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# A holomorphic structure on a homogeneous space of the diffeomorphism group of a circle

Hideo Doi

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Abstract. We consider a holomorphic structure on a homogenous space of the diffeomorphism group of a circle using the theory of quasiconformal mappings.

## Introduction

The geometric quantization provides interesting pictures of infinite dimensional homogeneous spaces of the Fréchet Lie group $\mathit{Diff}^+ S^1$ of orientation preserving diffeomorphisms of a circle [5], [11], [18]. From the structure of the Lie algebra of $\mathit{Diff}^+ S^1$, it follows that $T^1\backslash \mathit{Diff}^+ S^1$ and $PSU(1,1)\backslash \mathit{Diff}^+ S^1$ have invariant almost complex structures formally satisfying the integrability condition. This naturally leads to a question whether $T^1\backslash \mathit{Diff}^+ S^1$ and $PSU(1,1)\backslash \mathit{Diff}^+ S^1$ are homogeneous complex manifolds in a usual sense or not (cf. [13], [16]). In contrast with a finite dimensional Lie group or its loop group, the Lie group $\mathit{Diff}^+ S^1$ has no analytic structure and the exponential mapping cannot be a local isomorphism around the origin (cf. [12]). In order to overcome the undesirable property H. Omori introduced the concept of IHL-Lie groups and developed the infinite dimensional differential geometry [14], [15]. As he pointed out, the Frobenius theorem does not hold in the category of Fréchet manifolds in general. Thus in this paper we take a direct approach, that is to say, we construct a holomorphic coordinate system, which is closely related to the analytic realization of $\mathit{Diff}^+ S^1/T^1$ by A. A. Kirillov [10]. Our construction is based on the theory of quasiconformal mappings. Especially the variational formula by L. V. Ahlfors and L. Bers for the solutions of the Beltrami equation enables us to analyze the differential of a coordinate map.

In § 1 using the inverse function theorem of Nash and Moser, we consider a smooth structure of the homogeneous space. In § 2 as preliminaries, we review basic methods to solve the Beltrami equation and derive some

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formulas. In §3 we shall prove that $T^1\backslash Diff^+ S^1$ is diffeomorphic to an open submanifold of the Fréchet space $\{f \in C^\infty(S^1, \mathbf{C});\ f(e^{i\theta}) = \sum_{n<0} f_n e^{in\theta}\}$ and that the action of $Diff^+ S^1$ is holomorphic with respect to the natural complex structure. Via the projection: $T^1\backslash Diff^+ S^1 \to PSU(1,1)\backslash Diff^+ S^1$ we also show that $PSU(1,1)\backslash Diff^+ S^1$ is a homogeneous complex Fréchet manifold.

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## 1. The smooth structure

Let $G$ be a Lie group and $H$ its closed Lie subgroup. Let $U$ be a neighborhood of the unit element in $G$ and set $U_H = U \cap H$. We assume that there exists a regular submanifold $B \subset G$ such that $B \cdot B^{-1} \subset U$ and a mapping $U_H \times B \to G$ induced by the multiplication $(h, b) \mapsto h \cdot b$ is an open embedding. Let $h_i \in H$ and $b_i \in B$ $(i = 1, 2)$. If $h_1 \cdot b_1 = h_2 \cdot b_2$, then

$$h_2^{-1} h_1 = b_2 b_1^{-1} \in U_H \cap (B \cdot B^{-1}) = \{1\}.$$

Hence the canonical projection: $B \to H\backslash G$ is open and injective. Then we can define a smooth structure on the quotient space $H\backslash G$ with the action of $G$.

Elements of $PSU(1,1)$ act on $S^1$ as linear fractional transformations. Thus $PSU(1,1)$ can be regarded as a closed Lie subgroup of $Diff^+ S^1$. For

$$V = \left\{ v \in C^\infty(S^1, \mathbf{R});\ v(\theta) = v(e^{i\theta}) = \sum\nolimits_{|n|>1} v_n e^{in\theta} \right\},$$

we define a smooth map $\psi : V \to C^\infty(S^1, S^1)$ by

$$\psi(v)(e^{i\theta}) = e^{i(\theta + v(\theta))}$$

and we consider a neighborhood $B$ of $0 \in V$ such that $\psi(B) \subset Diff^+ S^1$.

Lemma 1.1. *Let $\Psi(h, b) = h \circ \psi(b)$ for $h \in PSU(1,1)$ and $b \in B$. Then $\Psi : PSU(1,1) \times B \to Diff^+ S^1$ gives a diffeomorphism of a neighborhood of $(1, 0)$ in $PSU(1,1) \times B$ onto a neighborhood of $1$ in $Diff^+ S^1$.*

Proof. The Fréchet manifold $Diff^+ S^1$ is modeled on $C^\infty(S^1, \mathbf{R})$ by definition. Since the tangent bundle of $S^1$ is trivial, we can identify the tangent bundle of $Diff^+ S^1$ with $Diff^+ S^1 \times C^\infty(S^1, \mathbf{R})$. Let $\Psi_{*(h,b)}$ denote the differential of $\Psi$ at $(h, b) \in PSU(1,1) \times B$. Then for $v \in V$, a tangent vector $\Psi_{*(h,b)}(0, v) = \partial_t \Psi(h, b + tv)|_{t=0}$ is computed by

$$-i\partial_t \log h(e^{i(\theta + b(\theta) + tv(\theta))})|_{t=0} = -i((\partial_\theta \log h) \circ \psi(b)) \cdot v.$$

