HIROSHIMA MATH. J.
30 (2000), 55–77

Uniqueness of double layer potentials for a domain with fractal boundary

Hisako WATANABE

(Received December 9, 1998)
(Revised April 15, 1999)

ABSTRACT. The double layer potentials for a bounded domain with fractal boundary depends on an extension operator on a space of functions on the boundary. We give a sufficient condition to define them uniquely and apply it to prove the injectivity of an operator with respect to the Dirichlet problem.

1. Introduction

Let D be a bounded smooth domain in $\mathbf{R}^d$ $(d \geq 3)$. The double layer potential Φg of $g \in L^p(\partial D)$ is defined by

$$\Phi g(x) = -\int_{\partial D} \langle \nabla_y N(x-y), n_y \rangle g(y) d\sigma(y), \tag{1.1}$$

where $N(x-y)$ is the Newton kernel, n_y is the unit outer normal to ∂D and σ is the surface measure on ∂D. The function Φg is harmonic in $\mathbf{R}^d \backslash \partial D$ and has a nontangential limit at σ-almost every boundary point.

If D is a domain with fractal boundary, then n_y and σ can not be considered and hence (1.1) is not defined. But we introduced double layer potentials in [W1] and [W3], in case $d \geq 2$ and ∂D is a β-set for β satisfying $d-1 \leq \beta < d$. According to A. Jonsson and H. Wallin we say that a closed set F is a β-set if there exist a positive Radon measure μ on F and positive real numbers b_1, b_2 such that

$$b_1 r^\beta \leq \mu(B(z,r) \cap F) \leq b_2 r^\beta \tag{1.2}$$

for all $z \in F$ and all $r \leq r_0$, where $B(z,r)$ stands for the open ball with center z and radius r in $\mathbf{R}^d$. Such a measure μ is called a β-measure.

We shall give some examples.

2000 *Mathematics Subject Classification.* 31B15
Key words and phrases. Fractal boundary, Layer potentials, Whitney decomposition, Extension operator.

1. If D is a bounded Lipschitz domain in $\mathbf{R}^d$, then ∂D is a $(d-1)$-set and the surface measure is a $(d-1)$-measure.

2. If ∂D consists of a finite number of self-similar sets, which satisfies the open set condition, and whose similarity dimensions are β, then ∂D is a β-set and the β-dimensional Hausdorff measure restricted to ∂D is a β-measure (cf. [**H**]). A typical example is the Von Koch snowflake in $\mathbf{R}^2$.

Let us fix a positive real number R such that $\bar{D} \subset B(0, R/2)$ and $R \geq 1$, and a β-measure μ on ∂D.

Since every function $f \in L^p(\mu)$ has an extension $\mathscr{E}(f)$ such that $\mathscr{E}(f)$ is a C^∞-function in $\mathbf{R}^d \setminus \partial D$, we define, for $f \in \Lambda_\alpha^p(\partial D)$, the double layer potential by

$$\Phi f(x) = \int_{\mathbf{R}^d \setminus \bar{D}} \langle \nabla_y \mathscr{E}(f)(y), \nabla_y N(x-y) \rangle dy \tag{1.3}$$

for $x \in D$ and

$$\Phi f(x) = -\int_D \langle \nabla_y \mathscr{E}(f)(y), \nabla_y N(x-y) \rangle dy \tag{1.4}$$

for $x \in \mathbf{R}^d \setminus \bar{D}$, where

$$N(x-y) = \begin{cases} \dfrac{1}{\omega_d (d-2)|x-y|^{d-2}} & \text{if } d \geq 3 \\ -\dfrac{5R}{2\pi} \log \dfrac{|x-y|}{5R} & \text{if } d = 2 \end{cases}$$

and ω_d stands for the surface area of the unit ball in $\mathbf{R}^d$. Here $\Lambda_\alpha^p(F)$ for a closed subset F of $\mathbf{R}^d$ is a Banach space defined by

$$\Lambda_\alpha^p(F) = \left\{ f \in L^p(\mu) : \iint \frac{|f(x) - f(y)|^p}{|x-y|^{\beta + p\alpha}} d\mu(x) d\mu(y) < \infty \right\}$$

with norm

$$\|f\|_{p,\alpha} = \left(\int |f|^p d\mu \right)^{1/p} + \left(\iint \frac{|f(x) - f(y)|^p}{|x-y|^{\beta + p\alpha}} d\mu(x) d\mu(y) \right)^{1/p}.$$

We saw in [W3] that Φf is harmonic in $\mathbf{R}^d \setminus \partial D$ and has a similar boundary behavior to that for an usual double layer potential.

Our definition of the double layer potentials depends on the choice of an extension operator. Under what conditions is the definition independent of an extension operator? In this paper we will give an answer to this problem.

Moreover we apply it to prove that an operator on the conjugate of $\Lambda_\alpha^p(\partial D)$ is injective. The operator plays an important role to solve the Dirichlet problem with boundary data in $\Lambda_\alpha^p(\partial D)$ by the layer potential method.

More precisely, denote by $\mathscr{V}(G)$ the Whitney decomposition of an open set G and by $V_k(G)$ the union of k-cubes in $\mathscr{V}(G)$. We shall mention the Whitney decomposition in §2. Pick an integer n_0 satisfying $2^{-n_0} > 100R\sqrt{d}$ and denote by $Q(n_0)$ the open cube with center 0 and common side-length 2^{-n_0}. Further put

$$(1.5)\qquad A_n := \bigcup\nolimits_{k\le n} V_k(D) \qquad \text{and} \qquad B_n = \bigcup\nolimits_{k\le n} V_k(Q(n_0)\backslash\bar{D})$$

for each natural number n.

Let $\tau > \beta - (d-1) \geq 0$ and $p > 1$. We denote by $\mathscr{U}_\tau^p(\bar{D})$ the family of all Borel measurable functions f defined on $\bar{D}$ having the following properties:

(i) f is of C^1-class in D,

(ii) There exists $n_1 \in \mathbf{N}$ such that

$$(1.6)\qquad \int_{\partial A_n} d\sigma_n(y) \int_{\{|y-w|\le b2^{-n}\}\cap \partial D} |f(y)-f(w)|^p d\mu(w) \le c_f(2^{-n})^{d-1+p\tau}$$

for every $n \geq n_1$, where $b = 6\sqrt{d}$, c_f is a constant independent of n, and σ_n stands for the surface measure on ∂A_n,

(iii)

$$(1.7)\qquad \int_D |\nabla f(y)|dy < \infty.$$

We also denote by $\mathscr{U}_\tau^p(\mathbf{R}^d\backslash D)$ the family of all measurable functions f on $\mathbf{R}^d\backslash D$ such that f has the properties (i)–(iii) (∂A_n and D are replaced with $\partial B_n \cap B(0,2R)$ and $\mathbf{R}^d\backslash\bar{D}$, respectively) and

(iv) $f(x)$ tends to 0 as $|x| \to \infty$.

Further we denote by $\mathscr{U}_\tau^p(\mathbf{R}^d)$ the family of all functions f defined on $\mathbf{R}^d$ such that $f|\bar{D} \in \mathscr{U}_\tau^p(\bar{D})$ and $f|(\mathbf{R}^d\backslash D) \in \mathscr{U}_\tau^p(\mathbf{R}^d\backslash D)$.

In §3 we will prove the following therem.

THEOREM 1. *Assume that D is a bounded domain in $\mathbf{R}^d$ $(d \geq 2)$ such that ∂D is a β-set. Further let $1 \geq \tau > \beta-(d-1) \geq 0$ and $p > 1$. We then have, for f_1, $f_2 \in \mathscr{U}_\tau^p(\bar{D})$ such that $f_1 = f_2$ μ-a.e. on ∂D,*

$$(1.8)\qquad \int_D \langle \nabla f_1(y), \nabla N(x-y)\rangle dy = \int_D \langle \nabla f_2(y), \nabla N(x-y)\rangle dy$$

for each $x \in \mathbf{R}^d\backslash\bar{D}$.

