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Measurability of multifractal measure functions and multifractal dimension functions

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ABSTRACT. For a Radon measure μ and q, $t \in \mathbf{R}$, let $\mathscr{H}_{\mu}^{q,t}$ and $\mathscr{P}_{\mu}^{q,t}$ denote the multifractal Hausdorff measure and the multifractal packing measure introduced in [L. Olsen, A Multifractal Formalism, *Advances in Mathematics* **116** (1996), 82–196]. Let $t \in \mathbf{R}$. We study the descriptive set theoretic complexity of the maps

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathscr{H}_{\mu}^{q,t}(K),$$

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathscr{P}_{\mu}^{q,t}(K),$$

and related multifractal measure and multifractal dimension maps; here $\mathscr{K}(\mathbf{R}^d)$ denotes the family of non-empty compact subsets of $\mathbf{R}^d$ equipped with the Hausdorff metric, and $\mathscr{M}(\mathbf{R}^d)$ denotes the family of Radon measures on $\mathbf{R}^d$ equipped with the weak topology.

CONTENTS

1. Introduction

Recently there has been a great interest in the multifractal structure of Borel measures μ on metric spaces. For $\alpha \geq 0$ write

$$\Delta_{\mu}(\alpha) = \left\{ x \in \operatorname{supp} \mu \mid \lim_{r \searrow 0} \frac{\log \mu B(x,r)}{\log r} = \alpha \right\}$$

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where $\operatorname{supp}\mu$ denotes the topological support of μ and $B(x,r)$ denotes the closed ball with center x and radius r, i.e. $\Delta_\mu(\alpha)$ denotes the set of those x for which $\mu B(x,r)$ behaves like r^α for r close to 0. The family $\{\Delta_\mu(\alpha)|\alpha \geq 0\}$ can be viewed as a "multifractal decomposition" of the support of μ into a family of (typically fractal) sets $\Delta_\mu(\alpha)$ indexed by α. The main problem in multifractal analysis is to compute the Hausdorff dimension and the packing dimension of $\Delta_\mu(\alpha)$, i.e. to compute

$$f_\mu(\alpha) = \dim \Delta_\mu(\alpha), \qquad F_\mu(\alpha) = \operatorname{Dim} \Delta_\mu(\alpha)$$

where dim and Dim denote Hausdorff dimension and packing dimension respectively. The functions f_μ and F_μ (and similar functions) are generically known as "the multifractal spectrum of μ". The function $f(\alpha) = f_\mu(\alpha)$ was first explicitly defined by the physicists Halsey et al. in 1986 in their seminal paper [HJKPS].

The multifractal spectra functions f_μ and F_μ have been computed for various classes of measures, cf. e.g. [AP, CM, EM, LN, MR, Ol1, Ol2, Ol3, Pey, Ra]. However, recently Olsen [Ol1], Pesin [Pes] and Peyriére [Pey] proposed a general multifractal formalism suited for analysing the properties of the spectra functions f_μ and F_μ for very general measures. This formalism is based on certain multifractal generalizations of the Hausdorff measure and the packing measure tailored for multifractal purposes, and has now been investigated further by a large number of authors, cf. Ben Nasr [BeN], Das [Da1, Da2], Levy–Vehel & Vojak [LV], Olsen [Ol2, Ol3, Ol4, Ol5], O'Neil [O'N1, O'N2] and Taylor [Ta]. For a Radon measure μ on a metric space X and $q, t \in \mathbf{R}$, let $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ denote the multifractal Hausdorff measure and the multifractal packing measure. For each $q \in \mathbf{R}$, using the measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$, we define, analogously to the Hausdorff dimension and the packing dimension, a multifractal Hausdorff dimension $\dim_\mu^q(E)$ and a multifractal packing dimension $\operatorname{Dim}_\mu^q(E)$ of subsets E of X (details will be given in the next section). It is natural to study the "smoothness" of the multifractal decomposition provided by the formalism in [Ol1, Pes, Pey]. We do this by studying the descriptive set theoretic complexity of the maps

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathscr{H}_\mu^{q,t}(K), \tag{1.1}$$

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathscr{P}_\mu^{q,t}(K), \tag{1.2}$$

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \dim_\mu^q(K), \tag{1.3}$$

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \operatorname{Dim}_\mu^q(K), \tag{1.4}$$

and related multifractal maps; here $\mathscr{K}(\mathbf{R}^d)$ denotes the family of non-empty compact subsets of $\mathbf{R}^d$ equipped with the Hausdorff metric, and $\mathscr{M}(\mathbf{R}^d)$ denotes the family of Radon measures on $\mathbf{R}^d$ equipped with the weak topology. We prove that the multifractal Hausdorff measure map (1.1) is measurable with respect to the σ-algebra generated by the analytic sets, and that the multifractal Hausdorff dimension map (1.3) (restricted to a suitable subspace determined by the family of doubling measures) is of Baire class 2. The measurability of the multifractal packing measure map (1.2) is discussed in Section 4, Remark (1), and we prove that the multifractal packing dimension map (1.4) (restricted to a suitable subspace determined by the family of doubling measures) is measurable with respect to the σ-algebra generated by the analytic sets. Some of our results can be viewed as multifractal extensions of the results in Mattila & Mauldin [MM]. For a Polish space X and a dimension function g, Mattila & Mauldin study the set theoretic complexity of the maps

$$\mathscr{K}(X) \to \bar{\mathbf{R}} : K \to \mathscr{H}^g(K),$$

$$\mathscr{K}(X) \to \bar{\mathbf{R}} : K \to \mathscr{P}^g(K),$$

and related measure and dimension maps; here $\mathscr{H}^g$ and $\mathscr{P}^g$ denote the Hausdorff measure and the packing measure generated by g, and $\mathscr{K}(X)$ denote the family of non-empty compact subsets of X.

A further reason for studying the measurability properties of the maps defined by (1.1) through (1.4) (and related multifractal maps) is provided by the following. The formalism based on the multifractal measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ leads to a multifractal geometry for measures which is analogous to the classical fractal geometry for sets, cf. [Ol4, Ol5, O'N2]. However, many arguments in [Ol4] and [Ol5] require (in order to apply Fubini's Theorem and Fatou's Lemma and other standard results from measure theory) that various maps defined in terms of $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ are (analytically) measurable. In fact, the measurability results in § 5 for slices of measures play an important part in the study of multifractal slices and negative dimensions in [Ol5].

We now give a brief description of the organization of the paper. In § 2 we collect the multifractal and topological (descriptive set theoretical) definitions and preliminary results we shall need. § 3 contains our analysis of the multifractal Hausdorff measure map (1.1) and the multifractal Hausdorff dimension map (1.3), and § 4 contains our analysis of the multifractal packing measure map (1.2) and the multifractal packing dimension map (1.4). Finally, in § 5 we apply the measurability results established in § 3 to study the measurability of multifractal slices.

2. The setting

2.1. Ordinary and multifractal Hausdorff measures and packing measures

We first recall the definition of the Hausdorff measure, the centered Hausdorff measure and the packing measure. Let X be a metric space, $E \subseteq X$ and $\delta > 0$. A countable family $\mathscr{B} = (B(x_i, r_i))_i$ of closed balls in X is called a centered δ-covering of E if $E \subseteq \bigcup_i B(x_i, r_i)$, $x_i \in E$ and $0 < r_i < \delta$ for all i. The family $\mathscr{B}$ is called a centered δ-packing of E if $x_i \in E$, $0 < r_i < \delta$ and $B(x_i, r_i) \cap B(x_j, r_j) = \varnothing$ for all $i \neq j$. Let $E \subseteq X$, $t \geq 0$ and $\delta > 0$. Now put

$$\mathscr{H}_\delta^t(E) = \inf \left\{ \sum_i \operatorname{diam}(E_i)^t \mid E \subseteq \bigcup_{i=1}^{\infty} E_i, \ \operatorname{diam} E_i < \delta \right\}.$$

The t-dimensional Hausdorff measure $\mathscr{H}^t(E)$ of E is defined by

$$\mathscr{H}^t(E) = \sup_{\delta > 0} \mathscr{H}_\delta^t(E).$$

Next we define the centered Hausdorff measure introduced by Raymond & Tricot in [RT]. Put

$$\bar{\mathscr{C}}_\delta^t(E) = \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^t \mid (B(x_i, r_i))_i \text{ is a centered } \delta\text{-covering of } E \right\}.$$

The t-dimensional centered pre-Hausdorff measure $\bar{\mathscr{C}}^t(E)$ of E is defined by

$$\bar{\mathscr{C}}^t(E) = \sup_{\delta > 0} \bar{\mathscr{C}}_\delta^t(E).$$

The set function $\bar{\mathscr{C}}^t$ is not necessarily monotone, and hence not necessarily an outer measure, c.f. [RT, pp. 137–138]. But $\bar{\mathscr{C}}^t$ gives rise to a Borel measure, called the t-dimensional centered Hausdorff measure $\mathscr{C}^t(E)$ of E, as follows

$$\mathscr{C}^t(E) = \sup_{F \subseteq E} \bar{\mathscr{C}}^t(F).$$

It is easily seen (c.f. [RT, Lemma 3.3]) that $2^{-t}\mathscr{C}^t \leq \mathscr{H}^t \leq \mathscr{C}^t$. We will now define the packing measure. Write

$$\bar{\mathscr{P}}_\delta^t(E) = \sup \left\{ \sum_{i=1}^{\infty} (2r_i)^t \mid (B(x_i, r_i))_i \text{ is a centered } \delta\text{-packing of } E \right\}.$$

The t-dimensional prepacking measure $\bar{\mathscr{P}}^t(E)$ of E is defined by

$$\bar{\mathscr{P}}^t(E) = \inf_{\delta > 0} \bar{\mathscr{P}}_\delta^t(E).$$

The set function $\bar{\mathscr{P}}^t$ is not necessarily countable subadditive, and hence not necessarily an outer measure, c.f. [TT]. But $\bar{\mathscr{P}}^t$ gives rise to a Borel measure, namely the t-dimensional packing measure $\mathscr{P}^t(E)$ of E, as follows

$$\mathscr{P}^t(E) = \inf_{E \subseteq \bigcup_{i=1}^{\infty} E_i} \sum_{i=1}^{\infty} \bar{\mathscr{P}}^t(E_i).$$

The packing measure was introduced by Taylor and Tricot in [TT] using centered δ-packings of open balls, and by Raymond and Tricot in [RT] using centered δ-packings of closed balls.

