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On the existence of solutions of nonlinear boundary value problems at resonance in Sobolev spaces of fractional order

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ABSTRACT. The purpose of this paper is to prove existence results for a class of degenerate boundary value problems for second-order elliptic operators in the framework of Sobolev spaces of fractional order. The proofs apply generalized solvability conditions of Landesman-Lazer type, Leray-Schauder degree arguments and maximum principles.

1. Introduction and main result

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Let

$$Au(x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + c(x)u(x)$$

be a second order elliptic differential operator with real C^∞ functions a_{ij}, c on $\overline{\Omega}$ satisfying the following properties:

(p1) $a_{ij}(x) = a_{ji}(x),\ i, j = 1, \dots, n,\ x \in \overline{\Omega}$.

(p2) There exists a positive constant C_0 such that for all $x \in \overline{\Omega}$ and all $\xi \in \mathbf{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq C_0|\xi|^2.$$

(p3) $c(x) \geq 0$ on $\overline{\Omega}$.

We consider the following class of degenerate boundary value problems for semilinear second-order elliptic differential operators

$$Au - \lambda_1 u = g(u) + f \quad \text{in } \Omega, \qquad Bu = a\frac{\partial u}{\partial \nu} + bu = 0 \quad \text{on } \partial\Omega \tag{P}$$

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in the framework of *real-valued* Bessel-potential spaces $H_p^s(\Omega)$, where B is a degenerate boundary operator. Here:

(p4) a and b are real-valued C^∞ functions defined on $\partial\Omega$.

(p5) $\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij} n_j \frac{\partial}{\partial x_i}$ is the conormal derivative corresponding with the operator A, where $n = (n_1, \dots, n_n)$ is the unit exterior normal to the boundary $\partial\Omega$.

Note that (P) is called to be *nondegenerate* if and only if either $a \neq 0$ on $\partial\Omega$ or $a \equiv 0$ and $b \neq 0$ on $\partial\Omega$. If $a \equiv 1$ and $b \equiv 0$, then we have the Neumann problem. The case when $a \equiv 0$ and $b \equiv 1$ hold coincides with the Dirichlet problem. Furthermore, if $a(x') \neq 0$ on $\partial\Omega$, then we get the third boundary problem (or Robin problem). We remark that the so-called Lopatinskij-Shapiro complementary condition does not hold at the points $x' = \partial\Omega$ with $a(x') = 0$. By the main theorem for elliptic boundary value problems, see J. Wloka [17, Hauptsatz 13.1] there exists an equivalence between the ellipticity of a boundary value problem and the Fredholm property if one uses the usual boundary value spaces of Besov type $B_{p,p}^{s-1/p}(\partial\Omega)$ for the boundary operators. To overcome these difficulties one introduces a subspace of $B_{p,p}^{1-1/p}(\partial\Omega)$ which is associated to our degenerate boundary operator B. For more details, we refer to K. Taira [10] and [7].

We make the following three conditions (H1)–(H3):

(H1) $a(x') \geq 0$ and $b(x') \geq 0$ on $\partial\Omega$.

(H2) $b(x') > 0$ on $M = \{x' \in \partial\Omega : a(x') = 0\}$.

(H3) $c(x) \geq 0$ in Ω, and $c \not\equiv 0$ in Ω.

Furthermore, g is a smooth real-valued function defined on $\mathbf{R}$ which satisfies a linear growth condition, and λ_1 denotes the first eigenvalue of A together with the homogeneous boundary condition $Bu = 0$. It is known that λ_1 is positive and simple, see Taira [13]. Let $\varphi_1 \in C^\infty(\bar{\Omega})$ be the associated eigenfunction satisfying $\varphi_1 > 0$ in Ω and $\|\varphi_1 | L_\infty\| = 1$. Thus we have $\ker_B(A - \lambda_1 \,\mathrm{id}) = \mathrm{span}\{\varphi_1\}$. Note that the boundary condition $Bu = 0$ on $\partial\Omega$ implies that

$$u = 0 \qquad \text{on } M = \{x' \in \partial\Omega : a(x') = 0\},$$

if $b > 0$ on M. Hence it holds

$$\varphi_1 = 0 \quad \text{on } M, \qquad \varphi_1 > 0 \quad \text{on } \bar{\Omega} \backslash M \quad \text{and} \quad \frac{\partial \varphi_1}{\partial \nu} < 0 \quad \text{on } M.$$

Boundary conditions of this type occur in multidimensional diffusion processes and Markov processes. We refer to K. Taira [10]. We treat solutions u of (P) in the Bessel-potential spaces $H_p^s(\Omega)$, $s > n/p$, $1 < p < \infty$. Recall that the spaces $H_p^s(\Omega)$ coincide with the classical Sobolev spaces $W_p^s(\Omega)$

if $s \in \mathbf{N}$. Throughout this paper, both u, f and g are assumed to be *real-valued*. Therefore we do not distinguish between a function spaces and its real part, and we use the same abbreviation.

