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Self-homotopy equivalences of $SO(4)$

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ABSTRACT. Let $\mathscr{E}(X)$ be the group consisting of all based homotopy classes of based self-homotopy equivalences on X. In this paper we shall study and determine the group $\mathscr{E}(X)$ for $X = SO(4)$. This is one of the problems proposed by M. Arkowitz [2].

1. Introduction

For a based spaces X and Y, let $[X, Y]$ denote the set consisting of all the based homotopy classes of the based maps $X \to Y$. If $X = Y$, the homotopy set $[X, X]$ becomes a monoid whose multiplication induced from the composition of maps. Let $\mathscr{E}(X)$ be the group consisting of all invertible elements of the monoid $[X, X]$ and it is called the group of self-homotopy equivalences of X. When X is a simply connected H-space of rank ≤ 2, the group $\mathscr{E}(X)$ is already determined by several authors in [8], [9], [10], [11], [12]. The author would like to study the group $\mathscr{E}(X)$ for non-simply connected H-spaces X.

PROBLEM (M. Arkowitz [2]). *Determine the group $\mathscr{E}(X)$ for non-simply connected H-spaces of rank 2. More specially, calculate the group $\mathscr{E}(X)$ for $X = \mathbf{RP}^i \times S^j$ (with $i = 3, 7$ and $j = 1, 3, 7$) or for $X = \mathbf{RP}^i \times \mathbf{RP}^k$ (with $i = 3, 7$ and $k = 3, 7$).*

In this paper we shall consider this problem for the case $X = SO(4) = S^3 \times SO(3) = S^3 \times \mathbf{RP}^3$.

DEFINITION 1.1. (i) Let $\mathrm{M}_2(\mathbf{R})$ be the ring consisting of all 2×2 real matrices and let $\mathrm{M}_2(\sqrt{2}) \subset \mathrm{M}_2(\mathbf{R})$ denote the subset of $\mathrm{M}_2(\mathbf{R})$ consisting of all 2×2 matrices A of the form

$$A = \begin{pmatrix} a_{1,1} & \sqrt{2}a_{1,2} \\ \sqrt{2}a_{2,1} & a_{2,2} \end{pmatrix} \qquad (\text{where } a_{i,j} \in \mathbf{Z}).$$

Clearly $\mathrm{M}_2(\sqrt{2})$ is a subring of $\mathrm{M}_2(\mathbf{R})$.

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(ii) For a ring R with unit 1, let $\mathrm{Inv}(R)$ denote the group consisting of all invertible elements $r \in R$.

We shall prove the following results.

THEOREM 1.2. *There is a short exact sequence of multiplicative groups*

$$1 \longrightarrow G_4 \xrightarrow{1+\tilde{q}^*} \mathscr{E}(SO(4)) \longrightarrow \mathrm{Inv}(\mathrm{M}_2(\sqrt{2})) \longrightarrow 1$$

where G_4 denotes the certain group of order $2^8 \cdot 3^2$.

THEOREM 1.3. *Let $\mu : [SO(4), SO(4)] \to \mathrm{End}(\pi_3(SO(4)))$ be the natural representation given by $\mu(f) = \pi_3(f)$. Then the map μ induces the multiplicative epimorphism $\tilde{\mu} : [SO(4), SO(4)] \to \mathrm{M}_2(\sqrt{2})$ with its kernel isomorphic to G_4.*

The main part of the proof is to use the product decomposition $SO(4) = S^3 \times SO(3)$ and is to compute several homotopy groups using the composition method [13] and classical homotopy technique [4], [5], [6], [7]. In section 2, we shall compute several homotopy groups and homotopy sets. In section 3, we shall give the proofs of Theorems 1.2 and 1.3.