Also recall that the Hausdorff dimension $\dim(E)$, the packing dimension $\operatorname{Dim}(E)$ and the pre-packing dimension $\Delta(E)$ of E is defined by

$$\dim(E) = \sup\{t \geq 0 \mid \mathscr{H}^t(E) = \infty\}$$

$$\operatorname{Dim}(E) = \sup\{t \geq 0 \mid \mathscr{P}^t(E) = \infty\}$$

$$\Delta(E) = \sup\{t \geq 0 \mid \bar{\mathscr{P}}^t(E) = \infty\}.$$

We refer the reader to [Tr] and [RT] for more information on the centered Hausdorff measure, the packing measure and the packing dimension.

Olsen [Ol1] suggested that some multifractal generealizations of the centered Hausdorff measure and the packing measure might be useful in multifractal analysis. Let $\mathscr{M}(X)$ denote the family of positive Radon measures on X. For $\mu \in \mathscr{M}(X)$, $E \subseteq X$, $q, t \in \mathbf{R}$ and $\delta > 0$ write

$$\bar{\mathscr{H}}^{q,t}_{\mu,\delta}(E) = \inf\left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \mid (B(x_i, r_i))_i \text{ is a centered } \delta\text{-covering of } E \right\}, \qquad E \neq \varnothing$$

$$\bar{\mathscr{H}}^{q,t}_{\mu,\delta}(\varnothing) = 0$$

$$\bar{\mathscr{H}}^{q,t}_{\mu}(E) = \sup_{\delta > 0} \bar{\mathscr{H}}^{q,t}_{\mu,\delta}(E)$$

$$\mathscr{H}^{q,t}_{\mu}(E) = \sup_{F \subseteq E} \bar{\mathscr{H}}^{q,t}_{\mu}(F).$$

We also make the dual definitions

$$\bar{\mathscr{P}}^{q,t}_{\mu,\delta}(E) = \sup\left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \mid (B(x_i, r_i))_i \text{ is a centered } \delta\text{-packing of } E \right\}, \qquad E \neq \varnothing$$

$$\bar{\mathscr{P}}^{q,t}_{\mu,\delta}(\varnothing) = 0$$

$$\bar{\mathscr{P}}^{q,t}_{\mu}(E) = \inf_{\delta>0} \bar{\mathscr{P}}^{q,t}_{\mu,\delta}(E)$$

$$\mathscr{P}^{q,t}_{\mu}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \bar{\mathscr{P}}^{q,t}_{\mu}(E_i).$$

It is proven in [Ol1] that $\mathscr{H}^{q,t}_{\mu}$ and $\mathscr{P}^{q,t}_{\mu}$ are measures on the family of Borel subsets of X. The measure $\mathscr{H}^{q,t}_{\mu}$ is of course a multifractal generalisation of the centered Hausdorff measure, whereas $\mathscr{P}^{q,t}_{\mu}$ is a multifractal generalisation of the packing measure. In fact, it is easily seen that the follwing holds for $t \geq 0$,

$$2^{-t}\mathscr{H}^{0,t}_{\mu} \leq \mathscr{H}^{t} \leq \mathscr{H}^{0,t}_{\mu}, \qquad \mathscr{P}^{t} = \mathscr{P}^{0,t}_{\mu}, \qquad \bar{\mathscr{P}}^{t} = \bar{\mathscr{P}}^{0,t}_{\mu}. \tag{2.1.1}$$

The next result shows that the measures $\mathscr{H}^{q,t}_{\mu}$, $\mathscr{P}^{q,t}_{\mu}$ and the pre-measure $\bar{\mathscr{P}}^{q,t}_{\mu}$ in the usual way assign a dimension to each subset E of X.

PROPOSITION 2.1.1. *There exist unique extended real valued numbers* $\Delta^{q}_{\mu}(E) \in [-\infty, \infty]$, $\mathrm{Dim}^{q}_{\mu}(E) \in [-\infty, \infty]$ *and* $\dim^{q}_{\mu}(E) \in [-\infty, \infty]$ *such that*

$$\bar{\mathscr{P}}^{q,t}_{\mu}(E) = \begin{cases} \infty & \text{for } t < \Delta^{q}_{\mu}(E) \\ 0 & \text{for } \Delta^{q}_{\mu}(E) < t \end{cases}$$

$$\mathscr{P}^{q,t}_{\mu}(E) = \begin{cases} \infty & \text{for } t < \mathrm{Dim}^{q}_{\mu}(E) \\ 0 & \text{for } \mathrm{Dim}^{q}_{\mu}(E) < t \end{cases}$$

$$\mathscr{H}^{q,t}_{\mu}(E) = \begin{cases} \infty & \text{for } t < \dim^{q}_{\mu}(E) \\ 0 & \text{for } \dim^{q}_{\mu}(E) < t \end{cases}$$

PROOF. See [Ol1, Proposition 1.1]. □

The number $\dim^{q}_{\mu}(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim(E)$ of E whereas $\mathrm{Dim}^{q}_{\mu}(E)$ and $\Delta^{q}_{\mu}(E)$ are obvious multifractal analogues of the packing dimension $\mathrm{Dim}(E)$ and the pre-packing dimension $\Delta(E)$ of E respectively. In fact, it follows immediately from the definitions that

$$\dim(E) = \dim^{0}_{\mu}(E), \qquad \mathrm{Dim}(E) = \mathrm{Dim}^{0}_{\mu}(E), \qquad \Delta(E) = \Delta^{0}_{\mu}(E). \tag{2.1.2}$$

We now define the family of doubling measures and list some useful properties of the measures $\mathscr{H}^{q,t}_{\mu}$ and $\mathscr{P}^{q,t}_{\mu}$ (see Proposition 2.1.2 below). For $\mu \in \mathscr{M}(X)$ and $a > 1$ write $T_a(\mu) := \limsup_{r\searrow 0}\left(\sup_{x\in \operatorname{supp}\mu} \dfrac{\mu B(x,ar)}{\mu B(x,r)}\right)$ and define the family $\mathscr{M}_0(X)$ of doubling measures on X by $\mathscr{M}_0(X) = \{\mu \in \mathscr{M}(X) \mid T_a(\mu) < \infty \text{ for some } a > 1\}$. It follows from [Ol1] that the definition of

$\mathscr{M}_0(X)$ is independent of the number $a > 1$, i.e. $T_a(\mu) < \infty$ for all $a > 1$ if and only if $T_a(\mu) < \infty$ for some $a > 1$. The results in Proposition 2.1.2 below will be used tactically in several of the proofs in §3 and §4. Let $\mathscr{P}(X)$ denote the family of subsets of X. Recall that a set function $D : \mathscr{P}(X) \to [-\infty, \infty]$ is called monotone if $D(E) \leq D(F)$ for all $E, F \subseteq X$ with $E \subseteq F$, and that D is called σ-stable if $D\left(\bigcup_{n \in \mathbf{N}} E_n\right) = \sup_{n \in \mathbf{N}} D(E_n)$ for all countable families $(E_n)_{n \in \mathbf{N}}$ of subsets of X.

PROPOSITION 2.1.2. *Let $\mu \in \mathscr{M}(\mathbf{R}^d)$ and $q, t \in \mathbf{R}$. Then*
(i) $\mathscr{H}_\mu^{q,t} \leq \mathscr{P}_\mu^{q,t}$ *for* $\mu \in \mathscr{M}_0(\mathbf{R}^d)$, *and* $\mathscr{P}_\mu^{q,t} \leq \bar{\mathscr{P}}_\mu^{q,t}$ *for* $\mu \in \mathscr{M}(\mathbf{R}^d)$.
(ii) $\dim_\mu^q \leq \mathrm{Dim}_\mu^q \leq \Delta_\mu^q$.
(iii) $\dim_\mu^q$ *and* Dim_μ^q *are monotone and σ-stable, and* Δ_μ^q *is monotone.*

PROOF. See [Ol1]. □

REMARK. The main importance of the measures $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$ to multifractal analysis is due to the following relationship between the spectra functions f_μ and F_μ, and the dimensions $\dim_\mu^q$ and Dim_μ^q. Define multifractal dimension functions $b_\mu, B_\mu : \mathbf{R} \to [-\infty, \infty]$ by $b_\mu(q) = \dim_\mu^q(\operatorname{supp} \mu)$ and $B_\mu(q) = \mathrm{Dim}_\mu^q(\operatorname{supp} \mu)$. It now follows from [Ol1, Theorem 2.17] that $f_\mu \leq b_\mu^*$ and $F_\mu \leq B_\mu^*$ where b_μ^* and B_μ^* denote the Legendre transform of b_μ and B_μ respectively (for a real valued function $f : \mathbf{R} \to \mathbf{R}$, we define the Legendre transform $f^* : \mathbf{R} \to [-\infty, \infty]$ of f by $f^*(x) = \inf_y(xy + f(y))$). These inequalities can be viewed as rigorous mathematical analogous of the so-called "Multifractal Formalism" in the physics literature (cf. [HJKPS]).

