HIROSHIMA MATH. J.
29 (1999), 245–253

Unitary groups and pairings of classifying spaces

Kenshi ISHIGURO and Maki TOKUZAWA

(Received May 25, 1998)

ABSTRACT. We consider the maps between classifying spaces of the form $BK \times BL \to BG$. If the restriction map $BL \to BG$ is a weak epimorphism, then the restriction on BK is known to factor through the classifying spaces of the center of the compact Lie group G. Replacing the weak epimorphism $BL \to BG$ by the map $BSU(n) \to BU(n)$, analogous results are obtained. The method of our proof is, however, different from the one used for the discussion about weak epimorphisms. Namely we will use not mapping spaces but admissible maps.

The first author [9] and [10] has studied the pairing problem of classifying spaces for weak epimorphisms. In this paper we will consider the problem for a map which is not a weak epimorphism. As a test map, we take the map $BSU(n) \to BU(n)$ induced from the inclusion $i : SU(n) \to U(n)$. More precisely, for a connected compact Lie group K, we determine a subset of the homotopy set $[BK, BU(n)]$, denoted by $(Bi)^{\perp}(BK, BU(n))$, which consists of the homotopy classes of maps $\alpha : BK \to BU(n)$ such that there exists a map (called *a pairing*) $\mu : BK \times BSU(n) \to BU(n)$ satisfying $\mu|_{BK} \simeq \alpha$ and $\mu|_{BSU(n)} \simeq Bi$. We notice that $[\alpha] \in (Bi)^{\perp}(BK, BU(n))$ if and only if, for some μ, the following diagram is homotopy commutative:

$$\begin{array}{ccc} BSU(n) & & \\ \downarrow & \searrow^{Bi} & \\ BK \times BSU(n) & \xrightarrow{\mu} & BU(n) \\ \uparrow & \nearrow_{\alpha} & \\ BK & & \end{array}$$

Our results will indicate that the group theoretical analog also holds for some maps other than weak epimorphisms.

1991 *Mathematics Subject Classification.* 55R35, 55P15, 55P60.

Key words and phrases. Classifying spaces, admissible maps, p-compact groups, Lie groups, pairing.

Theorem 1. *For the inclusion $i : SU(n) \to U(n)$, if a connected compact Lie group K is semi-simple, then any map in $(Bi)^{\perp}(BK, BU(n))$ is null homotopic:*

$$(Bi)^{\perp}(BK, BU(n)) = 0$$

Corollary 2. *Suppose $Z(U(n))$ denotes the center of $U(n)$. Then the following hold:*

(1) *If $\alpha \in (Bi)^{\perp}(BU(k), BU(n))$, the map α factors through $BZ(U(n))$ up to homotopy.*

(2) *Moreover, we have $(Bi)^{\perp}(BU(k), BU(n)) = Hom(U(k), Z(U(n)))$.*

We recall [9, Proposition 1.1] that $\alpha \in (Bi)^{\perp}(BK, BU(n))$ if and only if the map $Bi : BSU(n) \to BU(n)$ factors through $\mathrm{map}(BK, BU(n))_{\alpha}$. The group K can be replaced by any subgroup H of K, and if H is a p-toral group for a prime p, work of [6] and [16] shows that the mapping space $\mathrm{map}(BH, BU(n))_{\beta}$ with $\beta \simeq \alpha|_{BH}$ is mod p equivalent to the classifying space of the centralizer of a group homomorphism $\rho : H \to U(n)$ which induces the map β, that is $\beta \simeq B\rho$. In each of [9] and [10], the pairing problem is reduced to an argument of such mapping spaces. In this paper, however, we do not use these mapping spaces. Instead, admissible maps on the mod p cohomology will be used. We note that this method works for the connected compact Lie groups at certain primes or certain p-compact groups, and gives another proof for many cases discussed in [9] and [10].

The admissible maps for other cohomology theories as well as the realizability as maps between classifying spaces have been studied. See, for example, [1], [3], [7], [13] and [17, §2] etc.