We now denote by $\Lambda_\alpha^p(\partial D)'$ the space of all bounded linear functionals on $\Lambda_\alpha^p(\partial D)$ and write

$$\ll f, \psi \gg := \psi(f) \qquad \text{for } f \in \Lambda_\alpha^p(\partial D) \qquad \text{and} \qquad \psi \in \Lambda_\alpha^p(\partial D)'.$$

By the Hahn-Banach extension theorem, there exist, for each $\psi \in \Lambda_\alpha^p(\partial D)'$, $g_1 \in L^q(\mu)$ and $g_2 \in L^q(\mu \times \mu)$ such that

$$\ll f, \psi \gg = \int f g_1 d\mu + \iint \frac{f(x) - f(z)}{|x - z|^{\beta/p+\alpha}} g_2(x, z) d\mu(x) d\mu(z)$$

for every $f \in \Lambda_\alpha^p(\partial D)$, where $q = p/(p-1)$. We write $\psi = (g_1, g_2)$.

We then define an operator K on $\Lambda_\alpha^p(\partial D)$ by

$$(1.9) \qquad Kf(z) := \frac{1}{2}\int_{\mathbf{R}^d \setminus \bar{D}} \langle \nabla \mathscr{E}(f)(y), \nabla_y N(z - y) \rangle dy - \frac{1}{2}\int_D \langle \nabla \mathscr{E}(f)(y), \nabla_y N(z - y) \rangle dy$$

if it is well-defined and $Kf(z) = 0$ otherwise.

We saw in [W3] that

$$(1.10) \qquad Kf(z) + \frac{f(z)}{2} = \lim_{x \to z, x \in \Gamma_\tau(z)} \int_{\mathbf{R}^d \setminus \bar{D}} \langle \nabla \mathscr{E}(f)(y), \nabla_y N(x - y) \rangle dy = \int_{\mathbf{R}^d \setminus \bar{D}} \langle \nabla \mathscr{E}(f)(y), \nabla_y N(z - y) \rangle dy$$

for μ-a.e. $z \in \partial D$ and

$$(1.11) \qquad Kf(z) - \frac{f(z)}{2} = -\lim_{x \to z, x \in \Gamma_\tau^e(z)} \int_D \langle \nabla \mathscr{E}(f)(y), \nabla_y N(x - y) \rangle dy = -\int_D \langle \nabla \mathscr{E}(f)(y), \nabla_y N(z - y) \rangle dy$$

for μ-a.e. $z \in \partial D$, where

$$\Gamma_\tau(z) = \{y \in D; |y - z| < (1 + \tau)\delta(y)\}$$

and

$$\Gamma_\tau^e(z) = \{y \in (\mathbf{R}^d \setminus \bar{D}) \cap B(0, R); |y - z| < (1 + \tau)\delta(y)\}.$$

Here $\delta(y)$ stands for the distance of y from ∂D.

We also saw in [W2] that, if f is a Lipschitz function on ∂D, so is Kf. Using Theorem 1, we will prove the following theorem in §5.

THEOREM 3. *Assume that D is a bounded domain in $\mathbf{R}^d$ $(d \geq 3)$ such that $\mathbf{R}^d \setminus D$ is connected and ∂D is a β-set. Further assume that* $B(z, r) \cap \Gamma_\tau(z) \neq \varnothing$

and $B(z,r)\cap\Gamma_\tau^e(z)\neq\varnothing$ *for all* $z\in\partial D$ *and* $0<r\le r_0$. *If* $1<p\le 2$, $1-(d-\beta)/p>\alpha>\beta-(d-1)\ge 0$, $\psi\in\Lambda_\alpha^p(\partial D)'$ *and* $\ll Kf+f/2,\psi\gg\,=0$ *for every Lipschitz function* f *on* ∂D, *then* $\psi=0$.

2. Construction of an extension operator

Hereafter we assume that D is a bounded domain in $\mathbf{R}^d$ such that the boundary is a β-set satisfying $d-1\le\beta<d$ and $\bar{D}\subset B(0,R/2)$ $(R\ge 1)$. Fix a positive Radon measure μ on ∂D satisfying (1.2) for $F=\partial D$. We may assume that $r_0\ge 3R$.

To extend $f\in L^p(\mu)$, we use a Whitney decomposition.

More precisely, let G be an open set in $\mathbf{R}^d$. A cube Q is called a k-cube if it is of the form

$$[l_1 2^{-k},(l_1+1)2^{-k}]\times\cdots\times[l_d 2^{-k},(l_d+1)2^{-k}],$$

where k, $l_1,\dots,l_d$ are integers. We denote by $\mathscr{W}_k(G)$ the family of all k-cubes in G and set $\mathscr{W}(G)=\bigcup_{k=-\infty}^{\infty}\mathscr{W}_k(G)$. It is well-known that a Whitney decomposition of G can be chosen as follows (cf. [**HN**, p. 572]).

Theorem A. *Let* G *be a nonempty bounded open set in* $\mathbf{R}^d$. *Then there exists a family* $\mathscr{V}(G)=\{Q_j\}$ *of cubes in* $\mathscr{W}(G)$ *having the following properties:*

(i) $\bigcup_j Q_j=G$,

(ii) $\operatorname{int}Q_j\cap\operatorname{int}Q_k=\varnothing$ $(j\neq k)$,

(iii) $\operatorname{diam}Q_j\le\operatorname{dist}(Q_j,\mathbf{R}^d\backslash G)\le 4\operatorname{diam}Q_j$,

(iv) *If* $k\ge 1$ *and* $Q\in\mathscr{V}(G)\cap\mathscr{W}_k(G)$, *then each* k*-cube touching* Q *is contained in* G.

Here $\operatorname{int}A$, $\operatorname{diam}A$ *and* $\operatorname{dist}(A,B)$ *stand for the interior of* A, *the diameter of* A *and the distance between* A *and* B, *respectively.*

Let A_n, B_n be the sets defined by (1.5). We see by Theorem A that the boundaries of A_n and B_n consist of some surfaces of n-cubes in D and $Q(n_0)\backslash\bar{D}$, respectively.

Fix a positive real number η satisfying $\eta<1/4$ and choose a C^∞-function ϕ on $\mathbf{R}^d$ such that

$$\phi=1 \text{ on } Q_0,\qquad \operatorname{supp}\phi\subset(1+\eta)Q_0,\qquad 0\le\phi\le 1, \tag{2.1}$$

where Q_0 is the closed cube of unit length centered at the origin and $(1+\eta)Q_0$ stands for the set $\{(1+\eta)x : x\in Q_0\}$.

Let $\{Q_j\}$ be the family

$$\{Q; Q\in\mathscr{V}(D)\cup\mathscr{V}(Q(n_0)\backslash\bar{D}),\ Q \text{ is a } k\text{-cube for } k \text{ satisfying } 2^{-k}\le 4R\}.$$

Further let $q^{(j)}$, l_j be the center of Q_j and the common length of its sides, respectively. For each j pick a point $a^{(j)} \in \partial D$ satisfying $\operatorname{dist}(\partial D, Q_j) = \operatorname{dist}(a^{(j)}, Q_j)$ and fix it. Set

$$t(x) = \sum_j \phi\left(\frac{x - q^{(j)}}{l_j}\right) \qquad \text{and} \qquad \phi_j^*(x) = \frac{\phi((x - q^{(j)})/l_j)}{t(x)}.$$

Let $p \geq 1$ and $f \in L^p(\mu)$. We define

$$\mathscr{E}_0(f)(x) = \sum_j \frac{1}{\mu(B(a^{(j)}, \eta l_j))}\left(\int_{B(a^{(j)}, \eta l_j)} f(z) d\mu(z)\right)\phi_j^*(x)$$

if $x \in B(0, 3R)\backslash\partial D$ and $\mathscr{E}_0(f)(x) = f(x)$ if $x \in \partial D$. Noting that $x \in B(0, 3R)$ is contained in some k-cube satisfying $2^{-k} \leq 4R$, $\mathscr{E}_0(f)$ is a C^∞-function in $B(0, 3R)\backslash\partial D$. Choose a C^∞-function ϕ_0 satisfying

$$\phi_0 = 1 \text{ on } \overline{B(0, R)}, \qquad \operatorname{supp}\phi_0 \subset B(0, 2R), \qquad 0 \leq \phi_0 \leq 1$$

and define

$$\mathscr{E}(f)(x) = \begin{cases} \mathscr{E}_0(f)(x)\phi_0(x) & \text{if } x \in B(0, 3R) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathscr{E}(f)$ is a C^∞-function in $\mathbf{R}^d\backslash\partial D$ and supp $\mathscr{E}(f) \subset B(0, 2R)$.

