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The Palais-Smale condition for the energy of some semilinear parabolic equations

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ABSTRACT. In this paper we show that all the global solutions for some semilinear parabolic equations naturally contain a Palais-Smale sequence as a subsequence and then we apply a global compactness result due to Struwe [16] to the Palais-Smale sequence. Furthermore, the finite-time blowup problems are discussed.

1. Introduction

In this paper, we are concerned with the following mixed problem to semilinear parabolic equation:

$$u_t(t,x) - \Delta u(t,x) = |u(t,x)|^{p-1}u(t,x), \qquad (t,x) \in (0,T) \times \Omega, \tag{1}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{2}$$

$$u|_{\partial\Omega} = 0, \qquad t \in (0,T). \tag{3}$$

Here $1 < p \leq (N+2)/(N-2)$ and $\Omega \subset R^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$. In the case when $1 < p < (N+2)/(N-2)$ we can treat the lower dimensional case $N = 1, 2$, but for simplicity we restrict our attention to the above mentioned case. For large initial data u_0 in some sense, it is well-known that the solution $u(t,x)$ to the problem (1)–(3) blows up in a finite time (see Ikehata-Suzuki [9], Ishii [10], Levine [11], Ôtani [13], Tsutsumi [18], and Payne-Sattinger [14]), meanwhile for small initial data, exponentially decaying solutions are obtained (see [9] and the references therein). In this paper, we are interested in the solutions to (1)–(3) which neither blowup nor decay. We proceed our argument based on the following local well-posedness theorem due to [9] (see also Hoshino-Yamada [7]). In the following, $\|\cdot\|_q$ $(1 \leq q \leq \infty)$ means the usual real $L^q(\Omega)$-norm.

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Proposition 1.1. *For each* $u_0 \in H_0^1(\Omega)$, *there exists a maximal existence time* $T_m > 0$ *(possibly* $T_m = +\infty$*) such that the problem* (1)–(3) *has a unique solution* $u \in C([0, T_m); H_0^1(\Omega))$ *which becomes classical on* $(0, T_m)$. *Furthermore, if* $T_m < +\infty$, *then*

$$\lim_{t\uparrow T_m} \|u(t,\cdot)\|_\infty = +\infty,$$

and in particular, in the case when $1 < p < (N+2)/(N-2)$ *one also has*

$$\lim_{t\uparrow T_m} \|\nabla u(t,\cdot)\|_2 = +\infty.$$

Set

$$X = H_0^1(\Omega),$$

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

$$I(u) = \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1},$$

$$\mathcal{N} = \{v \in X\backslash\{0\} \mid I(v) = 0\},$$

$$d_p = \inf_{v\in\mathcal{N}} J(v) = \inf\left\{\sup_{\lambda\geq 0} J(\lambda v) \mid v \in X\backslash\{0\}\right\}.$$

It is easy to show that the potential depth d_p is positive (see Sattinger [15]) using the Sobolev continuous embedding $X \hookrightarrow L^{p+1}(\Omega)$. The stable and unstable sets are defined as usual:

$$W = \{u \in X \mid J(u) < d_p, I(u) > 0\} \cup \{0\},$$

$$V = \{u \in X \mid J(u) < d_p, I(u) < 0\}.$$

Furthermore, for later use we define the following notation.

$$E = \{u \in X \mid -\Delta u = |u|^{p-1}u \ \ in \ \ \Omega, u|_{\partial\Omega} = 0\},$$

$$E^* = \{u \in \mathcal{D}^{1,2}(R^N) \mid -\Delta u = |u|^{p-1}u \ \ in \ \ R^N\},$$

$$E_+^* = \{u \in E^* \mid u \geq 0 \ \ in \ \ R^N\},$$

$$J_*(u) = \frac{1}{2}\int_{R^N} |\nabla u(x)|^2 dx - \frac{1}{p+1}\int_{R^N} |u(x)|^{p+1} dx.$$

Here $\mathcal{D}^{1,2}(R^N)$ denotes the closure of $C_0^\infty(R^N)$ with respect to the norm $\|\nabla u\|_{L^2(R^N)}$. In the case when $p = (N+2)/(N-2)$, because of the Sobolev embedding $S\|u\|_{L^{p+1}(R^N)} \leq \|\nabla u\|_{L^2(R^N)}$ for $u \in \mathcal{D}^{1,2}(R^N)$, one also has

$$d^* = d_p = \inf\left\{\sup_{\lambda\geq 0} J_*(\lambda v) \mid v \in \mathscr{D}^{1,2}(R^N)\backslash\{0\}\right\} = \frac{1}{N}S^N > 0.$$