2.2. Topological definitions and preliminaries

The Hausdorff metric and $\mathscr{K}(X)$.

For a metric space (X, d), let $\mathscr{K}(X)$ denote the family of non-empty compact subsets of X. We will always equip $\mathscr{K}(X)$ with the topology generated by the Hausdorff metric D on $\mathscr{K}(X)$,

$$D(K, L) = \max\left(\sup_{x \in K} d(x, L), \sup_{x \in L} d(x, K)\right),$$

where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$ for $x \in X$ and $A \subseteq X$. It is well-known that $\mathscr{K}(X)$ is Polish if X is Polish (cf. e.g. [En, 4.5.23]). We also consider the Hausdorff metric $D_\varnothing$ on $\mathscr{K}(X) \cup \{\varnothing\}$,

$$D_\varnothing(K, L) = \begin{cases} 0 & \text{if } K = \varnothing \text{ and } L = \varnothing, \\ 1 & \text{if exactly one of the sets } K \text{ and } L \text{ is } \varnothing, \\ D(K, L) & \text{if } K \neq \varnothing \text{ and } L \neq \varnothing \end{cases}$$

and the space $\mathscr{K}(X) \cup \{\varnothing\}$will always be endowed with the topology generated by $D_\varnothing$.

The weak topology and $\mathscr{M}(X)$.

The weak topology on the space $\mathscr{M}(X)$ of positive Radon measures on a metric space X is the topology generated by the functionals

$$\mathscr{M}(X) \to \mathbf{R} : \mu \to \int \varphi \, d\mu,$$

where φ varies over the family of continuous non-negative functions on X with compact support. We remark that the weak topology on $\mathscr{M}(X)$ is also sometimes called the vague topology. The family $\mathscr{M}(X)$ of Radon measures will always be equipped with the weak topology. It is well-known that $\mathscr{M}(\mathbf{R}^d)$ is a Polish space (cf. e.g. [Ma2, Remark 14.15]). In particular, the space $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$ is Polish.

The Borel Hierarchy and Baire functions.

We will now briefly describe the Borel Hierarchy used in the classification of the smoothness of the maps in (1.1)–(1.4). Let X be a metric space. For an ordinal γ with $1 \leq \gamma < \omega_1$ (where ω_1 is the first uncountable cardinal) we define the Baire classes $\Sigma_\gamma^0(X) = \Sigma_\gamma^0$ and $\Pi_\gamma^0(X) = \Pi_\gamma^0$ inductively by

$$\Sigma_1^0(X) = \{G \subseteq X \,|\, G \text{ is open}\}, \qquad \Pi_1^0(X) = \{F \subseteq X \,|\, F \text{ is closed}\},$$

and

$$\Sigma_\gamma^0(X) = \left\{\bigcup\nolimits_{n=1}^\infty E_n \,|\, E_n \in \bigcup\nolimits_{\lambda<\gamma} \Pi_\lambda^0(X)\right\}, \quad \Pi_\gamma^0(X) = \left\{\bigcap\nolimits_{n=1}^\infty E_n \,|\, E_n \in \bigcup\nolimits_{\lambda<\gamma} \Sigma_\lambda^0(X)\right\}.$$

We then have the following diagram

$$\begin{array}{cccc} \Sigma_1^0(X) & \Sigma_2^0(X) & \Sigma_3^0(X) & \cdots \\ \Pi_1^0(X) & \Pi_2^0(X) & \Pi_3^0(X) & \cdots \end{array},$$

in which any Baire class is contained in any Baire class to the right of it; this is known as the Borel Hierarchy, cf. [Ke, p. 68]. Is is known that

$$\bigcup\nolimits_{\gamma<\omega_1} \Sigma_\gamma^0 = \bigcup\nolimits_{\gamma<\omega_1} \Pi_\gamma^0 = \mathscr{B}(X)$$

where $\mathscr{B}(X)$ denotes the Borel σ-algebra on X. The Borel hierarchy therefore gives a ramafication of the Borel sets in (at least) ω_1 levels. Sometimes we will use the traditional notation, $\mathscr{G}(X) = \mathscr{G}$, for the family of open subsets of X, and the traditional notation, $\mathscr{F}(X) = \mathscr{F}$, for the family of closed subsets of X. Hence,

$$\begin{array}{lllll} \Sigma_1^0(X) = \mathscr{G}, & \Sigma_2^0(X) = \mathscr{F}_\sigma, & \Sigma_3^0(X) = \mathscr{G}_{\delta\sigma}, & \Sigma_4^0(X) = \mathscr{F}_{\sigma\delta\sigma}, & \cdots, \\ \Pi_1^0(X) = \mathscr{F}, & \Pi_2^0(X) = \mathscr{G}_\delta, & \Pi_3^0(X) = \mathscr{F}_{\sigma\delta}, & \Pi_4^0(X) = \mathscr{G}_{\delta\sigma\delta}, & \cdots. \end{array}$$

A function $f : X \to Y$ between metric spaces X and Y is said to be of Baire class $n \in \mathbf{N} \cup \{0\}$ if f is $\Sigma^0_{n+1}(X)$-measurable, i.e. if $f^{-1}(G) \in \Sigma^0_{n+1}$ for every open subset G of Y. Hence, functions of Baire class 0 are continuous, functions of Baire class 1 are "1 step away from being continuous", et.c. It is well-known (see for example [Ke, Theorem 24.3]) that a function f is of Baire class $n \in \mathbf{N}$ if and only if f is the pointwise limit of a sequence of functions of Baire class $n - 1$.

Analytic sets.

Finally we recall the definition of an analytic set. A subset A of a Polish space is X is called analytic if it is the continuous image of a Polish space, i.e. if there exist a Polish space Y and a continuous map $f : Y \to X$ such that $f(Y) = A$. More generally, a subset A of a separable metric space X is called analytic if there exist a Polish space Y with $X \subseteq Y$ and an analytic subset B of Y such that $A = X \cap B$ (cf. [Ke, p. 197]). For a seperable metric space X, we let $\mathscr{A}(X)$ denote the family of analytic subsets of X. It is well-known that every Borel set is analytic, i.e.

$$\mathscr{B}(X) \subseteq \mathscr{A}(X).$$

In particular, we see that every Borel measurable map is $\sigma(\mathscr{A}(X))$-measurable, where $\sigma(\mathscr{A}(X))$ denotes the σ-algebra generated by the family, $\mathscr{A}(X)$, of analytic subsets of X.

Throughout the paper we will write $\bar{\mathbf{R}} = [-\infty, \infty]$.

3. Analysis of the multifractal Hausdorff measure and the multifractal Hausdorff dimension

The purpose of this section is to prove Theorem 3.4 and Theorem 3.5 regarding the set theoretic complexity of the multifractal Hausdorff measure map (1.1) and the multifractal Hausdorff dimension map (1.3). For $x \in \mathbf{R}^d$ and $r > 0$, $B(x, r)$ denotes the closed Euclidean ball with center x and radius r, and $U(x, r)$ denotes the open Euclidean ball with center x and radius r.

LEMMA 3.1. *Let $t, c \in \mathbf{R}$ and $\delta > 0$. Then $\{(K, \mu, q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \mid \overline{\mathscr{H}}^{q,t}_{\mu,\delta}(K) < c\}$ is open.*

PROOF. Let

$$\begin{aligned} G = \Big\{ (K, \mu, q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \mid \\ &\text{there exist } n \in \mathbf{N}, z_1, \dots, z_n \in K, \varepsilon_1, \dots, \varepsilon_n \in (0, \delta) \\ &\text{and } s_1, \dots, s_n > 0 \text{ with } s_1 + \cdots + s_n < c \text{ such that} \\ &\text{(i) } K \subseteq \textstyle\bigcup_{i=1}^n U(z_i, \varepsilon_i) \\ &\text{(ii) } \mu(U(z_i, \varepsilon_i))^q (2\varepsilon_i)^t < s_i \text{ for all } i = 1, \dots, n \Big\}. \end{aligned}$$

An easy compactness argument shows that

$$\{(K,\mu,q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \mid \overline{\mathscr{H}}^{q,t}_{\mu,\delta}(K) < c\} = G.$$

Write $F = (\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}) \backslash G$. We must now prove that F is closed. Let $(K,\mu,q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$ and let $(K_m,\mu_m,q_m)_m$ be a sequence in F with $(K_m,\mu_m,q_m) \to (K,\mu,q)$. We must prove that $(K,\mu,q) \in F$. Fix $n \in \mathbf{N}$, $z_1,\ldots,z_n \in K$, $\varepsilon_1,\ldots,\varepsilon_n \in (0,\delta)$ and $s_1,\ldots,s_n > 0$ with $s_1 + \cdots + s_n < c$. We must now show that

$$K \not\subseteq \bigcup_i U(z_i,\varepsilon_i) \tag{3.1}$$

or

$$\mu(U(z_i,\varepsilon_i))^q(2\varepsilon)^t \geq s_i \qquad \text{for some } i \in \{1,\ldots,n\}. \tag{3.2}$$

If (3.1) is satisfied, then we are done. We may therefore assume that (3.1) is not satisfied, i.e. we are assuming that

$$K \subseteq \bigcup_i U(z_i,\varepsilon_i) \tag{3.3}$$

Since K is compact, (3.3) implies that there is an $\eta_0 > 0$ such that

$$K \subseteq \bigcup_i U(z_i,\varepsilon_i - \eta) \qquad \text{for } 0 < \eta < \eta_0. \tag{3.4}$$

We now prove the following claim.

