Hiroshima Math. J.
29 (1999), 255–272

On punctured 3-manifolds in 5-sphere

Osamu Saeki

(Received May 8, 1998)

Abstract. Let M be a closed connected orientable 3-manifold with $H_1(M;\mathbf{Z})$ torsion free. In this paper, we classify up to isotopy the embedding maps of the punctured 3-manifold $M^\circ = M - \operatorname{Int} D^3$ into the 5-sphere S^5 in terms of the Euler classes of their normal bundles together with a certain kind of Seifert linking forms associated with such embeddings. It will be shown that the set of isotopy classes of all such embedding maps forms a finitely generated abelian group which depends only on the cohomology ring of M. We also discuss similar classifications for higher dimensions.

1. Introduction

Let M be a smooth closed connected orientable 3-manifold and set $M^\circ = M - \operatorname{Int} D^3$. It is known that M can always be smoothly embedded into S^5 and hence M° as well (for example, see [**W1**]).

Let us first consider the smooth embeddings of the closed 3-manifold M into S^5. When M is diffeomorphic to the standard 3-sphere, such embeddings are called 3-*knots* and have been studied to some extent (for example, see [**Le**]). However, when the 3-manifold M is not diffeomorphic to the 3-sphere, such embeddings have not been studied so much. Nevertheless, for those embeddings whose complements have the infinite cyclic fundamental group, some systematic studies have been possible, which are similar to those of simple 3-knots (see [**Le**]). For example, in [**Sa1, Sa2, Sa3**], the isotopy classes of such embeddings have been studied by using the Seifert linking matrices associated with simply connected 4-manifolds in S^5 bounded by the embedded 3-manifolds. Note that the normal bundle of a smooth embedding of M into S^5 is always trivial.

Let us now consider smooth embeddings of the punctured 3-manifold M° into S^5. Note that their complements in S^5 are always simply connected (for details, see Lemma 3.3 of the present paper), but that their normal bundles may

1991 *Mathematics Subject Classification.* Primary 57Q45; Secondary 57R40.

Keywords. Punctured 3-manifold, embedding, isotopy, Seifert matrix, intersection form, highly connected manifold.

The author has been partially supported by the Anglo-Japanese Scientific Exchange Programme, run by the Japan Society for the Promotion of Science and the Royal Society.

not necessarily be trivial in general. For lower dimensions, codimension two embeddings of punctured manifolds into spheres are not interesting so much: in fact, for each punctured 1- or 2-dimensional manifold, its embeddings are unique up to isotopy (for example, see [**Ti**]). However, for dimension three, it is surprising that, in most cases, there exist a lot of isotopy classes of embeddings of a given punctured 3-manifold, even if we restrict ourselves to those with trivial normal bundles.

Our purpose of this paper is to completely classify the embedding maps of M° into S^5 up to isotopy for M with $H_1(M;\mathbf{Z})$ torsion free. It will be shown that the set of isotopy classes of all such embedding maps forms a finitely generated abelian group which depends only on $H^2(M;\mathbf{Z})$ and the intersection form $\mu_M : H_2(M;\mathbf{Z}) \times H_2(M;\mathbf{Z}) \times H_2(M;\mathbf{Z}) \to \mathbf{Z}$ of M.

The strategy of our classification is as follows. In §2, we will define a skew symmetric bilinear linking form L^τ on $H_2(M;\mathbf{Z})$ when there exists a trivialization τ of the normal bundle. This linking form depends on the choice of the trivialization τ in general. However, we will show that the difference between two such forms is always given by the inner product of an element of $H_2(M;\mathbf{Z})$ and the intersection form μ_M of M (Lemma 2.2). This enables us to define an invariant of the isotopy class of an embedding map with trivial normal bundle as an element in the additive group $\bigwedge^2 H_2(M;\mathbf{Z})^*$ of all integral 2-forms on $H_2(M;\mathbf{Z})$ modulo the subgroup I_M generated by the inner products of μ_M. Note that the abelian group $\bigwedge^2 H_2(M;\mathbf{Z})^*/I_M$ depends only on the intersection form μ_M of M, or equivalently on the cohomology ring of M, and that this group can easily be calculated, once the intersection form μ_M of M is given (see Lemma 4.9).

In §3, we will show that this invariant is in fact complete and gives a bijection between the set of isotopy classes of the embedding maps of M° into S^5 with trivial normal bundles and the finitely generated abelian group $\bigwedge^2 H_2(M;\mathbf{Z})^*/I_M$ (Theorem 3.1).

In §4, we consider the general case where the normal bundle may not necessarily be trivial. If the normal bundle is not trivial, then we will see that its Euler class is always an even multiple of an element of $H^2(M;\mathbf{Z})$ (Lemma 4.1). In this case, we will twist the given embedding so that the resulting embedding has trivial normal bundle and that we can define the linking form as above. As a result, we will show that the set of isotopy classes of the embedding maps of M° into S^5 is in one-to-one correspondence with the finitely generated abelian group $H^2(M;\mathbf{Z}) \oplus (\bigwedge^2 H_2(M;\mathbf{Z})^*/I_M)$, where to each such embedding map corresponds one half of the Euler class of its normal bundle together with the linking form of the twisted embedding (Theorem 4.5). As a corollary, we will show that the isotopy classes of such embedding maps are unique if M has the same integral homology groups as S^3, and correspond bijectively to the set $\mathbf{Z}$ of the integers if M has the same integral homology groups as $S^1 \times S^2$ (Corollary 4.10).

In §5, we will consider embeddings of $(n-2)$-connected stably parallelizable $(2n-1)$-dimensional punctured manifolds M° into S^{2n+1} $(n \geq 3)$ (with trivial normal bundles if $n = 3$) and will give a complete classification similar to the three dimensional case. In fact, we will consider a classification up to an equivalence relation seemingly weaker than the isotopy (see Definition 5.3). We will see that for such higher dimensions, the effect of the intersection form disappears, and thus the set of equivalence classes of the embedding maps (with trivial normal bundles for $n = 3$) are in one-to-one correspondence with the finitely generated free abelian group $\bigwedge^2 H_n(M;\mathbf{Z})^*$ if n is even and to $\bigcirc^2 H_n(M;\mathbf{Z})^*$ if n is odd, where $\bigcirc^2 H_n(M;\mathbf{Z})^*$ denotes the set of all integral symmetric bilinear forms on $H_n(M;\mathbf{Z})$ (Theorem 5.4).

Throughout the paper, all manifolds and maps are of class C^∞. The homology and the cohomology groups are with integral coefficients unless otherwise indicated. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects.

The author would like to express his thanks to the people at the University of Liverpool for their hospitality during the preparation of the manuscript.