2. Homotopy groups

In this section we consider the cofibre sequence

$$S^1 \xrightarrow{2\iota_1} S^1 \xrightarrow{i} S^1 \cup_2 e^2 = \mathbf{RP}^2 \xrightarrow{q} S^2 \xrightarrow{2\iota_2} S^2 \xrightarrow{\Sigma i} \Sigma\mathbf{RP}^2 = S^2 \cup_2 e^3 \xrightarrow{\Sigma q} S^3. \tag{2.1}$$

Let $\rho : S^2 \to \mathbf{RP}^2$ denote the double covering projection. Since $SO(3) = \mathbf{RP}^3$, there is a cofibre sequence

$$S^2 \xrightarrow{\rho} \mathbf{RP}^2 \longrightarrow SO(3) \xrightarrow{\pi} S^3 \xrightarrow{\Sigma\rho} \Sigma\mathbf{RP}^2 = S^2 \cup_2 e^3 \tag{2.2}$$

and $SO(3)$ has the cell structure

$$SO(3) = \mathbf{RP}^2 \cup_\rho e^3 = S^1 \cup_2 e^2 \cup_\rho e^3. \tag{2.3}$$

Note the following fact:

LEMMA 2.4 ([13]).
(1) *If $J : \pi_1(SO(2)) \cong \pi_1(S^1) = \mathbf{Z}\{\iota_1\} \to \pi_3(S^2) = \mathbf{Z}\{\eta_2\}$ denotes the J-homomorphism, then J is an isomorphism and $J(\iota_1) = \eta_2$, where $\eta_2 \in \pi_3(S^2) = \mathbf{Z}\{\eta_2\}$ denotes the Hopf map.*
(2) *Let $\eta_n = \Sigma^{n-2}\eta_2 \in \pi_{n+1}(S^n)$ for $n \geq 3$. Then $\pi_{n+1}(S^n) = \mathbf{Z}/2\{\eta_n\}$ for $n \geq 3$.*

(3) *If we take* $\eta_n^2 = \eta_n \circ \eta_{n+1} \in \pi_{n+2}(S^2)$, $\pi_{n+2}(S^n) = \mathbf{Z}/2\{\eta_n^2\}$ *for* $n \geq 2$.
(4) *If* $\omega \in \pi_6(S^3)$ *denotes the Blakers-Massay element*, $\pi_6(S^3) = \mathbf{Z}/12\{\omega\}$.

LEMMA 2.5.

$$\pi_k(\Sigma\mathbf{RP}^2) = \begin{cases} \mathbf{Z}/2\{\Sigma i\} & (k=2) \\ \mathbf{Z}/4\{\Sigma i \circ \eta_2\} & (k=3). \end{cases}$$

PROOF. Let $\bar{\alpha} \in \pi_3(\Sigma\mathbf{RP}^2, S^2) = \mathbf{Z}\{\bar{\alpha}\}$ be the charactersitic map of the top cell e^3 in $\Sigma\mathbf{RP}^2 = S^2 \cup_2 e^3$ and consider the homotopy exact sequence

$$\mathbf{Z}\{\bar{\alpha}\} = \pi_3(\Sigma\mathbf{RP}^2, S^2) \xrightarrow{\partial_3} \pi_2(S^2) = \mathbf{Z}\{\iota_2\} \xrightarrow{\Sigma i_*} \pi_2(\Sigma\mathbf{RP}^2) \longrightarrow 0.$$

Since $\partial_3(\bar{\alpha}) = 2\iota_2$, ∂_3 is injective and $\pi_2(\Sigma\mathbf{RP}^2) = \mathbf{Z}/2\{\Sigma i\}$. Hence there is an exact sequence

$$\text{(i)}\qquad \begin{array}{ccccccc} \pi_4(\Sigma\mathbf{RP}^2, S^2) & \xrightarrow{\partial_4} & \pi_3(S^2) & \xrightarrow{\Sigma i_*} & \pi_3(\Sigma\mathbf{RP}^2) & \longrightarrow & 0 \\ \| & & \| & & & & \\ \bar{\alpha}_*\pi_4(D^3, S^2) \oplus \mathbf{Z}\{[\bar{\alpha}, \iota_2]_r\} & & \mathbf{Z}\{\eta_2\} & & & & \end{array}$$

where $[\,,]_r$ denotes the relative Whitehead product (cf. [3]). Since $[\iota_2, \iota_2] = 2\eta_2$,

$$\text{(ii)}\qquad \partial_4([\bar{\alpha}, \iota_2]_r) = -[\partial_3(\bar{\alpha}), \iota_2] = -[2\iota_2, \iota_2] = -2[\iota_2, \iota_2] = -4\eta_2.$$

