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Quantum deformations of certain prehomogeneous vector spaces III

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Abstract. We apply our previous result [14] to the classical groups, and construct quantum analogues of the coordinate algebras of certain prehomogeneous vector spaces as non-commutative algebras equipped with actions of the quantized enveloping algebras. We also give explicit descriptions of the non-commutative counterparts for the generators of the defining ideals of the closures of orbits including basic relative invariants. In particular, quantum analogues of a quadratic form and the determinant of a symmetric matrix are naturally obtained.

0. Introduction

Let L be a connected reductive algebraic group over the complex number field $\mathbf{C}$, and let $\mathfrak{l}$ be the Lie algebra of L. We denote by $U_q(\mathfrak{l})$ the quantum deformation of the enveloping algebra $U(\mathfrak{l})$ of $\mathfrak{l}$ constructed by Drinfel'd [1] and Jimbo [5]. It is a Hopf algebra over the rational function field $\mathbf{C}(q)$. By Lusztig [6] any finite dimensional $\mathfrak{l}$-module V has a quantum deformation V_q as a $U_q(\mathfrak{l})$-module. In order to investigate quantum analogues of results concerning geometric structure of V such as L-orbits, we need also a quantum deformation of the coordinate algebra $A(V)$. In this paper we shall construct a quantum deformation $A_q(V)$ of the coordinate algebra $A(V)$ for certain prehomogeneous vector spaces V, and give counterparts for the defining ideals of the closures of L-orbits on V and their canonical generator systems.

More generally, let X be an affine variety endowed with an action of L. Then $A(X)$ is a right $A(L)$-comodule whose coaction

$$\tau : A(X) \to A(X) \otimes A(L)$$

is an algebra homomorphism. Thus we obtain a locally finite left $U(\mathfrak{l})$-module structure on $A(X)$ satisfying

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$$u \cdot (mn) = \sum_i (u_i^{(1)} \cdot m)(u_i^{(2)} \cdot n)$$

for $u \in U(\mathrm{I})$, $m, n \in A(X)$ and $\Delta(u) = \sum_i u_i^{(1)} \otimes u_i^{(2)}$, where Δ is the comultiplication of $U(\mathrm{I})$.

Hence it is natural to define a quantum deformation $A_q(V)$ of the coordinate algebra $A(V)$ of an L-module V to be a $\mathbf{C}(q)$-algebra satisfying the following conditions:

(i) $A_q(V)$ is generated by the quantum deformation V_q^* of V^* satisfying quadratic homogeneous fundamental relations.

(ii) $A(V)$ is the limit of $A_q(V)$ when q tends to 1.

(iii) The action of $U_q(\mathrm{I})$ on V_q^* is uniquely extended to a $U_q(\mathrm{I})$-module structure on $A_q(V)$ satisfying

$$u \cdot (mn) = \sum_i (u_i^{(1)} \cdot m)(u_i^{(2)} \cdot n)$$

for $u \in U_q(\mathrm{I})$, $m, n \in A_q(V)$ and $\Delta(u) = \sum_i u_i^{(1)} \otimes u_i^{(2)}$.

In our previous paper [14], we gave a method to construct quantum deformations of $A_q(V)$ for prehomogeneous vector spaces V of parabolic types. We have also shown there that there exist counterparts for the defining ideals of the closures of L-orbits on V and their canonical generator systems inside $A_q(V)$. When V is a regular prehomogeneous vector space, the generator of the defining ideal of the closure of the one-codimensional orbit is the basic relative invariant, and we obtain a quantum deformation of the basic relative invariant in this case.

Our aim is to give the explicit descriptions of $A_q(V)$ for prehomogeneous vector spaces V of classical parabolic types by using the method of our previous paper. (For exceptional types, see Morita [15].)

In the case of a prehomogeneous vector space of parabolic type associated to the simple Lie algebra of type A where $L = GL_m(\mathbf{C}) \times GL_n(\mathbf{C})$, $V = M_{mn}(\mathbf{C})$ (the action is defined by $(l_1, l_2) \cdot v = l_1 v^t l_2$ for $(l_1, l_2) \in L$, $v \in V$), the quantum deformation $A_q(V)$ obtained by our method coincides with the object investigated by Hashimoto-Hayashi [2], Noumi-Yamada-Mimachi [8] and Taft-Towber [11]. If $m = n$, then V is regular and $f(v) = \det(v)$ $(v \in V)$ is a basic relative invariant. Its quantum deformation obtained by our method also coincides with the q-analogue of the determinant treated in [2], [8] and [11].

In the case of a prehomogeneous vector space of parabolic type associated to the simple Lie algebra of type D where $L = GL_n(\mathbf{C})$, $V = \{v \in M_n(\mathbf{C}) \mid {}^t v = -v\}$ (the action is defined by $l \cdot v = l v^t l$ for $l \in L$, $v \in V$), we obtain quantum deformations of $A(V)$ and a basic relative invariant Pfaffian which again coincide with the objects treated in Strickland [10].

There are two other cases.

(I) The regular prehomogeneous vector space of parabolic type associated to the simple Lie algebra of type B and D:
$L = SO_m(\mathbf{C}) \times \mathbf{C}^\times$, $V = M_{m1}(\mathbf{C}) = \mathbf{C}^m$, and the action of L is defined by

$$(l, z) \cdot v = zlv \qquad ((l, z) \in L, v \in V).$$

Under the realization

$$SO_m(\mathbf{C}) = \{l \in SL_m(\mathbf{C}) \mid {}^t lKl = K\},$$

where K is a symmetric non-singular matrix, the basic relative invariant is given by $f(v) = {}^t vKv$.

(II) The regular prehomogeneous vector space of parabolic type associated to the simple Lie algebra of type C:
$L = GL_n(\mathbf{C})$, $V = \{v \in M_n(\mathbf{C}) \mid {}^t v = v\}$, and the action of L is defined by

$$l \cdot v = lv\,{}^t l \qquad (l \in L, v \in V).$$

$f(v) = \det(v)$ is a basic relative invariant.

In these two cases we obtain the following results.

THEOREM 0.1. *For the regular prehomogeneous vector space V of type* (I), *a quantum deformation $A_q(V)$ of $A(V)$ is given by the following.*

(i) *Case of type B_n $(2n - 1 = m)$.*

(a) *$A_q(V)$ is an algebra over $\mathbf{C}(q)$ generated by Y_i $(1 \le i \le m)$ satisfying the fundamental relations*

$$Y_j Y_i = \begin{cases} q^{-2} Y_i Y_j & (i<j, i+j \neq 2n) \\ Y_i Y_j + \dfrac{q^{-2} - 1}{q + q^{-1}} Y_n^2 & (i=n-1, j=n+1) \\ Y_i Y_j + (-q^2)^{j-n-1} \dfrac{q^{-2}-1}{q+q^{-1}} Y_n^2 & \\ \qquad + (q^{-2} - q^2) \displaystyle\sum_{t=1}^{j-n-1} (-q^2)^{t-1} Y_{i+t} Y_{j-t} & (j > n+1, i+j=2n). \end{cases}$$

(b) *Let $K_r^{\pm 1}, E_s, F_s$ $(1 \le r \le n, 2 \le s \le n)$ be the canonical generators of the quantized enveloping algebra $U_q(\mathfrak{l})$. The action of $U_q(\mathfrak{l})$ on $A_q(V)$ is given as follows.*

$$K_r \cdot Y_i = \begin{cases} q^2 Y_i & (r = i + 1 \ \textit{or}\ n + 1 \le i = 2n - r < 2n - 1) \\ q^{-2} Y_i & (2 \le r = i \le n - 1 \ \textit{or}\ 1 = r < i < 2n - 1 \\ & \qquad \textit{or}\ n + 1 \le i = 2n + 1 - r) \\ q^{-4} Y_i & (r = i = 1) \\ Y_i & (\textit{otherwise}), \end{cases}$$

$$E_s \cdot Y_i = \begin{cases} Y_{i-1} & (1 \le i = s \le n-1 \ \ or \ \ n+1 \le i = 2n+1-s) \\ (q+q^{-1})\, Y_{i-1} & (i = n = s) \\ 0 & (otherwise), \end{cases}$$

$$F_s \cdot Y_i = \begin{cases} Y_{i+1} & (1 \le i = s-1 \le n-1 \ \ or \ \ n+1 \le i = 2n-s) \\ (q+q^{-1})\, Y_{i+1} & (i = n = s) \\ 0 & (otherwise). \end{cases}$$

(c) *The quantum deformation of the basic relative invariant of V is given by*

$$Y_n^2 - (q+q^{-1})(1+q^{-2}) \sum_{i=1}^{n-1} (-q^{-2})^{i-1}\, Y_{n-i} Y_{n+i}.$$

(ii) *Case of type D_n $(2n-2=m)$.*

(a) *$A_q(V)$ is an algebra over $\mathbf{C}(q)$ generated by Y_i $(1 \le i \le m)$ satisfying the fundamental relations*

$$Y_j Y_i = \begin{cases} q^{-1} Y_i Y_j & (j > i, i+j \ne 2n-1) \\ Y_i Y_j & (j = n, i = n-1) \\ Y_i Y_j + (q^{-1}-q) \displaystyle\sum_{t=1}^{j-n} (-q)^{t-1}\, Y_{i+t} Y_{j-t} & (j > n, i+j = 2n-1). \end{cases}$$

(b) *Let $K_r^{\pm 1}$, E_s, F_s $(1 \le r \le n, 2 \le s \le n)$ be the canonical generators of the quantized enveloping algebra $U_q(\mathfrak{l})$. The action of $U_q(\mathfrak{l})$ on $A_q(V)$ is given as follows.*

$$K_r \cdot Y_i = \begin{cases} q Y_i & (r = i+1 \ \ or \ \ r = n = i+2 \ \ or \ \ r = 2n-i-1 > 1) \\ q^{-1} Y_i & \begin{array}{l}(r = i > 1 \ \ or \ \ r = 2n-i < m \\ \qquad\qquad or \ \ r = n = i-1 \ \ or \ \ 1 = r < i < 2n-2)\end{array} \\ q^{-2} Y_i & (r = i = 1) \\ Y_i & (otherwise), \end{cases}$$

$$E_s \cdot Y_i = \begin{cases} Y_{i-1} & (1 \le i = s \le n-1 \ \ or \ \ n+1 \le i = 2n-s \le m) \\ Y_{n-1} & (i-1 = n = s) \\ Y_{n-2} & (i = n = s) \\ 0 & (otherwise), \end{cases}$$

$$F_s \cdot Y_i = \begin{cases} Y_{i+1} & (1 \le i = s-1 \le n-2 \ \ or \ \ n \le i = 2n-s-1 \le m) \\ Y_n & (i+2 = n = s) \\ Y_{n+1} & (i+1 = n = s) \\ 0 & (otherwise). \end{cases}$$

(c) *The quantum deformation of the basic relative invariant of V is given by*

$$\sum_{i=1}^{n-1}(-q^{-1})^{i-1}Y_{n-i}Y_{n+i-1}.$$

THEOREM 0.2. *For the regular prehomogeneous vector space V of type (II), a quantum deformation $A_q(V)$ of $A(V)$ is given by the following.*

(i) *$A_q(V)$ is an algebra over $\mathbf{C}(q)$ generated by Y_{ij} $(1 \le i \le j \le n)$ satisfying the fundamental relations*

$$Y_{ij}Y_{lm} = \begin{cases} qY_{lm}Y_{ij} & (l < i < m = j \text{ or } l = i < m < j) \\ q^2 Y_{lm}Y_{ij} & (l < i = m = j \text{ or } l = i = m < j) \\ Y_{lm}Y_{ij} & (l < i \le j < m) \\ Y_{lm}Y_{ij} + (q - q^{-1})Y_{lj}Y_{im} & (l < i < m < j) \\ qY_{lm}Y_{ij} + q(q - q^{-1})Y_{lj}Y_{ii} & (l < i = m < j) \\ Y_{lm}Y_{ij} + (q^2 - q^{-2})Y_{li}Y_{lj} & (l = m < i < j) \\ Y_{lm}Y_{ij} + (q - q^{-1})(Y_{li}Y_{mj} + qY_{lj}Y_{mi}) & (l < m < i < j) \\ Y_{lm}Y_{ij} + q^{-1}(q^2 - q^{-2})Y_{li}^2 & (l = m < i = j) \\ Y_{lm}Y_{ij} + (q^2 - q^{-2})Y_{li}Y_{mi} & (l < m < i = j). \end{cases}$$

(ii) *Let $K_r^{\pm 1}$, E_s, F_s $(1 \le r \le n, 1 \le s \le n-1)$ be the canonical generators of the quantized enveloping algebra $U_q(\mathfrak{l})$. The action of $U_q(\mathfrak{l})$ on $A_q(V)$ is given as follows.*

$$K_r \cdot Y_{ij} = \begin{cases} qY_{ij} & (r = i-1 < j-1 \text{ or } i < j-1 = r) \\ q^{-1}Y_{ij} & (r = i < j-1 \text{ or } i < j = r < n) \\ q^2 Y_{ij} & (r = i-1 = j-1) \\ q^{-2}Y_{ij} & (r = i = j < n \text{ or } i < j = r = n) \\ q^{-4}Y_{ij} & (i = j = r = n) \\ Y_{ij} & (\text{otherwise}), \end{cases}$$

$$E_s \cdot Y_{ij} = \begin{cases} Y_{i+1,j} & (s = i < j) \\ Y_{i,j+1} & (i < j = s) \\ (q + q^{-1})Y_{i,j+1} & (i = j = s) \\ 0 & (\text{otherwise}), \end{cases}$$

$$F_s \cdot Y_{ij} = \begin{cases} Y_{i-1,j} & (s+1 = i < j) \\ (q + q^{-1})Y_{i-1,j} & (s+1 = i = j) \\ Y_{i,j-1} & (i < j = s+1) \\ 0 & (\text{otherwise}). \end{cases}$$

(iii) *The quantum deformation of the basic relative invariant of V is given by*

$$\sum_{\sigma\in S_n}(-q^{-1})^{l(\sigma)}Y_{1,\sigma(1)}Y_{2,\sigma(2)}\cdots Y_{n,\sigma(n)},$$

where $l(\sigma)=\sharp\{(i,j)|1\le i<j\le n,\sigma(i)>\sigma(j)\}$, *and* $Y_{ji}=q^{-2}Y_{ij}$ *for* $i<j$.

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1 Generalities

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number filed $\mathbf{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta\subset\mathfrak{h}^*$ and $W\subset GL(\mathfrak{h})$ be the root system and the Weyl group respectively. We denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i\in I_0}$, where I_0 is an index set. For each $\alpha\in\Delta$ let $\mathfrak{g}_\alpha$ be the corresponding root space. We set $\mathfrak{n}^\pm=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_{\pm\alpha}$.

For $i\in I_0$ we denote the simple reflection corresponding to i by $s_i\in W$. Let $(\,,\,):\mathfrak{g}\times\mathfrak{g}\to\mathbf{C}$ be the invariant symmetric bilinear form such that $(\alpha,\alpha)=2$ for short roots α. Set $d_i=\dfrac{(\alpha_i,\alpha_i)}{2}$ $(i\in I_0), a_{ij}=\dfrac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ $(i,j\in I_0)$.

For a subset I of I_0 we set $\Delta_I=\Delta\cap\sum_{i\in I}\mathbf{Z}\alpha_i$, $W_I=\langle s_i|i\in I\rangle$, $\mathfrak{l}_I=\mathfrak{h}\oplus(\bigoplus_{\alpha\in\Delta_I}\mathfrak{g}_\alpha)$, and $\mathfrak{n}_I^\pm=\bigoplus_{\alpha\in\Delta^+\setminus\Delta_I}\mathfrak{g}_{\pm\alpha}$. Let L_I be the algebraic group corresponding to $\mathfrak{l}_I$.