For $X(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathfrak{su}(1,1)$,

$$\Psi_{*(h,b)}(R_{h*}X(\alpha,\beta),0) = (\partial_t \exp tX(\alpha,\beta)) \circ g|_{t=0} = 2\,Im\,(\alpha + \beta\bar{g}),$$

where $g = \Psi(h,b)$ and $R_h$ is the right translation on $PSU(1,1)$. By associating $(X(\alpha,\beta),v) \in \mathfrak{su}(1,1) \times V$ with a function:

$$S^1 \ni e^{i\theta} \mapsto 2\,Im\,(\alpha + \beta e^{-i\theta}) + v(e^{i\theta}) \in \mathbf{R},$$

we identify $\mathfrak{su}(1,1) \times V$ with the tangent space $C^\infty(S^1,\mathbf{R})$ hereafter. For $h \in PSU(1,1)$, $b \in B$, $\alpha \in i\mathbf{R}, \beta \in \mathbf{C}$, we define smooth functions on $S^1$ as follows:

$$\eta(h,b) = -i(\partial_\theta \log h) \circ \psi(b),$$

$$\delta(h,b) = 1/\eta(h,b) - 1,$$

$$\varepsilon(h,b,\alpha,\beta) = 2\,Im\,(\alpha + \beta e^{-i\theta})\delta(h,b)(\theta) + 2\,Im\,(\beta(\bar{g} - e^{-i\theta})/\eta(h,b)(\theta).$$

We set for $f \in C^0(S^1,\mathbf{R})$

$$F_{h,b}(f) = f + \varepsilon(h,b,\alpha,\beta)$$

with

$$\alpha = i\int_0^{2\pi} f(\theta)d\theta/4\pi, \qquad \beta = i\int_0^{2\pi} f(\theta)e^{i\theta}d\theta/2\pi.$$

Then $F_{h,b}(f) \in C^\infty(S^1,\mathbf{R})$ implies that $f \in C^\infty(S^1,\mathbf{R})$. Under the identification above of tangent vectors, we have

$$\Psi_{*(h,b)}(f) = \eta(h,b)F_{h,b}(f)$$

for $f \in C^\infty(S^1,\mathbf{R})$. Moreover if a neighborhood $U \times B$ of $(1,0) \in PSU(1,1) \times V$ is sufficiently small, we may assume that

$$F_{h,b} : C^\infty(S^1,\mathbf{R}) \to C^\infty(S^1,\mathbf{R})$$

is bijective for all $(h,b) \in U \times B$. Let $\|\phi\|_n = \sup \sum_{0 \le k \le n} |\partial_\theta^k \phi|$ for $\phi \in C^\infty(S^1,\mathbf{R})$. Then

$$\|F_{h,b}(f)\|_n \le c_n\|f\|_n$$

with constants $c_n$ independent of $f$ $(n = 0,1,2,\ldots)$. That is to say, $F_{h,b}$ is a tame linear map [9]. Because a composition of tame maps is also tame [9, Theorem 2.1.6], we see that $\Psi_{*(h,b)}(f) = \eta(h,b)F_{h,b}(f)$ is a smooth tame map: $U \times B \times C^\infty(S^1,\mathbf{R}) \to C^\infty(S^1,\mathbf{R})$. We have also

$$\|f\|_n \le \|F_{h,b}(f)\|_n + \|\varepsilon(h,b,\alpha,\beta)\|_n \le \|F_{h,b}(f)\|_n + c_n'\|f\|_0,$$

where $c_n'$ is a constant dependent on $U, B$ and $n$. If $U \times B$ is so small that $c_0' \leq 1/2$, then $\|f\|_0 \leq 2\|F(f)\|_0$. Therefore

$$\|f\|_n \leq (1 + 2c_n')\|F(f)\|_n.$$

Hence the family of inverses $(F_{h,b})^{-1} : U \times B \times C^\infty(S^1, \mathbf{R}) \to C^\infty(S^1, \mathbf{R})$ is tame. From this and [9, Theorem 3.1.1] it follows that the family of inverses $\Psi_{*(h,b)}^{-1} : U \times B \times C^\infty(S^1, \mathbf{R}) \to C^\infty(S^1, \mathbf{R})$ is a smooth tame map. Thus the inverse function theorem [9, Theorem III. 1.1.1] implies that $\Psi$ is locally invertible and each local inverse $\Psi^{-1}$ is a smooth tame map.

We now see that $PSU(1,1)\backslash \mathit{Diff}^+ S^1$ is a smooth Fréchet manifold. Also the same argument shows that $T^1 \backslash \mathit{Diff}^+ S^1$ is a smooth Fréchet manifold, where $T^1$ denotes the subgroup of $\mathit{Diff}^+ S^1$ consisting of rotations.

## 2. The quasiconformal mappings

To begin with, we review the formulas by L. V. Ahlfors and L. Bers [4] (cf. [3]). Let $\Omega$ denote the whole plane. We consider the operators:

$$(Pg)(z) = \frac{1}{2\pi i} \int_\Omega g(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta} \right) d\zeta d\bar{\zeta}$$

$$(Tg)(z) = \frac{1}{2\pi i} \int_\Omega \frac{g(\zeta) - g(z)}{(\zeta - z)^2} d\zeta d\bar{\zeta} \qquad \text{(the Cauchy principal value).}$$

Then $P$ and $T$ are well-defined on $L_p(\Omega)$ and that the relations:

$$\partial_{\bar{z}}(Pg) = g \qquad \text{and} \qquad \partial_z(Pg) = Tg$$

hold in the distributional sense, where $\partial_z$ and $\partial_{\bar{z}}$ denote the usual differential operators $\frac{1}{2}(\partial_x - i\partial_y)$ and $\frac{1}{2}(\partial_x + i\partial_y)$ with the standard coordinate $z = x + iy$ of $\Omega$, respectively [4, Lemma 3] (cf. [3, Chapter V]). Let $C_p$ denote the operator norm of $T$ on $L_p(\Omega)$. Then

$$\lim_{p \to 2} C_p = 1$$

[4, Lemma 4] (cf. [3, Chapter V, Section D]). Following [4], we introduce the Banach space $B_p$ of functions $\omega$, defined on the whole plane, which satisfy a global Hölder condition of order $1 - 2/p$, which vanish at the origin, and whose generalized derivatives $\partial_z \omega$ and $\partial_{\bar{z}} \omega$ exist and belong to $L_p(\Omega)$. The norm is defined by

$$\|\omega\|_{B_p} = \sup \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^{1-2/p}} + \|\partial_z \omega\|_p + \|\partial_{\bar{z}} \omega\|_p.$$