Consider the commutative diagram:

$$\begin{array}{ccc} \pi_4(D^3, S^2) & \xrightarrow[\cong]{\partial_4'} & \pi_3(S^2) = \mathbf{Z}\{\eta_2\} \\ \bar{\alpha}_* \downarrow & & \downarrow (2\iota_2)_* \\ \pi_4(\Sigma\mathbf{RP}^2, S^2) & \xrightarrow{\partial_4} & \pi_3(S^2) = \mathbf{Z}\{\eta_2\}. \end{array}$$

Because $[\iota_2, \iota_2] = 2\eta_2$, $h_0(\eta_2) = \iota_3$ and $[\eta_2, \iota_2] = 0$,

$$\text{(iii)}\qquad (2\iota_2) \circ \eta_2 = 2\eta_2 + \binom{2}{2}[\iota_2, \iota_2] \circ h_0(\eta_2) - \binom{3}{3}[[\iota_2, \iota_2], \iota_2] \circ h_1(\eta_2)$$

$$= 2\eta_2 + (2\eta_2) \circ \iota_3 - 2[\eta_2, \iota_2] \circ h_1(\eta_2) = 4\eta_2.$$

Hence it follows from the diagram (ii) and (iii) that the image of ∂_4 is $\mathbf{Z}\{4\eta_2\}$. Therefore, $\pi_3(\Sigma\mathbf{RP}^2) = \mathbf{Z}/4\{\Sigma i \circ \eta_2\}$.

LEMMA 2.6.
(1) $\Sigma p = \pm 2(\Sigma i \circ \eta_2) \in \pi_3(\Sigma\mathbf{RP}^2) = \pi_3(S^2 \cup_2 e^3) = \mathbf{Z}/4\{\Sigma i \circ \eta_2\}$.
(2) *There is a homotopy equivalence*

$$\Sigma^2\mathbf{RP}^3 = \Sigma^2 SO(3) \simeq \Sigma^2\mathbf{RP}^2 \vee S^5 = S^3 \cup_2 e^4 \vee S^5.$$

PROOF. Since $\Sigma^2\mathbf{RP}^3 = \Sigma^2 SO(3) = \Sigma^2(\mathbf{RP}^2 \cup_p e^3) = \Sigma^2\mathbf{RP}^2 \cup_{\Sigma^2 p} e^5$ and $2\eta_3 = 0$, it suffices to prove (1). It follows from the formula of James ((3.1) of [4]) that

$$\Sigma p = \pm\Sigma i \circ J(c(\xi)) = \pm\Sigma i \circ J(2\iota_1) = \pm 2(\Sigma i \circ \eta_2). \qquad \square$$

Consider the cofibre sequence

$$S^4 \xrightarrow{2\iota_4} S^4 \xrightarrow{\Sigma^3 i} \Sigma^3\mathbf{RP}^2 = S^4 \cup_2 e^5 \xrightarrow{\Sigma^3 q} S^5 \xrightarrow{2\iota_5} S^5. \tag{2.7}$$

Since $\eta_3 \circ 2\iota_4 = 0$, there is an extension $\bar{\eta}_3 \in [\Sigma^3\mathbf{RP}^2, S^3]$ of η_3 such that

$$\bar{\eta}_3 \circ \Sigma^3 i = \eta_3. \tag{2.8}$$

LEMMA 2.9.

$$[\Sigma^k\mathbf{RP}^2, S^3] = \begin{cases} \mathbf{Z}/2\{\Sigma q\} & (k=1) \\ \mathbf{Z}/2\{\eta_3 \circ \Sigma^2 q\} & (k=2) \\ \mathbf{Z}/4\{\bar{\eta}_3\} & (k=3). \end{cases}$$