The quantized enveloping algebra $U_q(\mathfrak{g})$ (Drinfel'd [1], Jimbo [5]) is an associative algebra over the rational function field $\mathbf{C}(q)$ generated by the elements $\{E_i,F_i,K_i^{\pm1}\}_{i\in I_0}$ satisfying the following relations:

$$K_iK_j=K_jK_i,$$

$$K_iK_i^{-1}=K_i^{-1}K_i=1,$$

$$K_iE_jK_i^{-1}=q_i^{a_{ij}}E_j,$$

$$K_iF_jK_i^{-1}=q_i^{-a_{ij}}F_j,$$

$$E_iF_j-F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\k\end{bmatrix}_{q_i}E_i^{1-a_{ij}-k}E_jE_i^k=0\qquad(i\ne j),$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\k\end{bmatrix}_{q_i}F_i^{1-a_{ij}-k}F_jF_i^k=0\qquad(i\ne j),$$

where $q_i = q^{d_i}$, and

$$[m]_t = \frac{t^m - t^{-m}}{t - t^{-1}}, \qquad [m]_t! = \prod_{k=1}^{m} [k]_t, \qquad \begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{[m]_t!}{[n]_t![m-n]_t!} \qquad (m \geq n \geq 0).$$

The Hopf algebra structure on $U_q(\mathfrak{g})$ is defined as follows. The comultiplication $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the algebra homomorphism satisfying

$$\Delta(K_i) = K_i \otimes K_i, \qquad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i.$$

The counit $\varepsilon : U_q(\mathfrak{g}) \to \mathbf{C}(q)$ is the algebra homomorphism satisfying

$$\varepsilon(K_i) = 1, \qquad \varepsilon(E_i) = \varepsilon(F_i) = 0.$$

The antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \qquad S(E_i) = -E_i K_i, \qquad S(F_i) = -K_i^{-1} F_i.$$

The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^{(1)} \otimes x_k^{(2)}$ and set $\mathrm{ad}(x)(y) = \sum_k x_k^{(1)} y S(x_k^{(2)})$. Then $\mathrm{ad} : U_q(\mathfrak{g}) \to \mathrm{End}_{\mathbf{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism.

We define subalgebras $U_q(\mathfrak{n}^{\pm})$, $U_q(\mathfrak{h})$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$U_q(\mathfrak{n}^+) = \langle E_i \mid i \in I_0 \rangle, \qquad U_q(\mathfrak{n}^-) = \langle F_i \mid i \in I_0 \rangle,$$

$$U_q(\mathfrak{h}) = \langle K_i^{\pm 1} \mid i \in I_0 \rangle, \qquad U_q(\mathfrak{l}_I) = \langle K_i^{\pm 1}, E_j, F_j \mid i \in I_0, j \in I \rangle.$$

For $i \in I_0$ we define an algebra automorphism T_i of $U_q(\mathfrak{g})$ (see Lusztig [6]) by

$$T_i(K_j) = K_j K_i^{-a_{ij}},$$

$$T_i(E_j) = \begin{cases} -F_i K_i & (i = j) \\ \displaystyle\sum_{k=0}^{-a_{ij}} (-q_i)^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (i \neq j), \end{cases}$$

$$T_i(F_j) = \begin{cases} -K_i^{-1} E_i & (i = j) \\ \displaystyle\sum_{k=0}^{-a_{ij}} (-q_i)^{k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (i \neq j), \end{cases}$$

where

$$E_i^{(k)} = \frac{1}{[k]_{q_i}!} E_i^k, \qquad F_i^{(k)} = \frac{1}{[k]_{q_i}!} F_i^k.$$

Then T_i satisfy the braid relations. In other words, if $s_is_j \in W$ has order m, then

$$\underbrace{T_iT_j\cdots}_{m} = \underbrace{T_jT_i\cdots}_{m}.$$

We often use the following formulas (see [6]):

$$T_iT_j(F_i) = F_j \qquad (a_{ij} = a_{ji} = -1), \tag{1.1}$$

$$T_iT_jT_i(F_j) = F_j, \qquad T_jT_iT_j(F_i) = F_i \qquad (a_{ij} = -1, a_{ji} = -2). \tag{1.2}$$

For $w \in W$ we choose a reduced expression $w = s_{i_1}\cdots s_{i_k}$, and set $T_w = T_{i_1}\cdots T_{i_k}$. It is known that T_w dose not depend on the choice of the reduced expression.

For $I \subset I_0$ let w_I be the longest element of W_I and set

$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-).$$

Let w_0 be the longest element of W and take a reduced expression $w_Iw_0 = s_{i_1}\cdots s_{i_r}$ of w_Iw_0. We set

$$\beta_k = s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad Y_{\beta_k} = T_{i_1}\cdots T_{i_{k-1}}(F_{i_k})$$

for $k = 1, \dots, r$. Then it is known that $\{\beta_k \mid 1 \le k \le r\} = \Delta^+ \backslash \Delta_I$, and that $\{Y_{\beta_1}^{d_1}\cdots Y_{\beta_r}^{d_r} \mid d_1, \dots, d_r \in \mathbf{Z}_{\ge 0}\}$ is a basis of $U_q(\mathfrak{n}_I^-)$. This basis depends on the choice of the reduced expression of w_Iw_0 in general. The subalgebra $U_q(\mathfrak{n}_I^-)$ is stable under the adjoint action of $U_q(\mathfrak{l}_I)$. For $\mu \in \sum_{\alpha\in\Delta}\mathbf{Z}\alpha$ we set

$$U_q(\mathfrak{n}_I^-)_\mu = \{y \in U_q(\mathfrak{n}_I^-) \mid \mathrm{ad}(K_i)y = q^{(\mu,\alpha_i)}y \text{ for all } i \in I_0\}.$$

Assume that $\mathfrak{n}_I^+ \ne \{0\}$ and $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. Then $(L_I, \mathfrak{n}_I^+)$ is a prehomogeneous vector space. Moreover, $\mathfrak{n}_I^+$ consists of finitely many L_I-orbits $C_0, C_1, \dots, C_t$ satisfying the closure relation

$$\{0\} = C_0 \subset \overline{C_1} \subset \cdots \subset \overline{C_t} = \mathfrak{n}_I^+.$$

Set $\mathscr{I}(\overline{C_p}) = \{f \in \mathbf{C}[\mathfrak{n}_I^+] \mid f(\overline{C_p}) = 0\}$. We denote by $\mathscr{I}^m(\overline{C_p})$ the subspace of $\mathscr{I}(\overline{C_p})$ consisting of homogeneous elements with degree m. For $p = 0, 1, \dots, t-1$, we have the following:
(i) $\mathscr{I}^m(\overline{C_p}) = 0$ for $m \le p$,
(ii) $\mathscr{I}^{p+1}(\overline{C_p})$ is an irreducible $\mathfrak{l}_I$-module,
(iii) $\mathscr{I}(\overline{C_p})$ is generated by $\mathscr{I}^{p+1}(\overline{C_p})$ as an ideal of $\mathbf{C}[\mathfrak{n}_I^+]$.
In this case we can regard $U_q(\mathfrak{n}_I^-)$ as a quantum deformation of the coordinate algebra $\mathbf{C}[\mathfrak{n}_I^+]$ of $\mathfrak{n}_I^+$. By $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = 0$ we have the following (see [14]).

Proposition 1.1. *The generators Y_β for $\beta \in \Delta^+ \backslash \Delta_I$ do not depend on the choice of the reduced expression of $w_I w_0$, and they satisfy quadratic fundamental relations.*

Since $\mathbf{C}[\mathfrak{n}_I^+]$ is a multiplicity free $\mathfrak{l}_I$-module, there exist unique $U_q(\mathfrak{l}_I)$-submodules $\mathscr{I}_q(\overline{C_p})$ and $\mathscr{I}_q^{p+1}(\overline{C_p})$ of $U_q(\mathfrak{n}_I^-)$ satisfying $\mathscr{I}_q(\overline{C_p})|_{q=1} = \mathscr{I}(\overline{C_p})$ and $\mathscr{I}_q^{p+1}(\overline{C_p})|_{q=1} = \mathscr{I}^{p+1}(\overline{C_p})$.

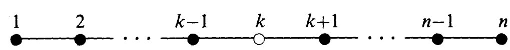
Theorem 1.2. (see [14]) $\mathscr{I}_q(\overline{C_p}) = U_q(\mathfrak{n}_I^-)\mathscr{I}_q^{p+1}(\overline{C_p}) = \mathscr{I}_q^{p+1}(\overline{C_p})U_q(\mathfrak{n}_I^-)$.

In the remainder of this paper we shall give explicit descriptions of $U_q(\mathfrak{n}_I^-)$ and $\mathscr{I}_q^{p+1}(\overline{C_p})$ for a classical simple Lie algebra $\mathfrak{g}$.

2. Quantum determinants of quantum square matrices

In this section we apply the method in [14] to the case where $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbf{C})$ and $\mathfrak{l}_I \simeq \{(g_1, g_2) \in \mathfrak{gl}_k(\mathbf{C}) \times \mathfrak{gl}_{n+1-k}(\mathbf{C}) \mid \operatorname{tr} g_1 + \operatorname{tr} g_2 = 0\}$. Since the quantum deformation we obtain by our method is not new, we shall only state the results and omit the proofs.

We label the vertices of the Dynkin diagram of $\mathfrak{g}$ as follows.



Hence we have $I_0 = \{1, 2, \ldots, n\}$. Set $I = I_0 \backslash \{k\}$, where $k - 1 \le n - k$. Then we have $\mathfrak{n}_I^+ \neq \{0\}$ and $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. There exist $k+1$ L_I-orbits $C_0, C_1, \ldots, C_k$ on $\mathfrak{n}_I^+$, where $C_p = \{x \in \mathfrak{n}_I^+ \mid \operatorname{rank}(x) = p\}$. Then we have the closure relation $\{0\} = C_0 \subset \overline{C_1} \subset \cdots \subset \overline{C_k} = \mathfrak{n}_I^+$.

We fix a reduced expression

$$w_I w_0 = (s_k s_{k+1} \cdots s_n)(s_{k-1} s_k \cdots s_{n-1}) \cdots (s_1 s_2 \cdots s_{n-k+1}).$$

For $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n+1-k$ we define $\beta_{ij} \in \Delta^+ \backslash \Delta_I$ by

$$\beta_{ij} = \alpha_{k-i+1} + \alpha_{k-i+2} + \cdots + \alpha_k + \alpha_{k+1} + \cdots + \alpha_{k+j-1},$$

and set

$$Y_{ij} = T^{(k)} T^{(k-1)} \ldots T^{(k-i+2)} T_{k-i+1} T_{k-i+2} \cdots T_{k-i+j-1}(F_{k-i+j}),$$

where we set $T^{(s)} = T_s T_{s+1} \cdots T_{n-k+s}$. Note that $Y_{ij} \in U_q(\mathfrak{n}_I^-)_{-\beta_{ij}}$.

By Yamane [13], we have

$$Y_{ij} Y_{lm} = \begin{cases} q Y_{lm} Y_{ij} & (i = l, j < m \text{ or } i < l, j = m) \\ Y_{lm} Y_{ij} & (i < l, j > m) \\ Y_{lm} Y_{ij} + (q - q^{-1}) Y_{lj} Y_{im} & (i < l, j < m). \end{cases} \tag{2.1}$$

Fix $p=0,1,\dots,k-1$. For two sequences $\{i_1,i_2,\dots,i_{p+1}\}$, $\{j_1,j_2,\dots,j_{p+1}\}\subset\mathbf{N}$ satisfying

$$1\le i_1<i_2<\cdots<i_{p+1}\le k,\qquad 1\le j_1<j_2<\cdots<j_{p+1}\le n+1-k, \tag{2.2}$$

we set

$$\begin{vmatrix} i_1 & i_2\cdots i_{p+1}\\ j_1 & j_2\cdots j_{p+1}\end{vmatrix}=\sum_{\sigma\in S_{p+1}}(-q)^{l(\sigma)}Y_{i_1,j_{\sigma(1)}}Y_{i_2,j_{\sigma(2)}}\cdots Y_{i_{p+1},j_{\sigma(p+1)}},$$

where $l(\sigma)=\sharp\{(i,j)|i<j,\sigma(i)>\sigma(j)\}$.

We can prove the following result. Details are omitted.

LEMMA 2.1. *We have*

$$\operatorname{ad}(F_r)\begin{vmatrix} i_1 & i_2 & \cdots & i_{p+1}\\ j_1 & j_2 & \cdots & j_{p+1}\end{vmatrix}$$

$$=\begin{cases}\begin{vmatrix} i_1 & i_2 & \cdots & i_t+1 & \cdots & i_{p+1}\\ j_1 & j_2 & \cdots & j_t & \cdots & j_{p+1}\end{vmatrix} & \text{if there exists } t\in\{1,2,\dots,p+1\} \text{ such that } i_t=k-r<i_{t+1}-1,\\ \begin{vmatrix} i_1 & i_2 & \cdots & i_t & \cdots & i_{p+1}\\ j_1 & j_2 & \cdots & j_t+1 & \cdots & j_{p+1}\end{vmatrix} & \text{if there exists } t\in\{1,2,\dots,p+1\} \text{ such that } j_t=r-k<j_{t+1}-1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\operatorname{ad}(E_r)\begin{vmatrix} i_1 & i_2 & \cdots & i_{p+1}\\ j_1 & j_2 & \cdots & j_{p+1}\end{vmatrix}$$

$$=\begin{cases}\begin{vmatrix} i_1 & i_2 & \cdots & i_t-1 & \cdots & i_{p+1}\\ j_1 & j_2 & \cdots & j_t & \cdots & j_{p+1}\end{vmatrix} & \text{if there exists } t\in\{1,2,\dots,p+1\} \text{ such that } i_{t-1}+1<i_t=k-r+1,\\ \begin{vmatrix} i_1 & i_2 & \cdots & i_t & \cdots & i_{p+1}\\ j_1 & j_2 & \cdots & j_t-1 & \cdots & j_{p+1}\end{vmatrix} & \text{if there exists } t\in\{1,2,\dots,p+1\} \text{ such that } j_{t-1}+1<j_t=r-k+1,\\ 0 & \text{otherwise}\end{cases}$$

for $r\in I$, *and*

$$\operatorname{ad}(K_r)\begin{vmatrix} i_1 & i_2 & \cdots & i_{p+1}\\ j_1 & j_2 & \cdots & j_{p+1}\end{vmatrix}=q^{-\sum_{t=1}^{p+1}(\beta_{i_t,j_t},\alpha_r)}\begin{vmatrix} i_1 & i_2 & \cdots & i_{p+1}\\ j_1 & j_2 & \cdots & j_{p+1}\end{vmatrix}$$

for $r\in I_0$.

We set

$$\psi_p = \begin{vmatrix} k-p & k-p+1 & \cdots & k \\ 1 & 2 & \cdots & p+1 \end{vmatrix},$$

$$J_{q,p} = \sum \mathbf{C}(q) \begin{vmatrix} i_1 & i_2 & \cdots & i_{p+1} \\ j_1 & j_2 & \cdots & j_{p+1} \end{vmatrix},$$

where the summation runs through the sequences $\{i_1, i_2, \dots, i_{p+1}\}$, $\{j_1, j_2, \dots, j_{p+1}\}$ satisfying (2.2).

COROLLARY 2.2. *For* $p = 0, 1, \dots, k-1$, $J_{q,p}$ *is an irreducible highest weight* $U_q(\mathfrak{l}_I)$*-module with highest weight vector* ψ_p.

PROOF. From Lemma 2.1, it is clear that $J_{q,p} = \mathrm{ad}(U_q(\mathfrak{l}_I))\psi_p$ and $\mathrm{ad}(E_r)\psi_p = 0$ for any $r \in I$. Since a finite dimensional highest weight module is irreducible, the statement holds. □

The highest weight of $J_{q,p}$ coincides with that of $\mathcal{J}^{p+1}(\overline{C_p})$. Therefore $J_{q,p}$ is a quantum deformation of $\mathcal{J}^{p+1}(\overline{C_p})$. By Theorem 1.2 we have $U_q(\mathfrak{n}_I^-)J_{q,p} = J_{q,p}U_q(\mathfrak{n}_I^-)$, and this two sided ideal of $U_q(\mathfrak{n}_I^-)$ is a quantum deformation of the defining ideal $\mathcal{J}(\overline{C_p})$ of the closure of C_p.