Claim(*). For each $0 < \eta < \eta_0$ there exists an $i(\eta) \in \{1,\ldots,n\}$ such that

$$s_{i(\eta)} \leq \begin{cases} \mu(U(z_{i(\eta)},\varepsilon_{i(\eta)} - \eta))^q(2(\varepsilon_{i(\eta)} - \frac{3}{4}\eta))^t & \text{for } q < 0 \\ \mu(U(z_{i(\eta)},\varepsilon_{i(\eta)}))^q(2(\varepsilon_{i(\eta)} - \frac{3}{4}\eta))^t & \text{for } 0 \leq q \end{cases}.$$

Proof of Claim (*). Fix $0 < \eta < \eta_0$. It follows from (3.4) and the fact that $K_m \to K$ that we can choose an integer M such that

$$K_m \subseteq \bigcup_i U(z_i,\varepsilon_i - \eta) \qquad \text{for } m \geq M, \tag{3.5}$$

and

$$K_m \cap U(z_i,\tfrac{1}{4}\eta) \neq \varnothing \qquad \text{for } m \geq M \text{ and } i \in \{1,\ldots,n\}.$$

Now fix $m \geq M$ and choose $z_{m,i} \in K_m \cap U(z_i,\frac{1}{4}\eta)$ for $i = 1,\ldots,n$. Observe that

$$U(z_i,\varepsilon_i - \eta) \subseteq U(z_{m,i},\varepsilon_i - \tfrac{3}{4}\eta) \subseteq U(z_i,\varepsilon_i - \tfrac{1}{2}\eta). \tag{3.6}$$

In particular,

$$K_m \subseteq \bigcup_i U(z_{m,i},\varepsilon_i - \tfrac{3}{4}\eta). \tag{3.7}$$

We infer from (3.7) and the fact that $(K_m, \mu_m, q_m) \in F$ that

$$\mu_m(U(z_{m,i(m)}, \varepsilon_{i(m)} - \tfrac{3}{4}\eta))^{q_m}(2(\varepsilon_{i(m)} - \tfrac{3}{4}\eta))^t \geq s_{i(m)} \quad \text{for some} \quad i(m) \in \{1, \dots, n\}. \tag{3.8}$$

Next choose $i = i(\eta) \in \{1, \dots, n\}$ such that there exists a strictly increasing sequence $(m_k)_k$ of positive integers with $i(m_k) = i$ for all k.

Since $\mu_{m_k} \to \mu$ weakly, (3.6) implies that

$$\begin{aligned}\mu(U(z_i, \varepsilon_i - \eta)) &\leq \liminf_k \mu_{m_k}(U(z_i, \varepsilon_i - \eta)) \leq \liminf_k \mu_{m_k}(U(z_{m_k,i}, \varepsilon_i - \tfrac{3}{4}\eta)) \\ &= \liminf_k \mu_{m_k}(U(z_{m_k,i(m_k)}, \varepsilon_{i(m_k)} - \tfrac{3}{4}\eta))\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\mu(U(z_i, \varepsilon_i)) &\geq \mu(B(z_i, \varepsilon_i - \tfrac{1}{2}\eta)) \\ &\geq \limsup_k \mu_{m_k}(B(z_i, \varepsilon_i - \tfrac{1}{2}\eta)) \geq \limsup_k \mu_{m_k}(U(z_{m_k,i}, \varepsilon_i - \tfrac{3}{4}\eta)) \\ &= \limsup_k \mu_{m_k}(U(z_{m_k,i(m_k)}, \varepsilon_{i(m_k)} - \tfrac{3}{4}\eta)).\end{aligned} \tag{3.10}$$

For k write $u_k = \mu_{m_k}(U(z_{m_k,i(m_k)}, \varepsilon_{i(m_k)} - \tfrac{3}{4}\eta))$, and observe that (3.9) and (3.10) imply that $(u_k)_k$ is a bounded sequence, whence $u_k^{q-q_{m_k}} \to 1$ as $k \to \infty$. Hence, if $q < 0$, then inequalities (3.8) and (3.9) imply that

$$\begin{aligned}\mu(U(z_i, \varepsilon_i - \eta))^q(2(\varepsilon_i - \tfrac{3}{4}\eta))^t &\geq \limsup_k u_k^{q_{m_k}} u_k^{q-q_{m_k}}(2(\varepsilon_i - \tfrac{3}{4}\eta))^t \\ &= \limsup_k u_k^{q_{m_k}}(2(\varepsilon_{i(m_k)} - \tfrac{3}{4}\eta))^t \\ &\geq \limsup_k s_{i(m_k)} = s_i,\end{aligned}$$

and if $0 \leq q$, then equations (3.8) and (3.10) imply that

$$\begin{aligned}\mu(U(z_i, \varepsilon_i))^q(2(\varepsilon_i - \tfrac{3}{4}\eta))^t &\geq \limsup_k u_k^{q_{m_k}} u_k^{q-q_{m_k}}(2(\varepsilon_i - \tfrac{3}{4}\eta))^t \\ &= \limsup_k u_k^{q_{m_k}}(2(\varepsilon_{i(m_k)} - \tfrac{3}{4}\eta))^t \\ &\geq \limsup_k s_{i(m_k)} = s_i.\end{aligned}$$

This completes the proof of Claim $(*)$.

It follows from Claim $(*)$ that there exist a sequence $(\eta_m)_m$ of positive reals and an $i \in \{1, \dots, n\}$ such that $\eta_m \to 0$ and $i(\eta_m) = i$ for all m, i.e.

$$s_i \leq \begin{cases} \mu(U(z_i, \varepsilon_i - \eta_m))^q (2(\varepsilon_i - \frac{3}{4}\eta_m))^t & \text{for } q < 0 \\ \mu(U(z_i, \varepsilon_i))^q (2(\varepsilon_i - \frac{3}{4}\eta_m))^t & \text{for } 0 \leq q \end{cases}.$$

Letting $m \to \infty$ yields (3.2). □

LEMMA 3.2. *Let $\mu \in \mathscr{M}(\mathbf{R}^d)$ and $q, t \in \mathbf{R}$.*
(i) $\overline{\mathscr{H}}_\mu^{q,t}(E) \leq \overline{\mathscr{H}}_\mu^{q,t}(\bar{E})$ *for* $E \subseteq \mathbf{R}^d$.
(ii) $\mathscr{H}_\mu^{q,t}(K) = \sup_{\substack{L \subseteq K \\ L \text{ compact}}} \overline{\mathscr{H}}_\mu^{q,t}(L)$ *for compact subsets* $K \subseteq \mathbf{R}^d$.

PROOF. (i) Let $\varepsilon, \delta > 0$ and let $(B(x_i, r_i))_i$ be a centered δ-covering of $\bar{E}$. Since $B(x_i, r_i + \eta) \searrow B(x_i, r_i)$ as $\eta \searrow 0$, there exists $0 < \eta_i < \delta$ such that $r_i + \eta_i < \delta$ and

$$a^q (2b)^t \leq \mu(B(x_i, r_i))^q (2r_i)^t + \frac{\varepsilon}{2^i} \qquad \text{for all } a, b \in \mathbf{R} \text{ satisfying}$$

$$\mu(B(x_i, r_i)) \leq a \leq \mu(B(x_i, r_i + \eta_i)) \quad \text{and} \quad r_i \leq b \leq r_i + \eta_i. \tag{3.11}$$

Now pick $x_i' \in B\left(x_i, r_i - \frac{\varepsilon}{2^i}\right) \cap E$, and observe that (3.11) implies that

$$\mu\left(B\left(x_i', r_i + \frac{1}{2}\eta_i\right)\right)^q \left(2\left(r_i + \frac{1}{2}\eta_i\right)\right)^t \leq \mu(B(x_i, r_i))^q (2r_i)^t + \frac{\varepsilon}{2^i}. \tag{3.12}$$

Since $(B(x_i', r_i + \frac{1}{2}\eta_i))_i$ is a centered δ covering of E, (3.12) shows that $\overline{\mathscr{H}}_{\mu,\delta}^{q,t}(E) \leq \sum_i \mu(B(x_i', r_i + \frac{1}{2}\eta_i))^q (2(r_i + \frac{1}{2}\eta_i))^t \leq \sum_i \mu(B(x_i, r_i))^q (2r_i)^t + \varepsilon$. Hence, $\overline{\mathscr{H}}_{\mu,\delta}^{q,t}(E) \leq \overline{\mathscr{H}}_{\mu,\delta}^{q,t}(\bar{E}) + \varepsilon$. Letting $\delta, \varepsilon \searrow 0$ now yields the desired result.
(ii) This follows easily from (i). □

LEMMA 3.3. *Let $\mu \in \mathscr{M}(\mathbf{R}^d)$ and $q, t \in \mathbf{R}$.*
(i) *If $q \leq 0$, then there exists a constant $c > 0$ such that $\mathscr{H}_\mu^{q,t} \leq c\overline{\mathscr{H}}_\mu^{q,t}$.*
(ii) *If $0 < q$ and $\mu \in \mathscr{M}_0(\mathbf{R}^d)$, then there exists a constant $c > 0$ such that $\mathscr{H}_\mu^{q,t} \leq c\overline{\mathscr{H}}_\mu^{q,t}$.*
(iii) *If $q \leq 0$, then* $\dim_\mu^q(E) = \inf\{s \in \mathbf{R} \mid \overline{\mathscr{H}}_\mu^{q,s}(E) = 0\} = \sup\{s \in \mathbf{R} \mid \overline{\mathscr{H}}_\mu^{q,s}(E) = \infty\}$ *for* $E \subseteq \mathbf{R}^d$.
(iv) *If $0 < q$ and $\mu \in \mathscr{M}_0(\mathbf{R}^d)$, then* $\dim_\mu^q(E) = \inf\{s \in \mathbf{R} \mid \overline{\mathscr{H}}_\mu^{q,s}(E) = 0\} = \sup\{s \in \mathbf{R} \mid \overline{\mathscr{H}}_\mu^{q,s}(E) = \infty\}$ *for* $E \subseteq \mathbf{R}^d$.