If $\sigma \in L_p(\Omega)$ and $\mu \in L_\infty(\Omega)$ with $\|\mu\|_\infty C_p < 1$, then the inhomogeneous Beltrami equation

$$\partial_{\bar{z}}\omega = \mu\partial_z\omega + \sigma \tag{2.1}$$

has a unique solution $\omega(\mu, \sigma) \in B_p$ [4, Theorem 1]. Using $T_\mu(g) = T(\mu g)$ for $g \in L_p(\Omega)$, we set $q = \sum_{n\geq 0} T_\mu^n T\sigma$. Then the solution above is given by

$$\omega(\mu, \sigma) = P(\mu q + \sigma)$$

[4, Theorem 4, Proof]. Furthermore the following lemma is an immediate consequence of the variational formula [4, Theorem 2].

LEMMA 2.2. *If $\mu_t \in L_\infty(\Omega)$ and $\sigma_t \in L_p(\Omega)$ are smooth functions of a real parameter $t$ and if $\|\mu_t\|_\infty C_p < 1$, then $\omega_t = \omega(\mu_t, \sigma_t) \in B_p$ is also a smooth function of $t$. Moreover its derivative is given by*

$$\partial_t\omega_t = \omega(\mu_t, \partial_t\mu_t \cdot q_t + \partial_t\sigma_t).$$

Let $\mu \in L_\infty(\Omega)$ with $\|\mu\|_\infty < 1$. A continuous solution of the Beltrami equation

$$\partial_{\bar{z}}f = \mu \cdot \partial_z f$$

is said to be $\mu$-conformal if $\partial_z f$ is locally of class $L_2$. Assume that $\mu \in L_\infty(\Omega)$ has compact support and $\|\mu\|_\infty C_p < 1$. Then we define $f^\mu$ by

$$f^\mu = z + \omega(\mu, \mu),$$

which is a unique solution of $\partial_{\bar{z}}f = \mu\partial_z f$ with $f(0) = 0$ and $\partial_z f - 1 \in L_q(\Omega)$ for some $q > 2$ [4, Theorem 4]. In particular, $f^\mu$ is a $\mu$-conformal mapping. Moreover $f^\mu$ is a homeomorphism of the whole plane onto itself [4, Lemma 8].

COROLLARY 2.3. *Let $K \subset \Omega$ be a closure of a bounded domain with the smooth boundary. Let $\mu \in L_\infty(\Omega)$ with* support $\mu \subset K$ *and $\|\mu\|_\infty C_p < 1$. Let $U$ be an open subset of $\Omega\backslash\{0\}$. If the restriction $\mu|U \cap K$ is smooth, then $f^\mu|K$ is also smooth on $U \cap K$.*

PROOF. Take a smooth one-parameter family $\phi_t$ of diffeomorphisms of $\Omega$ such that $\phi_t(K) = K$ and support $\phi_t \subset U$. We set

$$\mu_t = (\partial_{\bar{z}}\varphi_t + \mu \circ \varphi_t \cdot \overline{\partial_z\varphi_t})/(\partial_z\varphi_t + \mu \circ \varphi \cdot \partial_z\overline{\varphi_t}).$$

Then

$$\partial_{\bar{z}}(f^\mu \circ \varphi_t) = \mu_t \cdot \partial_z(f^\mu \circ \varphi_t).$$

Also we have $f^\mu \circ \varphi_t(0) = 0$ and $\partial_z(f^\mu \circ \varphi_t) - 1 \in L_q(\Omega)$. Therefore $f^\mu \circ \varphi_t =$

$f^{\mu_t}$. Hence $f^{\mu} \circ \varphi_t \in B_p$ is a smooth function of $t$ by Lemma 2.2. In particular, for $z \in \Omega$, $f^{\mu}(\varphi_t(z))$ depends smoothly on $t$.

LEMMA 2.4. *Let $\nu$ and $\mu \in L_\infty(\Omega)$ with a compact support. Let $0 < \kappa < 1$ satisfy $\kappa C_p < 1$ for some $p > 3$. Assume that $\|\nu\|_\infty, \|\mu\|_\infty, \|\nu + \mu\|_\infty \leq \kappa$ and set*

$$\lambda = \left(\frac{\nu}{1 - \bar{\mu}(\nu + \mu)} \cdot \frac{\partial_z f^{\mu}}{\overline{\partial_z f^{\mu}}}\right) \circ (f^{\mu})^{-1}.$$

*If $\|\lambda\|_\infty \leq \kappa$, then*

$$f^{\nu+\mu} = f^{\lambda} \circ f^{\mu}.$$

PROOF. Let $\varphi = f^{\lambda} \circ f^{\mu}$. Then $\partial_{\bar z}\varphi/\partial_z\varphi = \nu + \mu$. From [4, Lemma 10], it follows that $\partial_z\varphi$ is locally of class $L_r$ with $r = p^2/(2p-2) > 2$. Since $\partial_{\bar z}\omega(\mu, \mu) = 0$ near $\infty$ and $\partial_z\omega(\mu, \mu) \in L_p(\Omega)$ with $p > 2$, we see that $\omega(\mu, \mu)_z = O(1/|z|^2)$ as $z \to \infty$. Therefore $\partial_z\varphi = (\partial_z f^{\lambda} \circ f^{\mu}) \cdot \partial_z f^{\mu} = 1 + O(1/|z|^2)$ as $z \to \infty$. Hence $\partial_z\varphi - 1 \in L_r(\Omega)$. The lemma now follows from the uniqueness theorem for the solutions of the Beltrami equation.