Some of the results in this paper first appeared in the second author's master thesis written under the direction of Professor Toshio Yoshida. The second author would like to thank her advisor for his help and encouragement. She would also like to thank Yusuke Kawamoto for his suggestions.

1. Admissible Maps and the Pairing Problem

For connected compact Lie groups G and K together with maximal tori T_G and T_K respectively, suppose $H^*(BG; \mathbf{F}_p) \cong H^*(BT_G; \mathbf{F}_p)^{W(G)}$ and $H^*(BK; \mathbf{F}_p) \cong H^*(BT_K; \mathbf{F}_p)^{W(K)}$. Here $W(G)$ and $W(K)$ denote the Weyl groups. Recall that $H^*(BG; \mathbf{F}_p)$ is isomorphic to $H^*(BT_G; \mathbf{F}_p)^{W(G)}$, for instance, if p does not divide the order of $W(G)$. For any map $f : BG \to BK$ we have the commutative diagram

$$\begin{array}{ccc} H^*(BT_K;\mathbf{F}_p) & \xrightarrow{\phi} & H^*(BT_G;\mathbf{F}_p) \\ \uparrow & & \uparrow \\ H^*(BK;\mathbf{F}_p) & \xrightarrow[f^*]{} & H^*(BG;\mathbf{F}_p) \end{array}$$

Here ϕ is *admissible* [2] and [1]; namely for any $w \in W(G)$ we can find $w' \in W(K)$ such that $w\phi = \phi w'$.

Recall that $H^*(BT^n;\mathbf{F}_p)$ is a polynomial ring in n variables of degree 2. Hence the admissible map ϕ over the Steenrod algebra can be regarded as a $rank(G) \times rank(K)$ matrix. For instance, using the idea of [1, Proposition 2.16], one sees that the admissible self-maps for $H^*(BU(n);\mathbf{F}_p) \cong H^*(BT^n;\mathbf{F}_p)^{\Sigma_n}$ have the following types of $n \times n$ matrices:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ & & \cdots & \\ & & \cdots & \\ b & b & \cdots & a \end{pmatrix}$$

Of course, the symmetric group $\Sigma_n = W(U(n))$ acts on $H^*(BT^n;\mathbf{F}_p)$ by the permutation representation.

A p-compact group defined in [5] is a loop space X such that X is $\mathbf{F}_p$-finite and that the classifying space BX is $\mathbf{F}_p$-complete. The p-completion $G_p^\wedge$ of a compact Lie group G is a p-compact group if $\pi_0(G)$ is a p-group. For odd dimensional sphere S^{2n-1}, it is known that its p-completion has a loop structure if n divides $p-1$. This is an example of p-compact groups other than compact Lie groups. More examples are known as Clark-Ewing p-compact groups [15, §2]. We note here that $H^*(B(S^{2n-1})_p^\wedge;\mathbf{F}_p) \cong H^*(BT^1;\mathbf{F}_p)^{\mathbf{Z}/n}$, and that the admissible maps are similarly obtained for maps between classifying spaces of p-compact groups X with $H^*(BX;\mathbf{F}_p) \cong H^*(BT_X;\mathbf{F}_p)^{W(X)}$.

Proposition 3. *Let $i: SU(n) \to U(n)$ be the natural inclusion. Suppose that for an odd prime p a space X is a connected p-compact group with maximal torus T_X and Weyl group $W(X)$ such that the mod p cohomology $H^*(BX;\mathbf{F}_p)$ is isomorphic to the ring of invariants $H^*(BT_X;\mathbf{F}_p)^{W(X)}$. If $f = (Bi)_p^\wedge$ and $\alpha \in f^\perp(BX, BU(n)_p^\wedge)$, then $\alpha^*: H^*(BU(n);\mathbf{F}_p) \to H^*(BX;\mathbf{F}_p)$ factors through $H^*(BZ(U(n));\mathbf{F}_p)$ over the Steenrod algebra.*

