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Biharmonic extensions in Riemannian manifolds

I. BAJUNAID and V. ANANDAM

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ABSTRACT. We give a characterization of the hyperbolic Riemannian manifolds R in which for any biharmonic function b outside a compact set, there exists a biharmonic function B in R such that $B - b$ is bounded outside a compact set.

1. Introduction

Let R be a hyperbolic Riemannian manifold. It is known that given a harmonic function h outside a compact set, there always exists a harmonic function H in R such that $H - h$ is bounded outside a compact set. One method of proof of this is via the principal functions [11] making use of the potentials >0 in R, a potential in R being a superharmonic function $u \geq 0$ in R such that if h is a harmonic function satisfying $0 \leq h \leq u$, then $h \equiv 0$. To solve a similar problem for the biharmonic functions in R, Chung [8] uses a variant of these principal functions. But the result is not satisfactory.

We prove in this note that a biharmonic extension in R is possible if and only if R satisfies the following condition: There exist potentials $p > 0$ and $q > 0$ in R such that $\Delta q = p$ where $\Delta = -\sum_{i,j} g^{ij} \dfrac{\partial}{\partial x_i} \dfrac{\partial}{\partial x_j}$ is the Laplace-Beltrami operator, and q is bounded outside a compact set. We remark that this condition is verified in $\mathbf{R}^n$, $n \geq 5$.

The proof of this biharmonic extension depends on a lemma giving the representation of a biharmonic function defined outside a compact set in R by means of the difference of some special potentials in R.

2. Preliminaries

Let R be an oriented Riemannian manifold of dimension $n \geq 2$ with local parameters $x = (x^1, \ldots, x^n)$ and a C^∞ metric tensor g_{ij} such that $g_{ij}x^i x^j$ is positive definite. We denote the volume element by $dx = \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n$;

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$\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator which, acting on a function f, gives $\Delta f = -\text{div grad } f$. R is said to be hyperbolic or parabolic depending on the existence or the non-existence of a potential >0 on R respectively.

LEMMA 1. *Given any locally dx-integrable function f on an open set ω in R, there exists a δ-superharmonic function g on ω (that is, the difference of two superharmonic functions) such that $\Delta g = f$; in this case, g is said to be generated by f.*

PROOF. This can be deduced from Theorem 4.2 [4].

For an outerregular compact set K in R (that is, if ω is a relatively compact open set containing K, every point of ∂K is regular for the Dirichlet solution in $\omega \backslash K$), let $B_K f$ stand for the Dirichlet solution in $R \backslash K$ with boundary values f on ∂K and 0 at the point at infinity of R.

Terminology: The term "near infinity" is used to mean a set that is the complement of a compact set in R.

DEFINITION 2. *A biharmonic function b (that is, $\Delta^2 b = 0$) defined near infinity in R is said to be regular at infinity, if there exists an outerregular compact set K such that $B_K(\Delta b) = \Delta b$ in $R \backslash K$.*

REMARK 1. a) *In a hyperbolic manifold R, $B_K h = h$ in $R \backslash K$ if and only if there exists a Green potential p in R such that $|h| \leq p$ near infinity; and in a parabolic manifold $B_K h = h$ if and only if h is bounded near infinity.*

b) *In a hyperbolic manifold R, given a biharmonic function b near infinity, it is always possible (using Lemma* 1 *above and Remark* 23 [2]*) to find a biharmonic function B in R such that $(B - b)$ is regular at infinity; however $(B - b)$ may not be bounded near infinity (see Theorem* 13*). For example, if $b(x) = |x|$ in $|x| \geq 1$ in $\mathbf{R}^3$, we can take $B(x) \equiv 0$. In view of this example, the following proposition is interesting.*

PROPOSITION 3. *Let b be a biharmonic function defined near infinity in $\mathbf{R}^n$, $n \geq 5$. If b is regular at infinity, then there exists a harmonic function H in $\mathbf{R}^n$ such that $\lim_{|x| \to \infty} [b(x) - H(x)] = 0$.*