2. Seifert linking forms for embeddings with trivial normal bundles

Let M be a smooth closed connected orientable 3-manifold and set $M^\circ = M - \operatorname{Int} D^3$. Recall that every closed orientable 3-manifold embeds smoothly into S^5 (see [**W1**]), so that M° also embeds into S^5. Let $f : M^\circ \to S^5$ be a smooth embeddings. In this and the next sections, we assume that the normal bundle of f is trivial. We define the Seifert form of f as follows. We first fix an orientation of S^5. By our hypothesis, we can extend f to an embedding $\tau : M^\circ \times D^2 \to S^5$ such that $\tau | M^\circ \times \{0\} = f$, where we identify D^2 with the unit disk in $\mathbf{C}$. In the following, we call τ a *trivialization of the normal bundle of f*. Set $N = \tau(M^\circ \times D^2)$ and define $i^\tau : M^\circ \to \partial N$ by $i^\tau(x) = \tau(x, 1)$ $(x \in M^\circ)$. Then we define the bilinear form

$$L^\tau : H_2(M) \times H_2(M) \to \mathbf{Z}$$

by $L^\tau(\alpha, \beta) = \operatorname{lk}(f_*\alpha, i^\tau_*\beta)$, where lk denotes the linking number in S^5 and we naturally identify $H_2(M)$ with $H_2(M^\circ)$. Note that in general, L^τ depends on the choice of the trivialization τ of the normal bundle of f.

Lemma 2.1. *The bilinear form L^τ is skew symmetric.*

Proof. Define $j^\tau : M^\circ \to \partial N$ by $j^\tau(x) = \tau(x, -1)$ $(x \in M^\circ)$. Then for all $\alpha, \beta \in H_2(M)$, we have

$$\begin{aligned} L^{\tau}(\beta,\alpha) &= \mathrm{lk}(f_*\beta, i^{\tau}_*\alpha) \\ &= \mathrm{lk}(j^{\tau}_*\beta, f_*\alpha) \\ &= -\mathrm{lk}(f_*\alpha, j^{\tau}_*\beta) \\ &= -\mathrm{lk}(f_*\alpha, i^{\tau}_*\beta) \\ &= -L^{\tau}(\alpha,\beta), \end{aligned}$$

where the fourth equality follows from the fact that i^{τ} and j^{τ} are isotopic to each other as embedding maps from M° into $S^5 - f(M^{\circ})$. This completes the proof. ||

We call L^{τ} the *Seifert form* of f associated with the trivialization τ. We then have to clarify the changes on the Seifert form when we change the trivialization τ of the normal bundle of f.

Fix an orientation of M°. Note that by Poincaré duality, we have $H_2(M) \cong H^1(M)$. In the following, we identify these two groups by such an isomorphism.

Let

$$\mu_M : H^1(M) \times H^1(M) \times H^1(M) \to \mathbf{Z}$$

be the intersection form of M defined by $\mu_M(\alpha,\beta,\gamma) = \langle \alpha \smile \beta \smile \gamma, [M] \rangle$, where $[M] \in H_3(M)$ is the fundamental class of M corresponding to the fixed orientation. Note that μ_M is a skew symmetric 3-form on $H^1(M) = H_2(M)$ (for example, see [**Su**]).

Fix an embedding $\tau : M^{\circ} \times D^2 \to S^5$ which is a trivialization of the normal bundle of f. Then the other such trivializations correspond to homotopy classes of smooth maps $\varphi : M^{\circ} \to S^1 (= \partial D^2)$ up to isotopy. More precisely, the map φ corresponds to the isotopy class of the trivialization τ' given by $\tau'(x,z) = \tau(x, \varphi(x)z)$ $(z \in D^2 \subset \mathbf{C}, x \in M^{\circ})$. Thus the isotopy classes of embeddings $M^{\circ} \times D^2 \to S^5$ extending f are in one-to-one correspondence with the homotopy classes of maps of M° to S^1, which in turn are in one-to-one correspondence with the elements of $H^1(M) = H_2(M)$.

LEMMA 2.2. *Let $\tau' : M^{\circ} \times D^2 \to S^5$ be the embedding extending f which corresponds to the element $u \in H^1(M) = H_2(M)$ in the above correspondence. Then we have*

$$L^{\tau'}(\alpha,\beta) = L^{\tau}(\alpha,\beta) - \mu_M(u,\alpha,\beta)$$

for all $\alpha, \beta \in H^1(M) = H_2(M)$.

PROOF. We can represent the homology classes $u, \alpha, \beta \in H_2(M)$ by smoothly embedded oriented surfaces U, A and B in the interior of M° respectively. We may assume that U and B intersect transversely. Let $N(U)$ be a sufficiently small closed tubular neighborhood of U in the interior of M° and $\theta : U \times I \to N(U)$ ($I = [-1, 1]$) a diffeomorphism such that $\theta(x, 0) = x$ for all $x \in U$. Define the continuous map $\varphi : M^\circ \to S^1$ by

$$\varphi(x) = \begin{cases} 1 & (x \in M^\circ - \operatorname{Int} N(U)) \\ \rho \circ p_2 \circ \theta^{-1}(x) & (x \in N(U)), \end{cases}$$

where $p_2 : U \times I \to I$ is the projection to the second factor and $\rho : I \to S^1$ is the map defined by $\rho(t) = \exp((t+1)\pi\sqrt{-1})$. Note that the trivialization τ' corresponds to φ.

Then we have

$$\begin{aligned} L^{\tau'}(\alpha, \beta) - L^{\tau}(\alpha, \beta) &= \operatorname{lk}(f_*\alpha, i_*^{\tau'}\beta) - \operatorname{lk}(f_*\alpha, i_*^{\tau}\beta) \\ &= \operatorname{lk}(f_*\alpha, i_*^{\tau'}\beta - i_*^{\tau}\beta). \end{aligned}$$

Set $J = U \cap B$, which is a finite disjoint union of simple closed curves in M°, and set $D(J) = \tau(J \times D^2) (\subset \tau(M^\circ \times D^2))$, which is a finite disjoint union of embedded solid tori in S^5. We may assume that $N(U) \cap B = \theta(J \times I)$ by isotoping B if necessary. Then, by the construction of φ, we see that, with an appropriate orientation for $D(J)$,

$$i_*^{\tau'}\beta - i_*^{\tau}\beta = [\partial D(J)],$$

where $[\partial D(J)] \in H_2(S^5 - f(M^\circ))$ is the homology class represented by $\partial D(J)$. Thus we have

$$\begin{aligned} L^{\tau'}(\alpha, \beta) - L^{\tau}(\alpha, \beta) &= \operatorname{lk}(f_*\alpha, [\partial D(J)]) \\ &= (f_*\alpha) \cdot D(J) \\ &= \mu_M(\alpha, \mu, \beta) \\ &= -\mu_M(u, \alpha, \beta), \end{aligned}$$

where $(f_*\alpha) \cdot D(J)$ denotes the intersection number in S^5, which is equal to the intersection number of α and J in M. This completes the proof. ||

Let us denote by $\bigwedge^2 H_2(M)^*$ the finitely generated additive group of all integral 2-forms on $H_2(M)$. We set

$$I_M = \left\{ i(u)\mu_M \in \bigwedge^2 H_2(M)^* : u \in H_2(M) \right\},$$

where the *inner product* $i(u)\mu_M$ of u and μ_M is defined by $i(u)\mu_M(\alpha,\beta) = \mu_M(u,\alpha,\beta)$ $(\alpha,\beta \in H_2(M))$. Note that I_M is an additive subgroup of $\bigwedge^2 H_2(M)^*$ and hence that $\bigwedge^2 H_2(M)^*/I_M$ forms a finitely generated abelian group.