PROOF. Since the proofs of these cases are similar, we only prove the case $k=3$. Since $(2\iota_j)^* : \pi_j(S^3) \to \pi_j(S^3)$ is trivial for $j = 4, 5$, (2.7) induces the exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_5(S^3) = \mathbf{Z}/2\{\eta_3^2\} \xrightarrow{\Sigma^3 q^*} [S^4 \cup_2 e^5, S^3] \xrightarrow{\Sigma^3 i^*} \pi_4(S^3) \\ = \mathbf{Z}/2\{\eta_3\} \longrightarrow 0. \end{aligned} \tag{2.10}$$

Since $2\iota_4 \circ \eta_4 = 0$, there is a coextension $\tilde{\eta}_4 \in \pi_6(S^4 \cup_2 e^5)$ of the map η_4 such that $\Sigma^3 q \circ \tilde{\eta}_4 = \eta_5$. It is known that $\nu' = \bar{\eta}_3 \circ \tilde{\eta}_4 \in \pi_6(S^3)_{(2)} \cong \mathbf{Z}/4$ is the generator of 2-component ([13]). Hence the order of $\bar{\eta}_3$ is the multiple of 4 or infinite order. However, since the order of the homotopy set $[S^4 \cup_2 e^5, S^3]$ is 4 by (2.10), the order of $\bar{\eta}_3$ is divided by 4. Hence, from (2.8), $[\Sigma^3\mathbf{RP}^2, S^3] = [S^4 \cup_2 e^5, S^3] = \mathbf{Z}/4\{\bar{\eta}_3\}$. $\square$

COROLLARY 2.11.

$$[\Sigma^3 SO(3), S^3] \cong [\Sigma^3\mathbf{RP}^2, S^3] \oplus \pi_6(S^3) = \mathbf{Z}/4\{\bar{\eta}_3\} \oplus \mathbf{Z}/12\{\omega\}.$$

Let $\rho_3 : S^3 \to \mathbf{RP}^3 = SO(3)$ denote the double covering and consider the fibre sequence

$$S^3 \xrightarrow{\rho_3} \mathbf{RP}^3 = SO(3) \longrightarrow K(\mathbf{Z}/2, 1). \tag{2.12}$$

LEMMA 2.13. *There is an isomorphism*

$$(\rho_3)_* : [\Sigma^3 SO(3), S^3] \xrightarrow{\cong} [\Sigma^3 SO(3), SO(3)] \cong \mathbf{Z}/4 \oplus \mathbf{Z}/12.$$

Proof. This is because the sequence (2.12) induces the exact sequence

$$1 \longrightarrow [\Sigma^3 SO(3), S^3] \xrightarrow{(\rho_3)_*} [\Sigma^3 SO(3), SO(3)] \longrightarrow [\Sigma^3 SO(3), K(\mathbf{Z}/2, 1)] = 0.$$

□

Lemma 2.14.
(1) $[S^3, S^3] = \pi_3(S^3) = \mathbf{Z}\{\iota_3\}$.
(2) $[S^3, SO(3)] = \mathbf{Z}\{\rho_3\}$.
(3) $[SO(3), S^3] = \mathbf{Z}\{\pi\}$, *where* $\pi : SO(3) = \mathbf{RP}^3 \to S^3$ *denotes the pinch map to the top cell.*
(4) $[SO(3), SO(3)] = \mathbf{Z}\{\mathrm{id}\}$.
(5) $\rho_3 \circ \pi = 2 \cdot \mathrm{id} \in [SO(3), SO(3)]$.