PROOF. Since b is regular at infinity, for some a if $K = \{x : |x| \leq a\}$, $B_a(\Delta b) = B_K(\Delta b) = \Delta b$ in $|x| > a$. Since $B_K 1 = \left(\frac{a}{|x|}\right)^{n-2}$ in $|x| > a$, if $m = \max_{\partial K} |\Delta b|$, we see that $|\Delta b| \leq m \left(\frac{a}{|x|}\right)^{n-2} = \frac{m a^{n-2}}{2(n-4)} \Delta(|x|^{4-n})$ in $|x| > a$. This means that there exist in $|x| > a$, a superharmonic function $s(x)$ and a subharmonic function $t(x)$ such that $s(x) = b(x) + A|x|^{4-n}$ where $A = \dfrac{ma^{n-2}}{2(n-4)} > 0$ and $t(x) = b(x) - A|x|^{4-n}$.

Since $t(x) \le s(x)$, there exists a harmonic function $h(x)$ in $|x| > a$ such that $t(x) \le h(x) \le s(x)$. Then, if $r > a$ (following Remark 23 [2]), there exists a harmonic function $H(x)$ in $\mathbf{R}^n$ such that $|H - h| \le \alpha B_r 1$ in $|x| \ge r$ for some constant α. Consequently

$$\begin{aligned} |b - H| &\le |b - h| + |H - h| \ \text{ if } \ |x| > r \\ &\le A|x|^{4-n} + \alpha\left(\frac{r}{|x|}\right)^{n-2} \end{aligned}$$

and hence $\lim_{|x|\to\infty}[b(x) - H(x)] = 0$. Hence the proposition holds.

For a set A in R, R_1^A stands for the infimum of the family of positive superharmonic functions s on R such that $s \ge 1$ on A; let $\widehat{R_1^A}(x) = \liminf_{y\to x} R_1^A(y)$.

Definition 4. *A hyperbolic Riemannian manifold R is said to be a bipotential manifold if there exist potentials $p > 0$ and $q > 0$ in R such that $\Delta q = p$; q is called a bipotential.*

Using the integral representation of the potentials in R with respect to the Green kernel, or as in [6], we can prove the following:

Theorem 5. *In a hyperbolic Riemannian manifold R, the following are equivalent:*

1) *R is a bipotential manifold.*

2) *For some (and hence any) nonpolar compact set A, there exists a potential $q > 0$ in R such that $\Delta q = \widehat{R_1^A}$.*

3) *For any potential $p > 0$ in R with compact harmonic support (that is, p is a potential in R and is harmonic outside a compact set in R), there exists a potential $q > 0$ in R such that $\Delta q = p$.*

4) *For some (and hence any) y in R, if $G_y(x)$ is the Green function with pole $\{y\}$, there exists a bipotential $Q_y(x)$ called the biharmonic Green function with pole $\{y\}$, such that $\Delta Q_y(x) = G_y(x)$ in R.*

Suppose now that R is a parabolic Riemannian manifold. Fix $x_0 \in R$. Let ω_n be a fixed sequence of regular domains in R such that $x_0 \in \omega_n \subset \overline{\omega_n} \subset \omega_{n+1}$ and $R = \bigcup \omega_n$. Let $D_n f$ denote the Dirichlet solution in ω_n with boundary values f on $\partial\omega_n$. Let $E(x) = E(x_0, x)$ be a fixed Evans function for R (see Nakai [9] and Sario et al [12] p. 369).

Definition 6. *A superharmonic function s in a parabolic Riemannian manifold R is called a pseudo-potential if for some constant* α, $D_n(s - \alpha E)$ *tends locally uniformly to* 0 *when* $n \to \infty$.

Note: It can be proved in this case (see [3]) that every superharmonic function with compact harmonic support in R is the unique sum of a pseudo-potential and a harmonic function in R.