REMARK 1.1. *In the case when* $p = (N+2)/(N-2)$, *it is well-known* (Struwe [16]) *that the family* $\{u_\varepsilon^*(x)\}$ *defined by*

$$u_\varepsilon^*(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{[\varepsilon^2 + |x|^2]^{(N-2)/2}}, \qquad \varepsilon > 0$$

satisfies

$$-\Delta u = |u|^{p-1}u \qquad in\ R^N \tag{4}$$

so that $E_+^*\backslash\{0\} \neq \varnothing$.

We start with the following result which we showed quite recently in [9] with regard to the singularity of a global solution to the problem (1)–(3) under the assumptions below: let $u(t,x)$ be a solution to (1)–(3) as in Proposition 1.1. Furthermore, one assumes that

(A.1) $u_0 \geq 0$.

(A.2) $p = (N+2)/(N-2)$.

(A.3) $\Omega = \{x \in R^N \mid |x| < 1\}$.

(A.4) $u(t,x) = u(t,|x|), u_r(t,r) < 0$ on $0 < r \leq 1$ with $r = |x|$.

Finally, assume $T_m = +\infty$. For $1 < p \leq (N+2)/(N-2)$ set

$$C_0 = \frac{2(p+1)}{(p-1)} \lim_{t\to+\infty} J(u(t,\cdot)). \tag{5}$$

Note that $C_0 \geq 0$ if $T_m = +\infty$ (see [11]). Then, our results in [9] read as follows.

THEOREM 1.1 ([9]). *Assume* (A.1)–(A.4). *Let* $u(t,x)$ *be a solution to* (1)–(3) *on* $[0,T_m)$ *as in Proposition* 1.1. *Suppose* $T_m = +\infty$ *and* $C_0 > 0$. *Then, there exists a sequence* $\{t_n\}$ *with* $t_n \to +\infty$ *as* $n \to +\infty$ *such that*

(i) $|\nabla u(t_n,x)|^2 \to C_0\delta_0$ *(weakly*) in* $C_0(\Omega)^*$,

(ii) $u(t_n,x)^{p+1} \to C_0\delta_0$ *(weakly*) in* $C_0(\Omega)^*$,

as $n \to +\infty$. *Here,* δ_0 *stands for the usual Dirac measure having a unit mass at the origin.*

Since $C_0 > 0$ if and only if $u(t,\cdot) \notin (W \cup V)$ for all $t \geq 0$, this theorem states that a global orbit $u(t,\cdot)$ which neither decays nor blowups has a strong singularity at the origin if this kind of solution can be constructed.

In connection with this result, we notice that such a sequence $\{t_n\}$ constructed in Theorem 1.1, $\{u(t_n,\cdot)\}$ becomes a Palais-Smale sequence so that the global compactness result due to Struwe [17] can be applied to this functional sequence. So, our first result reads as follows (see also Cerami, Solimini and Struwe [4]):

THEOREM 1.2. *Let* $\{u(t_n,\cdot)\} \subset H_0^1(\Omega) \subset \mathscr{D}^{1,2}(R^N)$ *be a sequence as in Theorem* 1.1. *Then there exist a subsequence of* $\{u(t_n,\cdot)\}$, *relabelled again as* $\{u(t_n,\cdot)\}$, *an integer* $k \in N$, *a sequence of radii* $\{R_n^i\}$ *with* $\lim_{n\to+\infty} R_n^i = +\infty$ $(1 \le i \le k)$ *such that*

$$\lim_{n\to+\infty}\left\|\nabla\Big(u(t_n,\cdot)-\sum_{i=1}^{k}u_n^i\Big)\right\|_{L^2(R^N)} = 0,$$

$$\lim_{t\to+\infty} J(u(t,\cdot)) = \lim_{n\to+\infty} J(u(t_n,\cdot)) = kJ_*(\omega) = \frac{p-1}{2(p+1)}C_0 > 0,$$

$$\lim_{n\to+\infty}\|\nabla u(t_n,\cdot)\|_2^2 = k\|\nabla\omega\|_{L^2(R^N)}^2,$$

where

$$u_n^i(x) = (R_n^i)^{(N-2)/2}\omega(R_n^i x) \quad (1 \le i \le k), \qquad n = 1,2,\dots$$

together with $\omega(x) = u_1^*(x)$ *defined in Remark* 1.1.