In S. Ahmad [1, 2], S. B. Robinson and E. M. Landesman [5] and T. Runst and W. Sickel [8] the Dirichlet case was considered. Further results, by application of the bifurcation theory, may be found in the papers of A. Szulkin [9], K. Taira and K. Umezu [14], [8, 6.6] and the references therein.

Now we formulate an abstract solvability condition for problem (P) similar to that in [5], [8, 6.4.5]. Here $\lambda_2 > \lambda_1$ denotes the second eigenvalue.

THEOREM. *Assume that the conditions* (H1)–(H3) *are satisfied. Let* $s > \max(n/p, 1/p+1)$ *and* $\rho > -1$, *and let* $g \in C^\infty(\mathbf{R})$ *such that*

$$0 \leq \liminf_{|t|\to\infty} \frac{g(t)}{t} \leq \limsup_{|t|\to\infty} \frac{g(t)}{t} < \lambda_2 - \lambda_1. \tag{1}$$

Let $f \in H_p^{s-2}(\Omega) \cap B^\rho_{\infty,\infty}(\Omega)$. *Then* (P) *has a solution* $u \in H_p^s(\Omega)$ *if the function* f *satisfies the following generalized Landesman-Lazer condition* (GLL) *with respect to the kernel* $\ker_B(A - \lambda_1)$.

(GLL): *If* $\{u_k\}_{k=1}^\infty \subset H_p^s(\Omega)$ *such that* $\|u_k|L_\infty\| \to \infty$ *and* $u_k/\|u_k|L_\infty\| \to \varphi = \pm\varphi_1$ *in the* $C^1(\overline{\Omega})$ *norm, then there exists a number* $K > 0$ *such that*

$$\operatorname{sign}(\varphi) \int_\Omega (g(u_k(x)) + f(x))\varphi_1(x)dx \geq 0 \qquad \textit{for all } k \geq K.$$

Recall that $f \in B^\rho_{\infty,\infty}(\Omega)$, $\rho > -1$, means that $(-\Delta + \mathrm{id})^{-1} f$ belongs to the Hölder-Zygmund spaces $\mathscr{C}^{\rho+2}(\Omega) = B^{\rho+2}_{\infty,\infty}(\Omega)$ (Δ: Laplacian). We note that our result with $s = 2$ implies that (P) has a solution $u \in W_p^2(\Omega)$ for $f \in L_p(\Omega)$, if (GLL) and $p > n$ hold.

This theorem is a generalization of the paper S. B. Robinson and T. Runst [6], see also [8, Subsection 6.4.5, Theorem 1], to the degenerate case. Furthermore, we can show that further solvability conditions can be viewed as special cases of this abstract result.

For example, if the limits

$$\lim_{t\to\pm\infty} g(t) = g(\pm\infty)$$

exist or are infinite, then the solvability condition of Landesman-Lazer type

$$g(-\infty)\int_\Omega \varphi_1(x)dx < -\int_\Omega \varphi_1(x)f(x)dx < g(+\infty)\int_\Omega \varphi_1(x)dx$$

implies (GLL).

2. Preliminaries

Linear theory, mapping properties

Let $\Omega \subset \mathbf{R}^n$ be a bounded and smooth domain with boundary $\partial\Omega$. Let $f \in H_p^{s-2}(\Omega)$. We consider the corresponding linear problem

$$Au = f \quad \text{in } \Omega, \qquad Bu = 0 \quad \text{on } \partial\Omega \tag{1}$$

in the framework of Bessel-potential spaces $H_p^s(\Omega)$. As usual, let for $s \in \mathbf{R}$ and $1 < p < \infty$ the Bessel-potential space (or Sobolev spaces of fractional order) $H_p^s(\mathbf{R}^n)$ be given by

$$H_p^s(\mathbf{R}^n) = \{h \in \mathscr{S}'(\mathbf{R}^n) : \|h|H_p^s\| = \mathscr{F}^{-1}(1+|\xi|^2)^{s/2}\mathscr{F}h|L_p\| < \infty\},$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and its inverse, respectively, on the space of tempered distributions $\mathscr{S}'(\mathbf{R}^n)$. We assume that f belongs to a Bessel-potential space $H_p^{s-2}(\Omega)$, the space of restrictions to Ω of functions in $H_p^{s-2}(\mathbf{R}^n)$.