3. Representation of biharmonic functions near infinity

In this section, we give a method to express a given biharmonic function near infinity in a Riemannian manifold R, in terms of globally defined potentials (or pseudo-potentials) and biharmonic functions in R.

Lemma 7. *Let h be a harmonic function defined outside a compact set A in a Riemannian manifold R (hyperbolic or parabolic). Then there exist two finite continuous superharmonic functions* s_1 *and* s_2 *with compact harmonic support in R such that* $h(x) = s_1(x) - s_2(x)$ *near infinity.*

Proof. Let K be a compact set and ω a relatively compact domain such that $A \subset K^0 \subset K \subset \omega$ and $\omega \backslash K$ is a regular open set for the Dirichlet problem. For $y \in K^0$, let $s_y(x)$ denote a superharmonic function in R with harmonic point support at $\{y\}$.

Let Df denote the Dirichlet solution in ω with boundary values f on $\partial\omega$. Since $s_y > Ds_y$ in ω, it is possible to find $\alpha \geq 0$ such that $h - Dh \geq -\alpha(s_y - Ds_y)$ on ∂K. The minimum principle then implies that this inequality is valid on $\overline{\omega \backslash K}$.

Then

$$s_1 = \begin{cases} h + \alpha s_y & \text{in } R \backslash \omega \\ D(h + \alpha s_y) & \text{in } \omega \end{cases}$$

is a continuous superharmonic function with compact harmonic support in R and

$$s_2 = \begin{cases} \alpha s_y & \text{in } R \backslash \omega \\ D(\alpha s_y) & \text{in } \omega \end{cases}$$

is also a continuous superharmonic function with compact harmonic support in R.

Finally $h(x) = s_1(x) - s_2(x)$ in $R \backslash \omega$.

Notation: Let $\wp_0$ denote the cone of finite continuous potentials with compact harmonic support in R if it is hyperbolic (resp. the cone of continuous pseudo-potentials with compact harmonic support if R is parabolic). Let $\Im_0$ denote the cone of functions in R generated (Lemma 1) by the elements of $\wp_0$; that is, $s \in \Im_0$ if and only if $\Delta s \in \wp_0$. Let β denote the set of all biharmonic functions in R. Finally, let $\pounds_0$ be the smallest vector space containing $\wp_0$, $\Im_0$ and β.

THEOREM 8. *Let b be a biharmonic function defined near infinity in a Riemannian manifold R. Then there exists some $u \in \pounds_0$ such that $u = b$ near infinity.*

PROOF. Since Δb is harmonic outside a compact set, by Lemma 7, $\Delta b = s_1(x) - s_2(x)$ near infinity. Since $s_1(x)$ and $s_2(x)$ are finite continuous superharmonic functions with compact harmonic support, we can represent each of them as the unique sum of a continuous potential if R is hyperbolic (resp. a pseudo-potential if R is parabolic) and a harmonic function in R.

Thus, $\Delta b = u_1 - u_2 + h$ near infinity, where $u_i \in \wp_0$. Let $t_i \in \Im_0$ be generated by u_i and $B \in \beta$ be generated by h. Consequently, $b = t_1 - t_2 + B + v$ near infinity where v is a harmonic function outside a compact set in R and hence by Lemma 7, $v = p_1 - p_2 + H$, where $p_i \in \wp_0$ and H is harmonic in R.

Thus, outside a compact set in R, $b = (t_1 - t_2) + (p_1 - p_2) + (B + H)$ which is an element in $\pounds_0$.