Though the definition of $\mathscr{E}(f)$ is slightly different from that in [W3], they coincide eventually since $\mathscr{E}_0(f)$ defined above takes the same values in $B(0, 2R)$ as that in [W3].

We gave the following estimate for $|\nabla\mathscr{E}(f)|$ of $f \in \Lambda_\alpha^p(\partial D)$ in [W3, Lemma 2.2].

LEMMA B. *Let* $1 \geq \alpha > 0$, $p > 1$, $0 < r < 3R$, $\lambda \in \mathbf{R}$ *and* $f \in \Lambda_\alpha^p(\partial D)$. *If* $(\alpha - 1)p + d - \beta + p\lambda > 0$, *then*

$$\int_{\delta(y) \leq r} |\nabla\mathscr{E}(f)(y)|^p \delta(y)^{\lambda p} dy \leq c\|f\|_{p,\alpha}^p r^{(\alpha-1)p+d-\beta+p\lambda}.$$

To prove the above lemma, we need the following fundamental estimate for a bounded domain whose boundary is a β-set (cf. [W1, Lemma 2.3]).

LEMMA C. *Let* λ, k *be a real number. If* $d - \beta > \lambda$ *and* $d - \lambda + k > 0$, *then*

$$\int_{B(z,r)} \delta(y)^{-\lambda}|y - z|^k dy \leq cr^{d-\lambda+k}$$

for every $z \in \partial D$ *and* $0 < r \leq 3R$.

We shall often use the following lemma, which easily follows from the definition of a β-set and the fact that for $\varepsilon > 0$ the function $r \mapsto r^{\varepsilon}\log(r/5R)$ is bounded on $(0, 3R]$.

LEMMA D. *Let* $0 < r < 3R, k \in \mathbf{R}^d$ *and* $z \in \partial D$.
(i) *If* $k + \beta > 0$, *then*

$$\int_{B(z,r)\cap\partial D} |x - z|^k d\mu(x) \leq cr^{\beta+k}.$$

(ii) *If* $k + \beta < 0$, *then*

$$\int_{\partial D\setminus B(z,r)} |x - z|^k d\mu(x) \leq cr^{\beta+k}.$$

(iii) *If* $\varepsilon > 0$ *and* $k + \beta - \varepsilon > 0$, *then*

$$0 \leq -\int_{B(z,r)\cap\partial D} |x - z|^k \log \frac{|x - z|}{5R} d\mu(x) \leq cr^{\beta+k-\varepsilon}.$$

Here c *is a constant independent of* r, z.

3. Uniqueness of double layer potentials

Using the extension operator $\mathscr{E}$ in §2, we defined double layer potentials by (1.3) and (1.4). Similary we can define them by another extension operator having adequate properties. Under what conditions are double layer potentials uniquely defined independent of extension operators? In this section we will give an answer to this problem.

We begin with the following lemma.

LEMMA 3.1. *Let* k *be a natural number and* $0 < r \leq 2R$. *Suppose* z_0 *is a boundary point of* D. *Then the number* m *of* k*-cubes included in* $B(z_0, r)$ *is at most* $c2^{k\beta}r^{\beta}$, *where* c *is a constant independent of* k *and* r.

PROOF. We may assume $2^{-k} \leq r$ and set $l = 2^{-k}$. Let $\{Q_j\}_{j\in I}$ be a family of k-cubes included in $B(z_0, r)$. Then there exist points $z_1, \ldots, z_n \in \partial D$ such that $\bigcup_{i\in I} Q_i \subset \bigcup_{j=1}^{n} B(z_j, bl)$ and

$$\left(\bigcup_{i\in I} Q_i\right) \cap B(z_j, bl) \neq \varnothing \ \ (j = 1, 2, \ldots, n),$$

where $b = 6\sqrt{d}$. Using Vitali's covering lemma, we choose a subfamily $\{B_t\}_t$ of $\{B(z_j, bl)\}$ such that

$$\bigcup_{i\in I} Q_i \subset \bigcup_{t=1}^{n'} 5B_t \qquad \text{and} \qquad B_t \cap B_s = \varnothing \ \ (t \neq s),$$

where $5B_t$ stands for the ball having the same center as B_t but whose diameter is five times as large. Considering the d-dimensional Lebesgue measure of these sets and noting that ∂D is a β-set, we obtain

$$ml^d = \left|\bigcup_{i\in I} Q_i\right| \le \left|\bigcup_t 5B_t\right| \le c_1 n'(bl)^d \le c_2(bl)^{d-\beta}\mu\left(\bigcup_{t=1}^{n'} B_t\right)$$
$$\le c_2(bl)^{d-\beta}\mu(B(z_0, 13\sqrt{d}r)) \le c_3 l^{d-\beta} r^\beta.$$

This leads to the conclusion. □

COROLLARY 3.1. *Let $n \in \mathbf{N}$. Then*

$$\sigma_n(\partial A_n) \le c(2^{-n})^{d-1-\beta},$$

where c is a constant independent of n.

PROOF. Note that ∂A_n consists of some surfaces of n-cubes in D. Lemma 3.1 yields

$$\sigma_n(\partial A_n) \le c(2^{-n})^{d-1}2^{n\beta}.$$ □

LEMMA 3.2. *Let $0 < k < \beta$, $z_0 \in \partial D$ and $n \in \mathbf{N}$. Then*

$$\int_{\partial A_n} |y - z_0|^{-k} d\sigma_n(y) \le c(2^{-n})^{d-1-\beta}. \tag{3.1}$$

and

$$0 \le -\int_{\partial A_n} |y - z_0|^{-k} \log\frac{|y - z_0|}{5R} d\sigma_n(y) \le c(2^{-n})^{d-1-\beta}. \tag{3.2}$$

Here c is a constant independent of z_0 and n. We have the same estimates as (3.1) *and* (3.2) *for the surface integral over $\partial B_n \cap B(0, 2R)$.*

PROOF. Set, for each integer l,

$$E_l = \{y \in \partial A_n; |y - z_0|^{-k} > 2^l\}.$$

Then $|y - z_0| < 2^{-l/k}$ for $y \in E_l$. Put $r_l = 2^{-l/k}$. We note that

$$|y - z_0|^{-k} \le \sum_{l=l_0}^{l_1} 2^{l+1}(\chi_{B(z_0,r_l)}(y) - \chi_{B(z_0,r_{l+1})}(y))$$
$$\le \sum_{l=l_0}^{l_1}(2^{l+1} - 2^l)\chi_{B(z_0,r_l)}(y) + 2^{l_0}\chi_{B(z_0,r_{l_0})}(y)$$
$$\le 2\sum_{l=l_0}^{l_1} 2^l \chi_{B(z_0,r_l)}(y),$$

where

$$r_{l_1-1} > 2^{-n} \geq r_{l_1} \qquad r_{l_0} \geq R > r_{l_0+1}.$$

Using Lemma 3.1, we have

$$\text{(3.3)} \qquad \int_{\partial A_n} |y - z_0|^{-k} d\sigma_n(y)$$

$$\leq c_1 \sum_{l=l_0}^{l_1} 2^l 2^{n\beta} r_l^{\beta} (2^{-n})^{d-1} \leq c_2 (2^{-n})^{d-1-\beta} \sum_{l=l_0}^{l_1} 2^{(1-\beta/k)l}.$$

Since $1 - \beta/k < 0$, $\sum_{l=l_0}^{\infty} 2^{(1-\beta/k)l} < \infty$. So we see that (3.1) holds.