REMARK 1.2. *It is easy to see that* $J_*(\omega) = d^*$ *(least energy level) follows. Therefore, one has* $\frac{p-1}{2(p+1)}C_0 = kd^*$ *so that if, in particular,* $k = 1$, *then* $\lim_{t\to+\infty} J(u(t,\cdot)) = d^*$, *i.e., the energy* $J(u(t,\cdot))$ *for a solution* $u(t,\cdot)$ *of* (1)–(3) *may attain its least energy level as in the subcritical case. Similarly, since* $\|\nabla\omega\|_{L^2(R^N)}^2 = S^N$ *in the present case, from Lemma* 2.1 *below it follows that* $C_0 = kS^N$.

REMARK 1.3. *Under the assumptions* Ω = *star-shaped and* $u_0(x) \ge 0$, *one can get the similar results as in the radial case above with a slight modification. In the case when* u_0 *changes sign, however, even if* Ω *is star-shaped, one needs to modify the results above in accordance with the results in* [16] *(for more general case, see the proof of Proposition* 2.1*).*

The next result is concerned with the case when $1 < p < (N+2)/(N-2)$. It seems unknown that any global solutions to (1)–(3) naturally contain a subsequence which is relatively compact in X in the subcritical case. Our second result reads as follows:

THEOREM 1.3. *Let* $1 < p < (N+2)/(N-2)$ *and* $u(t,x)$ *be a solution on* $[0,T_m)$ *as in Proposition* 1.1. *If* $T_m = +\infty$, *then there exists a sequence* $\{t_n\}$ *with* $t_n \to +\infty$ *as* $n \to +\infty$ *such that* $\{u(t_n,\cdot)\}$ *becomes relatively compact in* X

so that there exists an element $u_\infty \in E$ *such that* $u(t_n, \cdot) \to u_\infty$ *in* X *as* $n \to +\infty$ *along a subsequence.*

REMARK 1.4. *If* $C_0 > 0$, *then one has* $u_\infty \in E\backslash\{0\}$ *in Theorem* 1.3. *Moreover, such a sequence* $\{t_n\}$ *is constructed in the same way as in Theorem* 1.2. *On the other hand, unfortunately, the results in Theorem* 1.3 *are weaker than that of* [3] *or* [13] *in the sense that their results state the relative compactness in* $H_0^1(\Omega)$ *of the trajectory* $\{u(t, \cdot)\}$.

2. Palais-Smale sequence

Reviewing some results concerning Theorem 1.1 due to [9] we shall construct some Palais-Smale sequences of a global solution to the problem (1)–(3), and then we will prove Theorems 1.2 and 1.3.

First, suppose $1 < p \leq (N+2)/(N-2)$ and $T_m = +\infty$ in Proposition 1.1. Since its solution satisfies the energy identity:

$$J(u(t, \cdot)) + \int_0^t \|u_t(s, \cdot)\|_2^2 ds = J(u_0) \qquad \text{all } t \geq 0, \tag{6}$$

this implies that the function $t \mapsto J(u(t, \cdot))$ is monotone decreasing so that $C_0 \geq 0$ (see (5)) is meaningfull. Letting $t \to +\infty$ in (6), the improper integral $\int_0^\infty \|u_t(s, \cdot)\|_2^2 ds$ is finitely determined. Therefore, there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n \to +\infty} \|u_t(t_n, \cdot)\|_2 = 0.$$

In fact this sequence $\{t_n\}$ is given in [9] for the proof of Theorem 1.1.

Next, multiplying the both sides of (1) by $u(t, x)$ and integrating it over Ω, we have

$$(u_t(t, \cdot), u(t, \cdot)) = -I(u(t, \cdot)), \tag{7}$$

where $(f, g) = \int_\Omega f(x)g(x)dx$. Due to [3], it is true that $\|u(t, \cdot)\|_2 \leq C$ for all $t \geq 0$ for some constant $C > 0$. Therefore, one has

$$|I(u(t, \cdot))| \leq C\|u_t(t_n, \cdot)\|_2 \qquad \text{for all } n \in N.$$

Letting $n \to +\infty$, it follows that

$$\lim_{n \to +\infty} I(u(t_n, \cdot)) = 0. \tag{8}$$

On the other hand, the identity holds:

$$J(u) = \frac{p-1}{2(p+1)}\|\nabla u\|_2^2 + \frac{1}{p+1}I(u). \tag{9}$$

So, from (9) with $u = u(t_n, \cdot)$ and (7)–(8) we find that

LEMMA 2.1. *Let $u(t,\cdot)$ be as in Proposition* 1.1. *If $T_m = +\infty$, then there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ as $n \to +\infty$ such that*

$$\lim_{n\to+\infty} \|u_t(t_n,\cdot)\|_2 = 0,$$

$$\lim_{n\to+\infty} \|\nabla u(t_n,\cdot)\|_2^2 = C_0,$$

$$\lim_{n\to+\infty} \|u(t_n,\cdot)\|_{p+1}^{p+1} = C_0.$$

From this lemma, one obtains the next one:

LEMMA 2.2. *Let $u(t,\cdot)$ be a local solution constructed in Proposition* 1.1. *If $T_m = +\infty$, then there exists a Palais-Smale sequence to the problem* (1)–(3).

PROOF. Let $\{t_n\}$ be as in Lemma 2.1. Then, it follows that

$$J(u_0) \geq J(u(t_n,\cdot)) \to \frac{p-1}{2(p+1)} C_0 \geq 0 \qquad \text{as } n \to +\infty. \tag{10}$$

Furthermore, for such a sequence, since $J \in C^1(X,R)$, by equation (1) we have

$$J'(u(t_n,\cdot))[v] = -(u_t(t_n,\cdot)),v)$$

for each $v \in X$, where $J'(u) \in X^*$ means the usual Fréchet-derivative of J at $u \in X$. By this equality and the Schwarz inequality together with the Poincaré inequality one gets:

$$|J'(u(t_n,\cdot))[v]| \leq C_1 \|u_t(t_n,\cdot))\|_2 \|\nabla v\|_2$$

which implies

$$\|J'(u(t_n,\cdot))\|_{H^{-1}(\Omega)} \to 0 \qquad (n \to +\infty), \tag{11}$$

where $C_1 > 0$ is a Poincaré constant. We find that $\{u(t_n,\cdot)\}$ is a Palais-Smale sequence because of (10) and (11). □

In particular, in the case when $1 < p < (N+2)/(N-2)$ one gets the following compactness result.

LEMMA 2.3. *Suppose $1 < p < (N+2)/(N-2)$. Let $u(t,\cdot)$ be a global (i.e., $T_m = +\infty$) solution to* (1)–(3) *as in Proposition* 1.1. *Then, the sequence $\{u(t_n,\cdot)\}$ constructed in Lemma* 2.1 *becomes relatively compact in X.*

PROOF. For simplicity, one sets $u_n = u(t_n,\cdot)$. Multiplying the both sides of (1) by $v \in X$ and integrating it over Ω, we have

$$|(\nabla u_n, \nabla v) - (f(u_n),v)| = |(u_t(t_n,\cdot),v)| \leq C_1 \|u_t(t_n,\cdot)\|_2 \|\nabla v\|_2, \tag{12}$$

where $f(v)(x) = |v(x)|^{p-1}v(x)$. From Lemma 2.1 it follows that for an arbitrary $\varepsilon > 0$, there exists a natural number N_0 such that for all $n \geq N_0$,

$$\|u_t(t_n, \cdot)\|_2 < \frac{\varepsilon}{C_1}. \tag{13}$$

Because of (12) and (13), we have

$$|(\nabla u_n, \nabla v) - (f(u_n), v)| \leq \varepsilon \|\nabla v\|_2 \leq \varepsilon^2 + \frac{1}{4}\|\nabla v\|_2^2. \tag{14}$$

On the other hand, it follows from the Hölder inequality that

$$|(f(u_n), v)| \leq \|u_n\|_{p+1}^p \|v\|_{p+1}. \tag{15}$$

By takin as $v = u_n - u_m$ in (14) and (15), we can proceed the following estimates:

$$\begin{aligned} \|\nabla u_n - \nabla u_m\|_2^2 &= \int_\Omega [\nabla u_n \nabla(u_n - u_m) - f(u_n)(u_n - u_m)]dx \\ &\quad - \int_\Omega [\nabla u_m \nabla(u_n - u_m) - f(u_m)(u_n - u_m)]dx \\ &\quad + \int_\Omega (f(u_n) - f(u_m))(u_n - u_m)dx \\ &\leq \frac{1}{2}\|\nabla u_n - \nabla u_m\|_2^2 + 2\varepsilon^2 + (\|u_n\|_{p+1}^p + \|u_m\|_{p+1}^p)\|u_n - u_m\|_{p+1} \end{aligned}$$

for all $m, n \geq N_0$. This implies

$$\|\nabla u_n - \nabla u_m\|_2^2 \leq 4\varepsilon^2 + 2(\|u_n\|_{p+1}^p + \|u_m\|_{p+1}^p)\|u_n - u_m\|_{p+1} \tag{16}$$

for all $m, n \geq N_0$.