LEMMA 2.5. *Let $K \subset \Omega$ be a closure of a bounded domain with the $C^1$-boundary. Let $\nu \in L_\infty(\Omega)$ with* support $\nu \subset K$ *satisfy $\|\nu\|_\infty < 1$ and $\|\nu\|_\infty C_p < 1$ for some $p > 3$. Assume that the restriction $\nu|K$ is of class $C^1$ around a point $m \in K$. Then the Jacobian $Jf^{\nu}$ does not vanish at $m$.*

PROOF. Set $\varphi = (f^{\nu})^{-1}$ and $\lambda = (-\nu \cdot \partial_z f^{\nu}/\overline{\partial_z f^{\nu}}) \circ \varphi$. Then $\varphi = f^{\lambda} \in L_p(\Omega)$ by Lemma 2.4. Since $\varphi(f^{\nu}(z)) = z$, we have

$$\begin{pmatrix} \partial_z f^{\nu}(z) & \partial_z \overline{f^{\nu}(z)} \\ \partial_{\bar z} f^{\nu}(z) & \partial_{\bar z} \overline{f^{\nu}(z)} \end{pmatrix} \begin{pmatrix} \partial_z \varphi(f^{\nu}(z)) \\ \partial_{\bar z} \varphi(f^{\nu}(z)) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore $Jf^{\nu}(z) \cdot \partial_z\varphi(f^{\nu}(z)) = \partial_{\bar z}\overline{f^{\nu}(z)}$. Also $Jf^{\nu} = |\partial_z f^{\nu}|^2 - |\partial_{\bar z} f^{\nu}|^2 = (1 - |\nu|^2)|\partial_z f^{\nu}|^2$. Hence, for an open subset $U \subset \Omega$,

$$\begin{aligned} \int_{f^{\nu}(U)} |\partial_z\varphi|^p dxdy &= \int_U |\partial_z\varphi(f^{\nu}(z))|^p Jf^{\nu}(z) dxdy \\ &= \int_U |\partial_z f^{\nu}|^p (Jf^{\nu})^{1-p} dxdy \\ &= \int_U (1 - |\nu|^2)^{1-p} |\partial_z f^{\nu}|^{2-p} dxdy. \end{aligned}$$

Thus we see that

$$(2.6) \qquad \int_U |\partial_z f^\nu|^{2-p}\,dxdy < \infty.$$

We now take a coordinate system $(U, x_1, x_2)$ around $m \in K$ such that $x_i(m) = 0$ $(i = 1, 2)$. Assume that $Jf^\nu(m) = 0$, then $\partial_z f^\nu(m) = \partial_{\bar z} f^\nu(m) = 0$. In view of Corollary 2.3, we have an expansion $\partial_z f^\nu = a_1 x_1 + a_2 x_2 + o(|x_1| + |x_2|)$ with complex constants $a_i$ $(i = 1, 2)$. But this contradicts (2.6) for $2 - p < -1$.

LEMMA 2.7. *Let $K \subset \Omega$ be a closure of a bounded domain with the smooth boundary. Let $\mu \in L_\infty(\Omega)$ with* support $\mu \subset K$ *and $\|\mu\|_\infty < 1$. If $\mu|K$ is smooth, then $f^\mu|\partial K$ is a smooth embedding.*

PROOF. Fix $p > 3$ and $0 < \kappa < 1$ satisfying $\kappa C_p < 1$. Taking an integer $N \geq \|\mu\|_\infty / \kappa(1 - \|\mu\|_\infty^2)$, we set $\nu = \mu/N$. Then $f^\nu|\partial K$ is a smooth embedding by Corollary 2.3 and Lemma 2.5. For an integer $n < N$, we assume that $f^{n\nu}|\partial K$ is a smooth embedding, and we set

$$\lambda = \left(\frac{\nu}{1 - n\bar\nu(n+1)\nu} \cdot \frac{\partial_z f^{n\nu}}{\overline{\partial_z f^{n\nu}}}\right) \circ (f^{n\nu})^{-1}.$$

Then we see that $f^\lambda | f^{n\nu}(\partial K)$ is smooth because $\lambda$ is smooth and $\|\lambda\|_\infty \leq \kappa$. Thus $f^{(n+1)\nu} = f^\lambda \circ f^{n\nu}$ is also a smooth embedding of $\partial K$.

LEMMA 2.8. *Let $K \subset \Omega$ be a closure of a bounded domain with the smooth boundary. Then*

$$\mu \mapsto f^\mu$$

*is a smooth mapping:* $\{\mu \in C^\infty(K, \mathbf{C});\ \sup|\mu| < 1\} \to C^\infty(K, \mathbf{C})$.

PROOF. Let $\mathcal{M}^{n,\alpha} = \{\mu \in C^{n,\alpha}(K, \mathbf{C});\ \sup|\mu| < 1\}$ with $0 < \alpha < 1$, where $C^{n,\alpha}(K, \mathbf{C})$ is the Banach space of functions of class $C^n$ in $K$ whose partial derivatives of order $n$ satisfy a Hölder condition with an exponent $\alpha$ (cf. [17]). We set

$$V = \left\{ f \in C^{n+1,\alpha}(K, \mathbf{C});\ \int_{\partial K} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 0 \text{ for } \forall z \in \operatorname{Int} K \right\}.$$

In this setting, C. J. Earle shows that there exist neighborhoods of zero $V_0 \subset V$ and $U_0 \subset \mathcal{M}^{n,\alpha}$ such that a mapping $V_0 \to U_0$:

$$f \mapsto \partial_{\bar z} f / (1 + \partial_z f)$$

is diffeomorphic [6, Theorem 1, Proof]. Because $\omega(\mu, \mu) = P(\mu q + \mu)$ with $q = \sum_{n \geq 0} T_\mu^n T\mu$ and $V = P(C^{n,\alpha}(K, \mathbf{C}))$, we see that if $\mu \in \mathcal{M}^{n,\alpha}$ is sufficiently small, then $\omega(\mu, \mu) \in V_0$. Therefore in view of Lemma 2.4, we see that the mapping $\mathcal{M}^{n,\alpha} \to C^{n+1,\alpha}(K, \mathbf{C})$ is smooth.