For a smooth embedding $f : M^\circ \to S^5$ with trivial normal bundle, we define the element $L(f) \in \bigwedge^2 H_2(M)^*/I_M$ as the class corresponding to L^τ for some trivialization τ. By Lemma 2.2, this element is independent of the choice of τ. In particular, $L(f)$ is an invariant of the isotopy class of f. We call $L(f)$ the *invariant Seifert form* of f.

3. Classification of embeddings with trivial normal bundles

In this section, we prove the following.

THEOREM 3.1. *Let M be a smooth closed connected orientable* 3*-manifold with $H_1(M)$ torsion free. Then the isotopy classes of smooth embedding maps $f : M^\circ \to S^5$ with trivial normal bundles correspond bijectively to the finitely generated abelian group $\bigwedge^2 H_2(M)^*/I_M$ by the correspondence $f \mapsto L(f)$.*

PROOF. First, let us prove the surjectivity. It is known that there exists a 4-dimensional handlebody $W = h^0 \cup h_1^2 \cup h_2^2 \cup \cdots \cup h_r^2$ with $\partial W \cong M$, where h^0 denotes a 0-handle and h_i^2 denotes a 2-handle $(i = 1, 2, \ldots, r)$ (for example, see [**Li**, Theorem 3]). Furthermore, we may assume that W is spin; i.e., the second Stiefel-Whitney class $w_2(W) \in H^2(W; \mathbf{Z}_2)$ of W vanishes (for example, see [**Ka**]). In the following, we identify M with ∂W. Consider the following exact sequence of homology of the pair $(W, \partial W)$:

$$H_3(W, \partial W) \to H_2(M) \to H_2(W) \to H_2(W, \partial W).$$

Note that $H_3(W, \partial W) = 0$ and that the other groups above are all finitely generated free abelian groups. Thus we may assume that $H_2(M)$ is a subgroup of $H_2(W)$. Then, we see that $H_2(M)$ is a direct summand of $H_2(W)$ by the above exact sequence.

Fix an orientation for W. Given an element L of $\bigwedge^2 H_2(M)^*$, let $\tilde{L}$ be a bilinear form on $H_2(W)$ which extends L on $H_2(M)$ such that

$$\tilde{L}(\alpha,\beta) + \tilde{L}(\beta,\alpha) = \alpha \cdot \beta$$

for all $\alpha,\beta \in H_2(W)$, where $\alpha \cdot \beta$ denotes the intersection number of α and β in W. Such a bilinear form $\tilde{L}$ exists, since the intersection form of W is of even type and the intersection form of W restricted to $H_2(M)$ is the zero form.

Then using a method of Kervaire [**Ke**, Chapitre II, §6], we can construct an embedding $\tilde{f} : W \to S^5$ such that its Seifert form coincides with $\tilde{L}$ (for the definition of a Seifert form, see [**Le**], for example). Then the embedding $f = \tilde{f}|M^\circ$ ($M^\circ \subset M = \partial W$) has trivial normal bundle and satisfies $L^\tau = L$ for some trivialization $\tau : M^\circ \times D^2 \to S^5$ such that $\tau(M^\circ \times [0,1]) = \tau(M^\circ \times D^2) \cap \tilde{f}(W)$. Thus every element of $\bigwedge^2 H_2(M)^*/I_M$ can be realized by some embedding $f : M^\circ \to S^5$ with trivial normal bundle as its invariant Seifert form. Note that for the surjectivity of the correspondence, we do not need the assumption that $H_1(M)$ should be torsion free, as the proof shows.

Let us prove the injectivity of the correspondence $f \mapsto L(f)$. Suppose that f and $g : M^\circ \to S^5$ are smooth embeddings with trivial normal bundles such that $L(f) = L(g) \in \bigwedge^2 H_2(M)^*/I_M$. Set $f(M^\circ) = K$ and $g(M^\circ) = K'$. By Lemma 2.2, there exist trivializations τ and $\tau' : M^\circ \times D^2 \to S^5$ of the normal bundles of f and g respectively such that $L^\tau = L^{\tau'}$. Let W be a 4-dimensional handlebody as in the first paragraph of the proof of Theorem 3.1.

LEMMA 3.2. *There exist smooth embeddings $\tilde{f}$ and $\tilde{g} : W \to S^5$ such that $\tilde{f}|M^\circ = f$ and $\tilde{g}|M^\circ = g$.*

For the proof of the above lemma, we need the following.

LEMMA 3.3. *The complements $S^5 - K$ and $S^5 - K'$ are simply connected.*

PROOF. Let $\psi : S^1 \to S^5 - K$ be an arbitrary continuous map. Since S^5 is simply connected, there exists a continuous map $\varphi : D^2 \to S^5$ such that $\varphi|\partial D^2 = \psi$. We may assume that φ is smooth and that it intersects K transversely. Then the intersection of $\varphi(D^2)$ with K consists of finitely many points. Then by a homotopy of φ fixing ∂D^2, we can "push out" the intersections towards ∂K so that, at the end of the homotopy, $\varphi(D^2)$ does not intersect K. Thus ψ is null homotopic in $S^5 - K$, and hence $S^5 - K$ is simply connected. The same argument can be applied to $S^5 - K'$. This completes the proof of Lemma 3.3. ||

PROOF OF LEMMA 3.2. Consider the dual handlebody decomposition

$$W = (M \times [0,1]) \cup (h_1^2)^* \cup \cdots \cup (h_r^2)^* \cup h^4$$

of W, where $(h_i^2)^*$ are the dual 2-handles of h_i^2 ($i = 1, \ldots, r$) and h^4 is the dual 4-handle of h^0. It is not difficult to show that the above handlebody is diffeomorphic to

$$(M^\circ \times [0,1]) \cup (h_1^2)^* \cup \cdots \cup (h_r^2)^*.$$

We identify W with the above handlebody. By using τ and τ', we can extend the embeddings f and g to embeddings of $M^\circ \times [0,1]$. Then, by using Lemma 3.3, we can extend them to embeddings f_1 and g_1 of $(M^\circ \times [0,1]) \cup D_1^2 \cup \cdots \cup D_r^2$, where D_i^2 are the core 2-disks of $(h_i^2)^*$

$(i = 1, \dots, r)$. Finally, by using the fact that W is spin, one can extend f_1 and g_1 to embeddings $\tilde{f}$ and $\tilde{g}$ of W respectively (for this, see the techniques used in [**Ke**, Chapitre II, §6], for example). This completes the proof of Lemma 3.2. ||

REMARK 3.4. Lemma 3.2 shows, in particular, that for every smooth closed connected orientable 3-manifold M, every smooth embedding of M° into S^5 with trivial normal bundle extends to a smooth embedding of M into S^5 such that the complement of the image has infinite cyclic fundamental group.

Next we will modify $\tilde{g}$ outside M° so that the Seifert forms of $\tilde{f}(W)$ and $\tilde{g}(W)$ coincide with each other.