If $k-1 = n-k$, the prehomogeneous vector space $(L_I, \mathfrak{n}_I^+)$ is regular, and the generator $\psi_{k-1} = \sum_{\sigma \in S_k} (-q)^{l(\sigma)} Y_{1,\sigma(1)} Y_{2,\sigma(2)} \cdots Y_{k,\sigma(k)}$ of the quantum deformation of $\mathcal{J}(\overline{C_{k-1}})$ is the quantum deformation of the basic relative invariant.

Hence we obtain the following known result by our method.

THEOREM 2.3. (Hashimoto-Hayashi [2], Noumi-Yamada-Mimachi [8], Taft-Towber [11])

(i) A quantum deformation $A_q(\mathfrak{n}_I^+)$ of the coordinate algebra $A(\mathfrak{n}_I^+)$ of $\mathfrak{n}_I^+$ is generated by Y_{ij} $(1 \le i \le k, 1 \le j \le n+1-k)$ satisfying the fundamental relations (2.1).

(ii) The action of $U_q(\mathfrak{l}_I)$ on $A_q(\mathfrak{n}_I^+)$ is given as follows. For $r \in I_0$ and $s \in I$,

$$K_r \cdot Y_{ij} = q^{-(\alpha_r, \beta_{ij})} Y_{ij},$$

$$E_s \cdot Y_{ij} = \begin{cases} Y_{i,j-1} & (s = k+j-1) \\ Y_{i-1,j} & (s = k-i+1) \\ 0 & (\textit{otherwise}), \end{cases}$$

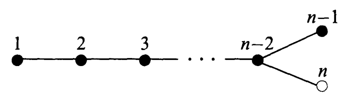
$$F_s \cdot Y_{ij} = \begin{cases} Y_{i,j+1} & (s = k+j) \\ Y_{i+1,j} & (s = k-i) \\ 0 & (\textit{otherwise}). \end{cases}$$

(iii) When $k-1=n-k$, $\sum_{\sigma\in S_k}(-q)^{l(\sigma)}Y_{1,\sigma(1)}Y_{2,\sigma(2)}\cdots Y_{k,\sigma(k)}$ is a quantum deformation of the basic relative invariant.

3. Quantum Pfaffians of quantum alternating matrices

In this section we apply the method in [14] to the case where $\mathfrak{g}=\mathfrak{o}_{2n}(\mathbf{C})$ and $\mathfrak{l}_I\simeq\mathfrak{gl}_n(\mathbf{C})$. Since the quantum deformation obtained by our method is not new, we shall only state the results and omit the proofs.

We label the vertices of the Dynkin diagram of $\mathfrak{g}$ as follows.



Hence we have $I_0=\{1,2,\dots,n\}$. Set $I=I_0\backslash\{n\}$. Then we have $\mathfrak{n}_I^+\neq\{0\}$ and $[\mathfrak{n}_I^+,\mathfrak{n}_I^+]=\{0\}$. Then there exist $\left[\dfrac{n}{2}\right]+1$ L_I-orbits $C_0,C_1,\dots,C_{[n/2]}$ on $\mathfrak{n}_I^+$, where $C_p=\{x\in\mathfrak{n}_I^+\mid \mathrm{rank}(x)=2p\}$. We have the closure relation $\{0\}=C_0\subset\overline{C_1}\subset\cdots\subset\overline{C_{[n/2]}}=\mathfrak{n}_I^+$.

We fix a reduced expression

$$w_Iw_0=(s_{\delta(1)}s_{n-2}\cdots s_1)(s_{\delta(2)}s_{n-2}\cdots s_2)\cdots(s_{\delta(n-2)}s_{n-2})s_{\delta(n-1)},$$

where

$$\delta(t)=\begin{cases} n & \text{if } t \text{ is odd,}\\ n-1 & \text{if } t \text{ is even.}\end{cases}$$

Let $1\le i<j\le n$. we define $\beta_{ij}\in\Delta^+\backslash\Delta_I$ by

$$\beta_{ij}=\begin{cases}\alpha_i+\alpha_{i+1}+\cdots+\alpha_{j-1} & \\ \qquad +2\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n & (j\le n-1)\\ \alpha_i+\alpha_{i+1}+\cdots+\alpha_{n-2}+\alpha_n & (j=n),\end{cases}$$

and set

$$Y_{ij}=T^{(1)}T^{(2)}\cdots T^{(n-j)}T_{\delta(n-j+1)}T_{n-2}\cdots T_{n-j+i+1}(F_{n-j+i})$$

where $T^{(s)}=T_{\delta(s)}T_{n-2}T_{n-3}\cdots T_s$ for $s=1,2,\dots,n-2$. If $j-i=1$, we set

$$T_{\delta(n-j+1)}T_{n-2}\cdots T_{n-j+i+1}(F_{n-j+i})=F_{\delta(n-j+1)}.$$

Note that $Y_{ij}\in U_q(\mathfrak{n}_I^-)_{-\beta_{ij}}$.

LEMMA 3.1. *We have*

$$Y_{ij}Y_{lm} = \begin{cases} qY_{lm}Y_{ij} & \begin{array}{l}(l<i<m=j\\ \quad \text{or } l<i=m<j\\ \quad \text{or } l=i<m<j)\end{array} \\ Y_{lm}Y_{ij} & (l<i<j<m) \\ Y_{lm}Y_{ij}+(q-q^{-1})Y_{lj}Y_{im} & (l<i<m<j) \\ Y_{lm}Y_{ij} \\ \quad +(q-q^{-1})(Y_{li}Y_{mj}-q^{-1}Y_{lj}Y_{mi}) & (l<m<i<j). \end{cases} \tag{3.1}$$

Fix $p=0,1,\dots,\left[\dfrac{n-2}{2}\right]$.

For the sequence $\{i_1,i_2,\dots,i_{2p+2}\}\subset \mathbf{N}$ satisfying

$$1\le i_1<i_2<\cdots<i_{2p+2}\le n, \tag{3.2}$$

we set

$$|i_1\ i_2\ \cdots\ i_{2p+2}| = \sum_{\sigma\in\hat{S}_{2p+2}} (-q^{-1})^{l(\sigma)} Y_{i_{\sigma(1)},i_{\sigma(2)}} Y_{i_{\sigma(3)},i_{\sigma(4)}} \cdots Y_{i_{\sigma(2p+1)},i_{\sigma(2p+2)}},$$

where $\hat{S}_{2p+2}=\{\sigma\in S_{2p+2}\mid \sigma(2k-1)<\sigma(2k+1),\sigma(2k-1)<\sigma(2k)$ for all $k\}$.

PROPOSITION 3.2. *We have*

$$\mathrm{ad}(F_r)|i_1\ i_2\ \cdots\ i_{2p+2}| = \begin{cases} |i_1\ i_2\ \cdots\ i_t-1\ \cdots\ i_{2p+2}| & \begin{array}{l}\textit{if there exists } t\in\{1,2,\dots,2p+2\}\\ \quad \textit{such that } i_{t-1}<r=i_t-1,\end{array} \\ 0 & \textit{otherwise,} \end{cases}$$

$$\mathrm{ad}(E_r)|i_1\ i_2\ \cdots\ i_{2p+2}| = \begin{cases} |i_1\ i_2\ \cdots\ i_t+1\ \cdots\ i_{2p+2}| & \begin{array}{l}\textit{if there exists } t\in\{1,2,\dots,2p+2\}\\ \quad \textit{such that } r=i_t<i_{t+1}-1,\end{array} \\ 0 & \textit{otherwise} \end{cases}$$

for $r\in I$, *and*

$$\mathrm{ad}(K_r)|i_1\ i_2\ \cdots\ i_{2p+2}| = q^{-\sum_{t=1}^{p+1}(\beta_{i_{2t-1},i_{2t}},\alpha_r)}|i_1\ i_2\ \cdots\ i_{2p+2}|$$

for $r\in I_0$.

We set

$$\psi_p = |n-2p-1 \;\; n-2p \;\; \cdots \;\; n|,$$

$$J_{q,p} = \sum \mathbf{C}(q)|i_1 \;\; i_2 \;\; \cdots \;\; i_{2p+2}|$$

where the summation runs through the sequence $\{i_1, i_2, \ldots, i_{2p+2}\} \subset \mathbf{N}$ satisfying (3.2).

By Proposition 3.2 we have the following.

PROPOSITION 3.3. *For* $p = 0, 1, \ldots, \left[\frac{n-2}{2}\right]$, $J_{q,p}$ *is an irreducible highest weight* $U_q(\mathfrak{l}_I)$*-module with highest weight vector* ψ_p.

The highest weight of $J_{q,p}$ coincides with that of $\mathscr{I}^{p+1}(\overline{C_p})$, hence $J_{q,p}$ is a quantum deformation of $\mathscr{I}^{p+1}(\overline{C_p})$. By Theorem 1.2, the two sided ideal $U_q(\mathfrak{n}_I^-)J_{q,p} = J_{q,p}U_q(\mathfrak{n}_I^-)$ is a quantum deformation of the defining ideal $\mathscr{I}(\overline{C_p})$ of the closure of C_p.

If n is even, the prehomogeneous vector space $(L_I, \mathfrak{n}_I^+)$ is regular, and the generator $\psi_{(n-2)/2}$ of the quantum deformation of $\mathscr{I}(\overline{C_{(n-2)/2}})$ is the quantum deformation of the basic relative invariant.

Therefore we have the following known result by our method.

THEOREM 3.4. (Strickland [10])

(i) A quantum deformation $A_q(\mathfrak{n}_I^+)$ of the coordinate algebra $A(\mathfrak{n}_I^+)$ of $\mathfrak{n}_I^+$ is generated by Y_{ij} $(1 \le i < j \le n)$ satisfying the fundamental relations (3.1).

(ii) The action of $U_q(\mathfrak{l}_I)$ on $A_q(\mathfrak{n}_I^+)$ is given as follows. For $r \in I_0$ and $s \in I$,

$$K_r \cdot Y_{ij} = q^{-(\alpha_r, \beta_{ij})} Y_{ij},$$

$$E_s \cdot Y_{ij} = \begin{cases} Y_{i+1,j} & (s = i, j > i+1) \\ Y_{i,j+1} & (s = j) \\ 0 & (\textit{otherwise}), \end{cases}$$

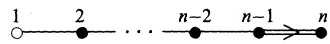
$$F_s \cdot Y_{ij} = \begin{cases} Y_{i-1,j} & (s = i-1) \\ Y_{i,j-1} & (s = j-1 > i) \\ 0 & (\textit{otherwise}). \end{cases}$$

(iii) When n is even, $\psi_{(n-2)/2}$ is the quantum deformation of the basic relative invariant.

4. Quantum quadratic forms on quantum vector spaces

In this section we apply the method in [14] to the case where $\mathfrak{g} = \mathfrak{o}_{m+2}(\mathbf{C})$ and $\mathfrak{l}_I \simeq \mathfrak{o}_m(\mathbf{C}) \times \mathbf{C}$.

Assume $m = 2n - 1$. We label the vertices of the Dynkin diagram of $\mathfrak{g}$ as follows.



Hence we have $I_0 = \{1, 2, \dots, n\}$. Set $I = I_0 \backslash \{1\}$. Then $\mathfrak{n}_I^+ \neq \{0\}$ and $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. There exist three L_I-orbits C_0, C_1, C_2 on $\mathfrak{n}_I^+$ satisfying the closure relation $\{0\} = C_0 \subset \overline{C_1} \subset \overline{C_2} = \mathfrak{n}_I^+$.

Fix the reduced expression $w_I w_0 = s_1 s_2 \cdots s_n s_{n-1} s_{n-2} \cdots s_1$, and for $i = 1, 2, \dots, m$ we define $\beta_i \in \Delta^+ \backslash \Delta_I$ and $Y_i = Y_{\beta_i} \in U_q(\mathfrak{n}_I^-)_{-\beta_i}$ as in Section 1. Note that

$$\beta_i = \begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_i & (1 \leq i \leq n) \\ \alpha_1 + \alpha_2 + \cdots + \alpha_{2n-i} + 2\alpha_{2n-i+1} + \cdots + 2\alpha_n & (n+1 \leq i \leq 2n-1). \end{cases}$$

Lemma 4.1. *For $r \in I$ we have*

$$\mathrm{ad}(F_r) Y_i = \begin{cases} Y_{i+1} & (1 \leq i \leq n-1, r = i+1 \\ & \quad \text{or } n+1 \leq i \leq 2n-1, r = 2n-i) \\ (q+q^{-1}) Y_{n+1} & (i = n = r) \\ 0 & (\text{otherwise}), \end{cases}$$

$$\mathrm{ad}(E_r) Y_i = \begin{cases} Y_{i-1} & (1 \leq i \leq n-1, r = i \\ & \quad \text{or } n+1 \leq i \leq 2n-1, r = 2n-i+1) \\ (q+q^{-1}) Y_{n-1} & (i = n = r) \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. Since $\bigoplus_i \mathbf{C}(q) Y_i$ is a $U_q(\mathfrak{l}_I)$-module (see [14]), we have $\mathrm{ad}(F_r) Y_i = 0$ if $\beta_i + \alpha_r \notin \Delta^+ \backslash \Delta_I$ and $\mathrm{ad}(E_r) Y_i = 0$ if $\beta_i - \alpha_r \notin \Delta^+ \backslash \Delta_I$. Therefore we have only to deal with the cases $r = i+1$ or $2n-i$ for F_r and $r = i$ or $2n-i+1$ for E_r.

Let $r = i+1 \in I$. We have $\mathrm{ad}(F_{i+1}) Y_i = F_{i+1} Y_i - q^2 Y_i F_{i+1}$. Since $F_{i+1} = T_1 T_2 \cdots T_{i-1}(F_{i+1})$, we obtain

$$\mathrm{ad}(F_{i+1}) Y_i = T_1 T_2 \cdots T_{i-1}(F_{i+1} F_i - q^2 F_i F_{i+1}) = T_1 T_2 \cdots T_{i-1} T_i (F_{i+1}) = Y_{i+1}.$$

Let $r = 2n - i \in I$ and $r \neq n$. We have $F_r = T_1 T_2 \cdots T_n T_{n-1} \cdots T_{2n-i+1} \cdot (F_{2n-i-1})$ by (1.1). By using this formula we obtain

$$\begin{aligned} \mathrm{ad}(F_r) Y_i &= F_r Y_i - q^2 Y_i F_r \\ &= T_1 T_2 \cdots T_n T_{n-1} \cdots T_{2n-i+1}(F_{2n-i-1} F_{2n-i} - q^2 F_{2n-i} F_{2n-i-1}) \\ &= T_1 T_2 \cdots T_n T_{n-1} \cdots T_{2n-i+1} T_{2n-i}(F_{2n-i-1}) \\ &= Y_{i+1}. \end{aligned}$$

If $r = 2n - i = n$, we have $\mathrm{ad}(F_r)Y_i = [F_n, Y_n] = T_1T_2\cdots T_{n-1}\cdot([T_{n-1}^{-1}(F_n), F_n])$. Since $T_{n-1}^{-1}(F_n) = F_{n-1}F_n - q^2F_nF_{n-1}$ (see [4]), we have

$$\begin{aligned}[T_{n-1}^{-1}(F_n), F_n] &= (q+q^{-1})(F_{n-1}F_n^{(2)} - qF_nF_{n-1}F_n + q^2F_n^{(2)}F_{n-1}) \\ &= (q+q^{-1})T_n(F_{n-1}).\end{aligned}$$

Therefore we obtain $\mathrm{ad}(F_n)Y_n = (q+q^{-1})Y_{n+1}$.