PROOF. (ii) Since μ satisfies the doubling condition, there exists a constant $c_1 > 0$ such that

$$c_1^{-1} \leq \left(\frac{\mu(B(x, 3r))}{\mu(B(x, r))}\right)^q \leq c_1 \qquad \text{for } x \in \mathbf{R}^d \text{ and } r > 0.$$

Let $c = 2^t c_1$. Fix $E \subseteq \mathbf{R}^d$ and let $F \subseteq E$. Let $\delta > 0$ and let $(B(x_i, r_i))_i$ be a centered δ-covering of E. Write $I = \{i \mid B(x_i, r_i) \cap F \neq \varnothing\}$. For each $i \in I$ choose $y_i \in B(x_i, r_i) \cap F$, and observe that $B(y_i, 2r_i) \subseteq B(x_i, 3r_i)$, whence $\mu(B(y_i, 2r_i))^q \leq \mu(B(x_i, 3r_i))^q$ (because $0 < q$). Also observe that $(B(y_i, 2r_i))_{i \in I}$ is a centered 2δ-covering of F. We therefore infer that

$$\begin{aligned}\overline{\mathscr{H}}^{q,t}_{\mu,2\delta}(F) &\leq \sum_{i \in I} \mu(B(y_i, 2r_i))^q (2 \cdot 2r_i)^t \leq 2^t \sum_i \mu(B(x_i, 3r_i))^q (2r_i)^t \\ &\leq c \sum_i \mu(B(x_i, r_i))^q (2r_i)^t. \end{aligned} \tag{3.13}$$

It follows from (3.13) that $\overline{\mathscr{H}}^{q,t}_{\mu,2\delta}(F) \leq c\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E)$. Letting $\delta \searrow 0$ now yields $\overline{\mathscr{H}}^{q,t}_{\mu}(F) \leq c\overline{\mathscr{H}}^{q,t}_{\mu}(E)$ for all $F \subseteq E$, whence $\mathscr{H}^{q,t}_{\mu}(E) \leq c\overline{\mathscr{H}}^{q,t}_{\mu}(E)$.

(i) The proof of (i) is very similar to the proof of (ii).

(iii)–(iv) Follows immediately from (i) and (ii) since $\overline{\mathscr{H}}^{q,t}_{\mu} \leq \mathscr{H}^{q,t}_{\mu}$. □

We are now ready to state and prove the main results in this section.

THEOREM 3.4. *Let $t \in \mathbf{R}$ and $\delta > 0$.*

(i) *The map*

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \overline{\mathscr{H}}^{q,t}_{\mu,\delta}(K)$$

is upper semi-continuous; in particular of Baire class 1.

(ii) *The map*

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \overline{\mathscr{H}}^{q,t}_{\mu}(K)$$

is of Baire class 2.

(iii) *The map*

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathscr{H}^{q,t}_{\mu}(K)$$

is $\sigma(\mathscr{A})$-measurable where $\sigma(\mathscr{A})$ denotes the σ-algebra generated by the family $\mathscr{A}$ of analytic subsets of $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$.

PROOF. (i) This follows immediately from Lemma 3.1.

(ii) Follows from (i) since $\overline{\mathscr{H}}^{q,t}_{\mu}(K) = \lim_n \overline{\mathscr{H}}^{q,t}_{\mu,(1/n)}(K)$ for all $(K, \mu, q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$.

(iii) We must prove that $\{(K, \mu, q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \mid \mathscr{H}^{q,t}_{\mu}(K) > c\}$ is analytic for all $c \in \mathbf{R}$. Fix $c \in \mathbf{R}$. Define the projection $\pi : \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \times \mathscr{K}(\mathbf{R}^d) \to \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$ by

$$\pi(K, \mu, q, L) = (K, \mu, q).$$

It now follows from Lemma 3.2 that

$$\{(K,\mu,q)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\,|\,\mathscr{H}_\mu^{q,t}(K)>c\}$$
$$=\{(K,\mu,q)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\,|$$
$$\text{there exists a compact subset } L \text{ of } K \text{ with } \overline{\mathscr{H}}_\mu^{q,t}(L)>c\}$$
$$=\pi(\{(K,\mu,q,L)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\times\mathscr{K}(\mathbf{R}^d)\,|\,L\subseteq K\}$$
$$\cap\{(K,\mu,q,L)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\times\mathscr{K}(\mathbf{R}^d)\,|\,\overline{\mathscr{H}}_\mu^{q,t}(L)>c\}). \quad (3.14)$$

Since the set $\{(K,\mu,q,L)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\times\mathscr{K}(\mathbf{R}^d)\,|\,L\subseteq K\}$ clearly is closed and the set $\{(K,\mu,q,L)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\times\mathscr{K}(\mathbf{R}^d)\,|\overline{\mathscr{H}}_\mu^{q,t}(L)>c\}$ is Borel (by (ii)), (3.14) shows that $\{(K,\mu,q)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\,|\,\mathscr{H}_\mu^{q,t}(K)>c\}$ is analytic. □

THEOREM 3.5.
(i) *The map*

$$\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\to\bar{\mathbf{R}}:(K,\mu,q)\to\dim_\mu^q(K)$$

is $\sigma(\mathscr{A})$-measurable where $\sigma(\mathscr{A})$ denotes the σ-algebra generated by the family $\mathscr{A}$ of analytic subsets of $\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}$.
(ii) *Write $\Gamma=\big(\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times(-\infty,0]\big)\cup\big(\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}_0(\mathbf{R}^d)\times\mathbf{R}\big)$. The map*

$$\Gamma\to\bar{\mathbf{R}}:(K,\mu,q)\to\dim_\mu^q(K)$$

is of Baire class 2 and not of Baire class 1.

PROOF. (i) Follows from Theorem 3.4.(iii).
(ii) It follows from Lemma 3.3 and Theorem 3.4.(ii) that if $s,t\in\mathbf{R}$, then

$$\{(K,\mu,q)\in\Gamma\,|\,s<\dim_\mu^q(K)<t\}$$
$$=\Gamma\cap\Big(\bigcup_n\{(K,\mu,q)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\,|\,1<\overline{\mathscr{H}}_\mu^{q,s+(1/n)}(K)\}$$
$$\cap\bigcup_n\{(K,\mu,q)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\,|\,\overline{\mathscr{H}}_\mu^{q,t-(1/n)}(K)<1\}\Big)$$
$$\in\mathscr{G}_{\delta\sigma}(\Gamma) \quad (3.15)$$

It follows from (3.15) that the map $\Gamma\to\bar{\mathbf{R}}:(K,\mu,q)\to\dim_\mu^q(K)$ is of Baire class 2.

We will now prove that the map $\Gamma\to\bar{\mathbf{R}}:(K,\mu,q)\to\dim_\mu^q(K)$ is not of Baire class 1. Since $\dim_\mu^0=\dim$ (cf. (2.1.2)), it suffices to show that dim :

$\mathscr{K}(\mathbf{R}^d) \to \bar{\mathbf{R}}$ is not of Baire class 1. Let $X = \{M \mid M \subseteq \mathbf{R}^d \text{ is finite}\}$ and $Y = \{L \cup M \mid L$ is a compact line segment in $\mathbf{R}^d$ of positive length, $M \subseteq \mathbf{R}^d$ is finite$\}$. Since X and Y are dense in $\mathscr{K}(\mathbf{R}^d)$ and $\dim M = 0$ for $M \in X$ and $\dim M = 1$ for $M \in Y$, dim is everywhere discontinuous and hence not of Baire class 1 by [Ke, Theorem 24.15]. □

Remarks. (1) In Theorem 3.5.(ii) we found the exact Baire class of the multifractal Hausdorff dimension map $\Gamma \to \bar{\mathbf{R}} : (K, \mu, q) \to \dim_\mu^q(K)$. However, we have not been able to determine the exact set theoretic complexity of the multifractal Hausdorff measure map $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathscr{H}_\mu^{q,t}(K)$. Theorem 3.4.(iii) shows that the multifractal Hausdorff measure map is measurable with respect to the the σ-algebra generated by the family of analytic subsets of $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$. It is natural to ask if this result can be improved. We do not believe that this is the case and make the following conjecture.

Conjecture 3.6. *The map $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathscr{H}_\mu^{q,t}(K)$ is, in general, not Borel measurable.*

It is instructive to consider the fractal counterpart of the multifractal Conjecture 3.6. For $t \geq 0$, $\mathscr{H}_\mu^{0,t} = \mathscr{C}^t$ where $\mathscr{C}^t$ denotes the t-dimensional centered Hausdorff measure. Hence, for $t \geq 0$, Theorem 3.4.(iii) shows that the map $\mathscr{K}(\mathbf{R}^d) \to \bar{\mathbf{R}} : K \to \mathscr{C}^t(K)$ is measurable with respect to the σ-algebra generated by the family of analytic subsets of $\mathscr{K}(\mathbf{R}^d)$. It is natural to ask if this result is the best possible. We believe that this is the case and make the following conjecture.