Proof. The assertions (1) and (2) are trivial and the other results are well-known. See for example [7], [11]. □

3. The multiplicative structure

In this section, we shall study the multiplicative structure of $[SO(4), SO(4)]$. First, recall the general property of multiplication induced from composition of maps. For example, if X is an H-space, the left distributive law

$$(3.1) \qquad (f + g) \circ h = f \circ h + g \circ h \qquad (\text{for } f, g \in [Y, X],\ h \in [Z, Y])$$

holds, but in general, the right distributive law does not necessarily hold. However, in our case, we can prove:

Lemma 3.2. *Let* $m, n \in \mathbf{Z}$ *be integers.*
(1) $(m\pi) \circ (n\rho_3) = 2mn \cdot \iota_3$.
(2) $(m\rho_3) \circ (n\pi) = 2mn \cdot \mathrm{id}$.

Proof. It follows from (3.1) that it suffces to prove the assertions (1) and (2) when $m = 1$. So from now on, assume $m = 1$. Note that $\pi \circ \rho_3 = 2 \cdot \iota_3$; in fact, since $\pi \circ \rho_3 \in \pi_3(S^3) = \mathbf{Z}\{\iota_3\}$, we can take $\pi \circ \rho_3 = y \cdot \iota_3$ for some $y \in \mathbf{Z}$. Since $\iota_3 = \Sigma \iota_2$, $y \cdot \rho_3 = \rho_3 \circ (y\iota_3)$. Hence using (2.14) and (3.1), we get $y = 2$, because

$$y \cdot \rho_3 = \rho_3 \circ (y\iota_3) = \rho_3 \circ (\pi \circ \rho_3) = (\rho_3 \circ \pi) \circ \rho_3 = (2 \cdot \mathrm{id}) \circ \rho_3 = 2\rho_3.$$

Since $\pi_* : \pi_3(SO(3)) \to \pi_3(S^3)$ is a homomorphism,

$$\pi \circ (n\rho_3) = \pi_*(n\rho_3) = n \cdot \pi_*(\rho_3) = n(\pi \circ \rho_3) = n \cdot (2\iota_3) = n \cdot (2\Sigma\iota_2) = 2n \cdot \iota_3$$

and the assertion (1) holds.

Since $\rho_3 \circ (n\pi) \in [SO(3), SO(3)] = \mathbf{Z}\{\mathrm{id}\}$, we can write $\rho_3 \circ (n\pi) = x \cdot \mathrm{id}$ for some $x \in \mathbf{Z}$. Then similarly,

$$x \cdot \rho_3 = (x \cdot \mathrm{id}) \circ \rho_3 = (\rho_3 \circ (n \cdot \pi)) \circ \rho_3 = \rho_3 \circ ((n \cdot \pi) \circ \rho_3) = \rho_3 \circ (2n \cdot \iota_3)$$
$$= (\rho_3) \circ (2n \cdot \Sigma\iota_2) = 2n \cdot (\rho_3 \circ \iota_3) = 2n \cdot \rho_3.$$

Hence $x = 2n$ and the assertion (2) is also proved. □

Next, recall the following elementary result due to A. J. Sieradski.

THEOREM 3.3 (Sieradski [11]). *Let X_1 and X_2 be homotopy associative H-spaces. If the homotopy set $[X_1 \vee X_2, X_1 \wedge X_2]$ is trivial, there is a short exact sequence of multiplicative group*

$$1 \longrightarrow [X_1 \wedge X_2, X_1 \times X_2] \xrightarrow{1+\tilde{q}^*} \mathscr{E}(X_1 \times X_2) \longrightarrow GL_2(\Lambda_{i,j}) \longrightarrow 1$$

where $\Lambda_{i,j} = [X_i, X_j]$ for $i, j = 1, 2$, $GL_2(\Lambda_{i,j})$ denotes the multiplicative group consisting of all invertible elements of the ring

$$[X_1 \vee X_2, X_1 \times X_2] = M_2(\Lambda_{I,j}) = \begin{pmatrix} [X_1, X_1] & [X_1, X_2] \\ [X_2, X_1] & [X_2, X_2] \end{pmatrix}$$

and $\tilde{q} : X_1 \times X_2 \to X_1 \wedge X_2$ denotes the projection map.

Now we shall prove Theorems 1.2 and 1.3.