Proof. For $\alpha \in f^\perp(BX, BU(n)_p^\wedge)$ there is a pairing map μ which gives the following commutative diagram

$$\begin{array}{ccc} BSU(n)_p^{\wedge} & & \\ \downarrow & \overset{f}{\searrow} & \\ BK \times BSU(n)_p^{\wedge} & \xrightarrow{\mu} & BU(n)_p^{\wedge} \\ \uparrow & \underset{\alpha}{\nearrow} & \\ BK & & \end{array}$$

Assume $rank(X) = k$ so that $H^*(BX; \mathbf{F}_p) \cong H^*(BT^k; \mathbf{F}_p)^{W(X)}$. We note that $H^*(BU(n); \mathbf{F}_p) \cong H^*(BT^n; \mathbf{F}_p)^{W(U(n))}$ and $H^*(BSU(n); \mathbf{F}_p) \cong H^*(BT^{n-1}; \mathbf{F}_p)^{W(SU(n))}$ for any odd prime p. Consider the admissible map ϕ which gives the commutative diagram

$$\begin{array}{ccc} H^*(BT^n; \mathbf{F}_p) & \xrightarrow{\phi} & H^*(BT^k; \mathbf{F}_p) \otimes H^*(BT^{n-1}; \mathbf{F}_p) \\ \uparrow & & \uparrow \\ H^*(BU(n); \mathbf{F}_p) & \xrightarrow[\mu^*]{} & H^*(BX; \mathbf{F}_p) \otimes H^*(BSU(n); \mathbf{F}_p) \end{array}$$

We recall that ϕ can be regarded as a $(k+n-1) \times n$ matrix. If ϕ_X is a $k \times n$ matrix expressing the admissible map which covers α^* and ϕ_S is a $(n-1) \times n$ matrix expressing the admissible map which covers f^*, then the $(k+n-1) \times n$ matrix ϕ is decomposed as follows:

$$\phi = \begin{pmatrix} \phi_X \\ \phi_S \end{pmatrix}$$

The $(n-1) \times n$ matrix ϕ_S is given by the following:

$$\phi_S = \begin{pmatrix} 1 & & 0 & -1 \\ & \ddots & & \vdots \\ 0 & & 1 & -1 \end{pmatrix}$$

and $W(SU(n))$ is isomorphic to the symmetric group Σ_n. The representation as a subgroup of $GL(n-1, \mathbf{F}_p)$ which makes $H^*(BSU(n); \mathbf{F}_p) \cong H^*(BT^{n-1}; \mathbf{F}_p)^{W(SU(n))}$ is generated by the permutation representation of Σ_{n-1} together with the following $(n-1) \times (n-1)$ matrix:

$$\begin{pmatrix} 1 & & 0 & -1 \\ & \ddots & & \vdots \\ 0 & & 1 & -1 \\ 0 & \cdots & 0 & -1 \end{pmatrix}$$

For instance, if $n = 4$, the subgroup of $GL(3, \mathbf{F}_p)$ isomorphic to the symmetric group is generated by the following matricies:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Since ϕ is admissible, for any $\sigma \in W(SU(n))$ we can find $\sigma' \in W(U(n))$ such that

$$\begin{pmatrix} I_k & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} \phi_X \\ \phi_S \end{pmatrix} = \begin{pmatrix} \phi_X \\ \phi_S \end{pmatrix} \cdot \sigma'$$

where I_k denotes the $k \times k$ identity matrix. For $\sigma_1, \sigma_2 \in W(SU(n))$, we see that $\sigma_1 \phi_S = \sigma_2 \phi_S$ implies $\sigma_1 = \sigma_2$. Hence the set of $(n-1) \times n$ matrices $\{\sigma \phi_S \mid \sigma \in W(SU(n))\}$ has $n!$ elements. Consequently the admissibility of ϕ tells us that $\phi_X \sigma' = \phi_X$ for any $\sigma' \in W(U(n))$. This implies that all column vectors are the same:

$$\phi_X = \begin{pmatrix} a_1 & \cdots & a_1 \\ \vdots & & \vdots \\ a_k & \cdots & a_k \end{pmatrix}$$