Conjecture 3.7. *The map $\mathscr{K}(\mathbf{R}^d) \to \bar{\mathbf{R}} : K \to \mathscr{C}^t(K)$ is, in general, not Borel measurable.*

Observe that the truth of Conjecture 3.7 implies the truth of Conjecture 3.6. The (conjectured) complicated set theoretic behaviour of the centered Hausdorff measure $\mathscr{C}^t$ is in sharp contrast to the simple set theoretic behaviour of the usual Hausdorff measure: for any Polish space X and any dimension function g, the map $\mathscr{K}(X) \to \bar{\mathbf{R}} : K \to \mathscr{H}^g(K)$ is of Baire class 2 [MM, Theorem 2.2]. In connection with Conjecture 3.7 we would like ask a slightly different question regarding the "smoothness" of $\mathscr{C}^t$.

Question 3.8. *Is the centered Hausdorff measure, $\mathscr{C}^t$, Borel regular, i.e. does the centered Hausdorff measure satisfy the following condition: For each $E \subseteq \mathbf{R}^d$, there exists a Borel set $B \subseteq \mathbf{R}^d$ such that $E \subseteq B$ and $\mathscr{C}^t(E) = \mathscr{C}^t(B)$. (It is a well-known fact that the ordinary Hausdorff measure $\mathscr{H}^t$ is Borel regular, cf.* [Ma2].*)*

Question 3.8 has been asked several times during the last 3 years by the author. The question also appears in [Ed, p. 64, Question 1.8.1]. See Note Added in Proof at the end of this paper.

(2) Theorem 3.5 shows that the multifractal Hausdorff dimension map restricted to the set Γ of doubling measures is of Baire class 2. It is natural to ask if the doubling conditon can be omitted. We therefore pose the following question.

Question 3.9. *Is the map $\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\to\bar{\mathbf{R}} : (K,\mu,q)\to\dim_\mu^q(K)$ of Baire class 2?*

4. Analysis of the multifractal packing measure and the multifractal packing dimension

The purpose of this section is to prove Theorem 4.7 and Theorem 4.8 regarding the set theoretic complexity of the multifractal packing measure map (1.2) and the multifractal packing dimension map (1.4). Recall that if $x\in\mathbf{R}^d$ and $r>0$, then $B(x,r)$ denotes the closed Euclidean ball with center x and radius r, and $U(x,r)$ denotes the open Euclidean ball with center x and radius r.

LEMMA 4.1. *Let $t,c\in\mathbf{R}$ and $\delta>0$. Then $\{(K,\mu,q)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R} \mid \bar{\mathscr{P}}_{\mu,\delta}^{q,t}(K)>c\}$ is open.*

PROOF. Let

$$\begin{aligned} G=\{(K,\mu,q)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R} \mid \\ &\text{there exist } n\in\mathbf{N}, z_1,\dots,z_n\in K, \varepsilon_1,\dots,\varepsilon_n\in(0,\delta) \\ &\text{and } s_1,\dots,s_n>0 \text{ with } s_1+\cdots+s_n>c \text{ such that} \\ &\text{(i) } B(z_i,\varepsilon_i)\cap B(z_j,\varepsilon_j)=\varnothing \text{ for } i\neq j \\ &\text{(ii) } \mu(B(z_i,\varepsilon_i))^q(2\varepsilon_i)^t>s_i \text{ for all } i=1,\dots,n\}. \end{aligned}$$

Is is easily seen that

$$\{(K,\mu,q)\in\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R} \mid \bar{\mathscr{P}}_{\mu,\delta}^{q,t}(K)>c\}=G.$$

The rest of the proof of Lemma 4.1 is very similar to the proof of Lemma 3.1 and is therefore omitted. □

LEMMA 4.2. *Let $E\subseteq\mathbf{R}^d$, $\mu\in\mathscr{M}(\mathbf{R}^d)$ and $q\in\mathbf{R}$.*
(i) $\mathrm{Dim}_\mu^q(E)=\inf_{E\subseteq\bigcup_{i=1}^\infty E_i}\sup_i \Delta_\mu^q(E_i)$.
(ii) *If $q\leq 0$, then $\Delta_\mu^q(E)=\Delta_\mu^q(\bar{E})$.*
(iii) *If $0<q$ and $\mu\in\mathscr{M}_0(\mathbf{R}^d)$, then $\Delta_\mu^q(E)=\Delta_\mu^q(\bar{E})$.*

PROOF. See [Ol3, Lemma 6.5.1 and Lemma 6.5.2]. □

LEMMA 4.3. *Let* $K \in \mathscr{K}(\mathbf{R}^d)$, $\mu \in \mathscr{M}(\mathbf{R}^d)$ *and* $q \in \mathbf{R}$. *If either:* $q \leq 0$ *or:* $0 < q$ *and* $\mu \in \mathscr{M}_0(\mathbf{R}^d)$, *then*

$$\operatorname{Dim}_\mu^q(K) = \inf_{\substack{K \subseteq \bigcup_{i=1}^\infty K_i \\ K_i \text{ compact}}} \sup_i \Delta_\mu^q(K_i).$$

PROOF. Follows easily from Lemma 4.2. □

LEMMA 4.4. *Let* $a \in \mathbf{R}$, $K \in \mathscr{K}(\mathbf{R}^d)$, $\mu \in \mathscr{M}(\mathbf{R}^d)$ *and* $q \in \mathbf{R}$ *with* $\mathscr{P}_\mu^{q,a}(K) > 0$ *(this holds, in particular, if* $\operatorname{Dim}_\mu^q(K) > a$*). Then there exists a subset* $L \subseteq K$ *such that*

(i) L *is compact and non-empty.*

(ii) *If* $U \subseteq \mathbf{R}^d$ *is open and* $L \cap U \neq \varnothing$, *then* $\operatorname{Dim}_\mu^q(L \cap U) \geq a$, *in particular* $\Delta_\mu^q(L \cap \bar{U}) \geq a$.

PROOF. Let ν be the restriction of $\mathscr{P}_\mu^{q,a}$ to K and put $L = \operatorname{supp} \nu$. Then clearly $L \subseteq K$ and L is therefore compact. Since $\nu(L) = \nu(\mathbf{R}^d) = \mathscr{P}_\mu^{q,a}(K) > 0$, we deduce that $L \neq \varnothing$. Finally, if $U \subseteq \mathbf{R}^d$ is open with $U \cap L \neq \varnothing$, then $\nu(U) > 0$ whence $\mathscr{P}_\mu^{q,a}(U \cap L) = \nu(U \cap L) = \nu(U) > 0$, which implies that $\operatorname{Dim}_\mu^q(U \cap L) \geq a$. (Incidentally, the set $L = \operatorname{supp} \nu$ is the largest subset L of K satisfying (i) and (ii).) □

LEMMA 4.5. *Let* $c \in \mathbf{R}$ *and let* Γ *be defined as in Theorem* 3.5.(ii). *Then*

$$\{(K, \mu, q) \in \Gamma \mid \operatorname{Dim}_\mu^q(K) \geq c\}$$

$= \{(K, \mu, q) \in \Gamma \mid$ *for all* $a < c$ *there exists a subset* $L \subseteq K$ *such that*

(i) L *is compact and non-empty*

(ii) *if* $U \subseteq \mathbf{R}^d$ *is open and* $L \cap U \neq \varnothing$, *then* $\Delta_\mu^q(L \cap \bar{U}) \geq a\}$.

PROOF. "$\subseteq$" Follows from the previous lemma.

"$\supseteq$" Let $(K, \mu, q) \in \Gamma$ and assume that if $a < c$, then there exists a non-empty subset $L \subseteq K$ such that L is compact and $\Delta_\mu^q(L \cap \bar{U}) \geq a$ for all open subsets $U \subseteq \mathbf{R}^d$ with $L \cap U \neq \varnothing$. We must now prove that $\operatorname{Dim}_\mu^q(K) \geq c$. Assume, in order to reach a contradiction, that $\operatorname{Dim}_\mu^q(K) < c$. Now pick a such that $\operatorname{Dim}_\mu^q(K) < a < c$. Since $a < c$, there exists a non-empty subset $L \subseteq K$ such that L is compact and $\Delta_\mu^q(L \cap \bar{U}) \geq a$ for all open subsets $U \subseteq \mathbf{R}^d$ with $L \cap U \neq \varnothing$. Also, since $\operatorname{Dim}_\mu^q(K) < a$, Lemma 4.3 shows that there exist compact sets $K_1, K_2, \ldots$ such that $K \subseteq \bigcup_n K_n$ and

$$\Delta_\mu^q(K_n) < a \qquad \text{for all } n. \tag{4.1}$$

The equation $L = \bigcup_n (L \cap K_n)$ and Baire's Category Theorem now imply that there is an open set U and an integer m with $\varnothing \neq L \cap U \subseteq L \cap K_m$. We may clearly choose an open set V such that $L \cap V \neq \varnothing$ and $\bar{V} \subseteq U$. It now follows from (4.1) that $a \leq \Delta_\mu^q(L \cap \bar{V}) \leq \Delta_\mu^q(L \cap U) \leq \Delta_\mu^q(L \cap K_m) \leq \Delta_\mu^q(K_m) < a$, which yields the desired contradiction. □

PROPOSITION 4.6. *Let $C \in \mathscr{K}(\mathbf{R}^d)$. The map*

$$\mathscr{K}(\mathbf{R}^d) \xrightarrow{I} \mathscr{K}(\mathbf{R}^d) \cup \{\varnothing\} : K \to K \cap C$$

is Borel measurable.