Consider the following Mayer-Vietoris exact sequence of the pair $(M^\circ \times [0,1], (h_1^2)^* \cup \cdots \cup (h_r^2)^*)$:

$$H_2(b_1 \cup \cdots \cup b_r) \xrightarrow{\iota_*} H_2(M^\circ \times [0,1]) \longrightarrow H_2(W)$$

$$\longrightarrow H_1(b_1 \cup \cdots \cup b_r) \xrightarrow{\iota_*} H_1(M^\circ \times [0,1]) \longrightarrow H_1(W), \qquad (3.5)$$

where $b_i = (h_i^2)^* \cap (M^\circ \times [0,1])$ is the solid torus along which the 2-handle $(h_i^2)^*$ is attached to $M^\circ \times [0,1]$ on $M^\circ \times \{1\}$ $(i = 1, \dots, r)$, and $\iota : b_1 \cup \cdots \cup b_r \to M^\circ \times [0,1]$ is the inclusion map. In the following, we identify $H_j\ (M^\circ \times [0,1])$ with $H_j(M)$ $(j = 1, 2)$. Note that $H_2(b_1 \cup \cdots \cup b_r) = 0 = H_1(W)$ and that $H_1(M)$ is free abelian by our assumption. Let $\{\gamma_1, \dots, \gamma_s\}$ be a basis of $H_2(M)$ over $\mathbf{Z}$. Recall that $H_2(M)$ is a direct summand of $H_2(W)$. Since $H_1(M)$ is free abelian, we see that $\ker(\iota_* : H_1(b_1 \cup \cdots \cup b_r) \to H_1(M^\circ \times [0,1]))$ is a direct summand of $H_1(b_1 \cup \cdots \cup b_r)$. Then by sliding the 2-handles $(h_i^2)^*$ if necessary, we may assume that $\{[\partial D_1^2], \dots, [\partial D_k^2]\}$ is a basis for $\ker \iota_*$, and that $\{\iota_*[\partial D_{k+1}^2], \dots, \iota_*[\partial D_r^2]\}$ is a basis for $H_1(M)$, where $s = r - k$ is the rank of $H_1(M)$ and $[\partial D_i^2] \in H_1(b_1 \cup \cdots \cup b_r)$ $(i = 1, \dots, r)$ are the homology classes represented by the boundary circles of the core 2-disks D_i^2 of $(h_i^2)^*$.

Since $\iota_*[\partial D_i^2] = 0$ in $H_1(M)$ for $i = 1, \dots, k$, there exist 2-chains c_i in M° such that $\partial c_i = -\partial D_i^2$. Let $\delta_i \in H_2(W)$ be the homology class represented by $D_i^2 \cup c_i$ $(i = 1, \dots, k)$. Then we see that $\{\gamma_1, \dots, \gamma_s, \delta_1, \dots, \delta_k\}$ is a basis of $H_2(W)$ by the exact sequence (3.5).

Let $L_{\tilde{f}}$ and $L_{\tilde{g}}$ denote the Seifert forms of $\tilde{f}(W)$ and $\tilde{g}(W)$ respectively. Since $L^\tau = L^{\tau'}$, we see that $L_{\tilde{f}}(\gamma_i, \gamma_j) = L_{\tilde{g}}(\gamma_i, \gamma_j)$ for all $1 \le i, j \le s$. Furthermore, by modifying the embedding $\tilde{g}$ on the 2-handles $(h_i^2)^*$ $(i = 1, \dots, k)$ if necessary, we can arrange so that $L_{\tilde{f}}(\delta_i, \delta_j) = L_{\tilde{g}}(\delta_i, \delta_j)$ for all $1 \le i, j \le k$ (for this, use the argument of Kervaire [**Ke**, Chapitre II, §6]).

Let $\alpha_i \in H_1(M, \partial(M^\circ))$ be the Poincaré dual of $\gamma_i^* \in H^2(M^\circ)$, where $\gamma_1^*, \dots, \gamma_s^* \in H^2(M^\circ) \cong \mathrm{Hom}(H_2(M^\circ), \mathbf{Z})$ are the dual basis of $\gamma_1, \dots, \gamma_s \in$

$H_2(M^\circ)$. We can represent α_i by a properly embedded arc a_i in M°. We may assume that a_i are mutually disjoint and that the attaching solid tori of $(h_j^2)^*$ $(1 \le j \le r)$ in $M^\circ \times \{1\} = M^\circ$ do not intersect a_i. Let $N(a_i)$ be a sufficiently small tubular neighborhood of a_i in M° and let $\eta_i : N(a_i) \to D^2 \times I$ $(I = [-1, 1])$ be a diffeomorphism such that $\eta_i(a_i) = \{0\} \times I$ and $\eta_i(N(a_i) \cap \partial(M^\circ)) = D^2 \times \{-1, 1\}$. Note that $N(a_i) \times [0, 1] (\subset M^\circ \times [0, 1])$ can be regarded as a 2-handle attached to

$$\overline{M^\circ - \bigcup_{j=1}^{s} N(a_j)} \times [0, 1]$$

with the core 2-disk D_i' identified with $\eta_i^{-1}(D^2 \times \{0\}) \times \{1/2\}$. Then, modifying the embedding $\tilde{g}$ on the 2-handles $(h_j^2)^*$ $(j = 1, \dots, k)$ so as to modify the linking numbers of D_i' and D_j^2, we can arrange so that $L_{\tilde{f}}(\gamma_i, \delta_j) = L_{\tilde{g}}(\gamma_i, \delta_j)$ for all i and j with $1 \le i \le s$ and $1 \le j \le k$ without changing $L_{\tilde{g}}(\delta_i, \delta_j)$ for $1 \le i, j \le k$ or $L_{\tilde{g}}(\gamma_i, \gamma_j)$ for $1 \le i, j \le s$ (for this, again use the argument of Kervaire [**Ke**, Chapitre II, §6]. See also Remark 3.6 below).

Thus we have shown that there exist smooth embeddings $\tilde{f}$ and $\tilde{g}$ of W into S^5 which are extensions of f and g respectively such that $L_{\tilde{f}} = L_{\tilde{g}}$. Then by using the same argument as in [**Le**, §§18–20], we see that $\tilde{f}$ and $\tilde{g}$ are isotopic to each other as embedding maps of W into S^5, since W has a handlebody decomposition consisting of one 0-handle and some 2-handles. Restricting this isotopy to $M^\circ \subset M = \partial W$, we see that f and g are isotopic as embedding maps into S^5. Thus the correspondence $f \mapsto L(f)$ is injective. This completes the proof of Theorem 3.1. ||

Remark 3.6. In the above construction, we can prove that

$$\Delta = \left(\overline{M^\circ - \bigcup_{i=1}^{s} N(a_i)} \times [0, 1]\right) \cup ((h_{k+1}^2)^* \cup \cdots \cup (h_r^2)^*)$$

is an integral homology 4-disk and that W can be obtained by attaching the 2-handles $N(a_i) \times [0, 1]$ $(i = 1, \dots, s)$ and $(h_j^2)^*$ $(j = 1, \dots, k)$ to Δ. Such an argument is possible only when $H_1(M)$ is torsion free.

4. Classification of general embeddings

Let $f : M^\circ \to S^5$ be a smooth embedding whose normal bundle may not necessarily be trivial. Let us denote by $e(f) \in H^2(M^\circ) \cong H^2(M)$ the Euler class of the normal bundle.