Then the following existence and uniqueness result for problem (1) holds (cf. K. Taira [10, 11, 13] and T. Runst [7]):

Proposition 1. *Let* (H1)–(H3) *be satisfied. Then the map*

$$A : H_{p,B}^s(\Omega) \to H_p^{s-2}(\Omega)$$

is an algebraic and topological isomorphism for all $s > 1 + 1/p$. *Here*

$$H_{p,B}^s(\Omega) = \{u \in H_p^s(\Omega) : Bu = 0 \text{ on } \partial\Omega\}.$$

We remark that this result was proved in [7] in the framework of the two scales of function spaces of Besov-Triebel-Lizorkin type, for definition and properties we refer to H. Triebel [16] and [8]. Especially, Proposition 1 holds in the case of Hölder-Zygmund spaces $\mathscr{C}^s$ for $s > 1$. Note that we have the continuous embedding

$$H_p^s(\Omega) \hookrightarrow \mathscr{C}^\varepsilon(\Omega) \hookrightarrow L_\infty(\Omega),$$

if $s - n/p > \varepsilon > 0$.

Now we consider the mapping properties for superposition (or Němytskiĭ) operator

$$T_g : u(x) \to g(u(x))$$

which may be found in [8, 5.3.4].

In our later considerations, the next proposition is sufficient. For the sake of simplicity, we suppose that the (real-valued) function $g : \mathbf{R} \to \mathbf{R}$ is smooth, i.e., $g \in C^\infty(\mathbf{R})$, but the results hold also under weaker smoothness assumptions. As usual an operator is called completely continuous if it is compact and continuous.

PROPOSITION 2. *Let g be a smooth function and $s > 0$.*

(a) *Then there exists a positive constant c_g such that*

$$\|g(u)|H_p^s(\Omega)\| \le c_g\|u|H_p^s(\Omega)\|(1 + \|u|L_\infty(\Omega)\|^{\max(0,s-1)}) \tag{2}$$

holds for all $u \in H_p^s(\Omega) \cap L_\infty(\Omega)$. Furthermore, T_g is continuous from $H_p^s(\Omega) \cap L_\infty(\Omega)$ into $H_p^s(\Omega)$.

(b) *Let $\varepsilon > 0$. Then T_g is a completely continuous map from $H_p^s(\Omega) \cap L_\infty(\Omega)$ into $H_p^{s-\varepsilon}(\Omega)$.*

We remark that part (b) is a consequence of (a), and the fact that the embedding

$$H_p^{s+\delta}(\Omega) \hookrightarrow H_p^s(\Omega), \qquad \delta > 0, \tag{3}$$

is compact.

Maximum principles

The next results are important for our further considerations. We start with the following assertion which is a consequence of K. Taira and K. Umezu [15, Lemma 2.1] and [8, 3.5.4]:

PROPOSITION 3. *Assume that (H1)–(H3) are satisfied. Let $v \in \bigcup_{\varepsilon>0} B^{1+\varepsilon}_{\infty,\infty}(\Omega)$. If $Av \ge 0$ in $\Omega, v \ge 0$ but $v \not\equiv 0$ in $\bar{\Omega}$, then v satisfies the following conditions*:

$$\text{(a) } v = 0 \qquad \text{on } M = \{x' \in \partial\Omega : a(x') = 0\}.$$

$$\text{(b) } v > 0 \qquad \text{in } \bar{\Omega}\setminus M.$$

$$\text{(c) } \frac{\partial v}{\partial \nu} < 0 \qquad \text{on } M.$$

(We use the symbol $\ge$ in the sense of distributions, see [8, Definition 3.5.4]). The next lemma will be useful in the proof of our theorem. Therefore we apply arguments which are essentially the same as that due to S. Ahmad [2, Lemma 2.2] and [6] for the Dirichlet boundary condition. We recall that for $\varepsilon > 0$ the continuous embedding

$$B^{1+\varepsilon}_{\infty,\infty}(\Omega) \hookrightarrow C^1(\bar{\Omega})$$

holds.

Lemma 1. *There exists a positive number $d, d > \lambda_1$, such that if $q \in C(\bar{\Omega})$ satisfies*

$$\lambda_1 \leq q \leq d \quad in\ \Omega, \tag{4}$$

and $v \in \bigcup_{\varepsilon>0} B^{1+\varepsilon}_{\infty,\infty}(\Omega)$ for which

$$Av = qv \quad in\ \Omega, \qquad Bv = 0 \quad on\ \partial\Omega, \tag{5}$$

and $v \not\equiv 0$, then either $v(x') = 0$ on $M = \{x' \in \partial\Omega : a(x') = 0\}, v > 0$ in $\bar{\Omega}\backslash M$ and $\dfrac{\partial v}{\partial \nu} < 0$ on M, or $v(x') = 0$ on $M, v < 0$ in $\bar{\Omega}\backslash M$ and $\dfrac{\partial v}{\partial \nu} > 0$ on M.