Let $r = i \in I$ or $r = 2n - i + 1 < n$. Since $\mathrm{ad}(E_r)Y_{i-1} = 0$ and $Y_i = \mathrm{ad}(F_r)Y_{i-1}$, we have

$$\begin{aligned}\mathrm{ad}(E_r)Y_i &= \mathrm{ad}(E_r)(\mathrm{ad}(F_r)Y_{i-1}) \\ &= (q_r - q_r^{-1})^{-1}\mathrm{ad}(K_r - K_r^{-1})Y_{i-1} + \mathrm{ad}(F_r)(\mathrm{ad}(E_r)Y_{i-1}) \\ &= (q_r - q_r^{-1})^{-1}(q^2 - q^{-2})Y_{i-1}.\end{aligned}$$

Since $q_r = q$ if $r = n$ and $q_r = q^2$ if $r < n$, the statement holds.

Let $r = 2n - i + 1 = n$. We have $\mathrm{ad}(E_n)Y_n = (q+q^{-1})Y_{n-1}$ and $Y_i = Y_{n+1} = (q+q^{-1})^{-1}\mathrm{ad}(F_n)Y_n$. Therefore we obtain

$$\begin{aligned}\mathrm{ad}(E_n)Y_{n+1} &= (q+q^{-1})^{-1}\mathrm{ad}(E_n)(\mathrm{ad}(F_n)Y_n) \\ &= (q+q^{-1})^{-1}(q_n - q_n^{-1})^{-1}\mathrm{ad}(K_n - K_n^{-1})Y_n \\ &\quad + (q+q^{-1})^{-1}\mathrm{ad}(F_n)(\mathrm{ad}(E_n)Y_n) \\ &= (q^2 - q^{-2})^{-1}(Y_n - Y_n) + \mathrm{ad}(F_n)Y_{n-1} \\ &= Y_n.\end{aligned}$$

□

PROPOSITION 4.2. *We have*

$$Y_jY_i = \begin{cases} q^{-2}Y_iY_j & (i < j, i+j \neq 2n) \\ Y_iY_j + \dfrac{q^{-2}-1}{q+q^{-1}}Y_n^2 & (i = n-1, j = n+1) \\ Y_iY_j + (-q^2)^{j-n-1}\dfrac{q^{-2}-1}{q+q^{-1}}Y_n^2 & \\ \quad +(q^{-2}-q^2)\displaystyle\sum_{t=1}^{j-n-1}(-q^2)^{t-1}Y_{i+t}Y_{j-t} & (i \leq n-2, i+j = 2n). \end{cases} \tag{4.1}$$

PROOF. Let $1 \leq i \leq n-1, i < j < 2n - i$ or $n \leq i < j \leq 2n-1$. We show $Y_jY_i = q^{-2}Y_iY_j$ by induction on $t = j - i$. If $t = 1$, we have

$$Y_{i+1}Y_i - q^{-2}Y_iY_{i+1} = T(T_k(F_{k'})F_k - q^{-2}F_kT_k(F_{k'}))$$

where $T = T_1 \cdots T_{i-1}$, $k = k' - 1 = i$ if $i \leq n-1$, and $T = T_1 \cdots T_{n-1} T_n \cdots T_{2n-i+1}$, $k = k' + 1 = 2n - i$ if $i \geq n$. Since we have

$$T_k(F_{k'})F_k - q^{-2} F_k T_k(F_{k'}) = c \sum_{s=0}^{1-a_{kk'}} (-1)^s \begin{bmatrix} 1 - a_{kk'} \\ s \end{bmatrix}_{q_k} F_k^{1-a_{kk'}-s} F_{k'} F_k^s$$

for some $c \in \mathbf{C}(q)$, we obtain $Y_{i+1} Y_i - q^{-2} Y_i Y_{i+1} = 0$. Assume that $t > 1$ and the statement is proved up to $t - 1$. We have

$$Y_j = (q + q^{-1})^{-\delta_{j,n+1}} \operatorname{ad}(F_l) Y_{j-1} = (q + q^{-1})^{-\delta_{l,n+1}} (F_l Y_{j-1} - q^{-(\beta_{j-1}, \alpha_l)} Y_{j-1} F_l)$$

where $l = j$ if $j \leq n$, and $l = 2n - j + 1$ if $j > n$. Since $t > 1$, we have $[F_l, Y_i] = \operatorname{ad}(F_l) Y_i = 0$ by Lemma 4.1. By the inductive hypothesis on t, we obtain

$$\begin{aligned} Y_j Y_i &= (q + q^{-1})^{-\delta_{l,n+1}} (F_l Y_{j-1} - q^{-(\beta_{j-1}, \alpha_l)} Y_{j-1} F_l) Y_i \\ &= q^{-2} (q + q^{-1})^{-\delta_{l,n+1}} Y_i (F_l Y_{j-1} - q^{-(\beta_{j-1}, \alpha_l)} Y_{j-1} F_l) \\ &= q^{-2} Y_i Y_j. \end{aligned}$$

Next we prove $Y_{n+1} Y_{n-1} = Y_{n-1} Y_{n+1} + \dfrac{q^{-2} - 1}{q + q^{-1}} Y_n^2$. From Lemma 4.1 we have $Y_{n+1} = (q + q^{-1})^{-1} [F_n, Y_n]$ and $Y_n = F_n Y_{n-1} - q^2 Y_{n-1} F_n$. By the preceding paragraph we have $Y_n Y_{n-1} = q^{-2} Y_{n-1} Y_n$. Hence

$$\begin{aligned} [Y_{n+1}, Y_{n-1}] &= (q+q^{-1})^{-1} [[F_n, Y_n], Y_{n-1}] \\ &= (q+q^{-1})^{-1} (q^{-2} (F_n Y_{n-1} - q^2 Y_{n-1} F_n) Y_n - Y_n (F_n Y_{n-1} - q^2 Y_{n-1} F_n)) \\ &= \frac{q^{-2} - 1}{q + q^{-1}} Y_n^2. \end{aligned}$$

Let $i + j = 2n, i \leq n - 2$. From Lemma 4.1 we have

$$Y_j = F_i Y_{j-1} - q^2 Y_{j-1} F_i, \qquad Y_{i+1} = F_i Y_i - q^2 Y_i F_i.$$

Since $Y_{j-1} Y_i = q^{-2} Y_i Y_{j-1}$, we have

$$\begin{aligned} [Y_j, Y_i] &= [F_i Y_{j-1} - q^2 Y_{j-1} F_i, Y_i] \\ &= q^{-2} (F_i Y_i - q^2 Y_i F_i) Y_{j-1} - q^2 Y_{j-1} (F_i Y_i - q^2 Y_i F_i) \\ &= q^{-2} Y_{i+1} Y_{j-1} - q^2 Y_{j-1} Y_{i+1}. \end{aligned} \tag{4.2}$$

By using (4.2) and $[Y_{n+1}, Y_{n-1}] = \dfrac{q^{-2} - 1}{q + q^{-1}} Y_n^2$, we obtain inductively

$$[Y_j, Y_i] = (-q^2)^{j-n-1} \frac{q^{-2} - 1}{q + q^{-1}} Y_n^2 + (q^{-2} - q^2) \sum_{s=1}^{j-n-1} (-q^2)^{s-1} Y_{i+s} Y_{j-s}.$$

Finally we prove $Y_j Y_i = q^{-2} Y_i Y_j$ for $1 \le i \le n-1, 2n-i < j \le 2n-1$ by induction on $t = j-(2n-i)$. Let $t=1$. Since $F_i Y_i - q^{-2} Y_i F_i = \mathrm{ad}(F_i) Y_i = 0$, $Y_j = \mathrm{ad}(F_i) Y_{2n-i} = F_i Y_{2n-i} - q^2 Y_{2n-i} F_i$ and (4.2), we have

$$\begin{aligned} Y_j Y_i - q^{-2} Y_i Y_j &= (F_i Y_{2n-i} - q^2 Y_{2n-1} F_i) Y_i - q^{-2} Y_i (F_i Y_{2n-i} - q^2 Y_{2n-i} F_i) \\ &= [F_i, [Y_{2n-i}, Y_i]] \\ &= [F_i, q^{-2} Y_{i+1} Y_{2n-i-1} - q^2 Y_{2n-i-1} Y_{i+1}]. \end{aligned}$$

From Lemma 4.1 $[F_i, Y_{2n-i-1}] = 0 = [F_i, Y_{i+1}]$, hence we obtain $Y_j Y_i - q^{-2} Y_i Y_j = 0$ for $t=1$. Assume that $t>1$ and the statement is proved up to $t-1$. Since $Y_j = F_{2n-j+1} Y_{j-1} - q^2 Y_{j-1} F_{2n-j+1}$ and $[F_{2n-j+1}, Y_i] = 0$, we obtain

$$\begin{aligned} Y_j Y_i &= (F_{2n-j+1} Y_{j-1} - q^2 Y_{j-1} F_{2n-j+1}) Y_i \\ &= q^{-2} Y_i (F_{2n-j+1} Y_{j-1} - q^2 Y_{j-1} F_{2n-j+1}) = q^{-2} Y_i Y_j. \end{aligned}$$ □

We set

$$\psi = Y_n Y_n - (q+q^{-1})(1+q^{-2}) \sum_{i=1}^{n-1} (-q^{-2})^{i-1} Y_{n-i} Y_{n+i}.$$

Since $\beta_{n-i} + \beta_{n+i} = 2\beta_n$ for any i, we have $\psi \in U_q(\mathfrak{n}_I^-)_{-2\beta_n}$.

PROPOSITION 4.3. *For $r \in I$ we have*

$$\mathrm{ad}(F_r)\psi = 0, \qquad \mathrm{ad}(E_r)\psi = 0.$$

PROOF. We shall prove $\mathrm{ad}(F_r)\psi = 0$. Let $r = n-k$. If $k=0$, we obtain

$$\begin{aligned} \mathrm{ad}(F_n)\psi &= (\mathrm{ad}(F_n) Y_n) Y_n + K_n Y_n K_n^{-1} (\mathrm{ad}(F_n) Y_n) \\ &\quad - (q+q^{-1})(1+q^{-2})(\mathrm{ad}(F_n) Y_{n-1}) Y_{n+1} \\ &= (q+q^{-1}) Y_{n+1} Y_n + (q+q^{-1}) Y_n Y_{n+1} - (q+q^{-1})(1+q^{-2}) Y_n Y_{n+1} \\ &= 0. \end{aligned}$$

If $k>0$, we obtain

$$\begin{aligned} \mathrm{ad}(F_{n-k})\psi &= -(q+q^{-1})(1+q^{-2})((-q^{-2})^k (\mathrm{ad}(F_{n-k}) Y_{n-k-1}) Y_{n+k+1} \\ &\quad + (-q^{-2})^{k-1} K_{n-k} Y_{n-k} K_{n-k}^{-1} (\mathrm{ad}(F_{n-k}) Y_{n+k})) \\ &= -(q+q^{-1})(1+q^{-2})((-q^{-2})^k Y_{n-k} Y_{n+k+1} \\ &\quad + (-q^{-2})^{k-1} q^{-2} Y_{n-k} Y_{n+k+1}) \\ &= 0. \end{aligned}$$

Similarly we can prove $\mathrm{ad}(E_r)\psi = 0$. □

By Lemma 4.1 and Proposition 4.3, we have the following:

Proposition 4.4. *$\sum_{1\le i\le 2n-1} \mathbf{C}(q)Y_i$ and $\mathbf{C}(q)\psi$ are irreducible highest weight $U_q(\mathfrak{l}_I)$-modules.*

The highest weight of $\mathbf{C}(q)\psi$ coincides with that of $\mathscr{I}^2(\overline{C_1})$. Hence, $\mathbf{C}(q)\psi$ is a quantum deformation of $\mathscr{I}^2(\overline{C_1})$. By Theorem 1.2 the two sided ideal $U_q(\mathfrak{n}_I^-)\psi = \psi U_q(\mathfrak{n}_I^-)$ is a quantum deformation of the defining ideal $\mathscr{I}(\overline{C_1})$ of the closure of C_1. Similarly, $\sum_{1\le i\le 2n-1}\mathbf{C}(q)Y_i$ is the quantum deformation of $\mathscr{I}^1(\overline{C_0})$. Moreover, the generator ψ of the quantum deformation of $\mathscr{I}(\overline{C_1})$ is the quantum deformation of the basic relative invariant.
Therefore we have the following.

Theorem 4.5. (i) *A quantum deformation $A_q(\mathfrak{n}_I^+)$ of the coordinate algebra $A(\mathfrak{n}_I^+)$ of $\mathfrak{n}_I^+$ is generated by Y_i $(1\le i\le 2n-1)$ satisfying the fundamental relations* (4.1).

(ii) *The action of $U_q(\mathfrak{l}_I)$ on $A_q(\mathfrak{n}_I^+)$ is given as follows. For $r\in I_0$ and $s\in I$,*

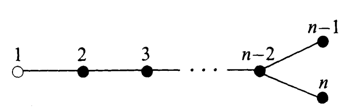
$$K_r\cdot Y_i = q^{-(\alpha_r,\beta_i)}Y_i,$$

$$E_s\cdot Y_i = \begin{cases} Y_{i-1} & (1\le i=s\le n-1 \text{ or } n+1\le i=2n+1-s)\\ (q+q^{-1})Y_{i-1} & (i=n=s)\\ 0 & (\text{otherwise}),\end{cases}$$

$$F_s\cdot Y_i = \begin{cases} Y_{i+1} & (1\le i=s-1\le n-1 \text{ or } n+1\le i=2n-s)\\ (q+q^{-1})Y_{i+1} & (i=n=s)\\ 0 & (\text{otherwise}).\end{cases}$$

(iii) *ψ is the quantum deformation of the basic relative invariant.*

Next we deal with the case $m=2n-2$. We label the vertices of the Dynkin diagram of $\mathfrak{g}$ as follows.



Hence we have $I_0=\{1,2,\dots,n\}$. Set $I=I_0\backslash\{1\}$. Then $\mathfrak{n}_I^+\neq\{0\}$ and $[\mathfrak{n}_I^+,\mathfrak{n}_I^+]=\{0\}$. There exist three L_I-orbits C_0,C_1,C_2 on $\mathfrak{n}_I^+$ satisfying the closure relation $\{0\}=C_0\subset\overline{C_1}\subset\overline{C_2}=\mathfrak{n}_I^+$.
We fix a reduced expression $w_Iw_0 = s_1s_2\cdots s_{n-1}s_ns_{n-2}s_{n-3}\cdots s_2s_1$. For $i=1,2,\dots,2n-2$ we define $\beta_i\in\Delta^+\backslash\Delta_I$ and $Y_i=Y_{\beta_i}\in U_q(\mathfrak{n}_I^-)_{-\beta_i}$ as in Section 1.

Note that

$$\beta_i = \begin{cases} \alpha_1 + \alpha_2 + \cdots + \alpha_i & (1 \le i \le n-1) \\ \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_n & (i = n) \\ \alpha_1 + \alpha_2 + \cdots + \alpha_{2n-i-1} & \\ \qquad + 2\alpha_{2n-i} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & (n+1 \le i \le 2n-2). \end{cases}$$

Since the arguments are simpler than and similar to the case $m = 2n-1$, we omit the proofs.

LEMMA 4.6. *For $r \in I$ we have*

$$\mathrm{ad}(F_r)\,Y_i = \begin{cases} Y_{i+1} & (1 \le i \le n-2, r = i+1 \ \ or \ \ n \le i \le 2n-2, r = 2n-i-1) \\ Y_n & (i = n-2, r = n) \\ Y_{n+1} & (i = n-1, r = n) \\ 0 & (otherwise), \end{cases}$$

$$\mathrm{ad}(E_r)\,Y_i = \begin{cases} Y_{i-1} & (1 \le i \le n-1, r = i \ \ or \ \ n+1 \le i \le 2n-2, r = 2n-i) \\ Y_{n-1} & (i = n+1, r = n) \\ Y_{n-2} & (i = n, r = n) \\ 0 & (otherwise). \end{cases}$$

PROPOSITION 4.7. *We have*

$$Y_j Y_i = \begin{cases} q^{-1} Y_i Y_j & (j > i, i+j \ne 2n-1) \\ Y_i Y_j & (j = n, i = n-1) \\ Y_i Y_j + (q^{-1} - q) \sum_{t=1}^{j-n} (-q)^{t-1} Y_{i+t} Y_{j-t} & (i \le n-2, i+j = 2n-1). \end{cases} \tag{4.3}$$

We set

$$\psi = \sum_{i=1}^{n-1} (-q^{-1})^{i-1} Y_{n-i} Y_{n+i-1}.$$

Since $\beta_{n-i} + \beta_{n+i-1}$ dose not depend on i, we have $\psi \in U_q(\mathfrak{n}_I^-)_{-(\beta_{n-1}+\beta_n)}$.