Lemma 4.1. *Suppose that $H_1(M)$ is torsion free. Then the Euler class $e(f)$ is always an even multiple of an element of $H^2(M)$; i.e., $e(f) \in 2H^2(M)$.*

Proof. By our assumption, $H^2(M)$ is isomorphic to $\mathrm{Hom}(H_2(M), \mathbf{Z})$ by the universal coefficient theorem. Thus we have only to show that $\langle e(f), \alpha \rangle$ is even for every $\alpha \in H_2(M)$. Let A be a closed oriented surface in M° representing α. Let ν denote the normal bundle of the embedding f. Then we have

$$\langle e(f), \alpha \rangle = \langle e(f), (i_A)_*[A] \rangle = \langle (i_A)^* e(f), [A] \rangle = \langle e(\nu|A), [A] \rangle,$$

where $i_A : A \to M^\circ$ denotes the inclusion map, $e(\nu|A)$ denotes the Euler class of the normal bundle ν restricted to A, and $[A] \in H_2(A)$ denotes the fundamental class. We may assume that the total space of ν is embedded in S^5 in a way compatible with f. Let A' be an embedded surface in S^5 which is obtained by perturbing the zero section A in the total space of $\nu|A$ and which intersects the zero section A transversely. Then $\langle e(\nu|A), [A] \rangle$ is equal to the intersection number of A' and A in the total space of $\nu|A$, which is equal to the intersection number of A' and $f(N(A))$ in S^5, where $N(A) \cong A \times [-1, 1]$ is a closed tubular neighborhood of A in M°. Set $A_+ = A \times \{1\}$ and $A_- = A \times \{-1\}$, which are identified with the boundary components of $N(A)$. Then we have

$$\begin{aligned}
\langle e(f), \alpha \rangle &= A' \cdot f(N(A)) \\
&= \mathrm{lk}(A', f(\partial N(A))) \\
&= \mathrm{lk}(A', f(A_+) - f(A_-)) \\
&= \mathrm{lk}(A', f(A_+)) - \mathrm{lk}(A', f(A_-)) \\
&= \mathrm{lk}(f(A), f(A_+)) + \mathrm{lk}(f(A_-), f(A)) \\
&= 2 \cdot \mathrm{lk}(f(A), f(A_+)).
\end{aligned}$$

Thus $\langle e(f), \alpha \rangle$ is even for every $\alpha \in H_2(M)$. This completes the proof.

Alternatively, we can prove the lemma by using the fact that $\nu|A$ is stably trivial. ||

Remark 4.2. In the above proof, the intersection number $A' \cdot f(N(A))$ is also equal to the intersection number $A' \cdot f(M^\circ)$, which is equal to the linking number of $f(A)$ and $f(\partial(M^\circ))$. Thus, for every $\alpha \in H_2(M)$, we have $\langle e(f), \alpha \rangle = \mathrm{lk}(f_* \alpha, f(\partial(M^\circ)))$.

Let us consider the twisting operation which converts f to an embedding with trivial normal bundle as follows. Let $\varepsilon \in H_1(M^\circ, \partial(M^\circ))$ be the Poincaré dual of $e(f)/2$. Then ε can always be represented by a properly embedded oriented arc a in M°. Let $N(a)$ be a closed tubular neighborhood of a in M° and fix a diffeomorphism $\eta : N(a) \to D^2 \times I$, where $I = [-1, 1]$, $\eta(a) = \{0\} \times I$ and $\eta(N(a) \cap \partial(M^\circ)) = D^2 \times \{-1, 1\}$. Set $D_a = \eta^{-1}(D^2 \times \{0\})$, which is a properly embedded 2-disk in $N(a)$. We can regard $N(a)$ as a 3-dimensional

2-handle attached to $\overline{M^\circ - N(a)}$ with the core 2-disk D_a. Note that the normal bundle of $f|D_a$ is trivial and we can extend $f|D_a$ to an embedding $g_a : D_a \times D^3 \to S^5$. We may assume that $g_a|(D_a \times I) = f \circ \eta^{-1} \circ \kappa$, where $\kappa : D_a \times I \to D^2 \times I$ is the natural identification, D^3 is identified with the unit 3-disk in $\mathbf{R}^3$ and $I = [-1, 1]$ is identified with the intersection of D^3 with the first coordinate axis. Let $\varphi : D_a \to \partial D^3 = S^2$ be a smooth map such that $\varphi(V) = (1, 0, 0)$ for some neighborhood V of ∂D_a in D_a and that it represents an element of $\pi_2(S^2) \cong \mathbf{Z}$ corresponding to -1. Then define the smooth embedding $\psi : D_a \times I \to D_a \times D^3$ by $\psi(x, t) = (x, t\varphi(x))$. Finally define the smooth embedding $f_a : M^\circ \to S^5$ by

$$f_a(x) = \begin{cases} f(x) & (x \notin N(a)) \\ g_a \circ \psi \circ \kappa^{-1} \circ \eta(x) & (x \in N(a)). \end{cases}$$

LEMMA 4.3. *The embedding f_a has trivial normal bundle.*

PROOF. Let $e(f_a) \in H^2(M^\circ)$ denote the Euler class of of the normal bundle of f_a. We have only to show that $\langle e(f_a), \alpha \rangle = 0$ for every $\alpha \in H_2(M^\circ)$. Let A be a closed oriented surface embedded in M° representing α. We may assume that A intersects a transversely at a finite number of points. Let k be the algebraic intersection number of a and A. Then, using the same notation as in the proof of Lemma 4.1, we have

$$\langle e(f_a), \alpha \rangle = 2 \cdot \mathrm{lk}(f_a(A), f_a(A_+)).$$

It is not difficult to show that

$$\mathrm{lk}(f_a(A), f_a(A_+)) = \mathrm{lk}(f(A), f(A_+)) - k$$

by the construction of f_a. On the other hand, by the proof of Lemma 4.1, we have

$$\mathrm{lk}(f(A), f(A_+)) = \langle e(f), \alpha \rangle / 2.$$

Furthermore, since a represents the element $\varepsilon \in H_1(M^\circ, \partial(M^\circ))$ Poincaré dual to $e(f)/2$, we have

$$k = \langle e(f)/2, \alpha \rangle = \langle e(f), \alpha \rangle / 2.$$

Combining the above equalities, we obtain $\langle e(f_a), \alpha \rangle = 0$. This completes the proof. ||

In the following, we say that f_a is an embedding obtained from f by the *twisting operation* with respect to the properly embedded oriented arc a in M°.

Remark 4.4. The isotopy class of the embedding f_a is uniquely determined by the isotopy class of the properly embedded oriented arc a in M° representing the Poincaré dual of $e(f)/2$. Let a and b be two such arcs. When a and b are not isotopic as properly embedded arcs in M°, we do not know if f_a and f_b are isotopic to each other.

In the following, for each element e of $H^2(M^\circ)$, we fix a properly embedded oriented arc a in M° Poincaré dual to e.

The main result of this section is the following.

Theorem 4.5. *Let M be a smooth closed connected orientable 3-manifold with $H_1(M)$ torsion free. Then the isotopy classes of smooth embedding maps $f: M^\circ \to S^5$ correspond bijectively to the finitely generated abelian group $H^2(M) \oplus (\bigwedge^2 H_2(M)^*/I_M)$ by the correspondence $f \mapsto e(f)/2 \oplus L(f_a)$, where a is the fixed properly embedded oriented arc in M° Poincaré dual to $e(f)/2$ and f_a is the embedding with trivial normal bundle obtained from f by the twisting operation with respect to a.*

Proof. The correspondence is well-defined by Lemmas 4.1 and 4.3 together with the results obtained in §2.