Proof. *Step 1:* First we consider the case, where $v \in \bigcup_{\varepsilon>0} B^{1+\varepsilon}_{\infty,\infty}(\Omega)$ is a solution of (5) such that $v \not\equiv 0$ and $v \geq 0$ in $\bar{\Omega}\backslash M$. If μ is a positive number large enough such that

$$\mu + q(x) > 0 \qquad \text{for all } x \in \Omega,$$

then

$$(A + \mu)v(x) \geq 0 \qquad \text{for } x \in \Omega.$$

Now the claim follows from Proposition 3. Similarly, if v is a solution of (5) with $v \not\equiv 0$ and $v \leq 0$ in $\bar{\Omega}\backslash M$, then $v < 0$ in $\bar{\Omega}\backslash M$ and $\dfrac{\partial v}{\partial \nu} > 0$ on M.

Step 2: If the assertion of Lemma 1 is false, then we can find a sequence $\{q_n\}_{n=1}^{\infty} \subset C(\bar{\Omega})$ with

$$c \leq q_n(x) \leq \lambda_1 + \frac{1}{n} \qquad \text{for all } x \in \Omega \tag{6}$$

and a corresponding sequence $\{v_n\}_{n=1}^{\infty} \subset \bigcup_{\varepsilon>0} B^{1+\varepsilon}_{\infty,\infty}(\Omega)$ such that $v_n \not\equiv 0$,

$$(Av_n)(x) = q_n(x)v_n(x) \quad \text{in } \Omega, \qquad Bv_n = 0 \quad \text{on } \partial\Omega, \tag{7}$$

and there exists a point $x_n \in \bar{\Omega}\backslash M$ such that $v_n(x_n) = 0$. Without loss of generality we may assume that $\|v_n|C^1\| = 1$ for all n. Applying the mapping properties of A, see Proposition 1 or [7], and compactness results of type (3), it follows that $v_n \to v_0$ as $n \to \infty$ in $C^1(\bar{\Omega})$ and $\|v_0|C^1\| = 1$.

Step 3: We show that there is $x_0 \in \bar{\Omega}$ such that either $x_0 \in \bar{\Omega}\backslash M$ and $v_0(x_0) = 0$ or $x_0 \in M$ and $\dfrac{\partial v_0}{\partial \nu}(x_0) = 0$. By (7) we have $Bv_0 = 0$ on $\partial\Omega$. If our claim is false, we have either $v_0(x) > 0$ for all $x \in \bar{\Omega}\backslash M$ and $\dfrac{\partial v_0}{\partial \nu} < 0$ on M, or $v_0(x) < 0$ for all $x \in \bar{\Omega}\backslash M$ and $\dfrac{\partial v_0}{\partial \nu} > 0$ on M. Applying continuity arguments this shows that v_n would have the same behaviour for n sufficiently large. This yields a contradiction.

Step 4: Using the boundedness of $\{q_n\}_{n=1}^{\infty}$ in $L_2(\Omega)$ and Mazur's theorem we may assume that $q_n \to q_0$ in $L_2(\Omega)$ (for a subsequence) which satisfies

$$c \leq q_0(x) \leq \lambda_1 \quad \text{a.e.} \quad \text{in } \Omega. \tag{8}$$

Applying similar arguments as in S. Ahmad [2, p. 150] then we can deduce from (7) that

$$(Av_0)(x) = q_0(x)v_0(x) \quad \text{in } \Omega, \qquad Bv_0 = 0 \quad \text{on } \partial\Omega \tag{9}$$

holds. Let φ_1 be as above. By the properties of v_0, i.e., $\|v_0|C^1\| = 1$, $v_0 \not\equiv 0$, we may assume that there is $x_1 \in \Omega$ with $v_0(x_1) > 0$. (If necessary, one has to replace v_0 by $-v_0$.) Furthermore, for sufficiently small $k > 0$ we get $\varphi_1(x) - kv_0(x) > 0$ for all $x \in \Omega$. Let k^* be the supremum of all such k. Now we define a function z by $z(x) = \varphi_1(x) - k^* v_0(x)$. Then we have $z(x) \geq 0$ for all $x \in \Omega$ and, by the properties of v_0 and φ_1, $\dfrac{\partial z}{\partial \nu} \leq 0$ on M. The definition of k^* shows that there is either a point $x^* \in \bar{\Omega}\backslash M$ such that $z(x^*) = 0$, or a point $x^* \in M$ with $\dfrac{\partial z}{\partial \nu}(x^*) = 0$. Finally, for $\gamma > 0$ so large that $\gamma + q_0 > 0$ a.e. in Ω,

$$(A + \gamma)z = (\gamma + q_0)z + (\lambda_1 - q_0)\varphi_1 \geq 0 \quad \text{in } \Omega, \qquad Bz = 0 \quad \text{on } \partial\Omega,$$

and maximum principle argument, see [8, 3.5.4], [11, Proposition 5.6] show that $z \equiv 0$. Hence Step 3 yields a contradiction to the properties of φ_1. The proof is finished. □