Recall that the center $Z(U(n))$ consists of the following diagonal matrices:

$$\begin{pmatrix} \zeta & & \\ & \ddots & \\ & & \zeta \end{pmatrix}$$

where $\zeta \in S^1$. So the admissible map ϕ_X which covers α^* is expressed as the product of two admissible maps:

$$\phi_X = \begin{pmatrix} a_1 & \cdots & a_1 \\ \vdots & & \vdots \\ a_k & \cdots & a_k \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} (1 \cdots 1)$$

Thus we obtain the desired factorization of homomorphism:

$$\begin{array}{ccc} H^*(BU(n); \mathbf{F}_p) & \xrightarrow{\alpha^*} & H^*(BX; \mathbf{F}_p) \\ \downarrow & \nearrow & \\ H^*(BZ(U(n)); \mathbf{F}_p) & & \end{array}$$

This completes the proof. □

REMARK. In Proposition 3, the prime p is assumed to be odd. When $p = 2$, the analogous result holds for $n \geq 3$, since in this case $H^*(BSU(n); \mathbf{F}_2) \cong H^*(BT^{n-1}; \mathbf{F}_2)^{W(SU(n))}$. If $n = 2$, however, we note that $H^*(BSU(2); \mathbf{F}_2)$ is a polynomial ring generated by the degree 4 element which is not isomorphic to the ring of invariants. This means that the admissibility won't work.

Note that Proposition 3 is a result about $\mathscr{A}_p$-maps, homomorphisms over the mod p Steenrod algebra $\mathscr{A}_p$. We claim that there is an $\mathscr{A}_2$-map

$$\varphi : H^*(BU(2); \mathbf{F}_2) \to H^*(BSU(2); \mathbf{F}_2) \otimes H^*(BSU(2); \mathbf{F}_2)$$

such that each of the restrictions $H^*(BU(2); \mathbf{F}_2) \to H^*(BSU(2); \mathbf{F}_2)$ is induced from the map $BSU(2) \to BU(2)$. The $\mathscr{A}_2$-map $H^*(BU(2); \mathbf{F}_2) \to H^*(BSU(2); \mathbf{F}_2)$ does not factor through $H^*(BZ(U(2)); \mathbf{F}_2)$ for degree reason. The existence of the $\mathscr{A}_2$-map φ is merely algebraic. Geometrically, using admissible maps on 2-adic K-theory, one can show that there is no map $BSU(2)_2^\wedge \times BSU(2)_2^\wedge \to BU(2)_2^\wedge$ whose restrictions are induced from the inclusion $SU(2) \to U(2)$.

Here we recall that $H^*(BO(2); \mathbf{F}_2) = \mathbf{F}_2[w_1, w_2]$ with $deg(w_i) = i$, and $H^*(BU(2); \mathbf{F}_2) = \mathbf{F}_2[c_1, c_2]$ with $deg(c_i) = 2i$. Turning back to the existence of φ, the $\mathscr{A}_2$-map is obtained from the following observation: Doubling the degree, we can see that the $\mathscr{A}_2$-algebra structures of $H^*(BO(2); \mathbf{F}_2)$ and $H^*(BSO(2); \mathbf{F}_2)$ are same as those of $H^*(BU(2); \mathbf{F}_2)$ and $H^*(BSU(2); \mathbf{F}_2)$ respectively. Since $SO(2)$ is abelian, there is a multiplication $BSO(2) \times BSO(2) \to BSO(2)$, and the composition with $BSO(2) \to BO(2)$ gives us a map

$$BSO(2) \times BSO(2) \to BO(2)$$

Doubling the degree of the $\mathscr{A}_2$-map obtained from this map produces the $\mathscr{A}_2$-maps φ.

2. Proof of the Main Result

Using Proposition 3, we will prove Theorem 1 and Corollary 2, which give some results about a map other than a weak epimorphism. For connected compact Lie groups L and G, a map $BL \to BG$ is called a *weak epimorphism* [11], if we have a fibration $F \to BL \to BG$ such that $H^*(\Omega F; \mathbf{Q})$ is a finite dimensional $\mathbf{Q}$-module. The map $BSU(n) \to BU(n)$ with $F = S^1$ can not be a weak epimorphism.