PROOF. For each positive integer k let $\mathbf{B}_k^d = B(0,k)$ and define the map I_k by $\mathscr{K}(\mathbf{B}_k^d) \xrightarrow{I_k} \mathscr{K}(\mathbf{B}_k^d) \cup \{\varnothing\} : K \to K \cap C$. It follows from [En, 3.12.28] that I_k is upper semi-continuous (see [Ku, §43, I] for the definition of an upper semi-continuous set valued map), and since $\mathbf{B}_k^d$ is compact, we now deduce from [Ku, §43, VII, Theorem 1] that I_k is Borel measurable (in fact, of Baire class 1). Hence, if $\mathscr{U}$ is an open subset of $\mathscr{K}(\mathbf{R}^d) \cup \{\varnothing\}$, then $I^{-1}(\mathscr{U}) = \bigcup_{k=1}^\infty I_k^{-1}(\mathscr{U} \cap (\mathscr{K}(\mathbf{B}_k^d) \cup \{\varnothing\}))$ is a Borel subset of $\mathscr{K}(\mathbf{R}^d)$. This shows that I is Borel measurable. □

We are now ready to state and prove the main results in this section.

THEOREM 4.7. *Let $t \in \mathbf{R}$ and $\delta > 0$.*

(i) *The map*

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \bar{\mathscr{P}}_{\mu,\delta}^{q,t}(K)$$

is lower semi-continuous; in particular of Baire class 1.

(ii) *The map*

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \bar{\mathscr{P}}_{\mu}^{q,t}(K)$$

is of Baire class 2 *and, in general, not of Baire class* 1.

PROOF. (i) This follows immediately from Lemma 4.1.

(ii) Since $\bar{\mathscr{P}}_\mu^{q,t}(K) = \lim_n \bar{\mathscr{P}}_{\mu,(1/n)}^{q,t}(K)$ for all $(K,\mu,q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$, part (i) shows that the map $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K,\mu,q) \to \bar{\mathscr{P}}_\mu^{q,t}(K)$ is of Baire class 2.

We will now prove that the map $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K,\mu,q) \to \bar{\mathscr{P}}_\mu^{q,t}(K)$ is not of Baire class 1. Since $\bar{\mathscr{P}}_\mu^{0,1/2} = \bar{\mathscr{P}}^{1/2}$ (cf. (2.1.1)), it suffices to show that $\bar{\mathscr{P}}^{1/2} : \mathscr{K}(\mathbf{R}^d) \to \bar{\mathbf{R}}$ is not of Baire class 1. Let X and Y be as in the proof of Theorem 3.5.(ii). Since $\bar{\mathscr{P}}^{1/2}(M) = 0$ for $M \in X$ and $\bar{\mathscr{P}}^{1/2}(M) = \infty$ for $M \in Y$, $\bar{\mathscr{P}}^{1/2}$ is everywhere discontinuous and hence not of Baire class 1 by [Ke, Theorem 24.15]. □

THEOREM 4.8. (i) *The map*

$$\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K,\mu,q) \to \Delta_\mu^q(K)$$

is of Baire class 2 *and not of Baire class* 1.

(ii) *Write* $\Gamma = (\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times (-\infty,0]) \cup \big(\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}_0(\mathbf{R}^d) \times \mathbf{R}\big)$. *The map*

$$\Gamma \to \bar{\mathbf{R}} : (K,\mu,q) \to \mathrm{Dim}_\mu^q(K)$$

is $\sigma(\mathscr{A}(\Gamma))$*-measurable where* $\sigma(\mathscr{A}(\Gamma))$ *denotes the* σ*-algebra generated by the family* $\mathscr{A}(\Gamma)$ *of analytic subsets of* Γ.

PROOF. (i) It follows from Theorem 4.7.(ii) that if $s,t \in \mathbf{R}$, then

$$\begin{aligned}
&\{(K,\mu,q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \mid s < \Delta_\mu^q(K) < t\}\\
&\quad = \textstyle\bigcup_n \{(K,\mu,q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \mid 1 < \bar{\mathscr{P}}_\mu^{q,s+(1/n)}(K)\}\\
&\qquad \cap \textstyle\bigcup_n \{(K,\mu,q) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \mid \bar{\mathscr{P}}_\mu^{q,t-(1/n)}(K) < 1\}\\
&\quad \in \mathscr{G}_{\delta\sigma}
\end{aligned} \tag{4.2}$$

It follows from (4.2) that the map $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K,\mu,q) \to \Delta_\mu^q(K)$ is of Baire class 2.

The proof of the fact that the map $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K,\mu,q) \to \Delta_\mu^q(K)$ is not of Baire class 1 is similar to the proof of Theorem 3.5.(ii) and Theorem 4.7.(ii) and is therefore omitted.

(ii) Fix $c \in \mathbf{R}$. Let $(x_i)_i$ be a countable dense subset of $\mathbf{R}^d$ and let $(r_i)_i$ be an enumeration of the positive rationals. For positive integers i,j,m write

$$F = \{(K,\mu,q,L) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \times \mathscr{K}(\mathbf{R}^d) \mid L \subseteq K\},$$

$$B_{ij} = \{(K,\mu,q,L) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \times \mathscr{K}(\mathbf{R}^d) \mid U(x_i,r_j) \cap L = \varnothing\},$$

$$C_{mij} = \left\{(K,\mu,q,L) \in \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \times \mathscr{K}(\mathbf{R}^d) \mid \Delta_\mu^q(L \cap B(x_i,r_j)) \geq c - \frac{1}{m}\right\},$$

and define the projection $\pi : \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \times \mathscr{K}(\mathbf{R}^d) \to \mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$ by

$$\pi(K,\mu,q,L) = (K,\mu,q).$$

It now follows from Lemma 4.5 that

$$\{(K,\mu,q)\in\Gamma \mid \mathrm{Dim}^q_\mu(K)\geq c\}$$

$$= \bigcap_m \Big\{(K,\mu,q)\in\Gamma \mid \text{there exists a subset } L\subseteq K \text{ such that}$$

$$\text{(i) } L \text{ is compact and non-empty}$$

$$\text{(ii) if } U\subseteq \mathbf{R}^d \text{ is open and } L\cap U\neq\varnothing,$$

$$\text{then } \Delta^q_\mu(L\cap \bar{U})\geq c-\frac{1}{m}\Big\}$$

$$= \bigcap_m \pi\Big(\{(K,\mu,q,L)\in\Gamma\times\mathscr{K}(\mathbf{R}^d)\mid L\subseteq K\}$$

$$\cap\Big\{(K,\mu,q,L)\in\Gamma\times\mathscr{K}(\mathbf{R}^d)\mid \text{if } i,j\in\mathbf{N} \text{ with } U(x_i,r_j)\cap L\neq\varnothing,$$

$$\text{then } \Delta^q_\mu(L\cap B(x_i,r_j))\geq c-\frac{1}{m}\Big\}\Big)$$

$$=\Gamma\cap\bigcap_m \pi\Big(F\cap\bigcap_{i,j}(B_{ij}\cup C_{mij})\Big)$$

$$=\Gamma\cap A,$$

where

$$A=\bigcap_m \pi\Big(F\cap\bigcap_{i,j}(B_{ij}\cup C_{mij})\Big)\subseteq\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}.$$

The sets F and B_{ij} are clearly closed, and it follows from Proposition 4.6 and part (i) of the theorem that the set C_{mij} is Borel. Hence, A is an analytic subset of $\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}$, and we therefore deduce that $\{(K,\mu,q)\in\Gamma \mid \mathrm{Dim}^q_\mu(K)\geq c\}=\Gamma\cap A$ is an analytic subset of Γ. □

REMARKS. **(1)** We have not been able to determine the complexity of the multifractal packing measure map

$$\mathscr{K}(\mathbf{R}^d)\times\mathscr{M}(\mathbf{R}^d)\times\mathbf{R}\to\bar{\mathbf{R}}:(K,\mu,q)\to\mathscr{P}^{q,t}_\mu(K). \tag{4.3}$$

Mattila & Mauldin [MM] have recently shown that if X is a Polish space and g is a dimension function satisfying the doubling condition (i.e. there exists $c>0$ such that $g(2t)\leq cg(t)$ for all $t\geq 0$), then the map

$$\mathscr{K}(X)\to\bar{\mathbf{R}}:K\to\mathscr{P}^g(K) \tag{4.4}$$

is $\sigma(\mathscr{A}(\mathscr{K}(X)))$-measurable where $\sigma(\mathscr{A}(\mathscr{K}(X)))$ denotes the σ-algebra generated by the family $\mathscr{A}(\mathscr{K}(X))$ of analytic subsets of $\mathscr{K}(X)$. Moreover, Mattila &

Mauldin also provide an example showing that the map in (4.4) is not necessarily Borel measurable. However, the ideas in [MM] do not apply to the multifractal case. In order to prove that the map in (4.4) is $\sigma(\mathscr{A}(\mathscr{K}(X)))$-measurable, Mattila & Mauldin use the fact that if g satisfies the doubling condition then $\mathscr{P}^g$ has "the subset of finite measure" property, i.e. if A is an analytic subset of X with $\mathscr{P}^g(A) = \infty$, then there exists a compact subset C of A with $0 < \mathscr{P}^g(C) < \infty$. It is not difficult to see that the multifractal packing measure $\mathscr{P}_\mu^{q,t}$ does not in general have "the subset of finite measure" property, and the method used in [MM] does therefore not work in the multifractal case. However, we do believe that the multifractal packing measure map in (4.3) is analytically measurable, and we therefore make the following conjecture.