Now, since $\{u_n\}$ is bounded in X, by the compact embedding of $X \hookrightarrow L^{p+1}(\Omega)$ we can assume that $u_n \to u_\infty$ in $L^{p+1}(\Omega)$ for some u_∞ as $n \to +\infty$. Together with (16), we find that $\{u_n\}$ becomes a Cauchy sequence in X. □

Now, we are in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Basically, this is a direct consequence of [16] (Theorem 3.1, p. 184) and Lemma 2.2. Under the framework of Theorem 1.1, however, one has $E = \{0\}$, $x_n^i = 0$ $(1 \leq i \leq k)$ in use of [16] and note that the solution $u(x)$ for the equation (4) is uniquely determined (up to scaling and translation) such as $u(x) = u_1^*(x) = \omega(x)$.

We shall state the outline of its proof. Indeed, set

$$Q_n(t) = \int_{|x|<t} (|\nabla u(t_n, x)|^2 + |u(t_n, x)|^{p+1})dx.$$

Then, for each $\nu \in (0, S^N)$, we can find a real number $R_n = R_n(\nu) > 1$ such that $Q_n\left(\frac{1}{R_n}\right) = \nu$. Set $u_n(x) = R_n^{-(N-2)/2} u(t_n, x/R_n)$. Then, since the embedding

$$\mathscr{D}^{1,2}_{rad}(R^N) \hookrightarrow L^\infty\left(\left\{\frac{1}{R} \le |x| \le R\right\}\right) \tag{17}$$

is compact for each $R > 1$, it will follow that $u_n \rightharpoonup \omega \in E_+^* \backslash \{0\}$ (weakly) in $\mathscr{D}^{1,2}_{rad}(R^N)$ as $n \to +\infty$ along a subsequence (c.f., [9] or [12]). Here, $\mathscr{D}^{1,2}_{rad}(R^N) = \{v \in \mathscr{D}^{1,2}(R^N) \mid v(x) = v(|x|)\}$.

In fact, if $\omega \equiv 0$, then it follows from Lemma 2.1 and the compact embedding (17) that

$$u_n(x)^{p+1} \to C_0\delta_0 \qquad (\text{weakly}^*) \text{ in } C_0(R^N)^*$$

as $n \to +\infty$. On the other hand, if we choose $\phi \in C_0^\infty(R^N)$, with $\phi = 1$ on $B_1(0)$ and $0 \le \phi \le 1$ on R^N, then one can estimate as follows:

$$0 \le \int_{R^N} \phi(x) u_n(x)^{p+1} dx \le \nu + \int_{|x| \ge 1} \phi(x) u_n(x)^{p+1} dx = \nu + o(1)$$

as $n \to +\infty$. This implies $C_0 \le \nu$ which contradicts the fact $\nu \in (0, S^N)$ and $C_0 \ge S^N$.

Next, set $v_n(x) = u(t_n, x) - R_n^{(N-2)/2}\omega(R_n x)$. By iterating this procedure for the sequence $\{v_n\} \subset \mathscr{D}^{1,2}_{rad}(R^N)$, one can prove Theorem 1.2 similarly to the usual global compactness argument (c.f. [16] or [17]). □

PROOF OF THEOREM 1.3. The first half is a direct consequence of Lemma 2.3. In order to prove $u_\infty \in E$, note the following estimates:

$$\|f(u) - f(v)\|_{1+(1/p)} \le p(\|u\|_{p+1} + \|v\|_{p+1})^{p-1} \|u - v\|_{p+1}$$
$$\text{for all } u, v \in L^{p+1}(\Omega),$$

and

$$|(f(u(t_n, \cdot)) - f(u_\infty), \phi)| \le \|f(u(t_n, \cdot)) - f(u_\infty)\|_{1+(1/p)} \|\phi\|_{p+1}$$
$$\text{for each } \phi \in C_0^\infty(\Omega),$$

where $\{u(t_n, \cdot)\}$ is a sequence constructed in the first half. By combining these estimates with Lemma 2.1 and the Sobolev embedding $X \hookrightarrow L^{p+1}(\Omega)$, one obtains the desired result. □

From the view point of the Palais-Smale condition, we have the following result.