For our further investigations, the following consequences of Lemma 1 suffices.

Corollary. *Let all assumptions of Lemma 1 be satisfied, and let* $v \in \bigcup_{\varepsilon > 0} B^{1+\varepsilon}_{\infty,\infty}(\Omega)$ *be a solution of* (5). *Then* $v \in \ker_B(A - \lambda_1 \,\mathrm{id})$.

Proof. By Lemma 1 we may conclude that either $v \equiv 0, v > 0$ in $\bar{\Omega}\backslash M$ and $\dfrac{\partial v}{\partial \nu} < 0$ on M, or $v < 0$ in $\bar{\Omega}$ and $\dfrac{\partial v}{\partial \nu} > 0$ on M. If $v \equiv 0$, then we are finished. Now we assume that $v > 0$ on $\bar{\Omega}\backslash M$. The other case can be investigated similarly. We choose $k > 0$ small enough such that $v - k\varphi_1 > 0$ in Ω. Now we use the same arguments as in the proof of Step 4 of Lemma 1. Thus the corollary is proved. □

Let d^* be the supremum of all numbers $d > \lambda_1$, such that if $q \in C(\bar{\Omega})$ satisfies (4), then Lemma 1 holds. Now we prove that

$$d^* = \lambda_2. \tag{10}$$

For it one applies some known results concerning eigenvalue problems with indefinite weight functions, which may be found in [8, Proposition 6.4.5]. We refer also to A. Manes and A. M. Micheletti [4].

Let $q \in C(\overline{\Omega})$. Then the eigenvalue problem $(\mathrm{P_q})$ with real parameter μ is given by

$$Av = \mu q v \quad \text{in } \Omega, \qquad Bv = 0 \quad \text{on } \partial\Omega. \tag{$\mathrm{P_q}$}$$

Now we are in position to prove (10).

LEMMA 2. *Let $0 < \lambda_1 < \lambda_2 \leq \cdots$ denote the eigenvalues, each appearing as often in the sequence as its multiplicity, of*

$$Au = \lambda u \quad \textit{in } \Omega, \qquad Bu = 0 \quad \textit{on } \partial\Omega. \tag{11}$$

Then $d^* = \lambda_2$ holds.

PROOF. Let $u_2 \in C^\infty(\overline{\Omega})$ be a nontrivial eigenfunction to the second eigenvalue. We know that φ_1 is positive everywhere in Ω_1. Hence u_2 has to change the sign on Ω. This gives $d^* \leq \lambda_2$. Now we suppose that d is an arbitrary number satisfying $\lambda_1 < d < \lambda_2$, $q \in C(\overline{\Omega})$ with $\lambda_1 \leq q \leq d$ in Ω, and that $v \in \bigcup_{\varepsilon>0} B^{1+\varepsilon}_{\infty,\infty}(\Omega)$ is a nontrivial solution of (9). Since $\mu = 1$ is a positive eigenvalue of $(\mathrm{P_q})$, [8, Proposition 6.4.5(i)] implies that q is positive on a set of positive Lebesgue measure and $\mu_k(q) = 1$ for some $k \geq 1$. It holds $\mu_k(\lambda_2) = \lambda_k/\lambda_2$ for $k \geq 1$. By our assumption $q \leq d < \lambda_2$ in Ω, we can conclude from [8, Proposition 6.4.5(iii)] that $1 = \mu_2(\lambda_2) < \mu_2(q)$ and $\mu_1(q) = 1$. Applying [8, Proposition 6.4.5(ii)] it follows that the corresponding nontrivial eigenfunction v is strictly positive (negative) on Ω. Now we choose a positive constant γ such that $\gamma + q > 0$ in Ω. We obtain

$$(A + \gamma)v = (q + \gamma)v$$

in Ω. Thus either v or $-v$ satisfies the hypotheses of Lemma 1. This shows $d^* \geq \lambda_2$.

□

3. Proof of the main result, generalizations

Proof of the main result

Applying the results from the last section we can prove our main results.

