by Theorem 7.2 of [17]. Then, by Lemma 2.4, we have

$$\Delta(\eta_{10}\bar{\nu}_{11}) = 2[\iota_7]_{10}\nu_7\bar{\nu}_{10} = 0 \qquad \text{and} \qquad \Delta(\eta_{10}\varepsilon_{11}) = 2[\iota_7]_{10}\nu_7\varepsilon_{10} = 0.$$

By (12.25) of [17], we know the relation $[\iota_{10}, \mu_{10}] = 2\sigma_{10}\zeta_{17}$. By Theorem 5.1 of [10], we have

$$\begin{aligned}\Delta\mu_{10} &\in \Delta\{\eta_{10}, 8\iota_{11}, 2\sigma_{11}\}_1\\ &\subset \{2[\iota_7]_{10}\nu_7, 8\iota_{10}, 2\sigma_{10}\}\\ &\supset 2[\iota_7]_{10}\{\nu_7, 8\iota_{10}, 2\sigma_{10}\}\\ &\ni 2[\iota_7]_{10}\zeta_7 \bmod 2[\iota_7]_{10}\nu_7 \circ \pi_{18}(S^{10}) + \pi_{11}(R_{10}) \circ 2\sigma_{11} = \{2[2[\iota_6, \iota_6]]_{10}\sigma_{11}\}.\end{aligned}$$

We assume that $\pi_{18}(R_{10}) \cong \mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$ or $\pi_{18}(R_{10}) \cong \mathbf{Z}_{32} \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2$. By use of $(18)_{10}$, we have $\pi_{18}(R_{11}) \cong G?\mathbf{Z}_2$, where the group G is not isomorphic to $\mathbf{Z}_4$. This contradicts the fact $\pi_{18}(R_{11}) \cong \mathbf{Z}_4?\mathbf{Z}_2$. Thus we have

$$\pi_{18}(R_{10}) \cong \mathbf{Z}_{32} \oplus \mathbf{Z}_8$$

and it is generated by $[[\iota_9, \iota_9]\eta_{17}]$ and $[\eta_9\varepsilon_{10}]$, where $2[\eta_9\varepsilon_{10}] = [\iota_7]_{10}\zeta_7 + 2[2[\iota_6, \iota_6]]_{10}\sigma_{11}$ and $2[[\iota_9, \iota_9]\eta_{17}] = [2[\iota_6, \iota_6]]_{10}\sigma_{11}$.

By [12], Lemma 3.9 and the group structure of $\pi_{18}(R_{10})$, we have

$$\Delta[\iota_{10}, \iota_{10}] = [[\iota_9, \iota_9]\eta_{17}] + 2[\eta_9\varepsilon_{10}],$$

$$\Delta\mu_{10} = 2([\iota_7]_{10}\zeta_7 + 2[2[\iota_6, \iota_6]]_{10}\sigma_{11})$$

and

$$J[[\iota_9, \iota_9]\eta_{17}] = \pm\sigma_{10}\zeta_{17}.$$

Finally we have the following.

LEMMA 3.10. (i) $\Delta\varepsilon_{10} = 0$.
(ii) $J[\varepsilon_{10}] \equiv \lambda' \bmod \Sigma\pi_{28}(S^{10})$.
(iii) $\pi_{18}(R_{11}) = \mathbf{Z}_8\{[\varepsilon_{10}]\}$.
(iv) $8[2\sigma_{11}] = 4[\varepsilon_{10}]_{12}$.

PROOF. Assume that $\Delta\bar{\nu}_{10} = 0$. Since $H(\lambda' + \xi') = \bar{\nu}_{21}$ by Proposition 3.8.(iii), we have $J[\bar{\nu}_{10}] \equiv \lambda' + \xi' \bmod \Sigma\pi_{28}(S^{10})$. Suppose that $[\bar{\nu}_{10}]$ is of order 8 and $2[\bar{\nu}_{10}] = [\eta_9\varepsilon_{10}]_{11}$. Applying the J homomorphism to this relation, we have $2(\lambda' + \xi') \equiv \Sigma\lambda'' \bmod 2\Sigma\pi_{28}(S^{10}) + \Sigma^2\pi_{27}(S^9)$. So, by Proposition 3.8, we have $2\xi' \in \{4\xi', \eta_{11}\mu_{12}\}$. This is a contradiction. Therefore we have $\pi_{18}(R_{11}) = \mathbf{Z}_4\{[\bar{\nu}_{10}]\} \oplus \mathbf{Z}_2\{[\bar{\nu}_{10}] \pm [\eta_9\varepsilon_{10}]_{11}\}$.

On the other hand, by the group structure of $\pi_{18}(R_{12})$ and by the exact sequence $(18)_{11}$, we have the relation $8[2\sigma_{11}] = [\bar{\nu}_{10}]_{12} \pm [\eta_9\varepsilon_{10}]_{12}$.