PROPOSITION 4.8. *For $r \in I$ we have*

$$\mathrm{ad}(F_r)\psi = 0, \qquad \mathrm{ad}(E_r)\psi = 0.$$

PROPOSITION 4.9. *$\sum_{1 \le i \le 2n-2} \mathbf{C}(q) Y_i$ and $\mathbf{C}(q)\psi$ are irreducible highest weight $U_q(\mathfrak{l}_I)$-modules.*

The $U_q(\mathfrak{l}_I)$-module $\sum_{1\le i\le 2n-2}\mathbf{C}(q)Y_i$ (resp. $\mathbf{C}(q)\psi$) is a quantum deformation of $\mathscr{I}^1(\overline{C_0})$ (resp. $\mathscr{I}^2(\overline{C_1})$).

THEOREM 4.10. (i) *A quantum deformation $A_q(\mathfrak{n}_I^+)$ of the coordinate algebra $A(\mathfrak{n}_I^+)$ of $\mathfrak{n}_I^+$ is generated by Y_i $(1\le i\le 2n-2)$ satisfying the fundamental relations* (4.3).

(ii) *The action of $U_q(\mathfrak{l}_I)$ on $A_q(\mathfrak{n}_I^+)$ is given as follows. For $r\in I_0$ and $s\in I$,*

$$K_r\cdot Y_i = q^{-(\alpha_r,\beta_i)}Y_i,$$

$$E_s\cdot Y_i=\begin{cases} Y_{i-1} & (1\le i=s\le n-1 \text{ or } n+1\le i=2n-s\le m)\\ Y_{n-1} & (i-1=n=s)\\ Y_{n-2} & (i=n=s)\\ 0 & (\text{otherwise}),\end{cases}$$

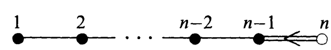
$$F_s\cdot Y_i=\begin{cases} Y_{i+1} & (1\le i=s-1\le n-2 \text{ or } n\le i=2n-s-1\le m)\\ Y_n & (i+2=n=s)\\ Y_{n+1} & (i+1=n=s)\\ 0 & (\text{otherwise}).\end{cases}$$

(iii) *ψ is the quantum deformation of the basic relative invariant.*

5. Quantum determinants of quantum symmetric matrices

In this section we apply the method in [14] to the case where $\mathfrak{g}=\mathfrak{sp}_{2n}(\mathbf{C})$ and $\mathfrak{l}_I\simeq\mathfrak{gl}_n(\mathbf{C})$.

We label the vertices of the Dynkin diagram of $\mathfrak{g}$ as follows.



Hence we have $I_0=\{1,2,\dots,n\}$. Set $I=I_0\backslash\{n\}$. Then $\mathfrak{n}_I^+\ne\{0\}$ and $[\mathfrak{n}_I^+,\mathfrak{n}_I^+]=\{0\}$. There exist $n+1$ L_I-orbits $C_0,C_1,\dots,C_n$ on $\mathfrak{n}_I^+$ satisfying the closure relation $\{0\}=C_0\subset\overline{C_1}\subset\cdots\subset\overline{C_n}=\mathfrak{n}_I^+$.

We fix a reduced expression

$$w_Iw_0=(s_ns_{n-1}\cdots s_1)(s_ns_{n-1}\cdots s_2)\cdots(s_ns_{n-1})s_n.$$

Let $1\le i\le j\le n$. We define $\beta_{ij}\in\Delta^+\backslash\Delta_I$ by

$$\beta_{ij}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_{j-1}+2\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_{n-1}+\alpha_n,$$

and set

$$Y_{ij}=c_{i,j}T^{(1)}T^{(2)}\cdots T^{(n-j)}T_nT_{n-1}\cdots T_{n-j+i+1}(F_{n-j+i}),$$

where $T^{(s)} = T_n T_{n-1} \cdots T_s$ for $s = 1, 2, \dots, n-1$ and

$$c_{i,j} = \begin{cases} q + q^{-1} & (1 \le i = j \le n) \\ 1 & (1 \le i < j \le n). \end{cases}$$

Note that $Y_{ij} \in U_q(\mathfrak{n}_I^-)_{-\beta_{ij}}$.

LEMMA 5.1. *For $r \in I$, we have*

$$\mathrm{ad}(F_r) Y_{ij} = \begin{cases} c_{i,j} Y_{i-1,j} & (r+1 = i \le j) \\ Y_{i,j-1} & (i < j = r+1) \\ 0 & (otherwise), \end{cases}$$

$$\mathrm{ad}(E_r) Y_{ij} = \begin{cases} Y_{i+1,j} & (r = i < j) \\ c_{i,j} Y_{i,j+1} & (i \le j = r) \\ 0 & (otherwise). \end{cases}$$

PROOF. Similarly to the proof of Lemma 4.1, we have only to deal with the cases $r+1 = i$ or $r+1 = j$ for F_r and $r = i$ or $r = j$ for E_r.

Let $r+1 = i \le j$. By using (1.1) we have

$$F_{i-1} = T^{(1)} T^{(2)} \cdots T^{(n-j)} T_n T_{n-1} \cdots T_{n-j+i+1} (F_{n-j+i-1}).$$

Hence, we obtain

$$\begin{aligned} \mathrm{ad}(F_{i-1}) Y_{ij} &= F_{i-1} Y_{ij} - q_{n-j+i} Y_{ij} F_i \\ &= c_{i,j} T^{(1)} T^{(2)} \cdots T^{(n-j)} T_n T_{n-1} \cdots T_{n-j+i+1} \\ &\quad \times (F_{n-j+i-1} F_{n-j+i} - q_{n-j+i} F_{n-j+i} F_{n-j+i-1}) \\ &= c_{i,j} T^{(1)} T^{(2)} \cdots T^{(n-j)} T_n T_{n-1} \cdots T_{n-j+i+1} T_{n-j+i} (F_{n-j+i-1}) \\ &= c_{i,j} Y_{i-1,j}. \end{aligned}$$

Let $i+1 < j = r+1$. By using (1.1) and (1.2) we have

$$\begin{aligned} F_{j-1} &= T^{(1)}(F_j) = T^{(1)} \cdots T^{(n-j)} (F_{n-1}) \\ &= T^{(1)} \cdots T^{(n-j)} T_n T_{n-1} T_n (F_{n-1}) \\ &= T^{(1)} \cdots T^{(n-j)} T_n T_{n-1} \cdots T_{n-j+i+1} T_n T_{n-1} \cdots T_{n-j+i+2} (F_{n-j+i+1}). \end{aligned}$$

Since

$$Y_{ij} = T^{(1)} \cdots T^{(n-j)} T_n T_{n-1} \cdots T_{n-j+i+1} T_n T_{n-1} \cdots T_{n-j+i+2} (F_{n-j+i}),$$

we have

$$\begin{aligned}
\mathrm{ad}(F_{j-1})Y_{ij} &= F_{j-1}Y_{ij} - qY_{ij}F_{j-1}\\
&= T^{(1)}\cdots T^{(n-j)}T_nT_{n-1}\cdots T_{n-j+i+1}\\
&\quad\times T_nT_{n-1}\cdots T_{n-j+i+2}(F_{n-j+i+1}F_{n-j+i} - qF_{n-j+i}F_{n-j+i+1})\\
&= T^{(1)}\cdots T^{(n-j)}T_nT_{n-1}\cdots T_{n-j+i+1}T_nT_{n-1}\cdots T_{n-j+i+2}T_{n-j+i}(F_{n-j+i+1}).
\end{aligned}$$

On the other hand, by the braid relations we have

$$\begin{aligned}
Y_{i,j-1} &= T^{(1)}\cdots T^{(n-j)}T^{(n-j+1)}T_nT_{n-1}\cdots T_{n-j+i+2}(F_{n-j+i+1})\\
&= T^{(1)}\cdots T^{(n-j)}T_nT_{n-1}\cdots T_{n-j+i+1}T_nT_{n-1}\cdots T_{n-j+i+2}\\
&\quad\times T_{n-j+i}T_{n-j+i-1}\cdots T_{n-j+1}(F_{n-j+i+1})\\
&= T^{(1)}\cdots T^{(n-j)}T_nT_{n-1}\cdots T_{n-j+i+1}T_nT_{n-1}\cdots T_{n-j+i+2}T_{n-j+i}(F_{n-j+i+1}).
\end{aligned}$$

Therefore we obtain $\mathrm{ad}(F_{j-1})Y_{ij} = Y_{i,j-1}$.

Assume $i+1=j=r+1$. Since $F_{j-1} = T^{(1)}\cdots T^{(n-j)}(F_{n-1})$, we have $\mathrm{ad}(F_{j-1})Y_{ij} = [F_{j-1}, Y_{ij}] = T^{(1)}\cdots T^{(n-j)}T_n[T_n^{-1}(F_{n-1}), F_{n-1}]$. On the other hand,

$$\begin{aligned}
Y_{i,j-1} &= (q+q^{-1})T^{(1)}T^{(2)}\cdots T^{(n-j)}T^{(n-j+1)}(F_n)\\
&= (q+q^{-1})T^{(1)}T^{(2)}\cdots T^{(n-j)}T_nT_{n-1}(F_n)\\
&= (q+q^{-1})T^{(1)}T^{(2)}\cdots T^{(n-j)}T_n(F_nF_{n-1}^{(2)} - qF_{n-1}F_nF_{n-1} + q^2F_{n-1}^{(2)}F_n)\\
&= T^{(1)}T^{(2)}\cdots T^{(n-j)}T_n([F_nF_{n-1} - q^2F_{n-1}F_n, F_{n-1}]).
\end{aligned}$$

Since $T_n^{-1}(F_{n-1}) = F_nF_{n-1} - q^2F_{n-1}F_n$ (see [4]), the statement holds.

Let $r=i<j$. We have $\mathrm{ad}(E_i)Y_{i+1,j}=0$ and $Y_{ij} = c_{i+1,j}^{-1}\mathrm{ad}(F_i)Y_{i+1,j}$, hence

$$\begin{aligned}
\mathrm{ad}(E_i)Y_{ij} &= c_{i+1,j}^{-1}\mathrm{ad}(E_i)(\mathrm{ad}(F_i)Y_{i+1,j})\\
&= c_{i+1,j}^{-1}(q_i-q_i^{-1})^{-1}\mathrm{ad}(K_i-K_i^{-1})Y_{i+1,j} + c_{i+1,j}^{-1}\mathrm{ad}(F_i)(\mathrm{ad}(E_i)Y_{i+1,j})\\
&= c_{i+1,j}^{-1}(q-q^{-1})^{-1}(q_{n-j-i-1} - q_{n-j-i-1}^{-1})Y_{i+i,j}\\
&= Y_{i+1,j}.
\end{aligned}$$

Let $i\le j=r$. We have

$$\begin{aligned}
\mathrm{ad}(E_j)Y_{ij} &= \mathrm{ad}(E_j)(\mathrm{ad}(F_j)Y_{i,j+1})\\
&= (q_j-q_j^{-1})^{-1}\mathrm{ad}(K_j-K_j^{-1})Y_{i,j+1} + \mathrm{ad}(F_j)(\mathrm{ad}(E_j)Y_{i,j+1}).
\end{aligned}$$

If $i < j$, $\mathrm{ad}(E_j)Y_{i,j+1} = 0$. Hence we obtain

$$\mathrm{ad}(E_j)Y_{ij} = (q - q^{-1})^{-1}(q - q^{-1})Y_{i,j+1} = Y_{i,j+1}.$$

If $i = j$, we have

$$\mathrm{ad}(F_i)(\mathrm{ad}(E_i)Y_{i,i+1}) = \mathrm{ad}(F_i)Y_{i+1,i+1} = (q + q^{-1})Y_{i,i+1},$$

$$\mathrm{ad}(K_i - K_i^{-1})Y_{i,i+1} = Y_{i,i+1} - Y_{i,i+1} = 0.$$

Hence we obtain $\mathrm{ad}(E_i)Y_{ii} = (q + q^{-1})Y_{i,i+1}$. $\square$

PROPOSITION 5.2. *We have*

$$Y_{ij}Y_{lm} = \begin{cases} q_{n-j+i}Y_{lm}Y_{ij} & (l < i \le m = j) \\ q_{n-m+l}Y_{lm}Y_{ij} & (l = i \le m < j) \\ Y_{lm}Y_{ij} & (l < i \le j < m) \\ Y_{lm}Y_{ij} + (q - q^{-1})Y_{lj}Y_{im} & (l < i < m < j) \\ qY_{lm}Y_{ij} + q(q - q^{-1})Y_{lj}Y_{ii} & (l < i = m < j) \\ Y_{lm}Y_{ij} + (q^2 - q^{-2})Y_{li}Y_{lj} & (l = m < i < j) \\ Y_{lm}Y_{ij} + (q - q^{-1})(Y_{li}Y_{mj} + qY_{lj}Y_{mi}) & (l < m < i < j) \\ Y_{lm}Y_{ij} + q^{-1}(q^2 - q^{-2})Y_{li}^2 & (l = m < i = j) \\ Y_{lm}Y_{ij} + (q^2 - q^{-2})Y_{li}Y_{mi} & (l < m < i = j). \end{cases} \tag{5.1}$$

PROOF. We shall prove $Y_{ij}Y_{lm} = q_{n-j+i}Y_{lm}Y_{ij}$ for $l < i \le m = j$ by induction on $t = i - l$. For $t = 1$ we have $Y_{lm} = c_{i,j}^{-1}\mathrm{ad}(F_{i-1})Y_{ij} = c_{i,j}^{-1}(F_{i-1}Y_{ij} - q_{n-j+i}Y_{ij}F_{i-1})$. Since $F_{i-1} = T^{(1)}T^{(2)}\cdots T^{(n-j)}T_nT_{n-1}\cdots T_{n-j+i+1}\cdot(F_{n-j+i-1})$, we obtain

$$\begin{aligned} &Y_{ij}Y_{lj} - q_{n-j+i}Y_{lj}Y_{ij} \\ &\quad = -c_{i,j}^{-1}q_{n-j+i}(Y_{ij}^2F_{i-1} - (q_{n-j+i}^{-1} + q_{n-j+i})Y_{ij}F_{i-1}Y_{ij} + F_{i-1}Y_{ij}^2) \\ &\quad = -c_{i,j}q_{n-j+i}T^{(1)}T^{(2)}\cdots T^{(n-j)}T_nT_{n-1}\cdots T_{n-j+i+1} \\ &\qquad \times (F_{n-j+i}^2F_{n-j+i-1} - (q_{n-j+i} + q_{n-j+i}^{-1})F_{n-j+i}F_{n-j+i-1}F_{n-j+i} + F_{n-j+i-1}F_{n-j+i}^2) \\ &\quad = 0. \end{aligned}$$

Assume that $t > 1$ and the statement is proved up to $t - 1$. From Lemma 5.1, we have $[F_l, Y_{ij}] = \mathrm{ad}(F_l)Y_{ij} = 0$. By the inductive hypothesis on t, we have $Y_{ij}Y_{l+1,j} = q_{n-j+i}Y_{l+1,j}Y_{ij}$. Therefore we obtain

$$\begin{aligned} Y_{ij}Y_{lj} &= Y_{ij}(\mathrm{ad}(F_l)Y_{l+1,j}) = Y_{ij}(F_lY_{l+1,j} - qY_{l+1,j}F_l) \\ &= q_{n-j+i}(F_lY_{l+1,j} - qY_{l+1,j}F_l)Y_{ij} = q_{n-j+i}Y_{lj}Y_{ij}. \end{aligned}$$