PROOF OF THEOREM 1.2. Note that $SO(4) = S^3 \times SO(3)$ and we take $(X_1, X_2) = (S^3, SO(3))$. It follows from the celluar approximation theorem that the homotopy set $[SO(3), \Sigma^3 SO(3)]$ and $\pi_3(\Sigma^3 SO(3))$ are trivial. Hence $[S^3 \vee SO(3), S^3 \wedge SO(3)] = 0$. So, using Theorem 3.3 and Lemma 2.14, there is a short exact sequence

$$1 \longrightarrow G_4 \xrightarrow{1+\tilde{q}^*} \mathscr{E}(SO(4)) \longrightarrow \mathrm{GL}_2(\Lambda_{i,j}) \longrightarrow 1 \tag{3.4}$$

where we take $G_4 = [\Sigma^3 SO(3), S^3 \times SO(3)] = [\Sigma^3 SO(3), S^3] \oplus [\Sigma^3 SO(3), SO(3)]$. It follows from lemma 2.14 that $G_4 \cong (\mathbf{Z}/4 \oplus \mathbf{Z}/12) \oplus (\mathbf{Z}/4 \oplus \mathbf{Z}/12)$. Hence the order of G_4 is $2^8 \cdot 3^2$. The multiplicative structure of G_4 may be different from the group $(\mathbf{Z}/4 \oplus \mathbf{Z}/12) \oplus (\mathbf{Z}/4 \oplus \mathbf{Z}/12)$.

Next we determine the group structure of $\mathrm{GL}_2(\Lambda_{i,j})$. For this purpose, consider the ring

$$\mathrm{M}_2(\Lambda_{i,j}) = \begin{pmatrix} [X_1, X_1] & [X_1, X_2] \\ [X_2, X_1] & [X_2, X_2] \end{pmatrix} = \begin{pmatrix} [S^3, S^3] & [S^3, SO(3)] \\ [SO(3), S^3] & [SO(3), SO(3)] \end{pmatrix}.$$

Let $A, B \in \mathrm{M}_2(\Lambda_{i,j})$ be elements

$$A = \begin{pmatrix} a_{1,1}\iota_3 & a_{1,2}\rho_3 \\ a_{2,1}\pi & a_{2,2}\mathrm{id} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{1,1}\iota_3 & b_{1,2}\rho_3 \\ b_{2,1}\pi & b_{2,2}\mathrm{id} \end{pmatrix} \qquad (\text{where } a_{i,j}, b_{i,j} \in \mathbf{Z}).$$

Then using (3.2), the product $A \cdot B$, which is induced from the composite of maps, is equal to

$$A \cdot B = \begin{pmatrix} a_{1,1}\iota_3 & a_{1,2}\rho_3 \\ a_{2,1}\pi & a_{2,2}\mathrm{id} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1}\iota_3 & b_{1,2}\rho_3 \\ b_{2,1}\pi & b_{2,2}\mathrm{id} \end{pmatrix}$$

$$= \begin{pmatrix} (a_{1,1}\iota_3) \circ (b_{1,1}\iota_3) + (a_{1,2}\rho_3) \circ (b_{2,1}\pi) & (a_{1,1}\iota_3) \circ (b_{1,2}\rho_3) + (a_{1,2}\rho_3) \circ (b_{2,2}\mathrm{id}) \\ (a_{2,1}\pi) \circ (b_{1,1}\iota_3) + (a_{2,2}\mathrm{id}) \circ (b_{2,1}\pi) & (a_{2,1}\pi) \circ (b_{1,2}\rho_3) + (a_{2,2}\iota_3) \circ (b_{2,2}\mathrm{id}) \end{pmatrix}$$

$$= \begin{pmatrix} (a_{1,1}b_{1,1} + 2a_{1,2}b_{2,1})\iota_3 & (a_{1,1}b_{1,2} + a_{1,2}b_{2,2})\rho_3 \\ (a_{2,1}b_{1,1} + a_{2,2}b_{2,1})\pi & (2a_{2,1}b_{1,2} + a_{2,2}b_{2,2})\mathrm{id} \end{pmatrix}.$$