To prove (3.2), choose $\varepsilon > 0$ satisfying $k + \varepsilon < \beta$. Noting that the function $y \mapsto |y - z_0|^{\varepsilon} \log \dfrac{|y - z_0|}{5R}$ is negative and bounded on $B(0, 2R)$, we get

$$-\int_{\partial A_n} |y - z_0|^{-k} \log \frac{|y - z_0|}{5R} d\sigma_n(y) \leq c_3 \int_{\partial A_n} |y - z_0|^{-k-\varepsilon} d\sigma_n(y),$$

which and (3.1) give (3.2). Similarly we can also obtain the same estimates for the integral over $\partial B_n \cap B(0, 2R)$. □

LEMMA 3.3. *Let $1 \geq \tau > \beta - (d-1) \geq 0$ and $p > 1$. Suppose f is a Borel measurable function on $\bar{D}$ and of C^1-class in D. If*

$$\text{(3.4)} \qquad \int_{\partial A_n} |f|^p d\sigma_n \leq c_f (2^{-n})^{d-1-\beta+p\tau}$$

for a constant c_f independent of n and

$$\int_D |\nabla f(y)| dy < \infty,$$

then

$$\int_D \langle \nabla f(y), \nabla N(x-y) \rangle dy = 0 \qquad \textit{for each } x \in \mathbf{R}^d \backslash \bar{D}.$$

PROOF. Let $x \in \mathbf{R}^d \backslash \bar{D}$. From the Green formula and Lemma 3.1 we deduce

$$\left| \int_{A_n} \langle \nabla f(y), \nabla N(x-y) \rangle dy \right|$$

$$= \left| \int_{\partial A_n} f(y) \langle \nabla N(x-y), n_y \rangle d\sigma_n(y) \right| \leq c_1 \delta(x)^{1-d} \int_{\partial A_n} |f(y)| d\sigma_n(y)$$

$$\leq c_2 \delta(x)^{1-d} \left(\int_{\partial A_n} |f(y)|^p d\sigma_n(y) \right)^{1/p} (2^{n\beta} 2^{-n(d-1)})^{1/q}$$

$$\leq c_3 \delta(x)^{1-d} (2^{-n})^{\tau+d-1-\beta},$$

where $q = p/(p-1)$. Since $\tau + d - 1 - \beta > 0$, we have the conclusion. □

PROOF of THEOREM 1. Let $y \in \partial A_n$ and set $f = f_1 - f_2$. Noting that $f(w) = 0$ μ-a.e. on ∂D and $B(y, b2^{-n}) \cap \partial D$ contains $B(a, \sqrt{d}2^{-n}) \cap \partial D$ for some $a \in \partial D$, we get

$$|f(y)| \leq c_1(2^{-n})^{-\beta} \sum_{j=1}^{2} \int_{B(y,b2^{-n}) \cap \partial D} |f_j(y) - f_j(w)| d\mu(w)$$

$$\leq c_2(2^{-n})^{-\beta/p} \sum_{j=1}^{2} \left(\int_{B(y,b2^{-n}) \cap \partial D} |f_j(y) - f_j(w)|^p d\mu(w) \right)^{1/p},$$

whence, together with (1.6),

$$\int_{\partial A_n} |f(y)|^p d\sigma_n(y) \leq c_3(2^n)^{p\tau + d - 1 - \beta}.$$

This shows (3.4). It is easy to see that f satisfies other assumptions of Lemma 3.3. Therefore Lemma 3.3 leads to the conclusion. □

Similarly we have

THEOREM 2. *Assume that* D, p *and* τ *satisfy the same assumptions as in Theorem* 1 *and let* $f_1, f_2 \in \mathscr{U}_\tau^p(\mathbf{R}^d \backslash D)$. *If* $f_1 = f_2$ μ-*a.e. on* ∂D, *then*

$$\int_{\mathbf{R}^d \backslash \bar{D}} \langle \nabla f_1(y), \nabla N(x-y) \rangle dy = \int_{\mathbf{R}^d \backslash \bar{D}} \langle \nabla f_2(y), \nabla N(x-y) \rangle dy$$

for each $x \in D$.

4. Examples of functions in $\mathscr{U}_\tau^p(\mathbf{R}^d)$

In this section we consider some examples of functions in $\mathscr{U}_\tau^p(\mathbf{R}^d)$.

The following two lemmas are well-known or proved by elementary calculations.

LEMMA E. *Let* $x, y, z \in B(0, 2R)$, $x \neq y$, $z \neq y$ *and* $0 \leq \varepsilon \leq 1$. *Then*

$$|N(x-y) - N(z-y)|$$
$$\leq c|x-z|^\varepsilon (|x-y|^{-\varepsilon} N(x-y) + |z-y|^{-\varepsilon} N(z-y))$$

and

$$|\nabla_y N(x-y) - \nabla_y N(z-y)| \leq c|x-z|^\varepsilon (|x-y|^{1-d-\varepsilon} + |z-y|^{1-d-\varepsilon}).$$

Here c *is a constant independent of* x, y, z.

LEMMA F. *Let* $x_j,\ y_k \in B(0,2R),\ x_j \neq y_k\ (j,k=1,2)$ *and* $0 \le \varepsilon_j \le 1$ $(j=1,2)$. *Then*

$$\left|\sum_{j=1}^{2}\sum_{k=1}^{2}(-1)^{j+k}N(x_j-y_k)\right|$$
$$\le c|x_1-x_2|^{\varepsilon_1}|y_1-y_2|^{\varepsilon_2}\sum_{j=1}^{2}\sum_{k=1}^{2}|x_j-y_k|^{-\varepsilon_1-\varepsilon_2}N(x_j-y_k),$$

where c *is a constant independent of* $x_j,\ y_k$.

LEMMA 4.1. *Let* $p>1,\ 1 \ge \alpha > \beta-(d-1) \ge 0$ *and* $f \in \Lambda_\alpha^p(\partial D)$. *Then* $\mathscr{E}(f) \in \mathscr{U}_\alpha^p(\mathbf{R}^d)$.

PROOF. We first show that $\mathscr{E}(f)$ satisfies (1.6). To do so, let Q be a n-cube with $Q \cap \partial A_n \neq \emptyset$. Further let $y \in Q \cap \partial A_n$ and $w \in \partial D$ such that $|y-w| \le b2^{-n}$. Suppose $Q \cap Q_j^* \neq \emptyset$, where Q_j^* is the cube with the same center as Q_j and with the common side-length $l_j(1+2\eta)$. Let $z \in B(a^{(j)}, \eta l_j)$. Then

$$|z-w| \le |y-z|+|y-w| \le 20\sqrt{d}2^{-n}.$$

Noting that $\mathscr{E}(f)=\mathscr{E}_0(f)$ on D and $\mathscr{E}_0(1)=1$ on D, we get

$$|\mathscr{E}(f)(y)-f(w)| \le c_1\sum_j \frac{\chi_{Q_j^*}(y)}{l_j^\beta}\int_{B(a^{(j)},\eta l_j)}|f(z)-f(w)|d\mu(z)$$
$$\le c_2(2^{-n})^\alpha\left(\int\frac{|f(z)-f(w)|^p}{|z-w|^{\beta+p\alpha}}d\mu(z)\right)^{1/p}.$$

Lemma 3.1 yields

$$\int_{\partial Q \cap \partial A_n} d\sigma_n(y)\int_{\{|y-w|\le b2^{-n}\}\cap\partial D}|\mathscr{E}(f)(y)-f(w)|^p d\mu(w)$$
$$\le c_3(2^{-n})^{\alpha p+d-1}\int_{\{|a_0-w|<b'2^{-n}\}\cap\partial D}d\mu(w)\int\frac{|f(z)-f(w)|^p}{|z-w|^{\beta+p\alpha}}d\mu(z),$$

where a_0 is a boundary point corresponding to Q in §2 and b' is a constant independent of n. We saw in [W3, Lemma 2.1] that each $z \in \partial D$ is contained in at most N numbers of the family $\{B(a^{(i)}, b'l_i) \cap \partial D\}_i$ corresponding to the n-cubes $Q_i \in \mathscr{V}(D)$, where N is a natural number independent of n. Using this, we get

$$\int_{\partial A_n} d\sigma_n(y) \int_{\{|y-w|\le b2^{-n}\}\cap\partial D} |\mathscr{E}(f)(y) - f(w)|^p d\mu(w)$$

$$\le c_4(2^{-n})^{\alpha p+d-1} \|f\|_{p,\alpha}^p.$$

This shows that $\mathscr{E}(f)$ satisfies (1.6) for $\tau = \alpha$.

Similarly we also obtain the estimate (1.6) in which ∂A_n is replaced with $\partial B_n \cap B(0, 2R)$.