## 3. The invariant holomorphic structures

Let $F$ denote a closed subspace:

$$\left\{\varphi \in C^{\infty}(S^1, \mathbf{C});\ \varphi(e^{i\theta}) = \sum_{n\leq 0} \varphi_n e^{in\theta},\ \varphi_n \in \mathbf{C}\right\}$$

of the Fréchet space $C^{\infty}(S^1, \mathbf{C})$. We also regard $\varphi(e^{i\theta}) = \sum_{n\leq 0} \varphi_n e^{in\theta} \in F$ as a holomorphic function $\varphi(z) = \sum_{n\leq 0} \varphi_n z^n$ on $D_{\infty} = \{z \in \Omega; |z| \geq 1\}$. We introduce here a smooth submanifold of $C^{\infty}(S^1, \mathbf{C})$, which plays an important role in our construction of a holomorphic coordinate system of $\mathit{Diff}^+ S^1$. Let $\mathscr{F}$ denote the set consisting of injective holomorphic functions $f$ on $D_{\infty}$ which satisfy the following conditions;

( i ) $f$ has a form $z + \varphi(z)$ with $\varphi \in F$,

(ii) $f|\partial D_{\infty}$ is an embedding,

(iii) $0 \notin f(D_{\infty})$.

Then $\mathscr{F}$ can be identified with an open submanifold of $F$.

Let $D_0 = \{z \in \Omega; |z| \leq 1\}$. For $g \in \mathit{Diff}^+ S^1$, we take a homeomorphism $\tilde{g}$ of $D_0$ satisfying $\tilde{g} = g$ on $\partial D_0$ and $\tilde{g}(0) = 0$. Moreover we assume that $\tilde{g}$ is differentiable almost everywhere and smooth on some collar neighborhood of $\partial D_0$. Let measurable functions $\lambda$ and $\mu$ satisfy

$$\tilde{g}^*|dz|^2 = \lambda|dz + \mu d\bar{z}|^2. \tag{3.1}$$

Putting $\mu = 0$ on $\Omega \backslash D_0$, we obtain $\mu \in L_{\infty}(\Omega)$ with $\|\mu\|_{\infty} < 1$. Let $f^{\mu}$ be the $\mu$-conformal mapping such that $f^{\mu}(0) = 0$ and $\partial_z f^{\mu} - 1 \in L_p(\Omega)$ for some $p > 2$. We set

$$g_0 = f^{\mu} \circ \tilde{g}^{-1} \qquad \text{and} \qquad g_{\infty} = f^{\mu}|D_{\infty}.$$

Then Lemma 2.7 with (3.1) implies that

(3.2) $\quad g_i \in C^{\infty}(D_i)$ $(i = 0, \infty)$ is injective and holomorphic on $\operatorname{Int} D_i$,
$\quad g_0(0) = 0,\ g_{\infty} \in \mathscr{F}$ and $g_0 \circ g = g_{\infty}$ on $\partial D_0 = S^1$.

Assume that $g_i' \in C^{\infty}(D_i, \mathbf{C})$, $i = 0, \infty$ satisfy (3.2) for $g \in \mathit{Diff}^+ S^1$. From $g_0'^{-1} \circ g_{\infty}' = g = g_0^{-1} \circ g_{\infty}$ on $S^1$, it follows that $g_{\infty}' \circ g_{\infty}^{-1} = g_0' \circ g_0^{-1}$ on $g_0(S^1)$. Because $g_i' \circ g_i^{-1}$ is holomorphic on $g_i(D_i)$ $(i = 0, \infty)$, we have a holomorphic automorphism $\iota$ of $\Omega$ such that

$$\iota|g_i(D_i) = g_i' \circ g_i^{-1} \qquad (i = 0, \infty).$$

Since $g_0' \circ g_0^{-1}(0) = 0$ and $g_{\infty}' \circ g_{\infty}^{-1}(z) = z + a_0 + a_1/z + a_2/z^2 + \cdots$, we see that $\iota(z) = z$. Thus $g_i' = g_i$ $(i = 0, \infty)$. In particular, $g_{\infty}$ is independent of the choice of $\tilde{g}$. For these reasons, we call the pair $(g_0, g_\infty)$ in (3.2) *LU-decomposition* of $g \in Diff^+ S^1$.

Let $(h_0, h_\infty)$ and $(g_0, g_\infty)$ be the LU-decomposition for $h$ and $g \in Diff^+ S^1$, respectively. We now assume that $h_\infty = g_\infty$. Then $h_0^{-1} \circ g_0$ is a holomorphic automorphism of $D_0$ and $h_0^{-1} \circ g_0(0) = 0$. Hence $h \circ g^{-1} = h_0^{-1} \circ g_0 \in T^1 = \{z \in \mathbf{C}; |z| = 1\}$ (the group of rotations). By virture of this, we can define an injection $\Phi : T^1 \backslash Diff^+ S^1 \to \mathscr{F}$ by

$$\Phi(g) = g_\infty$$

with the LU-decomposition $(g_0, g_\infty)$ of $g \in Diff^+ S^1$.