PROOF OF THEOREM. *Step* 1: From our assumptions we can conclude the existence of a positive number κ such that $\lambda_1 + \kappa < \lambda_2$. Thus $\lambda_1 + \kappa$ is not an eigenvalue of problem (11) in Section 2. For $\tau \in [0, 1]$ we define a family of boundary value problems

$$Au = (\lambda_1 + \tau\kappa)u + (1-\tau)(g(u) + f) \quad \text{in } \Omega, \qquad Bu = 0 \quad \text{on } \partial\Omega. \tag{P_τ}$$

The arguments in [8, Lemma 6.4.2] show that it is sufficient to prove the existence of a positive number R such that if u_τ is a solution of (P_τ), then

$$\|u_\tau | L_\infty\| \leq R, \tag{1}$$

where R is independent of $\tau \in [0,1]$. Therefore on applies Proposition 1 and Proposition 2. Afterwards we obtain that there is a constant $c > 0$ such that

$$\|u_\tau | H_p^s\| \leq c, \tag{2}$$

holds for all solutions u_τ of problem (P_τ), when $\tau \in [0,1]$. Recall that the definition of κ implies the invertibility of the linear map $T = \mathrm{id} - (\lambda_1 + \kappa)A^{-1}$ in $H_{p,B}^s(\Omega)$. Let c be given by (2). Since λ_1 is the principal eigenvalue of A under homogeneous boundary condition $Bu = 0$ we can deduce from the index formula for compact linear operators, see [8, Subsection 6.2.3, Theorem 7],

$$\mathrm{d}_{\mathrm{LS}}[\mathrm{id} - h(0,\cdot), B_{2c}, 0] = \mathrm{d}_{\mathrm{LS}}[\mathrm{id} - h(1,\cdot), B_{2c}, 0] = -1. \tag{3}$$

Here $h : [0,1] \times H_p^s(\Omega) \to H_p^s(\Omega)$ is the completely continuous operator which assigns to each $u \in H_p^s(\Omega)$ and $t \in [0,1]$ the unique solution $w \in H_p^s(\Omega)$ of the problem

$$Aw = (\lambda_1 + \tau\kappa)u + (1-\tau)(g(u) + f) \quad \text{in } \Omega, \qquad Bw = 0 \quad \text{on } \partial\Omega.$$

Finally, (3) and the properties of the Leray-Schauder degree imply the solvability of (P).

Step 2: It remains to prove (2). Assume the contrary. Then there exists a sequence of numbers $\{\tau_k\}_{k=1}^\infty \subset [0,1]$ and a corresponding sequence of functions $\{u_k\}_{k=1}^\infty \subset H_p^s(\Omega)$ such that u_k satisfies (P_{τ_k}) and $\|u_k | L_\infty\| \to \infty$ as $k \to \infty$. Without loss of generality we may suppose that $\|u_k | L_\infty\| > 0$ for all $k \in \mathbf{N}$. Now we define the functions w_k by $w_k = u_k / \|u_k | L_\infty\|$. Consequently, we obtain

$$Aw_k = q_k + f_k \quad \text{in } \Omega, \qquad Bw_k = 0 \quad \text{on } \partial\Omega. \tag{4}$$

Here we put

$$q_k = (\lambda_1 + \tau_k\kappa)w_k + (1-\tau_k)\frac{g(u_k)}{\|u_k | L_\infty\|}$$

and

$$f_k = (1-\tau_k)\frac{f}{\|u_k | L_\infty\|}.$$

We may assume that $\tau_k \to \tau \in [0,1]$. By our assumptions there exists $\sigma, -1 < \sigma < 0$, such that $f \in B^{\sigma}_{\infty,\infty}(\Omega)$. Now the linear growth condition on g and the mapping properties show that right-hand side of (4) is bounded in $B^{\sigma}_{\infty,\infty}(\Omega)$, independently of k. Note that $\|f_k|B^{\sigma}_{\infty,\infty}\| < c_1$ and $\|q_k|B^{\sigma}_{\infty,\infty}\| \le c'\|q_k|L_\infty\| \le c_2$. Thus we obtain the estimate $\|Aw_k|B^{\rho}_{\infty,\infty}\| < M$ for some $M > 0$, independently of $k \in \mathbf{N}$. Therefore, compactness arguments show that $w_k \to w$ as $k \to \infty$ in the $C^1(\bar{\Omega})$ norm by passing to a subsequence if necessary. Clearly, $\|w|L_\infty\| = 1$. Applying the arguments from the proof of [8, Subsection in 6.4.5, Theorem 1] we derive that there is a $q \in C(\bar{\Omega})$ which satisfies $\lambda_1 \le q < \lambda_2$ in Ω, and w satisfies $(\mathrm{P_q})$, i.e., we have

$$Aw = qw \quad \text{in } \Omega, \qquad Bw = 0 \quad \text{on } \partial\Omega.$$