We next see that (1.7) holds for $\mathscr{E}(f)$. Noting that $1 - \alpha - (d - \beta)/p < (d - \beta)/q$, we choose $\lambda > 0$ satisfying $1 - \alpha - (d - \beta)/p < \lambda < (d - \beta)/q$. Since $(\alpha - 1)p + d - \beta + \lambda p > 0$ and $\lambda q < d - \beta$, Lemmas B, C imply

$$\int_{\mathbf{R}^d\setminus\partial D} |\nabla\mathscr{E}(f)(y)|dy$$

$$\le \left(\int_{B(0,2R)\setminus\partial D} |\nabla\mathscr{E}(f)(y)|^p \delta(y)^{\lambda p} dy\right)^{1/p} \left(\int_{B(0,2R)\setminus\partial D} \delta(y)^{-\lambda q} dy\right)^{1/q} < \infty,$$

which gives (1.7) for $\mathscr{E}(f)$ and $\tau = \alpha$. Therefore we see that $\mathscr{E}(f) \in \mathscr{U}_\alpha^p(\mathbf{R}^d)$. □

Let $q > 1$, $g_1 \in L^q(\mu)$ and $g_2 \in L^q(\mu \times \mu)$. We define, for $y \in \mathbf{R}^d$,

$$S_1 g_1(y) = \begin{cases} -\int N(x - y) g_1(x) d\mu(x) & \text{if it is well-defined} \\ 0 & \text{otherwise} \end{cases}$$

and

$$S_2 g_2(y) = \begin{cases} -\iint \dfrac{N(x-y) - N(z-y)}{|x-z|^{(\beta/p)+\alpha}} g_2(x,z) d\mu(x) d\mu(z) & \text{if it is well-defined} \\ 0 & \text{otherwise,} \end{cases}$$

where $p = q/(q - 1)$ and $1 > \alpha > \beta - (d - 1) \ge 0$.

LEMMA 4.2. *Let* $1 < p \le 2$, $q = p/(p - 1)$ *and* $1 - (d - \beta)/p > \alpha > \beta - (d - 1) \ge 0$. *Then, for* $g_1 \in L^q(\mu)$, $g_2 \in L^q(\mu \times \mu)$, $S_1 g_1$, $S_2 g_2 \in \mathscr{U}_\alpha^p(\mathbf{R}^d) \cap \mathscr{U}_\alpha^q(\mathbf{R}^d)$ *if* $d \ge 3$ *and* $S_1 g_1$, $S_2 g_2 \in \mathscr{U}_\alpha^p(\bar{D}) \cap \mathscr{U}_\alpha^q(\bar{D})$ *if* $d = 2$.

PROOF. We will prove only that $S_2 g_2 \in \mathscr{U}_\alpha^q(\mathbf{R}^d)$ in the case $d \ge 3$, which means $S_2 g_2 \in \mathscr{U}_\alpha^p(\mathbf{R}^d)$ for $p \le q$. Let $y \in \partial A_n$, $w \in \partial D$ and $|y - w| \le b2^{-n}$. Further let ε be a sufficiently small positive number. With the aid of Lemma F we write

$$
\begin{aligned}
&|S_2g_2(y) - S_2g_2(w)| \\
&\quad \le \iint \frac{|\sum_{j=1}^{2}(-1)^j(N(x_j - y) - N(x_j - w))|}{|x_1 - x_2|^{\beta/p+\alpha}} |g_2(x_1, x_2)| d\mu(x_1) d\mu(x_2) \\
&\quad \le c_1 |y - w|^{\alpha} \iint |x_1 - x_2|^{-\beta/p+\varepsilon} |g_2(x_1, x_2)| \\
&\qquad \times \sum_{j=1}^{2} (|x_j - y|^{2-d-2\alpha-\varepsilon} + |x_j - w|^{2-d-2\alpha-\varepsilon}) d\mu(x_1) d\mu(x_2) \\
&\quad \equiv \sum_{j=1}^{2} (I_{j1} + I_{j2}).
\end{aligned}
$$

Then, by Lemma D,

$$
\begin{aligned}
I_{11} &\le c_2 |y - w|^{\alpha} \left(\iint |x_1 - x_2|^{-\beta+\varepsilon p} |x_1 - y|^{-\beta+\varepsilon p} d\mu(x_1) d\mu(x_2) \right)^{1/p} \\
&\quad \times \left(\iint |g_2(x_1, x_2)|^q |x_1 - y|^{q(2-d-2\alpha+\beta/p-2\varepsilon)} d\mu(x_1) d\mu(x_2) \right)^{1/q} \\
&\le c_3 |y - w|^{\alpha} \left(\iint |g_2(x_1, x_2)|^q |x_1 - y|^{q(2-d-2\alpha+\beta/p-2\varepsilon)} d\mu(x_1) d\mu(x_2) \right)^{1/q}
\end{aligned}
$$

The assumptions $1 - (d - \beta)/p > \alpha$ and $p \le 2$ imply $1 - (d - \beta)/2 > \alpha$ and hence $q(2 - d - 2\alpha + \beta/p) > -\beta$. So we can pick $\varepsilon > 0$ satisfying $q(2 - d - 2\alpha + \beta/p - 2\varepsilon) > -\beta$. Then, together with Lemma 3.2 and Lemma D,

$$
\begin{aligned}
&\int d\mu(w) \int_{\{|y-z| \le b2^{-n}\} \cap \partial A_n} I_{11}^q d\sigma_n(y) \\
&\quad \le c_4 (2^{-n})^{q\alpha} \iint |g_2(x_1, x_2)|^q d\mu(x_1) d\mu(x_2) \\
&\qquad \times \int_{\partial A_n} |x_1 - y|^{q(2-d-2\alpha+\beta/p-2\varepsilon)} d\sigma_n(y) \int_{\{|y-w| \le b2^{-n}\} \cap \partial D} d\mu(w) \\
&\quad \le c_5 (2^{-n})^{q\alpha+d-1} \|g_2\|_q^q,
\end{aligned}
$$

where

$$
\|g_2\|_q = \left(\iint |g_2(x_1, x_2)|^q d\mu(x_1) d\mu(x_2) \right)^{1/q}.
$$

Similarly we can estimate I_{21}.

We next estimate I_{12}. Since

$$I_{12} \le c_6|y-w|^{\alpha}\left(\iint |g_2(x_1,x_2)|^q|x_1-w|^{q(2-d-2\alpha+\beta/p-2\varepsilon)}d\mu(x_1)d\mu(x_2)\right)^{1/q},$$

we get, by Lemma 3.1 and Lemma D,

$$\begin{aligned}&\int d\mu(w)\int_{\{|y-w|\le b2^{-n}\}\cap\partial A_n} I_{12}^q d\sigma_n(y)\\ &\le c_7(2^{-n})^{q\alpha}\iint |g_2(x_1,x_2)|^q d\mu(x_1)d\mu(x_2)\\ &\quad\times\int |x_1-w|^{q(2-d-2\alpha+\beta/p-2\varepsilon)}d\mu(w)\int_{\{|y-w|\le b2^{-n}\}\cap\partial A_n} d\sigma_n(y)\\ &\le c_8(2^{-n})^{q\alpha+d-1}\|g_2\|_q^q.\end{aligned}$$

Similarly we obtain the same estimate for I_{22}.
Therefore we have

$$\begin{aligned}&\int d\mu(w)\int_{\{|y-w|\le b2^{-n}\}\cap\partial A_n}|S_2g_2(y)-S_2g_2(w)|^q d\sigma_n(y)\\ &\qquad\le c_9(2^{-n})^{q\alpha+d-1}\|g_2\|_q^q.\end{aligned}$$

Similarly we can also get

$$\begin{aligned}&\int d\mu(w)\int_{\{|y-w|\le b2^{-n}\}\cap\partial B_n\cap B(0,2R)}|S_2g_2(y)-S_2g_2(w)|^q d\sigma_n(y)\\ &\qquad\le c_{10}(2^{-n})^{q\alpha+d-1}\|g_2\|_q^q.\end{aligned}$$

We next estimate the volume integral of the gradient of S_2g_2. Let $y\in B(0,2R)\backslash\partial D$. We write, for a sufficiently small $\varepsilon>0$,

$$\begin{aligned}&\left|\frac{\partial S_2g_2}{\partial y_j}(y)\right|\\ &\quad\le c_{11}\iint |x-z|^{-\beta/p+\varepsilon}(|x-y|^{1-d-\alpha-\varepsilon}+|z-y|^{1-d-\alpha-\varepsilon})|g_2(x,z)|d\mu(x)\\ &\quad\equiv I_3+I_4.\end{aligned}$$

Then, by Lemma D,

$$I_3 \leq c_{12}\left(\iint |x-z|^{-\beta+p\varepsilon}|x-y|^{-\beta+p\varepsilon}d\mu(x)d\mu(z)\right)^{1/p}$$

$$\times \left(\iint |g_2(x,z)|^q|x-y|^{(1-d-\alpha-2\varepsilon+\beta/p)q}d\mu(x)d\mu(z)\right)^{1/q}$$

$$\leq c_{13}\left(\iint |g_2(x,z)|^q|x-y|^{(1-d-\alpha-2\varepsilon+\beta/p)q}d\mu(x)d\mu(z)\right)^{1/q}.$$