LEMMA 3.2. *$\Phi$ is a smooth mapping between the Fréchet manifolds.*

PROOF. Let us fix $\delta \in C^\infty(\mathbf{R}, [0,1])$ such that $\delta = 0$ on $(-\infty, 1/3]$ and $\delta = 1$ on $[2/3, \infty)$. Let $g \in Diff^+ S^1$. Since $\mathbf{R}$ is the universal covering space of $S^1 = \mathbf{R}/2\pi\mathbf{Z}$, we can take $\gamma \in Diff^+ \mathbf{R}$ satisfying $e^{i\gamma(\theta)} = g(e^{i\theta})$. For $\varepsilon \in C^\infty(\mathbf{R}, \mathbf{R})$ satisfying $\varepsilon(\theta + 2\pi) = \varepsilon(\theta)$ and $\partial_\theta(\gamma + \varepsilon) > 0$, we define $\varphi(\varepsilon) \in Diff^+ D_0$ by

$$\varphi(\varepsilon)(re^{i\theta}) = re^{i(\delta(r)(\gamma(\theta)+\varepsilon(\theta))+(1-\delta(r))\theta)}, \quad (re^{i\theta} \in D_0).$$

Then we can regard $\varphi(\cdot)$ as a smooth mapping from a neighborhood $U$ of $1 \in Diff^+ S^1$ into $Diff^+ D_0$.

Let $(ds)^2 = a(dx)^2 + 2b\,dxdy + c(dy)^2$ be a smooth Riemannian metric on $D_0$ with a standard real coordinate system $z = x + iy$. If we set $(ds)^2 = \lambda|dz + \mu d\bar{z}|^2$, then we have the following formulas;

$$\lambda = \frac{a + c + 2\sqrt{ac - b^2}}{4},$$

$$\mu = \frac{a - c + 2ib}{4\lambda}.$$

Hence, setting

$$\varphi(\varepsilon)^*|dz|^2 = \lambda(\varepsilon)|dz + \mu(\varepsilon)d\bar{z}|^2,$$

we obtain a smooth mapping $\mu(\cdot)$ from the neighborhood $U$ to $\mathscr{M} = \{\mu \in C^\infty(D_0, \mathbf{C}); \sup|\mu| < 1\}$. Therefore in view of Lemma 2.8, we see that the mapping: $U \to \mathscr{F}$ defined by

$$\varepsilon \mapsto (e^{i\varepsilon}g)_\infty = f^{\mu(\varepsilon)}|D_\infty$$

is smooth. Since $\varphi(\varepsilon)(e^{i\theta}) = e^{i\varepsilon(\theta)}g(e^{i\theta})$, the mapping $\Phi(e^{i\varepsilon}g) = f^{\mu(\varepsilon)}|D_\infty$ is also smooth.

Let $f \in \mathscr{F}$ and let $K$ be a compact subset of $\Omega$ with a smooth boundary $\partial K = f(S^1)$. Taking a diffeomorphism

$$\tilde{f} : D_0 \to K \qquad \text{with } \tilde{f}|\partial D_0 = f \quad \text{and} \quad \tilde{f}(0) = 0,$$

we define $\mu \in \mathscr{M}$ by $\tilde{f}^*|dz|^2 = \lambda|dz + \mu d\bar{z}|^2$. Then we have a unique $\mu$-conformal diffeomorphism $W^\mu : D_0 \to D_0$ such that $W^\mu(0) = 0$ and $W^\mu(1) = 1$ [4, Theorem 4] (cf. [6, Theorem 2]). We now set

$$g_0 = \tilde{f} \circ (W^\mu)^{-1},$$
$$g = g_0^{-1} \circ f|S^1 = W^\mu|S^1.$$

By definition, we see that $g_0$ is holomorphic and that $g \in \mathit{Diff}^+ S^1$. Also $(g_0, f)$ is the LU-composition for $g$ and so that we have $\Phi(g) = f$. Therefore the mapping: $f \mapsto g$ from $\mathscr{F}$ to $T^1\backslash \mathit{Diff}^+ S^1$ is the inverse of $\Phi$.

LEMMA 3.4. *$\Phi^{-1}$ is smooth.*

PROOF. As in the proof of Lemma 3.3, we may assume that the extension $f \mapsto \tilde{f}$ from $\mathscr{F}$ to $C^\infty(D_0, \mathbf{C})$ is locally smooth. Then the mapping: $\mathscr{F} \to \mathscr{M}$ defined by

$$f \mapsto \mu \qquad \text{with } \tilde{f}^*|dz|^2 = \lambda|dz + \mu d\bar{z}|^2$$

is also locally smooth. Hence [6, Theorem 2] implies that the mapping $\mu \mapsto W^\mu$ from $\mathscr{M}$ to $C^\infty(D_0, \mathbf{C})$ is smooth, with the composition method as in our proof of Lemma 2.8.

THEOREM 3.5. *$T^1\backslash \mathit{Diff}^+ S^1$ is a homogeneous complex Fréchet manifold.*

PROOF. We identify $C^\infty(S^1, \mathbf{R})$ with the Lie algebra of $\mathit{Diff}^+ S^1$. Let $\mathfrak{m} = \{v \in C^\infty(S^1, \mathbf{C});\ v(e^{i\theta}) = \sum_{n \neq 0} v_n e^{in\theta},\ \overline{v_n} = v_{-n}\} \subset C^\infty(S^1, \mathbf{R})$. Then $\mathfrak{m}$ is identified with the tangent space at the origin of $T^1\backslash \mathit{Diff}^+ S^1$. Let $J$ be an invariant almost complex structure on $T^1\backslash \mathit{Diff}^+ S^1$ whose $(1,0)$-tangent vectors are involutive. We set

$$\mathfrak{m}^+ = \left\{ v \in C^\infty(S^1, \mathbf{C});\ v(e^{i\theta}) = \sum_{n>0} v_n e^{in\theta} \right\} \subset \mathfrak{m} \otimes_{\mathbf{R}} \mathbf{C}$$

and $\mathfrak{m}^- = \overline{\mathfrak{m}^+}$. Then $J|\mathfrak{m}^- = \pm i$. We now choose $J|\mathfrak{m}^- = i$. Lemmas 3.3-4 imply that $\Phi : T^1\backslash \mathit{Diff}^+ S^1 \to \mathscr{F}$ is a diffeomorphism. Hence, if we prove that

$$\Phi_{*g}(R_{g*}Jv) = i\Phi_{*g}(R_{g*}v) \qquad \text{for } g \in \mathit{Diff}^+ S^1 \quad \text{and} \quad v \in \mathfrak{m}^-,$$

where $R_g$ denotes the right translation, then the proof is complete.