Let $l = i \leq m < j$. We show $Y_{lj} Y_{lm} = q_{n-m+l} Y_{lm} Y_{lj}$ by induction on $t = j - m$. For $t = 1$, from the proof of Lemma 5.1 we have

$$F_m = T^{(1)} \cdots T^{(n-m-1)}(F_{n-1}) = T(F_{n-m+l}),$$

where $T = T^{(1)} \cdots T^{(n-m-1)} T_n T_{n-1} \cdots T_{n-m+l} T_n T_{n-1} \cdots T_{n-m+l+1}$. If $l = m$, we have $Y_{mm} = \mathrm{ad}(F_m) Y_{m,m+1} = [F_m, Y_{m,m+1}]$ and $Y_{m,m+1} = T^{(1)} \cdots T^{(n-m-1)} \cdot T_n(F_{n-1})$. Hence we obtain

$$\begin{aligned}
Y_{mj} Y_{mm} - q_n Y_{mm} Y_{mj} &= -Y_{m,m+1}^2 F_m + (q^2+1) Y_{m,m+1} F_m Y_{m,m+1} - q^2 F_m Y_{m,m+1}^2 \\
&= T^{(1)} \cdots T^{(n-m-1)} T_n(-F_{n-1}^2 T_n^{-1}(F_{n-1}) \\
&\quad + (q^2+1) F_{n-1} T_n^{-1}(F_{n-1}) F_{n-1} - q^2 T_n^{-1}(F_{n-1}) F_{n-1}^2) \\
&= q^2 T^{(1)} \cdots T^{(n-m-1)} T_n (F_{n-1}^3 F_n - (q^2 + 1 + q^{-2}) F_{n-1}^2 F_n F_{n-1} \\
&\quad + (q^2 + 1 + q^{-2}) F_{n-1} F_n F_{n-1}^2 - F_n F_{n-1}^3) \\
&= 0.
\end{aligned}$$

If $l < m$, we have $Y_{lm} = \mathrm{ad}(F_m) Y_{l,m+1} = F_m Y_{l,m+1} - q Y_{l,m+1} F_m$ and $Y_{l,m+1} = T(F_{n-m+l-1})$. Hence we obtain

$$\begin{aligned}
Y_{lj} Y_{lm} - q_{n-m+l} Y_{lm} Y_{lj} &= -q(Y_{l,m+1}^2 F_m - (q + q^{-1}) Y_{l,m+1} F_m Y_{l,m+1} + F_m Y_{l,m+1}^2) \\
&= -q T(F_{n-m+l-1}^2 F_{n-m+l} - (q + q^{-1}) F_{n-m+l-1} F_{n-m+l} \\
&\quad \times F_{n-m+l-1} + F_{n-m+l} F_{n-m+l-1}^2) \\
&= 0.
\end{aligned}$$

Assume that $t > 1$ and the statement is proved up to $t - 1$. If $l = m$, we have $0 = \mathrm{ad}(F_m) Y_{mj} = F_m Y_{mj} - q^{-1} Y_{mj} F_m$. Therefore by the inductive hypothesis on t we obtain $Y_{mj} Y_{mm} = Y_{mj}[F_m, Y_{m,m+1}] = q^2 [F_m, Y_{m,m+1}] Y_{m,j} = q_n Y_{mm} Y_{mj}$. If $l < m$, we have $0 = \mathrm{ad}(F_m) Y_{mj} = [F_m, Y_{mj}]$. Hence we obtain $Y_{lj} Y_{lm} = Y_{lj}(F_m Y_{l,m+1} - q Y_{l,m+1} F_m) = q(F_m Y_{l,m+1} - q Y_{l,m+1} F_m) Y_{lj} = q_{n-m+l} Y_{lm} Y_{lj}$.

Let $l < i \leq j < m$. We show $Y_{ij} Y_{lm} = Y_{lm} Y_{ij}$ by induction on $t = m - j$. For $t = 1$, we have

$$[Y_{ij}, Y_{lm}] = c_{i,j} T([T_{n-j+l-1} \cdots T_{n-j} T_n \cdots T_{n-j+i+1}(F_{n-j+i}), F_{n-j+l-1}]),$$

where $T = T^{(1)} \cdots T^{(n-j-1)} T_n \cdots T_{n-j+l}$. By the braid relations, we have

$$\begin{aligned}
T_{n-j+l-1} \cdots T_{n-j} T_n \cdots T_{n-j+i+1}(F_{n-j+i}) &= T_n \cdots T_{n-j+i+1} T_{n-j+l-1} \cdots T_{n-j}(F_{n-j+i}) \\
&= T_n \cdots T_{n-j+i+1}(F_{n-j+i}).
\end{aligned}$$

Since $F_{n-j+l-1} = T_n \cdots T_{n-j+i+1}(F_{n-j+l-1})$ and $[F_{n-j+i}, F_{n-j+l-1}] = 0$, we obtain $[Y_{ij}, Y_{lm}] = 0$ for $t = 1$. Assume that $t > 1$ and the statement is proved up to $t-1$. Since $Y_{ij} = \mathrm{ad}(F_j) Y_{i,j+1}$ and $[F_j, Y_{lm}] = \mathrm{ad}(F_j) Y_{lm} = 0$, by the inductive hypothesis on t we obtain $Y_{ij} Y_{lm} = Y_{lm} Y_{ij}$.

Let $l < i < m < j$. We prove $[Y_{ij}, Y_{lm}] = (q - q^{-1}) Y_{lj} Y_{im}$ by the induction on $t = i - l$. When $t = 1$, we have $Y_{ls} = \mathrm{ad}(F_{i-1}) Y_{is} = F_{i-1} Y_{is} - q Y_{is} F_{i-1}$ for $s = m, j$. Since $Y_{ij} Y_{im} = q Y_{im} Y_{ij}$ and $Y_{im} Y_{lj} = Y_{lj} Y_{im}$, we obtain

$$\begin{aligned}
[Y_{ij}, Y_{lm}] &= [Y_{ij}, F_{i-1} Y_{im} - q Y_{im} F_{i-1}] \\
&= q Y_{im}(F_{i-1} Y_{ij} - q Y_{ij} F_{i-1}) - q^{-1}(F_{i-1} Y_{ij} - q Y_{ij} F_{i-1}) Y_{im} \\
&= (q - q^{-1}) Y_{lj} Y_{im}.
\end{aligned}$$

Assume that $t > 1$ and the statement is proved up to $t - 1$. By Lemma 5.1 we have $[F_l, Y_{is}] = 0$ for $s = m, j$. Hence we have

$$\begin{aligned}
[Y_{ij}, Y_{lm}] &= [Y_{ij}, \mathrm{ad}(F_l) Y_{l+1,m}] = [Y_{ij}, F_l Y_{l+1,m} - q Y_{l+1,m} F_l] \\
&= F_l [Y_{ij}, Y_{l+1,m}] - q [Y_{ij}, Y_{l+1,m}] F_l = (q - q^{-1})(F_l Y_{l+1,j} Y_{im} - q Y_{l+1,j} Y_{im} F_l) \\
&= (q - q^{-1})(F_l Y_{l+1,j} - q Y_{l+1,j} F_l) Y_{im} = (q - q^{-1})(\mathrm{ad}(F_l) Y_{l+1,j}) Y_{im} \\
&= (q - q^{-1}) Y_{lj} Y_{im}.
\end{aligned}$$

Similarly, we can prove $Y_{ij} Y_{lm} = q Y_{lm} Y_{ij} + q(q - q^{-1}) Y_{lj} Y_{ii}$ for $l < i = m < j$.

Let $l < m < i < j$. We prove the statement by induction on $t = i - m$. We have $[Y_{ij}, Y_{lm}] = [Y_{ij}, \mathrm{ad}(F_m) Y_{l,m+1}] = [Y_{ij}, F_m Y_{l,m+1} - q Y_{l,m+1} F_m]$. If $t = 1$, $Y_{mj} = F_m Y_{ij} - q Y_{ij} F_m$ and $Y_{ij} Y_{li} = q Y_{li} Y_{ij} + q(q - q^{-1}) Y_{lj} Y_{ii}$. Therefore $[Y_{ij}, Y_{lm}]$ is equal to

$$\begin{aligned}
& q Y_{li}(F_m Y_{ij} - q Y_{ij} F_m) - q^{-1}(F_m Y_{ij} - q Y_{ij} F_m) Y_{li} \\
&\qquad + q^{-1} F_m (Y_{ij} Y_{li} - q Y_{li} Y_{ij}) - q(Y_{ij} Y_{li} - q Y_{li} Y_{ij}) F_m \\
&\quad = q Y_{li} Y_{mj} - q^{-1} Y_{mj} Y_{li} + (q - q^{-1})(F_m Y_{lj} Y_{ii} - q^2 Y_{lj} Y_{ii} F_m).
\end{aligned}$$

Since $[F_m, Y_{lj}] = \mathrm{ad}(F_m) Y_{lj} = 0$, we have

$$\begin{aligned}
[Y_{ij}, Y_{lm}] &= q Y_{li} Y_{mj} - q^{-1} Y_{mj} Y_{li} + (q - q^{-1}) Y_{lj}(F_m Y_{ii} - q^2 Y_{ii} F_m) \\
&= q Y_{li} Y_{mj} - q^{-1} Y_{mj} Y_{li} + (q - q^{-1})(q + q^{-1}) Y_{lj} Y_{mi} \\
&= (q - q^{-1})(Y_{li} Y_{mj} + q Y_{lj} Y_{mi}).
\end{aligned}$$

We have used $[Y_{mj}, Y_{li}] = (q - q^{-1}) Y_{lj} Y_{mi}$ for the last step. Assume that $t > 1$ and the statement is proved up to $t - 1$. By $[F_m, Y_{ij}] = 0 = [F_m, Y_{ls}]$ for $s =$

i, j, we obtain

$$\begin{aligned}[Y_{ij}, Y_{lm}] &= F_m[Y_{ij}, Y_{l,m+1}] - q[Y_{ij}, Y_{l,m+1}]F_m \\ &= (q-q^{-1})(F_m(Y_{li}Y_{m+1,j} + qY_{lj}Y_{m+1,i}) - q(Y_{li}Y_{m+1,j} + qY_{lj}Y_{m+1,i})F_m) \\ &= (q-q^{-1})(Y_{li}(F_mY_{m+1,j} - qY_{m+1,j}F_m) + qY_{lj}(F_mY_{m+1,i} - qY_{m+1,i}F_m)) \\ &= (q-q^{-1})(Y_{li}Y_{mj} + qY_{lj}Y_{mi}).\end{aligned}$$

Here we have used the inductive hypothesis for the second equality.

Let $l = m < i < j$. We have $[Y_{ij}, Y_{ll}] = [Y_{ij}, [F_l, Y_{l,l+1}]]$. Assume $i - l = 1$. Then we have $Y_{lj} = F_lY_{ij} - qY_{ij}F_l$ and $Y_{ij}Y_{li} = qY_{li}Y_{ij} + q(q-q^{-1})Y_{lj}Y_{ii}$. Hence we have

$$\begin{aligned}[Y_{ij}, Y_{lm}] &= q^{-1}F_l(Y_{ij}Y_{li} - qY_{li}Y_{ij}) - (Y_{ij}Y_{li} - qY_{li}Y_{ij})F_l \\ &\quad - q^{-1}(F_lY_{ij} - qY_{ij}F_l)Y_{li} + Y_{li}(F_lY_{ij} - qY_{ij}F_l) \\ &= (q-q^{-1})(F_lY_{lj}Y_{ii} - qY_{lj}Y_{ii}F_l) - q^{-1}Y_{lj}Y_{li} + Y_{li}Y_{lj} \\ &= (q-q^{-1})(F_lY_{lj}Y_{ii} - qY_{lj}Y_{ii}F_l).\end{aligned}$$

By Lemma 5.1 we have $F_lY_{lj} - q^{-1}Y_{lj}F_l = \mathrm{ad}(F_l)Y_{lj} = 0$. Hence we obtain

$$\begin{aligned}[Y_{ij}, Y_{lm}] &= (q-q^{-1})q^{-1}Y_{lj}(F_lY_{ii} - q^2Y_{ii}F_l) = (q-q^{-1})q^{-1}Y_{lj}(\mathrm{ad}(F_l)Y_{ii}) \\ &= (q-q^{-1})q^{-1}(q+q^{-1})Y_{lj}Y_{li}\end{aligned}$$

Assume $i - l > 1$, then we have $[F_l, Y_{ij}] = 0$. By the preceding paragraph, $[Y_{ij}, Y_{l,l+1}] = (q-q^{-1})(Y_{li}Y_{l+1,j} + qY_{lj}Y_{l+1,i})$. Therefore we obtain

$$\begin{aligned}[Y_{ij}, Y_{lm}] &= [F_l, [Y_{ij}, Y_{l,l+1}]] = (q-q^{-1})[F_l, Y_{li}Y_{l+1,j} + qY_{lj}Y_{l+1,i}] \\ &= (q-q^{-1})(q^{-1}Y_{li}(\mathrm{ad}(F_l)Y_{l+1,j}) + Y_{lj}(\mathrm{ad}(F_l)Y_{l+1,i})) \\ &= (q-q^{-1})(q^{-1}Y_{li}Y_{lj} + Y_{lj}Y_{li}) = (q^2-q^{-2})Y_{li}Y_{lj}.\end{aligned}$$

Let $l < m < i = j$. We show the statement by induction on $t = i - m$. Assume $t = 1$. We have $Y_{ii}Y_{li} = q^2Y_{li}Y_{ii}$, hence $[Y_{ii}, Y_{lm}]$ is equal to

$$\begin{aligned}[Y_{ii}, F_mY_{li} - qY_{li}F_m] &= qY_{li}(F_mY_{ii} - q^2Y_{ii}F_m) - q^{-2}(F_mY_{ii} - q^2Y_{ii}F_m)Y_{li} \\ &= qY_{li}(\mathrm{ad}(F_m)Y_{ii}) - q^{-2}(\mathrm{ad}(F_m)Y_{ii})Y_{li} \\ &= (q+q^{-1})(qY_{li}Y_{mi} - q^{-2}Y_{mi}Y_{li}) \\ &= (q^2-q^{-2})Y_{li}Y_{mi}.\end{aligned}$$

Assume that $t > 1$ and the statement is proved up to $t-1$. By Lemma 5.1 we have $[F_m, Y_{si}] = 0$ for $s = l, i$. Therefore we obtain

$$\begin{aligned}[Y_{ii}, Y_{lm}] &= [Y_{ii}, F_m Y_{l,m+1} - qY_{l,m+1}F_m] = F_m[Y_{ii}, Y_{l,m+1}] - q[Y_{ii}, Y_{l,m+1}]F_m \\ &= (q^2 - q^{-2})(F_m Y_{li} Y_{m+1,i} - qY_{li} Y_{m+1,i} F_m) \\ &= (q^2 - q^{-2}) Y_{li}(F_m Y_{m+1,i} - qY_{m+1,i} F_m) \\ &= (q^2 - q^{-2}) Y_{li}(\mathrm{ad}(F_m) Y_{m+1,i}) = (q^2 - q^{-2}) Y_{li} Y_{mi}.\end{aligned}$$

Here we have used the inductive hypothesis on t for the third equality.