Noting that $(1-d-\alpha+\beta/p)q > -d$, we choose $\varepsilon > 0$ satisfying $(1-d-\alpha-2\varepsilon+\beta/p)q > -d$. Then

$$\int_{B(0,2R)\setminus\partial D} I_3^q dy$$

$$\leq c_{14}\iint |g_2(x,z)|^q d\mu(x)d\mu(z)\int_{B(0,2R)} |x-y|^{(1-d-\alpha-2\varepsilon+\beta/p)q}dy$$

$$\leq c_{15}\|g_2\|_q^q.$$

Since the same estimate for I_4 is obtained, we have

$$\int_{B(0,2R)\setminus\partial D} |\nabla S_2 g_2(y)|^q dy < \infty \tag{4.1}$$

and hence

$$\int_{B(0,2R)\setminus\partial D} |\nabla S_2 g_2(y)| dy < \infty.$$

Further it is easy to see that

$$\int_{\mathbf{R}^d\setminus B(0,2R)} |\nabla S_2 g_2(y)| dy < \infty.$$

Since $S_2 g_2$ is a C^1-function in $\mathbf{R}^d\setminus\partial D$, we conclude that $S_2 g_2 \in \mathscr{U}_\alpha^q(\mathbf{R}^d)$. □

5. Proof of Theorem 3

In this section we give the proof of Theorem 3. We prepare the following lemma.

LEMMA 5.1. *Let $q > 1$, $0 < \alpha < 1$ and $g_1 \in L^q(\mu)$, $g_2 \in L^q(\mu\times\mu)$. Then*

$$\lim_{x\to z,\, x\in\Gamma_\tau(z)} S_j g_j(x) = S_j g_j(z) \quad (j = 1,2) \tag{5.1}$$

and

$$\lim_{x \to z, x \in \Gamma_\tau^e(z)} S_j g_j(x) = S_j g_j(z) \quad (j = 1, 2) \tag{5.2}$$

for μ-*a.e.* $z \in \partial D$.

PROOF. Let $z \in \partial D$ and $x \in \Gamma_\tau(z) \cup \Gamma_\tau^e(z)$. Put

$$A = \{y \in \partial D; |y - z| \le 2|x - z|\}$$

and

$$B = \{y \in \partial D; |y - z| > 2|x - z|\}.$$

If $y \in A$, then

$$|x - y| \ge \delta(x) \ge \frac{|x - z|}{1 + \tau} \ge \frac{|y - z|}{2(1 + \tau)}.$$

If $y \in B$, then

$$|x - y| \ge |y - z| - |z - x| > \frac{|y - z|}{2}.$$

From these we get

(5.3) $|x - y| \ge c_1|y - z|$ for all $x \in \Gamma_\tau(z) \cup \Gamma_\tau^e(z)$ and for all $y \in \partial D$.

So

$$|S_1 g_1(x)| \le c_2 \int N(z - y)|g_1(y)|d\mu(y).$$

With the aid of Lemma E we also get

$$|S_2 g_2(x)|$$

$$\le c_3 \iint |y_1 - y_2|^{-\beta/p+\varepsilon} \sum_{j=1}^{2} |z - y_j|^{-\alpha-\varepsilon} N(z - y_j)|g_2(y_1, y_2)|d\mu(y_1)d\mu(y_2)$$

for a sufficiently small $\varepsilon > 0$ satisfying $2 - d - \alpha - 2\varepsilon > -\beta$. Therefore, by Lemma D, we get

$$\int \sup_{x \in \Gamma_\tau(z) \cup \Gamma_\tau^e(z)} |S_1 g_1(x)|^q d\mu(z) \le c_4 \|g_1\|_q^q \tag{5.4}$$

and

$$\int \sup_{x \in \Gamma_\tau(z) \cup \Gamma_\tau^e(z)} |S_2 g_2(x)|^q d\mu(z) \le c_5 \|g_2\|_q^q. \tag{5.5}$$

Especially if g_1 and g_2 are bounded on ∂D and $\partial D \times \partial D$, respectively, we get

$$(5.6) \qquad \lim_{x \to z, x \in \Gamma_\tau(z) \cup \Gamma_\tau^e(z)} S_j g_j(x) = S_j g_j(z) \qquad (j = 1, 2)$$

for every $z \in \partial D$. From (5.4), (5.5) and (5.2) we deduce (5.1) by the usual method. □

LEMMA 5.2. *Let* $g_1 \in L^q(\mu)$ *and* $g_2 \in L^q(\mu \times \mu)$. *Under the same conditions as in Lemma* 4.2

$$(5.7) \quad \lim_{x \to z, x \in \Gamma_\tau^e(z)} \int_D \langle \nabla(S_j g_j)(y), \nabla_y N(x-y) \rangle dy = \int_D \langle \nabla(S_j g_j)(y), \nabla_y N(z-y) \rangle dy$$

and

$$\lim_{x \to z, x \in \Gamma_\tau(z)} \int_{R^d \setminus \bar{D}} \langle \nabla(S_j g_j)(y), \nabla_y N(x-y) \rangle dy = \int_{R^d \setminus \bar{D}} \langle \nabla(S_j g_j)(y), \nabla_y N(z-y) \rangle dy$$

for μ*-a.e.* $z \in \partial D$ *and for* $j = 1, 2$.

PROOF. We will show (5.7) only for $S_2 g_2$. Let $z \in \partial D$, $x \in \Gamma_\tau^e(z)$ and $y \in D$. Writing, for a sufficiently small $\varepsilon > 0$,

$$|\nabla S_2 g_2(y)| \le c_1 \iint |x_1 - x_2|^{-\beta/p+\varepsilon} \left(\sum_{j=1}^2 |x_j - y|^{1-d-\alpha-\varepsilon} \right) |g_2(x_1, x_2)| d\mu(x_1) d\mu(x_2)$$
$$\equiv I_1 + I_2,$$

we have, by Lemma D,

$$I_j \le c_1 \left(\iint |x_1 - x_2|^{-\beta+\varepsilon p} |x_j - y|^{-\beta+\varepsilon p} d\mu(x_1) d\mu(x_2) \right)^{1/p}$$
$$\times \left(\iint |g_2(x_1, x_2)|^q |x_j - y|^{(1-d-\alpha+\beta/p-2\varepsilon)q} d\mu(x_1) d\mu(x_2) \right)^{1/q}$$
$$\le c_2 \left(\iint |g_2(x_1, x_2)|^q |x_j - y|^{(1-d-\alpha+\beta/p-2\varepsilon)q} d\mu(x_1) d\mu(x_2) \right)^{1/q},$$

whence, together with (5.3),

$$\left| \int_D \langle \nabla_y S_2 g_2(y), \nabla_y N(x-y) \rangle dy \right| \le c_3 \int_D |\nabla_y S_2 g_2(y)| \, |z-y|^{1-d} dy$$
$$\le c_4 \sum_{j=1}^2 \int_D I_j |z-y|^{1-d} dy \equiv J_1 + J_2.$$

Noting that $(1-d+\beta/q)p+d=(1-(d-\beta)/q)p>0$ and $(1-d-\alpha+\beta/p)q+d=(1-(d-\beta)/p-\alpha)q>0$, we can choose $\varepsilon>0$ satisfying $(1-d+\beta/q-\varepsilon)p+d>0$ and $(1-d-\alpha+\beta/p-2\varepsilon)q+d>0$. Since

$$J_j \le c_4\left(\iint_D I_j^q|z-y|^{-\beta+\varepsilon q}dy\right)^{1/q}\left(\int_D |z-y|^{(1-d+\beta/q-\varepsilon)p}dy\right)^{1/p}$$
$$\le c_5\left(\iint_D I_j^q|z-y|^{-\beta+\varepsilon q}dy\right)^{1/q},$$

we get, by Lemma D,

$$\int J_j^q d\mu(z) \le c_6 \iint |g_2(x_1,x_2)|^q d\mu(x_1)d\mu(x_2)$$
$$\int_D |x_j-y|^{(1-d-\alpha+\beta/p-2\varepsilon)q}dy \int_D |z-y|^{-\beta+\varepsilon q}d\mu(z) \le c_7\|g_2\|_q^q,$$

whence

$$(5.8)\qquad \int\left(\sup_{x\in\Gamma_\tau^\varepsilon(z)}\left|\int_D \langle\nabla_y S_2g_2(y), \nabla_y N(x-y)\rangle dy\right|^q\right)d\mu(z) \le c_8\|g_2\|_q^q.$$

We next consider a bounded continuous function g_2 on $\partial D\times\partial D$. We claim that (5.7) for S_2g_2 holds for every $z\in\partial D$.