Let us show that the correspondence is surjective. Take an element $e \oplus L \in H^2(M) \oplus (\bigwedge^2 H_2(M)^*/I_M)$. By Theorem 3.1, L is realized by an embedding $g: M^\circ \to S^5$ with trivial normal bundle as its invariant Seifert form. Let a be the properly embedded oriented arc in M° Poincaré dual to e. Then by applying the reverse operation of the twisting operation to g with respect to the arc a, we can construct an embedding f of M° into S^5 such that the Euler class of its normal bundle is equal to $2e$. Thus, we have $e(f)/2 = e$. Since f_a is isotopic to g, we have $L(f_a) = L(g) = L$. Hence f is an embedding which realizes the given element $e \oplus L$ of $H^2(M) \oplus (\bigwedge^2 H_2(M)^*/I_M)$.

Let us show the injectivity. Let f and g be embeddings of M° into S^5 with $e(f) = e(g)$ and $L(f_a) = L(g_a)$ (note that, since $e(f) = e(g)$, we can use the same arc a for both f and g). Then by Theorem 3.1, f_a and g_a are isotopic. Since f and g are obtained by applying the reverse operations of the twisting operation to f_a and g_a respectively with respect to a, we see that f and g are also isotopic. This completes the proof. ||

Remark 4.6. Let us consider the isotopy classes of the *images* of the embeddings of M° into S^5, not as the embedding maps. Then such isotopy classes of embeddings with *trivial normal bundles* are in one-to-one correspondence with the elements of $(\bigwedge^2 H_2(M)^*/I_M)/G_M$, where G_M is the group of self-diffeomorphisms of M° which acts on $\bigwedge^2 H_2(M)^*/I_M$ in a natural way. In general, such a set does not inherit a structure of an abelian group. As to the general case where the normal bundle may not necessarily be trivial, the group G_M does act on the group $H^2(M) \oplus (\bigwedge^2 H_2(M)^*/I_M)$; however, we do not know if such isotopy classes are in one-to-one correspondence with the elements of $(H^2(M) \oplus (\bigwedge^2 H_2(M)^*/I_M))/G_M$, since the choice of the properly embedded arc a is not canonical. For example, if $h : M^\circ \to M^\circ$ is a self-diffeomorphism of M° which acts on the homology and the cohomology groups trivially such that the properly embedded arcs a and $h(a)$ are not isotopic to each other, then we do not know if $L(f_a) = L(f_{h(a)})$.

REMARK 4.7. Let us denote by $\mathrm{Emb}^\infty(M^\circ, S^5)$ the set of all smooth embedding maps of M° into S^5 endowed with the usual C^∞ topology (or equivalently, the Whitney C^∞ topology, since M^0 is compact) (for example, see [**H**, Chapter 2]). Then Theorem 4.5 implies that the connected components of $\mathrm{Emb}^\infty(M^\circ, S^5)$ are in one-to-one correspondence with the elements of $H^2(M) \oplus (\bigwedge^2 H_2(M)^*/I_M)$.

REMARK 4.8. Sullivan [**Su**] has shown that every integral 3-form μ on an arbitrary finitely generated free abelian group H can be realized as the intersection form of some smooth closed oriented 3-manifold. Furthermore, such 3-manifolds are constructed in such a way that their first homology groups are torsion free. Let I_μ denote the subgroup of $\bigwedge^2 H^*$ which consists of the inner products of the elements of H and μ. Then, every abelian group of the form $H^* \oplus (\bigwedge^2 H^*/I_\mu)$ can be realized as the set of isotopy classes of the embedding maps of M^0 into S^5 for some smooth closed orientable 3-manifold M.

Note that, once the intersection form μ_M is given, the group $\bigwedge^2 H_2(M)^*/I_M$ can easily be calculated. For example, we have the following.

LEMMA 4.9. (1) *If* $\mu_M = 0$, *then* $\bigwedge^2 H_2(M)^*/I_M$ *is isomorphic to* $\bigwedge^2 H_2(M)^*$, *which is a free abelian group of rank* $b_1(M)(b_1(M)-1)/2$, *where* $b_1(M)$ *denotes the first Betti number of* M.

(2) *If* $b_1(M) = 0$ *or* 1, *then the group* $\bigwedge^2 H_2(M)^*/I_M$ *is trivial.*

(3) *If* $b_1(M) = 2$, *then the group* $\bigwedge^2 H_2(M)^*/I_M$ *is an infinite cyclic group.*

(4) *If* $b_1(M) = 3$, *then* $\bigwedge^2 H_2(M)^*/I_M$ *is isomorphic to* $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_d$, *where* $d = |\mu_M(\alpha_1, \alpha_2, \alpha_3)|$ *for a basis* $\{\alpha_1, \alpha_2, \alpha_3\}$ *of* $H_2(M)$.

PROOF. (1) is obvious. (2) and (3) follow from (1) together with the fact that $\mu_M = 0$ unless $b_1(M) \geq 3$. For (4), it is not difficult to see that $\bigwedge^2 H_2(M)^*$ is isomorphic to $\mathbf{Z}^3$ and that I_M is generated by $(0, 0, \pm d)$, $(0, \pm d, 0)$ and $(\pm d, 0, 0)$ with respect to an appropriate basis for $\bigwedge^2 H_2(M)^*$. Then the result follows immediately. This completes the proof. ||

Using the argument similar to the proof of Lemma 4.9 (4), we can calculate the abelian group $\bigwedge^2 H_2(M)^*/I_M$ from the intersection form μ_M.

As a corollary to the above calculations and Theorem 4.5, we obtain the following.

COROLLARY 4.10. (1) *Let M be an integral homology 3-sphere. Then the embedding maps of M° into S^5 are unique up to isotopy.*

(2) *Let M be a closed 3-manifold with the same integral homology groups as $S^1 \times S^2$. Then the isotopy classes of embedding maps of M° into S^5 correspond bijectively to $H^2(M) \cong \mathbf{Z}$, where to an embedding corresponds one half of its Euler class of the normal bundle.*

(3) *Let M be a closed 3-manifold with the same integral homology groups as $S^1 \times S^2 \# S^1 \times S^2$. Then the isotopy classes of embedding maps of M° into S^5 are in one-to-one correspondence with $H^2(M) \oplus \mathbf{Z} \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. Here, to an embedding $f : M^\circ \to S^5$ corresponds one half of the Euler class in $H^2(M)$ and the linking number $\mathrm{lk}(\alpha', \beta')$ in S^5, where $\{\alpha, \beta\}$ is a fixed basis of $H_2(M), \alpha' = (f_a)_*\alpha, \beta'$ is the homology class in $S^5 - f_a(M^\circ)$ which is obtained by pushing off $(f_\alpha)_*\beta$ into a normal direction of $f_a(M^\circ)$, and f_a is the embedding with trivial normal bundle obtained from f by the twisting operation with respect to a properly embedded oriented arc a in M° Poincaré dual to $e(f)/2$.*

REMARK 4.11. We do not know if a classification as in Theorem 4.5 is also possible for 3-manifolds M with $H_1(M)$ not necessarily torsion free. Note that the argument in §2 works completely for such general cases as well, and thus the invariant Seifert form can be defined also in such cases.