Let $z = re^{i\theta}$ be the polar coordinates on $\Omega$. For measurable functions $\lambda$, $\rho \geq 0$ and $\mu$, if it holds that $(dr)^2 + (\rho r d\theta)^2 = \lambda |dz + \mu d\bar{z}|^2$, then

$$\mu(re^{i\theta}) = e^{2i\theta}(1-\rho)/(1+\rho). \tag{3.6}$$

Let $g \in \mathit{Diff}^+ S^1$ and $\gamma \in \mathit{Diff}^+ \mathbf{R}$ satisfy $g(e^{i\theta}) = e^{i\gamma(\theta)}$. For $v \in C^\infty(S^1, \mathbf{R})$, we set $\chi_t(\theta) = \theta + t \cdot v(\theta)$ and $\gamma_t = \chi_t \circ \gamma \in C^\infty(\mathbf{R}, \mathbf{R})$. Thus we define a smooth 1-parameter family $g_t \in \mathit{Diff}^+ S^1$ by

$$g_t = e^{i\gamma_t} = e^{i\chi_t} \circ g.$$

In addition we extend $g_t$ to a homeomorphisms $\tilde{g}_t$ of $D_0$ as $\tilde{g}_t(re^{i\theta}) = rg_t(e^{i\theta})$. Setting $\tilde{g}_t^*|dz|^2 = \lambda_t |dz + \mu_t d\bar{z}|^2$, we have

$$\mu_t(re^{i\theta}) = e^{2i\theta}\frac{1 - \partial_\theta \gamma_t}{1 + \partial_\theta \gamma_t}$$

by (3.6). In view of Lemma 2.4, we write $f^{\mu_t} = f^{\varepsilon_t} \circ f^\mu$ with $\mu = \mu_t|_{t=0}$. We also introduce two functions $v$ and $\delta$ on $D_0$ as follows;

$$v(re^{i\theta}) = -2e^{2i\theta}\frac{\partial_\theta(v \circ \gamma)}{(1 + \partial_\theta \gamma)^2},$$

$$\delta(re^{i\theta}) = \left(\frac{v}{1 - |\mu|^2}\frac{\partial_z f^\mu}{\overline{\partial_z f^\mu}}\right) \circ (f^\mu)^{-1}.$$

Then a direct calculation shows that $\partial_t \mu_t|_{t=0} = v(re^{i\theta})$ and $\partial_t \varepsilon_t|_{t=0} = \delta(re^{i\theta})$ (cf. [4, Lemma 21]). From [4, Theorem 2] it follows that

$$\partial_t f^{\varepsilon_t}|_{t=0} = \omega(0, \delta) = P\delta = \frac{1}{2\pi i}\int_{|\zeta|<1} \delta(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right) d\zeta d\bar{\zeta}.$$

Since the Jacobian of $\mu$-conformal mapping $f^\mu$ is $(1 - |\mu|^2)|\partial_z f^\mu|^2$, after a change of variables, we have

$$\Phi_{*g}(R_{g*}v) = \frac{1}{2\pi i}\int_{|\xi|<1} v(\xi)\partial_z f^\mu(\xi)^2 \left(\frac{1}{f^\mu(\xi) - f^\mu(z)} - \frac{1}{f^\mu(\xi)}\right) d\xi d\bar{\xi}.$$

Let $\tilde{g} = \tilde{g}_t|_{t=0}$ and $\xi = \tilde{g}^{-1}(\zeta)$ with $\zeta = re^{i\theta}$. Setting $\psi = f^\mu \circ \tilde{g}^{-1}$, we see that the integral above equals

$$\frac{-1}{\pi}\int_{|\zeta|<1} v(\gamma^{-1}(\theta))\partial_z f^\mu(\tilde{g}^{-1}(\zeta))^2 \left(\frac{1}{\psi(\zeta) - f^\mu(z)} - \frac{1}{\psi(\zeta)}\right) \partial_\theta \gamma^{-1}(\theta) r dr d\theta.$$

Since $\partial_z \tilde{g}(re^{i\theta}) = \frac{1}{2}e^{-i\theta}e^{i\gamma(\theta)}(1 + \partial_\theta \gamma(\theta))$ and $\psi(z)$ is holomorphic on $|z| < 1$, we have

$$\partial_z f^\mu(\tilde{g}^{-1}(\zeta)) = \tfrac{1}{2}\partial_z\psi(\zeta)e^{-i\gamma^{-1}(\theta)}e^{i\theta}(1+\partial_\theta\gamma(\gamma^{-1}(\theta))).$$

Since $\partial_\theta v = \partial_\theta(v\circ\gamma)\circ\gamma^{-1}\cdot\partial_\theta\gamma^{-1}$, we have

$$v(\gamma^{-1}(\theta)) = -2e^{2i\gamma^{-1}(\theta)}\frac{\partial_\theta v(\theta)}{(1+\partial_\theta\gamma(\gamma^{-1}(\theta)))^2\partial_\theta\gamma^{-1}(\theta)}.$$

Therefore

$$\Phi_{*g}(R_{g*}v) = \frac{1}{2\pi}\int_{|\zeta|<1}\partial_\theta v(\theta)\partial_z\psi(\zeta)^2 e^{2i\theta}\left(\frac{1}{\psi(\zeta)-f^\mu(z)}-\frac{1}{\psi(\zeta)}\right)rdrd\theta.$$

Putting $v = e^{in\theta}$ with $n \geq 0$, we see that the integral is zero, because $\psi(\zeta) - f^\mu(z) \neq 0$ for $|z| \geq 1$ and $\psi(\zeta) = \psi_1\zeta + \psi_2\zeta^2 + \cdots$ with $\psi_1 \neq 0$. Hence $\Phi_{*g}(R_{g*}v)$ is a linear combination of $v_n e^{in\theta}$ with $n < 0$.