To show this, let $y\in D$. Then

$$|\nabla S_2g_2(y)|$$
$$\le c_9\|g_2\|_\infty \iint |x_1-x_2|^{-\beta/p+\varepsilon}\left(\sum_{j=1}^2 |x_j-y|^{1-d-\alpha-\varepsilon}\right)d\mu(x_1)d\mu(x_2),$$

where $\|g_2\|_\infty=\sup\{|g_2(x_1,x_2)|; x_1,x_2\in\partial D\}$.

Choose $\lambda>0$ satisfying $d-\beta>\lambda>\alpha-(\beta-d+1)$. Since $(1-d-\alpha+\beta/q+\lambda)p+\beta=(1-d-\alpha+\beta+\lambda)p>0$, we pick $\varepsilon>0$ satisfying $(1-d-\alpha+\beta/q+\lambda-2\varepsilon)p>-\beta$. Then, by Lemma D,

$$\left|\frac{\partial S_2g_2}{\partial y_k}(y)\right|\delta(y)^\lambda$$
$$\le c_{10}\|g_2\|_\infty\sum_{j=1}^2\left(\iint |x_1-x_2|^{-\beta+\varepsilon p}|x_j-y|^{(1-d-\alpha+\beta/q-2\varepsilon+\lambda)p}d\mu(x_1)d\mu(x_2)\right)^{1/p}$$
$$\times\left(\iint |x_j-y|^{-\beta+q\varepsilon}d\mu(x_1)d\mu(x_2)\right)^{1/q} \le c_{11}\|g_2\|_\infty,$$

Let $w \in \Gamma_\tau^e(z)$. Noting that

$$|w - y| \geq c_{12}|z - y| \qquad \text{for all } y \in D$$

and using Lemmas E and C, we get

$$\left|\int_D \langle \nabla_y S_2 g_2(y), \nabla_y(N(w-y) - N(z-y))\rangle dy\right| \tag{5.9}$$
$$\leq c_{13}|w - z|^{\varepsilon_1}\|g_2\|_\infty \int_D \delta(y)^{-\lambda}|z - y|^{1-d-\varepsilon_1} dy$$
$$\leq c_{14}|w - z|^{\varepsilon_1}\|g_2\|_\infty,$$

where we picked $\varepsilon_1 > 0$ satisfying $1 - \lambda - \varepsilon_1 > 0$. Thus we see that the claim is true.

Using the claim and (5.8), we can show (5.7) for $S_2 g_2$. It is easy to show (5.7) for $S_1 g_1$. □

LEMMA 5.3. *Let $g_1 \in L^q(\mu)$, $g_2 \in L^q(\mu \times \mu)$ and $\{\phi_t\}_{0<t<1}$ be a mollifier on $\mathbf{R}^d$ such that supp $\phi_t \subset B(0, t)$. Under the same assumtions as in Lemma* 4.2 *we get*

$$\int_{\delta(y)<2t}\left|\frac{\partial}{\partial y_i}\phi_t * S_j g_j\right|^q dy \leq c\|g_j\|_q^q \tag{5.10}$$

for $j = 1, 2$, where c is a constant independent of t and g_j.

PROOF. We will prove (5.10) only for $j = 2$. Suppose $\delta(y) < 2t$ and put

$$F_1 := \{(v, w) \in \partial D \times \partial D; |y - v| \leq 4t, |y - w| \leq 4t\},$$
$$F_2 := \{(v, w) \in \partial D \times \partial D; |y - v| > 4t, |y - w| \leq 2t\},$$
$$F_3 := \{(v, w) \in \partial D \times \partial D; |y - v| \leq 2t, |y - w| > 4t\},$$
$$F_4 := \{(v, w) \in \partial D \times \partial D; |y - v| > 2t, |y - w| > 2t\}.$$

We write, for $x \in B(y, t)$,

$$S_2 g_2(x) = -\iint_{(\partial D\times\partial D)\backslash F_4} \frac{N(v-x) - N(w-x)}{|v-w|^{(\beta/p)+\alpha}} g_2(v, w)d\mu(v)d\mu(w)$$
$$-\iint_{F_4} \frac{N(v-x) - N(w-x)}{|v-w|^{(\beta/p)+\alpha}} g_2(v, w)d\mu(v)d\mu(w) \equiv J_1(x) + J_2(x).$$

Noting that $(\partial D \times \partial D)\backslash F_4 \subset F_1 \cup F_2 \cup F_3$, we write again

$$\begin{aligned}\left|\frac{\partial}{\partial y_i}(\phi_t * J_1)(y)\right| &= \left|\int \frac{\partial \phi_t}{\partial y_i}(y-x)J_1(x)\right|dx \\ &\le c_1 t^{-d-1}\sum_{k=1}^{3}\int_{|y-x|<t} dx \\ &\quad \times \int_{F_k} \frac{|N(v-x)-N(w-x)|}{|v-w|^{(\beta/p)+\alpha}}|g_2(v,w)|d\mu(v)d\mu(w) \\ &\equiv J_{11}(y)+J_{12}(y)+J_{13}(y).\end{aligned}$$

Let us first estimate J_{11}. Lemma E implies

$$\begin{aligned}&J_{11}(y) \\ &\le c_2 t^{-d-1}\int_{|y-x|<t} dx \int_{F_1} \frac{|v-x|^{2-d-\alpha-\varepsilon}+|w-x|^{2-d-\alpha-\varepsilon}}{|v-w|^{(\beta/p)-\varepsilon}}|g_2(v,w)|d\mu(v)d\mu(w) \\ &\equiv J_{111}(y)+J_{112}(y).\end{aligned}$$

Since $q(2-d-\alpha+\beta/p)+d>0$, we choose $\varepsilon>0$ satisfying $q(2-d-\alpha+(\beta/p)-2\varepsilon)>0$. Then

$$\begin{aligned}J_{111}(y) &\le c_2 t^{-d-1}\left(\int_{|y-x|<t} dx \iint_{F_1} |v-w|^{-\beta+\varepsilon p}|v-x|^{-\beta+\varepsilon p}d\mu(v)d\mu(w)\right)^{1/p} \\ &\times \left(\int_{|y-x|<t} dx \iint_{F_1} |v-x|^{q(2-d-\alpha+\beta/p-2\varepsilon)}|g_2(v,w)|^q d\mu(v)d\mu(w)\right)^{1/q}.\end{aligned}$$

Since a similar estimate for J_{112} is also obtained, Lemmas C and D lead to

$$\int_{\delta(y)<2t} J_{11}(y)^q dy \le c_3 t^{q(1-\alpha-(d-\beta)/p)}\|g_2\|_q^q.$$

We next estimate J_{12}. Noting that $(v,w)\in F_2$ and $|x-y|<t$ imply $|v-w|>2t$ and $|v-x|>3t\ge|w-x|$. Using Lemma E, we have

$$\begin{aligned}J_{12} &\le c_4 t^{-d-1}\int_{|y-x|<t} dx \int_{F_2} \frac{|v-x|^{2-d-\alpha+\varepsilon}+|w-x|^{2-d-\alpha+\varepsilon}}{|v-w|^{(\beta/p)+\varepsilon}}|g_2(v,w)|d\mu(v)d\mu(w) \\ &\le c_5 t^{-d-1}\int_{|y-x|<t} dx \int_{F_2} |v-w|^{-\beta/p-\varepsilon}|w-x|^{2-d-\alpha+\varepsilon}|g_2(v,w)|d\mu(v)d\mu(w) \\ &\le c_5 t^{-d-1}\left(\int_{|y-x|<t} dx \iint_{F_2} |v-w|^{-\beta-\varepsilon p}|w-x|^{-\beta+\varepsilon p}d\mu(v)d\mu(w)\right)^{1/p} \\ &\times \left(\int_{|y-x|<t} dx \iint_{F_2} |w-x|^{q(2-d-\alpha+\beta/p)}|g_2(v,w)|^q d\mu(v)d\mu(w)\right)^{1/q}.\end{aligned}$$

With the aid of Lemmas C and D we conclude

$$(5.11) \qquad \int_{\delta(y)\leq 2t} J_{12}(y)^q dy \leq c_6 t^{q(1-\alpha-(d-\beta)/p)} \|g_2\|_q^q.$$

We also obtain the estimate (5.11) for J_{13} by exchanging the roles of v and w.