So we have $8J[2\sigma_{11}] = J[\bar{\nu}_{10}]_{12} \pm J[\eta_9\varepsilon_{10}]_{12} \equiv (\lambda' + \xi') \pm \Sigma\lambda''$ $\bmod \Sigma\pi_{28}(S^{10})$. By stabilizing this relation, we have $0 \equiv 2\Sigma^\infty\lambda + 2\xi \pm 4\Sigma^\infty\lambda$ $\bmod \{4\Sigma^\infty\lambda, 4\xi, \eta\bar{\mu}\}$ by Proposition 3.8. Since $\Sigma^\infty\lambda = 2\nu^* = -2\xi$, we have $4\xi + 2\xi \in \{4\xi, \eta\bar{\mu}\}$. This is a contradiction and hence we have (i).

Since $H(\lambda') = \varepsilon_{21}$, we have $J[\varepsilon_{10}] \equiv \lambda' \bmod \Sigma\pi_{28}(S^{10})$. So $[\varepsilon_{10}]$ is of order 8. This leads us to (ii) and (iii).

Since $[2\sigma_{11}]$ is of order 16, we have (iv) by the exact sequence $(18)_{11}$. This completes the proof. □

By the exact sequence $(18)_{11}$ and Lemma 3.10, we have the group $\pi_{18}(R_{12})$. We have $\pi_{18}(R_{13}) = \mathbf{Z}_8\{[\nu_{12}^2]\}$, where $2[\nu_{12}^2] = 4([\varepsilon_{10}]_{13} - 2[2\sigma_{11}]_{13})$. Since $H(\lambda) = \nu_{25}^2$ by Proposition 3.8.(v), we have $J[\nu_{12}^2] \equiv \lambda \bmod \Sigma\pi_{30}(S^{12}) = \{\xi_{13}, 2\lambda, \eta_{13}\bar{\mu}_{14}\}$.

The rest of Theorem 2 is easily obtained. This completes the proof of Theorem 2.

REMARK 1. Odd primary components of $\pi_{n+k}(R_n)$ for $16 \le k \le 18$ are easily obtained from [8] and its method. These results with [5], Theorems 1 and 2 lead us to the following table. Here $m + (r)^k$ means $\mathbf{Z}_m \oplus \mathbf{Z}_r \oplus \cdots \oplus \mathbf{Z}_r$ ($k+1$ direct sums).

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\pi_{16}(R_n)$	6	$(6)^2$	$(2)^8$	$504+(2)^4$	$(2)^7$	$(2)^{11}$	$(2)^6$	$240+(2)^2$	$(2)^2$	2	2	$12+2$	$(2)^2$	$(2)^3$	$(2)^2$	2	2	2
$\pi_{17}(R_n)$	30	$(30)^2$	40	$40+(2)^3$	$(8)^2+(2)^2$	$24+(8)^2+(2)^3$	$8+(2)^2$	$4+2$	$(2)^2$	2	2	2	$(2)^2$	$(2)^3$	$(2)^2$	$\infty+2$	2	2
$\pi_{18}(R_n)$	30	$(30)^2$	$2520+2$	$2520+12+2$	$15120+8+2$	$15120+504+8+(2)^2$	$45360+8+2$	$90720+8$	8	$240+4$	8	8	8	$24+8$	8	4	2	0

REMARK 2. As for direct proofs of Lemmas 2.2 and 3.5, we know the following fact (cf. [10]); Let $n \geq 2$ and $k \geq 5$. Assume that $\Delta\iota_6 \circ \beta = 0$ and $n\beta = 0$ for $\beta \in \pi_k(S^5)$. Then, for any element $\delta \in \{\Delta\iota_6, \beta, n\iota_k\} \subset \pi_{k+1}(R_6)$, there exists an element $\varepsilon \in \pi_{k+1}(R_7)$ such that $p_*\varepsilon = \Sigma\beta$ and $i_*\delta = n\varepsilon$.

We recall the relation $2\bar{\nu}_6\nu_{14} = \nu_6\bar{\nu}_9 = \nu_6\varepsilon_9$. We set $\beta = \nu_5\bar{\nu}_8$. Then we have

$$\begin{aligned}\{\Delta\iota_6, \nu_5\bar{\nu}_8, 2\iota_6\} &\supset \{\Delta\iota_6 \circ \nu_5, \bar{\nu}_8, 2\iota_{16}\}\\ &\supset [\nu_5]\{2\iota_8, \bar{\nu}_8, 2\iota_{16}\}\\ &\ni [\nu_5]\bar{\nu}_8\eta_{16}\\ &= [\nu_5]\nu_8^3 \bmod \Delta\iota_6 \circ \pi_{17}(S^5) + 2\pi_{17}(R_6) = \{2[\nu_4^2]_6\sigma_{10}\}.\end{aligned}$$

We take $\delta = [\nu_5]\nu_8^3$. Then, ε is taken as $2[\bar{\nu}_6 + \varepsilon_6]\nu_{14}$ mod {elements of $i_*\pi_{17}(R_6)$ of order 2 or 4}. This leads us to the first assertion of Lemma 2.2.(ii).

By a parallel argument to the above, we have the assertion of Lemma 3.5.

REMARK 3. We hope that direct proofs of the results $\Delta\varepsilon_{10} = 0$ and $p_*\Delta[\iota_{10}, \iota_{10}] = [\iota_9, \iota_9]\eta_{17}$ can be found.

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