Since $\|w|L_\infty\| = 1$, it follows from Corollary in Section 2 that $w = \pm\varphi_1$. Thus we can apply condition (GLL) to $u_k/\|u_k|L_\infty\|$. Because of the definition of w_k and the properties of φ_1, we may assume that for all $k \ge K > 0$ the function u_k is either strictly positive and $\lim_{k\to\infty} u_k = +\infty$ for all $x \in \Omega$, or strictly negative and $\lim_{k\to\infty} u_k = -\infty$ for all $x \in \Omega$. We suppose that the first alternative holds, the other case can be handled similarly. Now we compute the L_2 inner product of P_{τ_k} with φ_1 and simplify. Then we get

$$0 = \tau_k \kappa \int_\Omega u_k(x)\varphi_1(x)dx + (1-\tau_k)\int_\Omega (g(u_k(x) + f(x))\varphi_1(x)dx. \tag{5}$$

It follows that

$$0 > \int_\Omega (g(u_k(x) + f(x))\varphi_1(x)dx \tag{6}$$

which contradicts (GLL).

A careful look at our arguments reveals that an a priori bound has been established for $\tau \in (0,1)$ and that is trivial to include the case $\tau = 0$. However, it is possible that the solution set corresponding to $\tau = 1$ is unbounded, as it is in the linear case, where $g \equiv 0$ and $\int_\Omega g(x)\varphi_1(x)dx = 0$. Thus we are left with the possibilities that there are infinitely many solutions, and the proof is finished, or that there is an a priori bound on the solutions for all $\tau \in (0,1)$. Thus (1) is proved, and by the first step we can finish the proof of our theorem.

Some remarks and examples

REMARK 1. In S. Ahmad [1], the following two point boundary value problem was considered

$$-u''(x) - u(x) = g(u(x)) + f(x), \quad x \in (0,\pi), \qquad u(0) = u(\pi) = 0, \tag{7}$$

where $f \in L_1(0,\pi)$. It was proved that if g satisfies a linear growth condition of the type

$$|g(t)| \leq c_1 + c_2|t|,$$

where $c_1 > 0$ and $0 < c_2 < 3$, then (7) is solvable if the following Landesman-Lazer condition is satisfied:

$$g_- \int_0^\pi \sin x dx < - \int_0^\pi f(x) \sin x dx < g_+ \int_0^\pi \sin x dx, \qquad \text{(LL}^*\text{)}$$

where the finite or infinite values g_- and g_+ are defined by

$$\limsup_{t \to -\infty} g(t) = g_-, \qquad \liminf_{t \to +\infty} g(t) = g_+.$$

Since the boundary value problem

$$-u''(x) - u(x) = 3u(x) + \sin 2x, \quad x \in (0,\pi), \qquad u(0) = u(\pi) = 0,$$

has no solution, the growth condition (1) in Section 1 is sharp. Observe that in this case $\lambda_2 - \lambda_1 = 3$, where λ_1 and λ_2 are the first two eigenvalues of

$$-u''(x) = \lambda u(x), \quad x \in (0,\pi), \qquad u(0) = u(\pi) = 0,$$

i.e., the distance between λ_2 and λ_1 limits the linear growth of the nonlinear term g, see also P. Drábek [3].

REMARK 2. The n-dimensional analogue of this assertion was proved by Ahmad [2]. Consider the condition of Landesman-Lazer type

$$g_- \int_\Omega \varphi_1(x) dx < - \int_\Omega f(x)\varphi_1(x) dx < g_+ \int_\Omega \varphi_1(x) dx, \qquad \text{(LL}^{**}\text{)}$$

where $g_\pm$ are defined as before. Assume that there is a constant $r_0 > 0$ such that

$$\frac{g(t)}{t} < \lambda_2 - \lambda_1 \qquad \text{if } |t_0| \geq r_0. \tag{8}$$

It is not hard to check that these conditions which are used in [2] imply (GLL) in the nondegenerate case. Thus we can extend the Landesman-Lazer condition (LL**) to degenerate boundary conditions. Note that the lower bound $\liminf_{|t| \to \infty} g(t)/t \geq 0$ is implicit in (LL**), but not in (GLL).

REMARK 3. One can prove that if $g_\pm$ exist or are infinite, and

$$g_- < g(t) < g_+ \qquad \text{for all real } t,$$

then (LL**) is also necessary for the solvability of (P).