We finally estimate J_2. Noting that

$$\frac{\partial}{\partial y_i}(\phi_t * J_2)(y) = \left(\phi_t * \left(\frac{\partial}{\partial y_i} J_2\right)\right)(y),$$

we get

$$\int_{\delta(y)\leq 3t} \left|\frac{\partial}{\partial y_i}(\phi_t * J_2)(y)\right|^q dy \leq c_7 \int \left(\mathscr{M}\left(\frac{\partial}{\partial y_i} J_2\right)\right)^q dy \leq c_8 \|g_2\|_q^q$$

by the same method as in the proof of (4.1), where $\mathscr{M}(f)$ is the Hardy-Littlewood maximal function of f on $\mathbf{R}^d$. Thus we have the conclusion for $S_2 g_2$. □

Proof of Theorem 3. We define, for $\psi = (g_1, g_2)$,

$$S\psi = S_1 g_1 + S_2 g_2.$$

With the aid of (4.1) we also get $\int_{B(0,2R)} |\nabla S\psi|^2 dy < \infty$ and hence $\int_{\mathbf{R}^d} |\nabla S\psi|^2 dy < \infty$ because of $q \geq 2$. Let $\{\phi_t\}_{t>0}$ be a mollifier on $\mathbf{R}^d$ such that supp $\phi_t \subset B(0,t)$ and set $h_t := \phi_t * S\psi$. Then h_t is a C^1-function on $\mathbf{R}^d$ and

$$\frac{\partial h_t}{\partial y_i}(y) = \left(\phi_t * \frac{\partial S\psi}{\partial y_i}\right)(y) \qquad \text{for every } y \text{ with } \delta(y) > 2t.$$

Since h_t is a Lipschitz function on $\mathbf{R}^d$, $h_t|(\mathbf{R}^d \backslash D) \in \mathscr{U}_\alpha^p(\mathbf{R}^d \backslash D)$. Using Theorem 2 and Lemma 5.2 we get

$$\int_{\mathbf{R}^d\backslash\bar{D}} \langle \nabla h_t(y), \nabla_y N(z-y)\rangle dy = \int_{\mathbf{R}^d\backslash\bar{D}} \langle \nabla \mathscr{E}(h_t|\partial D)(y), \nabla_y N(z-y)\rangle dy$$

for every $z \in \partial D$ and $t > 0$. Since $\langle\!\langle Kf + f/2, \psi \rangle\!\rangle = 0$ for every Lipschitz function f, we have

$$\int_{\mathbf{R}^d\backslash\bar{D}} \langle \nabla h_t(y), \nabla_y S\psi(y)\rangle dy = \int_{\mathbf{R}^d\backslash\bar{D}} \langle \nabla \mathscr{E}(h_t|\partial D)(y), \nabla_y S\psi(y)\rangle dy$$

$$= \left\langle\!\!\left\langle Kh_t + \frac{h_t}{2}, \psi \right\rangle\!\!\right\rangle = 0.$$

On the other hand we note that

$$|\nabla h_t(y)| \le \sum_{i=1}^{d} \left| \left(\phi_t * \frac{\partial S\psi}{\partial y_i} \right)(y) \right| \le c_1 \sum_{i=1}^{d} \mathscr{M}\left(\frac{\partial S\psi}{\partial y_i} \right)(y)$$

for all y with $\delta(y) > 2t$. Since

$$\int_{\mathbf{R}^d} \mathscr{M}\left(\frac{\partial S\psi}{\partial y_i} \right)^2 dy \le c_2 \int_{\mathbf{R}^d} |\nabla S\psi|^2 dy < \infty$$

and

$$\lim_{t \to 0} \langle \nabla h_t(y), \nabla_y S\psi(y) \rangle = |\nabla S\psi(y)|^2$$

for every $y \in \mathbf{R}^d \backslash \partial D$, we have

$$\int_{\mathbf{R}^d \backslash \bar{D}} |\nabla S\psi(y)|^2 dy = \lim_{t \to 0} \int_{\mathbf{R}^d \backslash \bar{D}} \langle \nabla h_t(y), \nabla_y S\psi(y) \rangle dy = 0.$$

From this we deduce

$$S\psi = const. \text{ on } \mathbf{R}^d \backslash \bar{D}.$$

Noting that $\lim_{y \to \infty} S\psi(y) = 0$, we have

$$S\psi = 0 \qquad \text{in } \mathbf{R}^d \backslash \bar{D}.$$

Lemma 5.1 yields

$$S\psi(z) = \lim_{y \to z,\, y \in \Gamma_\tau^e(z)} S\psi(y) = 0 \qquad \text{for } \mu\text{-a.e. } z \in \partial D.$$

Since $S\psi \in \mathscr{U}_\alpha^p(\bar{D})$ by Lemma 4.2, we have, by Theorem 1,

$$\text{(5.12)} \qquad \int_D \langle \nabla S\psi(y), \nabla N(x-y) \rangle dy = \int_D \langle \nabla \mathscr{E}(S\psi|\partial D), \nabla N(x-y) \rangle dy = 0$$

for all $x \in \mathbf{R}^d \backslash \bar{D}$. Using Lemma 5.2 and (1.11), we see that (5.12) holds for μ-a.e. $x \in \partial D$. Therefore we have

$$\int_D |\nabla S\psi|^2 dy = 0,$$

whence $S\psi = \text{const.}$ in D. Hence, by (1.11),

$$\left\langle\!\!\left\langle Kf - \frac{f}{2}, \psi \right\rangle\!\!\right\rangle = -\int_D \langle \nabla \mathscr{E}(f)(y), \nabla_y S\psi(y) \rangle dy = 0$$

for every Lipschitz function f on ∂D. Since, by (1.10),

$$\left\langle\!\!\left\langle Kf + \frac{f}{2}, \psi \right\rangle\!\!\right\rangle = \int_{\mathbf{R}^d \setminus \bar{D}} \langle \nabla \mathscr{E}(f)(y), \nabla_y S\psi(y) \rangle dy = 0,$$

we get

$$\ll f, \psi \gg = \left\langle\!\!\left\langle Kf + \frac{f}{2}, \psi \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle Kf - \frac{f}{2}, \psi \right\rangle\!\!\right\rangle = 0$$

for every Lipschitz function f on ∂D. Since the family of the Lipschitz functions is dense in $\Lambda_\alpha^p(\partial D)$, we conclude that $\psi = 0$. □

Similarly we can also prove the following theorem.

THEOREM 4. *Assume that D, p and α satisfy the same assumptions as in Theorem 3. If $\psi \in \Lambda_\alpha^p(\partial D)'$, $\ll 1, \psi \gg = 0$ and $\ll Kf - f/2, \psi \gg = 0$ for every Lipschitz function f on ∂D, then $\psi = 0$.*

References

[H] J. E. Hutchinson, Fractals and selfsimilarity, Indiana Univ. Math. J. **30** (1981), 713–747.

[HN] J. Harrison and A. Norton, Geometric integration on fractal curves in the plane, Indiana Univ. Math. J. **40** (1991), 567–594.

[JW] A. Jonsson and H. Wallin, A Whitney extension theorem in L_p and Besov spaces, Ann. Inst. Fourier, Grenoble **28**, 1 (1978), 139–192.

[S] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton- New Jersey, 1970.

[W1] H. Watanabe, The double layer potentials for a bounded domain with fractal boundary, Potential Theory-ICPT94, 463–471, Walter de Gruyter, Berlin-New York, 1996.

[W2] H. Watanabe, Layer pontetials for a bounded domain with fractal boundary, Sci. Bull. Josai Univ. **1** (1997), 33–44.

[W3] H. Watanabe, Double layer potentials of functions in a Besov space for a bounded domain with fractal boundary, Proceedings of the Fifth International Colloquim on Complex Analysis, 337–343, Beijing, 1997.

Department of Mathematics
Faculty of Science
Ochanomizu University
Bunkyou-ku, Tokyo, 112-8610 Japan