Let $l = m < i = j$. Since $Y_{ll} = \mathrm{ad}(F_l) Y_{l,l+1} = [F_l, Y_{l,l+1}]$, we have $[Y_{ii}, Y_{ll}] = [Y_{ii}, [F_l, Y_{l,l+1}]]$. If $i - l = 1$, then $Y_{ii} Y_{l,l+1} = q^2 Y_{l,l+1} Y_{ii}$. Hence, $[Y_{ii}, Y_{ll}]$ is equal to

$$\begin{aligned}&Y_{l,l+1}(F_l Y_{ii} - q^2 Y_{ii} F_l) - q^{-2}(F_l Y_{ii} - q^2 Y_{ii} F_l) Y_{l,l+1} \\ &\quad = Y_{li}(\mathrm{ad}(F_l) Y_{ii}) - q^{-2}(\mathrm{ad}(F_l) Y_{ii}) Y_{li} \\ &\quad = (q + q^{-1})(1 - q^{-2}) Y_{li}^2 = q^{-1}(q^2 - q^{-2}) Y_{li}^2.\end{aligned}$$

If $i - l > 1$, we have $[F_l, Y_{ii}] = \mathrm{ad}(F_l) Y_{ii} = 0$. From the preceding paragraph we have $[Y_{ii}, Y_{l,l+1}] = (q^2 - q^{-2}) Y_{li} Y_{l+1,i}$, hence

$$\begin{aligned}[Y_{ii}, Y_{ll}] &= [F_l, [Y_{ii}, Y_{l,l+1}]] = (q^2 - q^{-2})[F_l, Y_{li} Y_{l+1,i}] \\ &= (q^2 - q^{-2})((F_l Y_{li} - q^{-1} Y_{li} F_l) Y_{l+1,i} + q^{-1} Y_{li}(F_l Y_{l+1,i} - qY_{l+1,i} F_l)) \\ &= (q^2 - q^{-2})((\mathrm{ad}(F_l) Y_{li}) Y_{l+1,i} + q^{-1} Y_{li}(\mathrm{ad}(F_l) Y_{l+1,i})) \\ &= (q^2 - q^{-2}) q^{-1} Y_{li}^2.\end{aligned}$$

□

For $j > i$ we define Y_{ji} by $Y_{ji} = q^{-2} Y_{ij}$. For $p = 0, 1, \dots, n-1$ we set

$$\psi_p^- = \sum_{\sigma \in S_{p+1}} (-q^{-1})^{l(\sigma)} Y_{1,\sigma(1)} Y_{2,\sigma(2)} \cdots Y_{p+1,\sigma(p+1)} \in U_q(\mathfrak{n}_I^-)_{-\sum_{t=1}^{p+1} \beta_{tt}},$$

$$\psi_p^+ = \sum_{\sigma \in S_{p+1}} (-q^{-1})^{l(\sigma)} Y_{i_1, i_{\sigma(1)}} Y_{i_2, i_{\sigma(2)}} \cdots Y_{i_{p+1}, i_{\sigma(p+1)}} \in U_q(\mathfrak{n}_I^-)_{-\sum_{t=0}^{p} \beta_{n-t n-t}}$$

where $i_s = n - p - 1 + s$.

PROPOSITION 5.3. *Let $r \in I$. For $p = 0, 1, \dots, n-1$, we have*

$$\mathrm{ad}(F_r)\psi_p^- = 0, \qquad \mathrm{ad}(E_r)\psi_p^+ = 0.$$

In order to prove Proposition 5.3 we need the following lemma.

LEMMA 5.4. *Let* $t = 1, 2, \dots, p$ *and* $a_{t+2}, a_{t+3}, \dots, a_{p+1} \in \{1, 2, \dots, p+1\}$. *We set* $A = \{\sigma \in S_{p+1} \mid \sigma(t) \le t-1, \sigma(t+1) \ge t, \sigma(s) = a_s (s > t+1)\}$. *For* $1 \le i_1 < i_2 < \cdots < i_{p+1} \le n$, *we have*

$$\sum_{\sigma \in A} (-q^{-1})^{l(\sigma)} Y_{i_1, i_{\sigma(1)}} \cdots Y_{i_{t-1}, i_{\sigma(t-1)}} Y_{i_{\sigma(t)}, i_{\sigma(t+1)}} = 0. \tag{5.2}$$

PROOF. We prove the statement by induction on t. Set

$$A(t; a_{t+2}, \dots, a_{p+1}) = \{\sigma \in S_{p+1} \mid \sigma(t) \le t-1, \sigma(t+1) \ge t, \sigma(s) = a_s (s > t+1)\},$$

$$f(t, \sigma) = (-q^{-1})^{l(\sigma)} Y_{i_1, i_{\sigma(1)}} \cdots Y_{i_{t-1}, i_{\sigma(t-1)}} Y_{i_{\sigma(t)}, i_{\sigma(t+1)}},$$

and for a subset B of S_{p+1}

$$f(t, B) = \sum_{\sigma \in B} f(t, \sigma).$$

For $t = 1$ we have $A(1; a_3, \dots, a_{p+1}) = \varnothing$, hence the statement holds.

For $t = 2$ we have $f(2, A(2; a_4, \dots, a_{p+1})) = f(2, A_1) + f(2, A_2)$, where

$$A_1 = \{\sigma \in A(2; a_4, \dots, a_{p+1}) \mid 1 < \sigma(1) < \sigma(3)\},$$

$$A_2 = \{\sigma \in A(2; a_4, \dots, a_{p+1}) \mid 1 < \sigma(3) < \sigma(1)\}.$$

For $\sigma \in A_2$ set $\tau = \sigma(1, 3)$, then $\tau \in A_1$, $l(\tau) = l(\sigma) - 1$ and

$$Y_{i_1, i_{\sigma(1)}} Y_{i_1, i_{\sigma(3)}} = q Y_{i_1, i_{\sigma(3)}} Y_{i_1, i_{\sigma(1)}} = q Y_{i_1, i_{\tau(1)}} Y_{i_1, i_{\tau(3)}}.$$

Hence we obtain $f(2, A_2) = -f(2, A_1)$, and the statement holds.

Assume that $t > 2$ and the statement is proved up to $t - 1$. We have

$$f(t, A(t; a_{t+2}, \dots, a_{p+1})) = \sum_{j=1}^{7} f(t, B_j),$$

where B_j is the subset of $A(t; a_{t+2}, \dots, a_{p+1})$ given by

$$B_1 = \{\sigma | \sigma(t) = t-1, t \le \sigma(t+1) < \sigma(t-1)\},$$

$$B_2 = \{\sigma | \sigma(t) = t-1, t \le \sigma(t-1) < \sigma(t+1)\},$$

$$B_3 = \{\sigma | \sigma(t) = t-1, \sigma(t-1) < t-1\},$$

$$B_4 = \{\sigma | \sigma(t) < t-1, t \le \sigma(t+1) < \sigma(t-1)\},$$

$$B_5 = \{\sigma | \sigma(t) < t-1, t \le \sigma(t-1) < \sigma(t+1)\},$$

$$B_6 = \{\sigma | \sigma(t) < \sigma(t-1) \le t-1\},$$

$$B_7 = \{\sigma | \sigma(t-1) < \sigma(t) < t-1\}.$$

For $\sigma \in B_2$, set $\tau = \sigma(t-1, t+1)$. Then we have $\tau \in B_1$, $l(\tau) = l(\sigma) + 1$ and

$$Y_{i_{t-1}, i_{\sigma(t-1)}} Y_{i_{\sigma(t)}, i_{\sigma(t+1)}} = q^{-1} Y_{i_{\sigma(t)}, i_{\sigma(t+1)}} Y_{i_{t-1}, i_{\sigma(t-1)}} = q^{-1} Y_{i_{t-1}, i_{\tau(t-1)}} Y_{i_{\tau(t)}, i_{\tau(t+1)}}.$$

Hence we obtain $f(t, B_2) = \sum_{\sigma \in B_2} f(t, \sigma) = -\sum_{\tau \in B_1} f(t, \tau) = -f(t, B_1)$.

Let $\sigma \in B_3$. We set $s = \sigma(t+1)$, then $\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1})$ and

$$f(t, \sigma) = q^{-2} Y_{i_1, i_{\sigma(1)}} \cdots Y_{i_{t-2}, i_{\sigma(t-2)}} Y_{i_{\sigma(t-1)}, i_{t-1}} Y_{i_{\sigma(t)}, i_s} = q^{-2} f(t-1, \sigma) Y_{i_{t-1}, i_s}.$$

Hence we obtain

$$f(t, B_3) = q^{-2} \sum_{s=t}^{p+1} f(t-1, C_1(s)) Y_{i_{t-1}, i_s},$$

where $C_1(s) = \{\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1}) \mid \sigma(t) = t-1\}$.

For $\sigma \in B_4$ we set $\tau = \sigma(t-1, t)$ and $\rho = \tau(t, t+1)$. Then we have $\tau \in A(t-1; s, a_{t+2}, \dots, a_{p+1})$, and $\rho \in A(t-1; r, a_{t+2}, \dots, a_{p+1})$, where $s = \sigma(t+1)$ and $r = \sigma(t-1)$. Since $l(\tau) = l(\sigma) - 1$, $l(\rho) = l(\sigma) - 2$ and

$$\begin{aligned} Y_{i_{t-1}, i_{\sigma(t-1)}} Y_{i_{\sigma(t)}, i_{\sigma(t+1)}} &= Y_{i_{\sigma(t)}, i_{\sigma(t+1)}} Y_{i_{t-1}, i_{\sigma(t-1)}} + (q - q^{-1}) Y_{i_{\sigma(t)}, i_{\sigma(t-1)}} Y_{i_{t-1}, i_{\sigma(t+1)}} \\ &= Y_{i_{\rho(t-1)}, i_{\rho(t)}} Y_{i_{t-1}, i_r} + (q - q^{-1}) Y_{i_{\tau(t-1)}, i_{\tau(t)}} Y_{i_{t-1}, i_s}, \end{aligned}$$

we have

$$f(t, \sigma) = q^{-2} f(t-1, \rho) Y_{i_{t-1}, i_r} + (q^{-2} - 1) f(t-1, \tau) Y_{i_{t-1}, i_s}.$$

Hence we have

$$f(t, B_4) = q^{-2} \sum_{s=t}^{p+1} f(t-1, B_1'(s)) Y_{i_{t-1}, i_s} + (q^{-2} - 1) \sum_{s=t}^{p+1} f(t-1, B_2'(s)) Y_{i_{t-1}, i_s},$$

where $B_1'(s) = \{\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1}) | t \le \sigma(t) < s\}$ and $B_2'(s) = \{\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1}) \mid s < \sigma(t)\}$. Similarly, for $\sigma \in B_5$ we have $\rho \in B_2'(s)$ and $f(t, \sigma) = f(t-1, \rho) Y_{i_{t-1}, i_s}$, where $s = \sigma(t-1)$. Therefore we have

$$f(t, B_5) = \sum_{s=t}^{p+1} f(t-1, B_2'(s)) Y_{i_{t-1}, i_s}.$$

Hence we obtain

$$f(t, B_4) + f(t, B_5) = q^{-2} \sum_{s=t}^{p+1} f(t-1, C_2(s)) Y_{i_{t-1}, i_s},$$

where $C_2(s) = \{\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1}) \mid t \le \sigma(t)\} = B_1'(s) \cup B_2'(s)$ (disjoint).

Let $\sigma \in B_6$. We set $\tau = \sigma(t, t+1)$, $\rho = \tau(t-1, t+1)$, $s = \sigma(t-1)$ and $r = \sigma(t)$. Then we have $\rho \in A(t-1; s, a_{t+2}, \dots, a_{p+1})$, $\tau \in A(t-1; r, a_{t+2}, \dots, a_{p+1})$, $l(\rho) = l(\sigma) + 2$, $l(\tau) = l(\sigma) + 1$, and $f(t, \sigma) = q^2 f(t-1, \rho) Y_{i_{t-1}, i_s} + (q^2 - 1) f \cdot (t-1, \tau) Y_{i_{t-1}, i_r}$. Hence we obtain

$$f(t, B_6) = q^2 \sum_{s=1}^{t-2} f(t-1, B_3'(s)) Y_{i_{t-1}, i_s} + (q^2 - 1) \sum_{r=1}^{t-2} f(t-1, B_4'(s)) Y_{i_{t-1}, i_s},$$

where $B_3'(s) = \{\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1}) \mid s < \sigma(t-1), t \le \sigma(t)\}$ and $B_4'(s) = \{\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1}) \mid \sigma(t-1) < s, t \le \sigma(t)\}$. Similarly, for $\sigma \in B_7$ we have $\rho \in B_4'(s)$ and $f(t, \sigma) = f(t-1, \rho) Y_{i_{t-1}, i_s}$, where $s = \sigma(t-1)$. Hence we have

$$f(t, B_7) = \sum_{s=1}^{t-1} f(t-1, B_4'(s)) Y_{i_{t-1}, i_s}.$$

We set $C_3(s) = \{\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1}) \mid t \le \sigma(t)\}$. Since $C_3(s) = B_3'(s) \cup B_4'(s)$ (disjoint) and $B_4'(t-1) = A(t-1; t-1, a_{t+2}, \dots, a_{p+1})$, we obtain

$$\begin{aligned} f(t, B_6) + f(t, B_7) &= q^2 \sum_{s=1}^{t-2} f(t-1, C_3(s)) Y_{i_{t-1}, i_s} + f(t-1, B_4'(t-1)) Y_{i_{t-1}, i_{t-1}} \\ &= q^2 \sum_{s=1}^{t-2} f(t-1, C_3(s)) Y_{i_{t-1}, i_s}. \end{aligned}$$

We have used the inductive hypothesis on t for the last step.

Therefore we obtain

$$\begin{aligned} f(t, A(t; a_{t+2}, \dots, a_{p+1})) = q^{-2} &\sum_{s=t}^{p+1} (f(t-1, C_1(s)) + f(t-1, C_2(s))) Y_{i_{t-1}, i_s} \\ &+ q^2 \sum_{s=1}^{t-2} f(t-1, C_3(s)) Y_{i_{t-1}, i_s}. \end{aligned}$$

Since $A(t-1; s, a_{t+2}, \dots, a_{p+1}) = C_1(s) \cup C_2(s)$ (disjoint), by the inductive hypothesis on t, we have $f(t-1, C_1(s)) + f(t-1, C_2(s)) = f(t-1, A(t-1; s, a_{t+2}, \dots, a_{p+1})) = 0$. Hence we have only to show $\sum_{s=1}^{t-2} f(t-1, C_3(s)) Y_{i_{t-1}, i_s} = 0$.

We set $C(\gtrless s) = \{\sigma \in A(t-1; s, a_{t+2}, \dots, a_{p+1}) \mid \sigma(t) = t-1, \sigma(t-1) \gtrless s\}$ for $1 \le s \le t-2$. Then we have $A(t-1; s, a_{t+2}, \dots, a_{p+1}) = C(>s) \cup C(<s) \cup C_3(s)$ (disjoint). By using inductive hypothesis on t, we have

$$f(t-1, C_3(s)) = -f(t-1, C(>s)) - f(t-1, C(<s)).$$

For $\sigma \in C(<s)$ set $\tau = \sigma(t-1, t+1)$ and $r = \sigma(t-1)$. Then we have $l(\tau) = l(\sigma) + 1$ and $\tau \in C(>r)$. Since $Y_{i_{\sigma(t-1)}, i_{\sigma(t)}} Y_{i_{t-1}, i_s} = q^{-1} Y_{i_{\tau(t-1)}, i_{\tau(t)}} Y_{i_{t-1}, i_r}$, we have $f(t-1, \sigma) Y_{i_{t-1}, i_s} = f(t-1, \tau) Y_{i_{t-1}, i_r}$. Therefore we obtain

$$\begin{aligned}\sum_{s=1}^{t-2} f(t-1, C_3(s)) Y_{i_{t-1}, i_s} &= -\sum_{s=1}^{t-2} f(t-1, C(>s)) Y_{i_{t-1}, i_s} - \sum_{s=1}^{t-2} f(t-1, C(<s)) Y_{i_{t-1}, i_s} \\ &= -\sum_{s=1}^{t-2} f(t-1, C(>s)) Y_{i_{t-1}, i_s} + \sum_{r=1}^{t-2} f(t-1, C(>r)) Y_{i_{t-1}, i_r} \\ &= 0. \end{aligned}$$ □

Let us show Proposition 5.3.

By Lemma 5.1, it is clear that $\mathrm{ad}(F_r)\psi_p^- = 0$ for $r > p$ and $\mathrm{ad}(E_r)\psi_p^+ = 0$ for $r < n-p$.