We would like to close this paper by discussing a holomorphic structure of $PSU(1,1)\backslash Diff^+ S^1$.

LEMMA 3.7. *Let $(g_0, g_\infty)$ be the LU-decomposition for $g \in Diff^+ S^1$. Let $\sigma \in PSU(1,1)$. Then $\Phi(\sigma\circ g) = \Phi(g) - g_0\circ\sigma^{-1}(0)$.*

PROOF. Note that $g_\infty = g_0\circ g = g_0\circ\sigma^{-1}\circ\sigma\circ g$. We set

$$h_0 = g_0\circ\sigma^{-1} - g_0\circ\sigma^{-1}(0),$$

$$h_\infty = g_\infty - g_0\circ\sigma^{-1}(0).$$

Then $h_\infty \in \mathcal{M}$, $h_0\circ\sigma\circ g = h_\infty$ and $h_0$ is holomorphic. Hence the uniqueness of the LU-decomposition implies that $\Phi(\sigma\circ g) = h_\infty$.

We define a smooth mapping $\pi$: $\mathcal{M} \to C^\infty(S^1, \mathbf{C})$ by

$$z + \phi_0 + \phi_1/z + \phi_2/z^2 + \cdots \mapsto \phi_1/z + \phi_2/z^2 + \cdots.$$

Let $(g_0, g_\infty)$ be the LU-decomposition for $g \in Diff^+ S^1$. Take $\varepsilon > 0$ such that $\{z \in \Omega; |z| < \varepsilon\} \subset g_0(D_0)$ and set

$$U = \{h \in \mathcal{M}; \lim_{z\to\infty}|\Phi(h) - \Phi(g)| < \varepsilon/2\}.$$

Then for $g_1, g_2 \in U$,

$$\pi(g_1) = \pi(g_2) \iff \sigma\circ g_1 = g_2 \qquad \text{for some } \sigma \in PSU(1,1).$$

Hence

$$\pi\circ\Phi : PSU(1,1)\backslash Diff^+ S^1 \to \left\{\varphi \in C^\infty(S^1, \mathbf{C});\ \varphi(e^{i\theta}) = \sum_{n<0}\varphi_n e^{in\theta}\right\}$$

is a local diffeomorphism. Thus we have proved

THEOREM 3.6. *$PSU(1,1)\backslash Diff^+ S^1$ is a homogeneous complex manifold such that the canonical projection:*

$$T^1\backslash Diff^+ S^1 \to PSU(1,1)\backslash Diff^+ S^1$$

*is holomorphic.*

## References


[1] Ahlfors, L. V.: Conformality with respect to Riemannian metrics, Ann. Acad. Sci. Fenn. Ser. A1 **226** (1955), 1–22.

[2] Ahlfors, L. V.: Some remarks on Teichmüller's space of Riemann surfaces, Ann. of Math. **74** (1961), 171–191.

[3] Ahlfors, L. V.: Lectures on quasiconformal mappings, Van Nostrand, Princeton (1966).

[4] Ahlfors, L. V. and L. Bers: Riemann's mapping theorem for variable metrics, Ann. of Math. **72** (1960), 385–404.

[5] Bowick, M. J. and S. G. Rajeev: The holomorphic geometry of closed bosonic string and $Diff\, S^1/S^1$, Nucl. Phys. **B293** (1987), 348–384.

[6] Earle, C. J.: Some remarks on the Beltrami equation, Math. Scand. **36** (1975), 44–48.

[7] Earle, C. J. and A. Schatz: Teichmüller theory for surfaces with boundary, J. Diff. Geometry **4** (1970), 169–185.

[8] Gardiner, F.: An analysis of the group operation in universal Teichmüller space, Trans. Amer. Math. Soc. **132** (1968), 471–486.

[9] Hamilton, R. S.: The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. **7** (1982), 65–222.

[10] Kirillov, A. A.: Kähler structures on $K$-orbits of the group of diffeomorphisms of a circle, Funkts. Anal. Prilozhen. **21** No.2 (1987), 42–45.

[11] Mickelsson, J.: String quantization on group manifolds and the holomorphic geometry of $Diff^+ S^1/S^1$, Comm. Math. Phys. **112** (1987), 653–661.

[12] Milnor, J. W.: Remarks on infinite-dimensional Lie group, In: Relativity, Groups and Topology II, Les Houches Session XL (1983) edited by B. S. de Witt and R. Stora, North-Holland, Amsterdam (1984).

[13] Nag, S. and A. Verjovsky: $\mathrm{Diff}(S^1)$ and the Teichmüller spaces, Comm. Math. Phys. **130** (1990), 123–138.

[14] Omori, H.: Local structures of groups of diffeomorphisms, J. Math. Soc. Japan **20** (1972), 60–88.

[15] Omori, H.: Infinite-dimensional Lie groups, A.M.S. Translations of Mathematical Monographs **158** (1996).

[16] Pressley, A. and G. Segal: Loop groups, Clarendon Press, Oxford (1986).

[17] Vekua, I. N.: Generalized analytic functions, Pergamon Press, Oxford (1962).

[18] Witten, E.: Coadjoint orbits of the Virasoro group, Comm. Math. Phys. **114** (1988), 1–53.



*Department of Mathematics*
*Faculty of Science*
*Hiroshima University*
*Higashi-Hiroshima 739-8526, Japan*