5. Higher dimensional cases

Let M be a smooth closed $(2n-1)$-dimensional $(n-2)$-connected manifold $(n \geq 2)$ which is stably parallelizable. When $n = 2$, this is equivalent to that M is a smooth closed connected orientable 3-manifold; in other words, the above class of $(2n-1)$-dimensional manifolds can be regarded as a generalization of such a class of 3-manifolds. In this section, we assume that $H_{n-1}(M)$ is torsion free and consider the classification of the embedding maps of the punctured manifold $M^\circ = M - \mathrm{Int}\, D^{2n-1}$ into S^{2n+1} for $n \geq 3$ under a certain equivalence relation.

Recall that the manifold M admits a handlebody decomposition which consists of one 0-handle, some $(n-1)$- and n-handles and one $(2n-1)$-handle (see [**Sm1**, §6] for $n \geq 4$ and [**Sm2**] for $n = 3$). Furthermore, there exists a smooth closed $(2n-1)$-dimensional manifold M_1 which bounds a smooth compact $2n$-dimensional $(n-1)$-connected parallelizable manifold W such that $M - \mathrm{Int}\, D^{2n-1}$ is diffeomorphic to $M_1 - \mathrm{Int}\, D^{2n-1}$ (see [**KM**, Theorem 6.6] or

[**W1**, § 13]). Since we will consider only the punctured manifold $M^\circ = M - \operatorname{Int} D^{2n-1}$, we may assume, from the beginning, that $M = M_1$. Note that the manifold W admits a handlebody decomposition which consists of one 0-handle and some n-handles. Thus, in particular, W can be embedded into S^{2n+1} and hence M° can also be embedded into S^{2n+1} (see [**Ke**, Chapitre II, § 6]).

Such a $(2n-1)$-dimensional manifold as M above arises, for example, as the link of an isolated hypersurface singularity in $\mathbf{C}^{n+1}$ (see [**M**]). In this case, the manifold W corresponds to the Milnor fiber.

REMARK 5.1. For classifications of smooth closed $(2n-1)$-dimensional $(n-2)$-connected manifolds ($n \geq 3$) which are stably parallelizable, see [**Ta, D, Sm2**].

Let $f : M^\circ \to S^{2n+1}$ be a smooth embedding. Suppose that there exists an embedding $\tau : M^\circ \times D^2 \to S^{2n+1}$ such that $\tau | M^\circ \times \{0\} = f$. We call such an embedding τ a *trivialization of the normal bundle of* f. Then we define the bilinear form

$$L^\tau : H_n(M) \times H_n(M) \to \mathbf{Z}$$

by the same way as in the case $n = 2$ (see § 2).

It is easy to see that, if $n \geq 4$, then the normal bundle of f is always trivial, since $H^2(M) = 0$. Furthermore, when the normal bundle of f is trivial, its trivialization is unique up to homotopy, if $n \geq 3$, since $H^1(M) = 0$. Thus we obtain the following.

LEMMA 5.2. *Suppose that the embedding* $f : M^\circ \to S^{2n+1}$ *has trivial normal bundle. If* $n \geq 3$, *then the Seifert form* L^τ *is independent of the choice of the trivialization* $\tau : M^\circ \times D^2 \to S^{2n+1}$ *of the normal bundle of* f *and is an invariant of the isotopy class of* f.

In the above situation, we denote the Seifert form L^τ by $L(f)$ and we call it the *invariant Seifert form* of f. Note that $L(f)$ is a skew symmetric bilinear form on $H_n(M)$ if n is even, while it is a symmetric bilinear form on $H_n(M)$ if n is odd (see the proof of Lemma 2.1). We denote by $\bigwedge^2 H_n(M)^*$ the space of all integral skew symmetric bilinear forms on $H_n(M)$ and by $\bigcirc^2 H_n(M)^*$ the space of all integral symmetric bilinear forms on $H_n(M)$. Thus we have $L(f) \in \bigwedge^2 H_n(M)^*$ if n is even and $L(f) \in \bigcirc^2 H_n(M)^*$ if n is odd.

DEFINITION 5.3. Let f and $g : M^\circ \to S^{2n+1}$ be two smooth embeddings. We say that f and g are *almost isotopic* if there exists an orientation preserving diffeomorphism $h : M^\circ \to M^\circ$ which induces the identity on the homology groups $H_*(M^\circ)$ such that f and $g \circ h$ are isotopic as embedding maps.

It is easy to see that the invariant Seifert form $L(f)$ of an embedding f is an invariant of the almost isotopy class of f. Then by using arguments similar to §3, we obtain the following.

THEOREM 5.4. *Let M be a smooth closed $(n-2)$-connected $(2n-1)$-dimensional manifold which is stably parallelizable $(n \geq 3)$. Suppose that $H_{n-1}(M)$ is torsion free.*

(1) *If $n = 3$, then the almost isotopy classes of smooth embedding maps $f : M^{\circ} \to S^7$ with trivial normal bundles correspond bijectively to the finitely generated free abelian group $\bigcirc^2 H_3(M)^*$ by the correspondence $f \mapsto L(f)$.*

(2) *If $n > 3$, then the almost isotopy classes of smooth embedding maps $f : M^{\circ} \to S^{2n+1}$ correspond bijectively to the finitely generated free abelian group $\bigwedge^2 H_n(M)^*$ (or $\bigcirc^2 H_n(M)^*$) by the correspondence $f \mapsto L(f)$ if n is even (resp. odd).*

PROOF. By Lemma 5.2, the correspondence is well defined.

Let us first show that the correspondence is surjective. Let L be an arbitrary element of $\bigwedge^2 H_n(M)^*$ (or $\bigcirc^2 H_n(M)^*$) when n is even (resp. odd). By the same argument as in the proof of Theorem 3.1, we can regard $H_n(M)$ as a direct summand of $H_n(W)$, where W is the $2n$-dimensional manifold as in the second paragraph of this section and we identify M with ∂W. Furthermore, the intersection form on $H_n(W)$ restricted to $H_n(M)$ vanishes and the bilinear form $L + (-1)^n {}^tL$ also vanishes, where ${}^tL$ is the form defined by ${}^tL(\alpha, \beta) = L(\beta, \alpha)$ $(\alpha, \beta \in H_n(M))$. Thus, since W is parallelizable, there exists a bilinear form $\tilde{L}$ on $H_n(W)$ which extends L such that $\tilde{L} + (-1)^n({}^t\tilde{L})$ coincides with the intersection form of W. Then by using the method of Kervaire [**Ke**, Chapitre II, §6], we can construct an embedding of W into S^{2n+1} whose Seifert form coincides with $\tilde{L}$. Then restricting this embedding to $M^{\circ}(\subset M = \partial W)$, we obtain a desired embedding.