PROOF OF THEOREM 1. For $\beta \in (Bi)^\perp(BK, BU(n))$, if $f = (Bi)_p^\wedge$ and $\alpha = \beta_p^\wedge : BK_p^\wedge \to BU(n)_p^\wedge$, then

$$\alpha \in f^\perp(BK_p^\wedge, BU(n)_p^\wedge).$$

Recall that if p does not divide the order of the Weyl group $W(K)$, then $H^*(BK;\mathbf{F}_p) \cong H^*(BT_K;\mathbf{F}_p)^{W(K)}$. In fact, it is known that if p is odd and K is p-torsion free, $H^*(BX;\mathbf{F}_p)$ is isomorphic to the ring of invariants. Since $K_p^\wedge$ is a connected p-compact group, Proposition 3 implies that $\alpha^* : H^*(BU(n);\mathbf{F}_p) \to H^*(BK_p^\wedge;\mathbf{F}_p)$ factors through $H^*(BZ(U(n));\mathbf{F}_p)$. Notice that the restriction $\beta|_{BT_K}$ is induced from a homomorphism from T_K into a maximal torus T^n of $U(n)$. Notice also that the homotopy set $[BT_K, BT^n]$ is completely determined by matrices whose entries are integer. Consequently $\beta|_{BT_K}$ must factor through $BZ(U(n))$. Since the connected compact Lie group K is semi-simple, its universal covering $\tilde{K}$ is a product group $\tilde{K}_1 \times \tilde{K}_2 \times \cdots \tilde{K}_r$ where each 1-connected Lie group $\tilde{K}_i$ $(1 \leq i \leq r)$ is simple. Let $q : \tilde{K} \to K$ be the projection. A result of [8] shows $[B\tilde{K}_i, BG] = 0$ for any connected compact Lie group G with $\operatorname{rank}(\tilde{K}_i) > \operatorname{rank}(G)$. Since $\beta|_{BT_K}$ factors through $BZ(U(n))$, we can show that each of the maps $B\tilde{K}_i \to BU(n)$ factors through $BZ(U(n))$, using the fibration $BZ(U(n)) \to BU(n) \to B(U(n)/Z(U(n)))$. Hence, if $\operatorname{rank}(\tilde{K}_i) \geq 2$, the restriction $\beta \cdot Bq|_{B\tilde{K}_i}$ is null homotopic, since $Z(U(n)) \cong S^1$. If $\tilde{K}_i = S^3 = SU(2)$ and $\xi = \beta \cdot Bq|_{B\tilde{K}_i}$, then $\xi \in (Bi)^\perp(BSU(2), BU(n))$. For an odd prime p, an argument analogous to the one we used in the proof of Proposition 3 is applicable:

$$\begin{array}{ccc} H^*(BT^n;\mathbf{F}_p) & \xrightarrow{\ \phi\ } & H^*(BS^1;\mathbf{F}_p) \otimes H^*(BT^{n-1};\mathbf{F}_p) \\ \uparrow & & \uparrow \\ H^*(BU(n);\mathbf{F}_p) & \xrightarrow[\mu^*]{} & H^*(BS^3;\mathbf{F}_p) \otimes H^*(BSU(n);\mathbf{F}_p) \end{array}$$

For ϕ_X with $\phi = \begin{pmatrix} \phi_X \\ \phi_S \end{pmatrix}$ in the proof of Proposition 3 taking $X = (S^3)_p^\wedge$, we see $\phi_X = (a \cdots a)$ for some integer a. Note that the Weyl group of S^3 is $\mathbf{Z}/2 = \{\pm 1\}$, and we have

$$\begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} \phi_X \\ \phi_S \end{pmatrix} = \begin{pmatrix} \phi_X \\ \phi_S \end{pmatrix} \cdot \sigma'$$

for some $\sigma' \in W(U(n))$. This implies $-a = a$ so that $a = 0$. Consequently $\beta \cdot Bq|_{B\tilde{K}_i} 0$ for any i. Therefore $\beta|_{BT_K} \simeq 0$ and hence $\beta \simeq 0$ by a result of [12]. □