Corollary 2.1. *Let* $1 < p \le (N+2)/(N-2)$ *and* $u(t,x)$ *be a global solution constructed in Proposition* 1.1, *i.e.,* $T_m = +\infty$. *If* $C_0 = 0$, *then the sequence* $\{u(t_n, \cdot)\}$ *given in Lemma* 2.1 *becomes relatively compact, and in fact,* $u(t, \cdot) \to 0$ *in* X *as* $t \to +\infty$.

Proof. If $C_0 = 0$, then, from Lemma 2.1 it follows that $\lim_{n\to+\infty} \|\nabla u(t_n, \cdot)\|_2 = 0$. On the other hand, it is well- known that the stable set W is a bounded neighbourhood of 0 in X. Thus, $u(t_{n_0}, \cdot) \in W$ for some t_{n_0}. This implies that $\|\nabla u(t, \cdot)\|_2 = O(e^{-\alpha t})$ as $t \to +\infty$ (see [9]). □

From Theorem 1.1 and corollary 2.1 with $p = (N+2)/(N-2)$, one can say that it depends on the least energy level $(p-1)C_0/2(p+1)$ whether the Palai-Smale condition holds or not to the sequence $\{u(t_n, \cdot)\}$ in Lemma 2.1.

Now, we apply Theorem 1.3 and Lemma 2.2 for the finite time blowup problem concerning (1)–(3). First, as a consequence of [16] one obtains the following lemma.

Lemma 2.4. *Let* Ω *be a bounded smooth domain and* $p = (N+2)/(N-2)$. *Then, for all* $v \in E$, *one has* $J(v) \in \{0\} \cup (d^*, +\infty)$, *and also, for each* $w \in E^* \backslash \{0\}$, *one has* $J_*(w) \in \{d^*\} \cup (2d^*, +\infty)$.

The following result gives a kind of alternative proof of [13] concerning blowup problem.

Proposition 2.1. *Let* $1 < p \le (N+2)/(N-2)$ *and* $u(t,x)$ *be a local solution of* (1)–(3) *on* $[0, T_m)$ *constructed in Proposition* 1.1. *If* $u(t_0, \cdot) \in V$ *for some* $t_0 \in [0, T_m)$, *then* $T_m < +\infty$.

Proof. First, we shall deal with the case when $1 < p < (N+2)/(N-2)$. Suppose $T_m = +\infty$. Then, it follows from Theorem 1.3 that there exists a Palais-Smale sequence $\{u(t_n, \cdot)\}$ to the problem (1)–(3) and $u_\infty \in E$ such that $u(t_n, \cdot) \to u_\infty$ in X. On the other hand, it is well-known (see [8]) that $u(t, \cdot) \in V$ for all $t \in [t_0, \infty)$. If $u_\infty = 0$, then $u(t_m, \cdot) \in W$ with some t_m since W is a neighbourhood of 0 in X and this contradicts the fact that $W \cap V = \emptyset$. Thus, $u_\infty \in E \backslash \{0\}$. Since the function $t \mapsto J(u(t, \cdot))$ is monotone, one obtains $J(u(t_n, \cdot)) \ge J(u_\infty) \ge d_p$ which contradicts $u(t_n, \cdot) \in V$ with large t_n.

Next, we are concerned with the critical case $p = (N+2)/(N-2)$. Suppose $T_m = +\infty$. Obviously, $C_0 > 0$. Then, from Lemma 2.2 and Theorem 3.1 of [16], p. 184 there exist a Palais-Smale sequence $\{u(t_n, \cdot)\}$, $k \in N$, $u^0 \in E$, and $u^i \in E^* \backslash \{0\}$ $(1 \le i \le k)$ such that

$$\lim_{n\to+\infty} J(u(t_n, \cdot)) = \lim_{t\to+\infty} J(u(t, \cdot)) = J(u^0) + \sum_{i=1}^{k} J_*(u^i).$$

By Lemma 2.4 and the monotone decreasingness of a function $t \mapsto J(u(t,\cdot))$, one finds that

$$J(u(t,\cdot)) \geq d^* \qquad \text{for all } t \geq 0.$$

This contradicts also $u(t,\cdot) \in V$ for all $t \geq t_0$. □

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