Define the additive map $\phi : \mathrm{M}_2(\Lambda_{i,j}) \to \mathrm{M}_2(\sqrt{2})$ by

$$\phi\left(\begin{pmatrix} a_{1,1}\iota_3 & a_{1,2}\rho_3 \\ a_{2,1}\pi & a_{2,2}\mathrm{id} \end{pmatrix}\right) = \begin{pmatrix} a_{1,1} & \sqrt{2}a_{1,2} \\ \sqrt{2}a_{2,1} & a_{2,2} \end{pmatrix} \qquad (\text{where } a_{i,j} \in \mathbf{Z}).$$

Then it follows from (1.1) and the above computation that $\phi : \mathrm{M}_2(\Lambda_{i,j}) \xrightarrow{\cong} \mathrm{M}_2(\sqrt{2})$ is a ring isomorphism. Hence $\mathrm{GL}_2(\Lambda_{i,j}) = \mathrm{Inv}(\mathrm{M}_2(\Lambda_{i,j})) \cong \mathrm{Inv}(\mathrm{M}_2(\sqrt{2}))$. So (3.4) reduces to the exact sequence

$$1 \longrightarrow G_4 \xrightarrow{1+\tilde{q}^*} \mathscr{E}(SO(4)) \longrightarrow \mathrm{Inv}(\mathrm{M}_2(\sqrt{2})) \longrightarrow 1$$

and this completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Consider the representation

$$\mu : [SO(4), SO(4)] \to \mathrm{End}(\pi_3(SO(4)))$$

given by $\mu(f) = \pi_3(f)$. Since each $\Lambda_{i,j} = [X_i, X_j]$ and $\pi_3(SO(4))$ are torsion free, $\mu([SO(4), SO(4)]) = \mathrm{M}_2(\Lambda_{i,j})$ and the assertion follows from Theorem 1.2. □

References

[1] Y. Ando and K. Yamaguchi, *Homotopy self-equivalences of the product $A \times B$*, Proc. Japan Acad. **58** (1982), 323–325.

[2] D. W. Kahn, *Some research problems on homotopy-self-equivalences*, Lecture Notes in Math. **1425** (1990), Springer-Verlag, 204–214.

[3] I. M. James, *On the homotopy groups of certain pairs and triads*, Quart. J. Math. Oxford **5** (1954), 260–270.

[4] I. M. James, *On sphere bundles over spheres*, Comment. Math. Helv. **35** (1961), 126–135.

[5] I. M. James and J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres (I)*, Proc. London Math. Soc. **4** (1954), 196–218.

[6] I. M. James and J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres (II)*, Proc. London Math. Soc. **5** (1955), 148–166.
[7] C. A. McGibbon, *Self maps of projective spaces*, Trans. A. M. S. **271** (1982), 325–346.
[8] M. Mimura and N. Sawashita, *On the group of self-homotopy equivalences of H-spaces of rank* 2, J. Math. Kyoto Univ. **21** (1981), 331–349.
[9] S. Oka, N. Sawashita and M. Sugawara, *On the group of self-equivalences of a mapping cone*, Hiroshima Math. J. **4** (1974), 9–28.
[10] N. Sawashita, *On the group of self-equivalences of the product of spheres*, Hiroshima Math. J. **5** (1975), 69–86.
[11] A. J. Sieradski, *Twisted self-homotopy equivalences*, Pacific J. Math. **34** (1970), 789–802.
[12] J. W. Rutter, *The group of self-homotopy equivalences of principal three sphere bundles over the seven sphere*, Math. Proc. Camb. Phil. Soc. **84** (1978), 303–311.
[13] H. Toda, *Composition methods in homotopy groups of spheres*, vol. 49, Ann. of Math. Studies, Princeton Univ. Press, 1962.
[14] G. W. Whitehead, *Elements of homotopy theory*, vol. 61, Graduate Text in Math., Springer-Verlag, 1978.

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