Let $r \leq p$. We shall show $\mathrm{ad}(F_r)\psi_p^- = 0$. We set for $\sigma \in S_{p+1}$ and $y \in U_q(\mathfrak{n}_I^-)$

$$g(\sigma, j_1, y, j_2) = (-q^{-1})^{l(\sigma)} K_r Y_{i,\sigma(1)} \cdots Y_{j_1, \sigma(j_1)} K_r^{-1} y Y_{j_2, \sigma(j_2)} \cdots Y_{p+1, \sigma(p+1)},$$

and for a subset A of S_{p+1}

$$g(A, j_1, y, j_2) = \sum_{\sigma \in A} g(\sigma, j_1, y, j_2).$$

We have

$$\begin{aligned}\mathrm{ad}(F_r)\psi_p^- &= \sum_{j=i}^{p+1} g(S_{p+1}, j-1, \mathrm{ad}(F_r) Y_{j,\sigma(j)}, j+1) \\ &= (q+q^{-1}) g(A_1, r, Y_{r,r+1}, r+2) + q^{-2} g(A_2, r, Y_{r,r}, r+2) \\ &\quad + g(B, r, Y_{r,\sigma(r+1)}, r+2) + \sum_{j \neq r+1} g(C(j), j-1, Y_{j,r}, j+1)\end{aligned}$$

where $A_1 = \{\sigma | \sigma(r+1) = r+1\}$, $A_2 = \{\sigma | \sigma(r+1) = r\}$, $B = \{\sigma | \sigma(r+1) \neq r, r+1\}$, and $C(j) = \{\sigma | \sigma(j) = r+1\}$.

Let $\sigma \in A_2$. If $\sigma(r) = r+1$, we have $l(\sigma) = l(\tau) + 1$ and $Y_{r,\sigma(r)} K_r^{-1} Y_{rr} = q^4 Y_{r,\tau(r)} K_r^{-1} Y_{r,r+1}$, where $\tau = \sigma(r, r+1)$. On the other hand, we have $Y_{r,\sigma(r)} K_r^{-1} Y_{rr} = Y_{\sigma(r),\sigma(r+1)} Y_{rr} K_r^{-1}$ if $\sigma(r) \leq r-1$, and $Y_{r,\sigma(r)} K_r^{-1} Y_{rr} = q^4 Y_{\sigma(r),\sigma(r+1)} Y_{rr} K_r^{-1}$ if $\sigma(r) \geq r+2$. Therefore we obtain

$$\begin{aligned} q^{-2} g(A_2, r, Y_{rr}, r+2) = -q g(A_1', r, Y_{r,r+1}, r+2) \\ + q^{-2} g(A_2', r-1, K_r Y_{\sigma(r),\sigma(r+1)} Y_{rr} K_r^{-1}, r+2) \\ + q^2 g(A_3', r-1, K_r Y_{\sigma(r),\sigma(r+1)} Y_{rr} K_r^{-1}, r+2),\end{aligned}$$

where $A_1' = \{\sigma|\sigma(r+1) = r+1, \sigma(r) = r\}$, $A_2' = \{\sigma|\sigma(r+1) = r, \sigma(r) \leq r-1\}$ and $A_3' = \{\sigma|\sigma(r+1) = r, \sigma(r) \geq r+2\}$.

For $\sigma \in B$ set $\tau = \sigma(r, r+1)$. We define subsets B_j $(1 \leq j \leq 10)$ of B as follows:

$$B_1 = \{\sigma(r) < \sigma(r+1) < r\}, \qquad B_2 = \{\sigma(r+1) < \sigma(r) < r\},$$

$$B_3 = \{\sigma(r) < r < r+1 < \sigma(r+1)\}, \qquad B_4 = \{\sigma(r+1) < r < r+1 < \sigma(r)\},$$

$$B_5 = \{r+1 < \sigma(r) < \sigma(r+1)\}, \qquad B_6 = \{r+1 < \sigma(r+1) < \sigma(r)\},$$

$$B_7 = \{\sigma(r+1) < \sigma(r) = r\}, \qquad B_8 = \{\sigma(r) = r < r+1 < \sigma(r+1)\},$$

$$B_9 = \{\sigma(r+1) < r < r+1 = \sigma(r)\}, \qquad B_{10} = \{r+1 = \sigma(r) < \sigma(r+1)\}.$$

If $\sigma \in B_1$, we have $\tau \in B_2$, $l(\tau) = l(\sigma) + 1$, and

$$Y_{r,\sigma(r)} K_r^{-1} Y_{r,\sigma(r+1)} = q^{-1} Y_{r,\tau(r)} K_r^{-1} Y_{r,\tau(r+1)}.$$

Therefore we obtain

$$g(B_1, r, Y_{r,\sigma(r+1)}, r+2) = -g(B_2, r, Y_{r,\tau(r+1)}, r+2).$$

Similarly, we have

$$g(B_5, r, Y_{r,\sigma(r+1)}, r+2) = -g(B_6, r, Y_{r,\tau(r+1)}, r+2),$$

$$\begin{aligned} g(B_4, r, Y_{r,\sigma(r+1)}, r+2) &= -g(B_3, r, Y_{r,\tau(r+1)}, r+2) \\ &\quad + (q^{-2}-1) g(B_3, r-1, K_r Y_{\tau(r),\tau(r+1)} Y_{rr} K_r^{-1}, r+2), \end{aligned}$$

$$g(B_7, r, Y_{r,\sigma(r+1)}, r+2) = -g(A_2', r-1, K_r Y_{\tau(r),\tau(r+1)} Y_{rr} K_r^{-1}, r+2),$$

$$g(B_8, r, Y_{r,\sigma(r+1)}, r+2) = -q^2 g(A_3', r-1, K_r Y_{\tau(r),\tau(r+1)} Y_{rr} K_r^{-1}, r+2),$$

$$\begin{aligned} g(B_9, r, Y_{r,\sigma(r+1)}, r+2) &= -qg(A_4', r, Y_{r,r+1}, r+2) \\ &\quad + (q^{-2}-1) g(A_4', r-1, K_r Y_{\tau(r),\tau(r+1)} Y_{rr} K_r^{-1}, r+2), \end{aligned}$$

$$g(B_{10}, r, Y_{r,\sigma(r+1)}, r+2) = -qg(A_5', r, Y_{r,r+1}, r+2),$$

where $A_4' = \{\tau|\tau(r) < r < r+1 = \tau(r+1)\}$ and $A_5' = \{\tau|\tau(r+1) = r+1 < \tau(r)\}$.

Here, we set $A(r) = \{\sigma|\sigma(r) < r \leq \sigma(r+1)\}$. Since $A_1 = A_1' \cup A_4' \cup A_5'$ (disjoint) and $A(r) = A_2' \cup B_3 \cup A_4'$ (disjoint), we obtain

$$\begin{aligned} &q^{-2} g(A_2, r, Y_{r,r}, r+2) + g(B, r, Y_{r,\sigma(r+1)}, r+2) \\ &\quad = -qg(A_1, r, Y_{r,r+1}, r+2) \\ &\qquad + (q^{-2}-1) g(A(r), r-1, K_r Y_{\sigma(r),\sigma(r+1)} Y_{rr} K_r^{-1}, r+2). \end{aligned}$$

For $j \neq k$ we set $C(j,k) = \{\sigma | \sigma(j) = r+1, \sigma(k) = r\}$. Let $\sigma \in C(j,k)$, and set $\tau = \sigma(j,k) \in C(k,j)$. If $k = r+1$ and $j \leq r$, we have $l(\tau) = l(\sigma) - 1$ and $K_r^{-1} Y_{j,r} Y_{j+1,\sigma(j+1)} \cdots Y_{r,\sigma(r)} Y_{r+1,\sigma(r+1)} = Y_{j,\tau(j)} Y_{j+1,\tau(j+1)} \cdots Y_{r,\tau(r)} K_r^{-1} Y_{r,r+1}$. Hence we obtain

$$g(C(j,r+1), j-1, Y_{j,r}, j+1) = -q^{-1} g(C(r+1,j), r, Y_{r,r+1}, r+2).$$

Similarly, we obtain for $j \geq r+2$

$$g(C(j,r+1), j-1, Y_{j,r}, j+1) = -q^{-1} g(C(r+1,j), r, Y_{r,r+1}, r+2).$$

If $1 \leq j < k \leq r$, we have $l(\tau) = l(\sigma) - 1$ and $K_r^{-1} Y_{j,r} Y_{j+1,\sigma(j+1)} \cdots Y_{k-1,\sigma(k-1)} \cdot Y_{k,\sigma(k)} = q Y_{j,\tau(j)} Y_{j+1,\tau(j+1)} \cdots Y_{k-1,\tau(k-1)} K_r^{-1} Y_{k,r}$. Therefore we obtain

$$\sum_{1 \leq j < k \leq r} g(C(j,k), j-1, Y_{j,r}, j+1) = -\sum_{1 \leq j < k \leq r} g(C(k,j), k-1, Y_{k,r}, k+1).$$

Similarly, we have

$$\sum_{\substack{1 \leq j \leq r \\ r+2 \leq k \leq p+1}} g(C(j,k), j-1, Y_{j,r}, j+1) = -\sum_{\substack{1 \leq j \leq r \\ r+2 \leq k \leq p+1}} g(C(k,j), k-1, Y_{k,r}, k+1),$$

$$\sum_{r+2 \leq j < k \leq p+1} g(C(j,k), j-1, Y_{j,r}, j+1) = -\sum_{r+2 \leq j < k \leq p+1} g(C(k,j), k-1, Y_{k,r}, k+1).$$

Therefore we obtain

$$\begin{aligned} \sum_{j \neq r+1} g(C(j), j-1, Y_{j,r}, j+1) &= \sum_{\substack{j \neq r+1 \\ k \neq j}} g(C(j,k), j-1, Y_{j,r}, j+1) \\ &= -q^{-1} \sum_{j \neq r+1} g(C(r+1,j), r, Y_{r,r+1}, r+2) \\ &= -q^{-1} g(A_1, r, Y_{r,r+1}, r+2). \end{aligned}$$

Here we have used for the last step that $A_1 = \bigcup_{j \neq r+1} C(r+1,j)$ (disjoint).

Hence we have

$$\mathrm{ad}(F_r)\psi_p^- = (q^{-2} - 1) g(A(r), r-1, K_r Y_{\sigma(r),\sigma(r+1)} Y_{r,r} K_r^{-1}, r+2).$$

We can write

$$g(A(r), r-1, K_r Y_{\sigma(r),\sigma(r+1)}) = \sum_{a_{r+2},\dots,a_{p+1}} K_r f(r, A'(r)) Y_{r,r} K_r^{-1} Y_{r+2,a_{r+2}} \cdots Y_{p+1,a_{p+1}},$$

where $A'(r) = \{\sigma \in S_{p+1} \,|\, \sigma(r) \leq r-1,\ \sigma(r+1) \geq r,\ \sigma(s) = a_s (s > r+1)\}$. Hence, by Lemma 5.4 we obtain $\mathrm{ad}(F_r)\psi_p^- = 0$ for $r \leq p$.

Let $r \geq n-p$ and $i_s = n-p-1+s$. Then there exists t such that $r = i_t$. Similarly to the case $r \leq p$ for F_r we have

$$\mathrm{ad}(E_{i_t})\psi_p^+ = -q^{-3}(q-q^{-1}) \sum_{a_{t+2},\dots,a_{p+1}} f(t, A'(t))\, Y_{i_{t+1}, i_{t+i}} K_{i_t}^{-1} Y_{i_{t+2}, i_{a_{t+2}}} \cdots Y_{i_{p+1}, i_{a_{p+1}}} K_{i_t}.$$

By Lemma 5.4 we obtain $\mathrm{ad}(E_{i_t})\psi_p^+ = 0$. $\square$

We denote $\psi_{n-1}^- = \psi_{n-1}^+$ by ψ_{n-1}. By Lemma 5.1 and Proposition 5.3 we have the following:

PROPOSITION 5.5. $\mathbf{C}(q)\psi_{n-1}$ *and* $\sum_{1 \leq i \leq j \leq n} \mathbf{C}(q) Y_{ij}$ *are irreducible highest weight* $U_q(\mathfrak{l}_I)$*-modules.*

The highest weight of $\mathbf{C}(q)\psi_{n-1}$ coincides with that of $\mathscr{I}^n(\overline{C_{n-1}})$. Hence, $\mathbf{C}(q)\psi_{n-1}$ is a quantum deformation of $\mathscr{I}^n(\overline{C_{n-1}})$. By Theorem 1.2 we have $U_q(\mathfrak{n}_I^-)\psi_{n-1} = \psi_{n-1} U_q(\mathfrak{n}_I^-)$, and this two sided ideal is a quantum deformation of the defining ideal $\mathscr{I}(\overline{C_{n-1}})$ of the closure of C_{n-1}. Similarly, $\sum \mathbf{C}(q) Y_{ij}$ is the quantum deformation of $\mathscr{I}^1(\overline{C_0})$. Moreover, the generator ψ_{n-1} of the quantum deformation of $\mathscr{I}(\overline{C_{n-1}})$ is the quantum deformation of the basic relative invariant.

Therefore we have the following.

THEOREM 5.6. (i) *A quantum deformation* $A_q(\mathfrak{n}_I^+)$ *of the coordinate algebra* $A(\mathfrak{n}_I^+)$ *of* $\mathfrak{n}_I^+$ *is generated by* Y_{ij} $(1 \leq i \leq j \leq n)$ *satisfying the fundamental relations* (5.1).

(ii) *The action of* $U_q(\mathfrak{l}_I)$ *on* $A_q(\mathfrak{n}_I^+)$ *is given as follows. For* $r \in I_0$ *and* $s \in I$,

$$K_r \cdot Y_{ij} = q^{-(\alpha_r, \beta_{ij})} Y_{ij},$$

$$E_s \cdot Y_{ij} = \begin{cases} Y_{i+1,j} & (s = i < j) \\ Y_{i,j+1} & (i < j = s) \\ (q+q^{-1}) Y_{i,j+1} & (i = j = s) \\ 0 & (otherwise), \end{cases}$$

$$F_s \cdot Y_{ij} = \begin{cases} Y_{i-1,j} & (s+1 = i < j) \\ (q+q^{-1}) Y_{i-1,j} & (s+1 = i = j) \\ Y_{i,j-1} & (i < j = s+1) \\ 0 & (otherwise). \end{cases}$$

(iii) ψ_{n-1} *is the quantum deformation of the basic relative invariant.*

We also obtain the explicit description of quantum deformation of $\mathscr{I}^2(\overline{C_1})$ as follows. Let $1 \leq i_1 < i_2 \leq n$, $1 \leq j_1 < j_2 \leq n$ satisfying $i_1 \leq j_1, i_2 \leq j_2$. Set

$$\begin{vmatrix} i_1 & i_2 \\ j_1 & j_2 \end{vmatrix} = \begin{cases} Y_{i_1,j_1}Y_{i_2,j_2} - Y_{i_1,j_2}Y_{i_2,j_1} & (i_1 < j_1 < i_2 \leq j_2) \\ Y_{i_1,j_1}Y_{i_2,j_2} - q^{-1}Y_{i_1,j_2}Y_{i_2,j_1} & (otherwise). \end{cases}$$

Then we can show that $\sum \mathbf{C}(q)\begin{vmatrix} i_1 & i_2 \\ j_1 & j_2 \end{vmatrix}$ is an irreducible highest weight $U_q(\mathfrak{l}_I)$-module with highest weight vector $\begin{vmatrix} n-1 & n \\ n-1 & n \end{vmatrix}$ (we omit the proof). This module is a quantum deformation of $\mathscr{I}^2(\overline{C_1})$.

For $2 \leq p \leq n-2$, we have not yet obtained the explicit description of the quantum deformation of $\mathscr{I}^{p+1}(\overline{C_p})$ as in the case $p = 0, 1, n-1$. The difficulty mainly comes from the fact that the $\mathfrak{l}_I$-module $\mathscr{I}^{p+1}(\overline{C_p})$ is not a multiplicity free $\mathfrak{h}$-module. It would be an interesting problem to define a quantum deformation of the non-principal minors of a symmetric matrix, and to develop an analogue of the classical invariant theory for symmetric matrices.

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