Conjecture 4.10. *The map* $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathscr{P}_\mu^{q,t}(K)$ *is measurable with respect to the* σ*-algebra generated by the analytic subsets of* $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$.

(2) Theorem 4.9 shows that the multifractal packing dimension map restricted to the set Γ of doubling measures is measurable with respect to the σ-algebra generated by the analytic subsets of Γ. It is natural to ask if the doubling conditon can be omitted. We therefore pose the following question.

Question 4.11. *Is the map* $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R} \to \bar{\mathbf{R}} : (K, \mu, q) \to \mathrm{Dim}_\mu^q(K)$ *measurable with respect to the* σ*-algebra generated by the analytic subsets of* $\mathscr{K}(\mathbf{R}^d) \times \mathscr{M}(\mathbf{R}^d) \times \mathbf{R}$*?*

5. An example: Measurability of multifractal slices

We will now apply the results in §3 to study the measurability of multifractal slices. The results in this section play an important part in the study of multifractal slices and negative dimensions in [Ol5].

Fix positive integers n and m with $m \le n$. Let $G(n,m)$ denote the Grassmannian manifold of m-dimensional linear subspaces of $\mathbf{R}^n$. For $\Pi \in G(n,m)$, let $\Pi^\perp$ denote the orthogonal complement of Π, and let π_Π denote the orthogonal projection onto Π. Furthermore, for $\Pi \in G(n,m)$, $x \in \mathbf{R}^n$ and $\delta > 0$ write $B(\Pi + x, \delta) = \{z \in \mathbf{R}^n \mid \mathrm{dist}(z, \Pi + x) \le \delta\}$. For a measure space $(X, \mathscr{E}, \mu)$ and $E \in \mathscr{E}$, let $\mu \llcorner E$ denote the restriction of μ to E, i.e. $(\mu \llcorner E)(F) = \mu(E \cap F)$ for all $F \in \mathscr{E}$. Now we define the slices of a Radon measure μ in $\mathbf{R}^n$ by m-dimensional planes. Fix $\mu \in \mathscr{M}(\mathbf{R}^n)$ and $\Pi \in G(n,m)$. For $\mathscr{H}^{n-m}$ almost all $x \in \Pi^\perp$ the following limit Radon measure exists

$$\mu_{\Pi,x} = \text{weak-lim}_{\delta \searrow 0}\, (2\delta)^{-(n-m)} \mu \llcorner B(\Pi + x, \delta),$$

where weak-lim denotes limit with respect to the weak topology, cf. [Ma2, Chapter 10]. If $\Pi \in G(n,m)$ and $x \in \mathbf{R}^n$ are such that $\mu_{\Pi,y}$ exists for $y = \pi_{\Pi^\perp}(x)$, then we write $\mu_{\Pi,x} = \mu_{\Pi,y}$. The measure $\mu_{\Pi,x}$ is the slice of μ by the plane $\Pi + x$. We are now ready to state and prove the main result in this section. Recall that $\mathscr{F}(\mathbf{R}^n)$ denotes the family of closed subsets of $\mathbf{R}^n$.

THEOREM 5.1. *Let $t \in \mathbf{R}$, $\mu \in \mathscr{M}(\mathbf{R}^n)$ and $F \in \mathscr{F}(\mathbf{R}^n)$. The maps*

$$G(n,m) \times \mathbf{R}^n \times \mathbf{R} \to \bar{\mathbf{R}} : (\Pi, x, q)$$
$$\to \begin{cases} \mathscr{H}^{q,t}_{\mu_{\Pi,x}}(F \cap (\Pi + x)) & \text{if } \mu_{\Pi,x} \text{ exists} \\ 0 & \text{if } \mu_{\Pi,x} \text{ does not exist} \end{cases}, \tag{5.1}$$

$$G(n,m) \times \mathbf{R}^n \times \mathbf{R} \to \bar{\mathbf{R}} : (\Pi, x, q)$$
$$\to \begin{cases} \dim^q_{\mu_{\Pi,x}}(F \cap (\Pi + x)) & \text{if } \mu_{\Pi,x} \text{ exists} \\ 0 & \text{if } \mu_{\Pi,x} \text{ does not exist} \end{cases}, \tag{5.2}$$

are $\sigma(\mathscr{A})$-measurable where $\sigma(\mathscr{A})$ denotes the σ-algebra generated by the family $\mathscr{A}$ of analytic subsets of $G(n,m) \times \mathbf{R}^n \times \mathbf{R}$.

PROOF. If X is a Polish space, we write $\mathscr{A}(X)$ for the family of analytic subsets of X. Let h denote the map in (5.1) and let d denote the map in (5.2). Write $\Sigma = \{(\Pi, x) \in G(n,m) \times \mathbf{R}^n \mid \mu_{\Pi,x} \text{ exists}\}$ and observe that it follows from [Ma1] that Σ is Borel. Fix $k \in \mathbf{N}$ and $\delta > 0$ and define maps by

$$\mathscr{K}(\mathbf{R}^n) \times \mathscr{M}(\mathbf{R}^n) \times \mathbf{R} \xrightarrow{H} \bar{\mathbf{R}} : (K, \nu, q) \to \mathscr{H}^{q,t}_\nu(K),$$

$$\mathscr{K}(\mathbf{R}^n) \times \mathscr{M}(\mathbf{R}^n) \times \mathbf{R} \xrightarrow{D} \bar{\mathbf{R}} : (K, \nu, q) \to \dim^q_\nu(K),$$

$$G(n,m) \times \mathbf{R}^n \xrightarrow{L_k} \mathscr{K}(\mathbf{R}^n) : (\Pi, x) \to F \cap (\Pi + x) \cap B(0,k),$$

$$G(n,m) \times \mathbf{R}^n \xrightarrow{S_\delta} \mathscr{M}(\mathbf{R}^n) : (\Pi, x) \to (2\delta)^{-(n-m)} \mu \llcorner B(\Pi + x, \delta),$$

$$G(n,m) \times \mathbf{R}^n \xrightarrow{S} \mathscr{M}(\mathbf{R}^n) : (\Pi, x) \to \begin{cases} \mu_{\Pi,x} = \text{weak-lim}_{\delta \searrow 0} S_\delta(\Pi, x) & \text{if } (\Pi, x) \in \Sigma \\ \mathscr{H}^n & \text{if } (\Pi, x) \notin \Sigma \end{cases}.$$

The map L_k is clearly Borel, and since S_δ is Borel, S is Borel. It therefore follows from [Ke, 37.3] that the map

$$G(n,m) \times \mathbf{R}^n \times \mathbf{R} \xrightarrow{T_k} \mathscr{K}(\mathbf{R}^n) \times \mathscr{M}(\mathbf{R}^n) \times \mathbf{R} : (\Pi, x, q) \to (L_k(\Pi, x), S(\Pi, x), q)$$

is $\sigma(\mathscr{A}(G(n,m) \times \mathbf{R}^n \times \mathbf{R}))$-$\sigma(\mathscr{A}(\mathscr{K}(\mathbf{R}^n) \times \mathscr{M}(\mathbf{R}^n) \times \mathbf{R}))$-measurable, and Theorem 3.4 and Theorem 3.5 therefore show that the maps $h = 1_{\Sigma \times \mathbf{R}} \sup_k H \circ T_k$ and $d = 1_{\Sigma \times \mathbf{R}} \sup_k D \circ T_k$ are $\sigma(\mathscr{A}(G(n,m) \times \mathbf{R}^n \times \mathbf{R}))$-measurable. □

Remark. For $t \geq 0$, $\mathscr{H}_\mu^{0,t} = \mathscr{C}^t$ where $\mathscr{C}^t$ denotes the t-dimensional centered Hausdorff measure (cf. (2.1.1)). Hence, for $t \geq 0$, Theorem 5.1 shows that if $F \in \mathscr{F}(\mathbf{R}^n)$, then the map $G(n,m) \times \mathbf{R}^n \to \bar{\mathbf{R}} : (\Pi, x) \to \mathscr{C}^t(F \cap (\Pi + x))$ is measurable with respect to the σ-algebra generated by the family of analytic subsets of $G(n,m) \times \mathbf{R}^n$. It is natural to ask if this result is the best possible. We believe that this is the case and make the following conjecture.

Conjecture 5.2. *Let $F \in \mathscr{F}(\mathbf{R}^n)$. The map $G(n,m) \times \mathbf{R}^n \to \bar{\mathbf{R}} : (\Pi, x) \to \mathscr{C}^t(F \cap (\Pi + x))$ is, in general, not Borel measurable.*

Dellacherie [De] proved that if T and X are compact metric spaces, B is an analytic subset of $T \times X$ and $s \geq 0$, then the map $T \to \bar{\mathbf{R}} : t \to \mathscr{H}^s(B_t)$, where $B_t = \{x \in X \mid (t,x) \in B\}$, is measurable with respect to the σ-algebra generated by the family of analytic subsets T. Theorem 5.1 can thus be viewed as a natural multifractal extension of classical measurability results for sections of sets.

Note Added in Proof. Question 3.8 has recently been answered affirmatively by A. Schechter [On the centred Hausdorff measure, Bull. of the Lond. Math. Soc., to appear]. In fact, A. Schechter proved that if μ is a Radon measure on $\mathbf{R}^d$ satisfying the doubling condition, then the multifractal Hausdorff measure $\mathscr{H}_\mu^{q,t}$ is Borel regular for all q, $t \in \mathbf{R}$. Moreover, he also constructed a Radon measure ν in the plane not satisfying the doubling condition for which $\mathscr{H}_\nu^{1/2,0}$ is not Borel regular.

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