Let us now prove the injectivity. Let f and $g : M^{\circ} \to S^{2n+1}$ be two smooth embeddings with trivial normal bundles such that $L(f) = L(g)$. Set $K = f(M^{\circ})$ and $K' = g(M^{\circ})$. Since M° admits a handlebody decomposition consisting of m-handles with $m \leq n$, we see that both $\pi_{n-1}(S^{2n+1} - K)$ and $\pi_{n-1}(S^{2n+1} - K')$ vanish, by using an argument similar to that in the proof of Lemma 3.3. Then by an argument similar to that in the proof of Lemma 3.2, we see that there exist smooth embeddings $\tilde{f}$ and $\tilde{g} : W \to S^{2n+1}$ such that $\tilde{f}|M^{\circ} = f$ and $\tilde{g}|M^{\circ} = g$.

If $n \geq 4$, then by [**Sm1**, Theorem 6.3], M° admits a handlebody decomposition consisting of a 0-handle $(h')^0$, $(n-1)$-handles $(h')_1^{n-1}, \dots, (h')_s^{n-1}$, and n-handles $(h')_1^n, \dots, (h')_s^n$, where $s = \operatorname{rank} H_n(M^{\circ})$, since $H_{n-1}(M^{\circ})$ is torsion free by our assumption. If $n = 3$, then the existence of such a handlebody decomposition follows directly from the classification theorem of spin 5-manifolds due to Smale [**Sm2**] (in fact, for $n = 3$, M is diffeomorphic to the connected sum of $S^2 \times S^3$). Let a_j be the cocore $(n-1)$-disk of $(h')^n_j$ and let $N(a_j)$ be a sufficiently small tubular neighborhood of a_j in $M^\circ (j = 1, \ldots, s)$ (in fact, $N(a_j)$ can be identified with the n-handle $(h')^n_j$).

Furthermore, by an argument similar to that in the proof of Lemma 3.2, we see that W has the handlebody decomposition

$$W = (M^\circ \times [0,1]) \cup (h^n_1)^* \cup \cdots \cup (h^n_r)^*$$

for some $r(\geq s)$, where $(h^n_i)^*$ are dual n-handles attached to $M^\circ \times [0,1]$ on $M^\circ \times \{1\}$. We may assume that the image of the attaching map of $(h^n_i)^*$, which is diffeomorphic to $S^{n-1} \times D^n$, does not intersect $N(a_j) \times \{1\}$ for all i and j. Furthermore, as in the proof of Theorem 3.1 (see also Remark 3.6), we may assume that

$$\Delta = \left(\overline{M^\circ - \bigcup_{j=1}^s N(a_j)} \times [0,1]\right) \cup ((h^n_{k+1})^* \cup \cdots \cup (h^n_r)^*)$$

is an integral homology $2n$-disk and that W can be obtained by attaching the n-handles $N(a_i) \times [0,1]$ $(i = 1, \ldots, s)$ and $(h^n_j)^*$ $(j = 1, \ldots, k)$ to Δ. In fact, by using the generalized Poincaré conjecture proved by Smale [**Sm1**, Theorem 5.1], it is not difficult to show that Δ is diffeomorphic to the $2n$-dimensional disk.

Then, by using the method of Kervaire [**Ke**, Chapitre II, §6], we can modify the embedding $\tilde{g}$ so that the Seifert forms of $\tilde{f}$ and $\tilde{g}$ coincide with each other as in the proof of Theorem 3.1.

Then, again by using the argument of Kervaire [**Ke**, Chapitre II, §6], we can show that the images of $\tilde{f}$ and $\tilde{g}$ are isotopic to each other. Thus there exists a diffeomorphism $\tilde{h} : W \to W$ such that $\tilde{f}$ and $\tilde{g} \circ \tilde{h}$ are isotopic as embedding maps and that $\tilde{h}$ induces the identity on $H_n(W)$. Furthermore, we may assume that $\tilde{h}$ is orientation preserving and that $\tilde{h}|\Delta$ is the identity map. Set $h = \tilde{h}|M^\circ$. Then we see that $h|\overline{(M^\circ - \bigcup_{j=1}^s N(a_j))}$ is the identity map and, in particular, $h_* : H_{n-1}(M^\circ) \to H_{n-1}(M^\circ)$ is the identity map. Thus we see that f and $g \circ h$ are isotopic as embedding maps, that h is orientation preserving, and that $h_* : H_j(M^\circ) \to H_j(M^\circ)$ is the identity map for $j = n-1$ and n. Thus f and g are almost isotopic. Hence the correspondence $f \mapsto L(f)$ is injective. This completes the proof of Theorem 5.4. ||

Remark 5.5. We do not know if almost isotopic embeddings are always isotopic as embedding maps. As the result in §3 shows, this is true for $n = 2$, provided that the normal bundles are trivial.

Remark 5.6. When $n = 3$, we do not know if we can obtain a result similar to Theorem 4.5 concerning a classification of general embedding maps with not necessarily trivial normal bundles.

References

[D] R. De Sapio, On $(k-1)$-connected $(2k+1)$-manifolds, Math. Scand. **25** (1969), 181–189.

[H] M. W. Hirsch, "Differential topology," Graduate Texts in Math., vol. 33, Springer, New York, Heidelberg, Berlin, 1976.

[Ka] S. Kaplan, Constructing framed 4-manifolds with given almost framed boundaries, Trans. Amer. Math. Soc. **254** (1979), 237–263.

[Ke] M. Kervaire, Les nœuds de dimensions supérieures, Bull. Soc. Math. France **93** (1965), 225–271.

[KM] M. A. Kervaire and J. W. Milnor, Groups of homotopy spheres: I, Ann. of Math. **77** (1963), 504–537.

[Le] J. Levine, An algebraic classification of some knots of codimension two, Comment. Math. Helv. **45** (1970), 185–198.

[Li] W. B. R. Lickorish, A representation of orientable combinatorial 3-manifolds, Ann. of Math. **76** (1962), 531–540.

[M] J. Milnor, "Singular points of complex hypersurfaces," Ann. of Math. Stud., vol. 61, Princeton Univ. Press, Princeton, N. J., 1968.

[Sa1] O. Saeki, On simple fibered knots in S^5 and the existence of decomposable algebraic 3-knots, Comment. Math. Helv. **62** (1987), 587–601.

[Sa2] O. Saeki, Knotted homology 3-spheres in S^5, J. Math. Soc. Japan **40** (1988), 65–75.

[Sa3] O. Saeki, Theory of fibered 3-knots in S^5 and its applications, preprint, Hiroshima University, 1996.

[Sm1] S. Smale, On the structure of manifolds, Amer. J. Math. **84** (1962), 387–399.

[Sm2] S. Smale, On the structure of 5-manifolds, Ann. of Math. **75** (1962), 38–46.

[Su] D. Sullivan, On the intersection ring of compact three manifolds, Topology **14** (1975), 275–277.

[Ta] I. Tamura, Classifications des variétés différentiables, $(n-1)$-connexes, sans torsion, de dimension $2n+1$, Séminaire Henri Cartan **15** (1962/63), Exp. 16 to 19.

[Ti] R. Tindell, Classifying bounded 2-manifolds in S^4, Osaka J. Math. **7** (1970), 173–177.

[W1] C. T. C. Wall, All 3-manifolds imbed in 5-space, Bull. Amer. Math. Soc. **71** (1965), 564–567.

[W2] C. T. C. Wall, Classification problems in differential topology—VI, classification of $(s-1)$-connected $(2s+1)$-manifolds, Topology **6** (1967), 273–296.

Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan
E-mail address: saeki@math.sci.hiroshima-u.ac.jp