REMARK 4. Note that the growth condition

$$\limsup_{|t|\to\infty} g(t)/t < \lambda_2 - \lambda_1$$

cannot be improved. This follows from the fact that

$$Au - \lambda_2 u = f \quad \text{in } \Omega, \qquad Bu = 0 \quad \text{on } \partial\Omega$$

is solvable if and only if the Fredholm condition $\int_\Omega f(x)\varphi_2(x)dx = 0$ for every eigenfunction $\varphi_2 \in \ker_B(A - \lambda_2)$ holds. Now we choose $g(t) = (\lambda_2 - \lambda_1)t$.

Furthermore, one can give examples for which the set of function f satisfying (LL**) may be empty. The next result is an analogue to [8, Subsection 6.4.5, Theorem 2], and can be proved similarly.

COROLLARY. *Let $s > n/p$, $\rho > -1$, and let g be the smooth function from Theorem which satisfies the following additional properties.*
(i) *The finite limits $G_- = \liminf_{t\to-\infty} tg(t)$ and $G_+ = \liminf_{t\to+\infty} tg(t)$ exist.*
(ii) $G_\pm > 0$.

Let $f \in H_p^{s-2}(\Omega) \cap B_{\infty,\infty}^{\rho}(\Omega)$ with $\int_\Omega f(x)\varphi_1(x)dx = 0$. Then (P) *has at least one solution $u \in H_p^s(\Omega)$.*

REMARK 5. Let $r_0 > 0$ be a constant. Suppose that $g(t)t \geq 0$ for all $|t| \geq r_0$. Then the proof shows that one can replace (ii) by $G_\pm \geq 0$.

REMARK 6. Finally, we remark that one can prove analogous results in the framework of the two scales of function spaces of Besov-Triebel-Lizorkin type which cover many classical function spaces. We refer to [6] and [8, 6.4], where it was done in the case of nondegenerate boundary value problems.

References

[1] Ahmad, S., A resonance problem in which the nonlinearity may grow linearly, Proc. Amer. Math. Soc. **92** (1984), 381–384.
[2] Ahmad, S., Nonselfadjoint resonance problems with unbounded perturbations, Nonlinear Analysis TMA **10** (1986), 147–156.
[3] Drábek, P., On the resonance problem with nonlinearity which has arbitrary linear growth, J. Math. Anal. Appl. **127** (1987), 435–442.
[4] Manes, A., Micheletti, A. M., Un' estensione della teoria variazionale classica degli autovalori per operatori elliticci del secondo ordine, Boll. Un. Mat. Ital. **7** (1973), 285–301.
[5] Robinson, S. B., Landesman, E. M., A general approach to solvability conditions for semilinear elliptic boundary value problems at resonance, Diff. Int. Equations **8** (1995), 1555–1569.
[6] Robinson, S. B., Runst, T., Solvability conditions for semilinear elliptic boundary value problems at resonance with bounded and unbounded nonlinear terms, Advances in Diff. Equations **3** (1998), 595–624.

[7] Runst, T., Degenerate boundary value problems in Sobolev spaces of fractional order, Forschungserg. FSU Jena, Math/Inf/98/03 (1998).

[8] Runst, T., Sickel, W., Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. (de Gruyter Series in Nonlinear Analysis and Applications, **3**) Berlin-New York: de Gruyter 1996.

[9] Szulkin, A., On the number of solutions of some semilinear elliptic boundary value problems, Nonlinear Analysis TMA **6** (1982), 95–116.

[10] Taira, K., Diffusion processes and partial differential equations. Boston-San Diego-New York-Berkeley-London-Sydney-Tokyo-Toronto: Academic Press 1988.

[11] Taira, K., Analytic semigroups and semilinear initial boundary value problems. (London Mathematical Society Lecture Notes Series, **223**) London-New York: Cambridge University Press 1995.

[12] Taira, K., The Yamabe problem and nonlinear boundary value problems, J. Differential Equations **122** (1995), 316–372.

[13] Taira, K., Bifurcation for nonlinear elliptic boundary value problems I, Collectanea Mathematica **47** (1996), 207–229.

[14] Taira, K., Umezu, K., Bifurcation for nonlinear elliptic boundary value problems II, Tokyo J. Math. **19** (1996), 381–396.

[15] Taira, K., Umezu, K., Bifurcation for nonlinear elliptic boundary value problems III, Advances in Diff. Equations **1** (1996), 709–727.

[16] Triebel, H., Theory of function spaces. Basel Boston Stuttgart: Birkhäuser 1983; and Leipzig: Akad. Verlagsges. Geest & Portig 1983

[17] Wloka, J., Partielle Differentialgleichungen. Stuttgart: Teubner 1982.

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