Proof of Corollary 2. Suppose $\alpha \in (Bi)^\perp(BU(k), BU(n))$. If $j : SU(k) \to U(k)$ is the inclusion, we see that the composite map $\alpha \cdot B_j$ is contained in $(Bi)^\perp(BSU(k), BU(n))$. Since $SU(k)$ is simple, Theorem 1 implies $\alpha \cdot Bj \simeq 0$. Thus we see that the map α factors through BS^1 up to homotopy:

$$\begin{array}{ccc} BSU(k) & & \\ {\scriptstyle Bj}\downarrow & \searrow^{\simeq 0} & \\ BU(k) & \xrightarrow{\alpha} & BU(n) \\ {\scriptstyle B\det}\downarrow & \nearrow & \\ BS^1 & & \end{array}$$

where the map $BS^1 \to BU(n)$ is induced from the identification of S^1 with the center of $U(n)$. An argument analogous to the one used in [9] and [10] implies the desired result. □

References

[1] J. F. Adams and Z. Mahmud, Maps between classifying spaces, Inventiones Math. **35**, (1976), 1-41.

[2] J. F. Adams and C. W. Wilkerson, Finite H-spaces and algebras over the Steenrod algebra, Ann. of Math. **111** (1980), 95–143.

[3] J. F. Adams and Z. Wojtkowiak, Maps between p-completed classifying spaces, Proc. of the Royal Soc. of Edinburgh **112A** (1989), 231–235.

[4] A. Bousfield and D. Kan, Homotopy limits, completions and localisations, LNM **304** (1972).

[5] W. G. Dwyer and C. W. Wilkerson, Homotopy fixed-point methods for Lie groups and finite loop spaces, Ann. of Math. **139(2)**, (1994), 395–442.

[6] W. G. Dwyer and A. Zabrodsky, Maps between classifying spaces, Proc. 1986 Barcelona conference, LNM **1298** (1987), 106–119

[7] K. Ishiguro, Classifying spaces and p-local irreducibility, J. of Pure and Applied Algebra **49** (1987), 254–258.

[8] K. Ishiguro, Classifying spaces of compact simple Lie groups and p-tori, Proc. of 1990 Barcelona Conf., LNM **1509** (1992), 210–226.

[9] K. Ishiguro, Classifying spaces and homotopy sets of axes of pairings, Proc. of AMS **124** (1996), 3897-3903.

[10] K. Ishiguro, Pairings of p-compact groups and H-structures on the classifying spaces of finite loop spaces, Preprint.

[11] K. Ishiguro and D. Notbohm, Fibrations of classifying spaces, Transaction of AMS **343(1)** (1994), 391–415.

[12] S. Jackowski, J. E. McClure and B. Oliver, Homotopy classification of self-maps of BG via G-actions Part I and Part II, Ann. of Math. **135** (1992), 183–226, 227–270.

[13] S. Jackowski, J. E. McClure and B. Oliver, Maps between classifying spaces revisited, Proc. of Cech Centennial Conf., Contemporary Math. **181** (1995), 263–298.

[14] H. Miller, The Sullivan conjecture on maps from classifying spaces, Ann. of Math. **120** (1984), 39–87.

[15] J. Møller, Homotopy Lie groups, Bull. of AMS **32(4)** (1995), 413–428.

[16] D. Notbohm, Maps between classifying spaces, Math. Z. **207** (1991), 153–168.

[17] D. Notbohm, Classifying spaces of compact Lie groups and finite loop spaces, Handbook of Algebraic Topology, North-Hollad (1995), 1049–1094.
[18] M. Tokuzawa, Pairing problem of classifying spaces for A-type compact Lie groups (in Japanese), Master thesis, Hiroshima University (1998).

Department of Applied Mathematics
Faculty of Science
Fukuoka University
Fukuoka 814-0180, Japan
e-mail: kenshi@ssat.fukuoka-u.ac.jp
and
Department of Mathematics
Faculty of Science
Hiroshima University
Hiroshima 739-8526, Japan
e-mail: m0871022@sci.hiroshima-u.ac.jp