REMARK 2. a) *In the above theorem, suppose that R is a bipotential manifold. Then a biharmonic function b near infinity can be represented as $b = (q_1 - q_2) + (p_1 - p_2) + B$ where q_i, Δq_i and p_i are potentials in R, with the finite continuous functions Δq_i and p_i having compact harmonic supports and B is a uniquely determined biharmonic function in R. For, in a bipotential Riemannian manifold R, if p is a potential with compact harmonic support, there exists a potential q such that $\Delta q = p$ (Theorem 5).*

To show that B is unique: Suppose $(q_1 - q_2) + (p_1 - p_2) + B = (q_1' - q_2') + (p_1' - p_2') + B'$ outside a compact set. Then near infinity, $\Delta(B - B') = \Delta(q_1' + q_2) - \Delta(q_1 + q_2')$; in this equality the right side is the difference of two potentials and the left side is a harmonic function in R. Hence $\Delta(B - B') = 0$; that is $B' = B + h$ where h is harmonic in R. Then again, near infinity h is the difference of two potentials in R and hence $h \equiv 0$.

b) *We can easily deduce from the above theorem that in a hyperbolic Riemannian manifold R, given a biharmonic function b outside a compact set, there exists a biharmonic function B in R such that $b - B$ is bounded near infinity if and only if for any given $t \in \Im_0$, there exists a biharmonic function u on R such that $t + u$ is bounded near infinity.*

In the next section, we give a sufficient condition expressed in different equivalent forms on R so that such a biharmonic extension is possible on R.

4. Biharmonic extension in a manifold

In this section, we study the manifolds R with the following property: Given a biharmonic function b near infinity in R, there exists a biharmonic function B in R such that $B - b$ is bounded near infinity.

DEFINITION 9. *A hyperbolic manifold R is said to be tapered if there exist potentials p and q, q being bounded outside a compact set such that $\Delta q = p > 0$ in R.*

THEOREM 10. *In a hyperbolic Riemannian manifold R, the following are equivalent:*

1) *There exists a bounded function s outside a compact set such that Δs is a superharmonic function >0.*

2) *For any harmonic function h defined outside a compact set K such that $|h| \leq R_1^K$, there exists a bounded function b near infinity such that $\Delta b = h$.*

3) *R is tapered.*

PROOF. 1) $\Rightarrow$ 2): Let s be a bounded function outside a compact set A such that $\Delta s = t > 0$ is superharmonic. Clearly s can be assumed to be positive. Let h be a harmonic function defined outside a compact set K such that $|h| \leq R_1^K$ in K^c. Since R_1^K is an increasing function of K, we can assume $A \subset K^0$. Then $R_1^K \leq \alpha t$ in K^c where $\alpha = (\min_{\partial k} t)^{-1}$ and by Lemma 1, there exist superharmonic functions u_1 and u_2 in K^c such that $\Delta u_1 = R_1^K$ and $\Delta u_2 = \alpha t - R_1^K$ so that in $K^c, u_1 + u_2 = \alpha s + v$ where v is a harmonic function.

Since $s \geq 0$, u_1 has a subharmonic minorant in K^c and hence in the equation $\Delta u_1 = R_1^K$ we can assume u_1 is a potential and so is u_2. Consequently, the potential $u_1 + u_2$ majorizes the harmonic function v; this implies $v \leq 0$ and hence $u_1 + u_2 \leq \alpha s$. Since s is bounded, so are u_1 and u_2. Thus, there exists a bounded function u_1 in K^c such that $\Delta u_1 = R_1^K$. Since $h^+ \leq R_1^K$ in K^c, the same argument shows that there exists a bounded function v^+ in K^c such that $\Delta v^+ = h^+$; and similarly another bounded function v^- in K^c such that $\Delta v^- = h^-$.

Thus, if $b = v^+ - v^-$, b is a bounded function in K^c and $\Delta b = h$.

2) $\Rightarrow$ 3): Let K be a nonpolar compact set and let $\Delta q = \widehat{R_1^K}$ in R.

Now, by the assumption (2), there exists a bounded function b near infinity such that $\Delta b = \widehat{R_1^K}$ (taking $h = \widehat{R_1^K}$ in K^c). Hence, outside a compact set, $q = b+$ a harmonic function u; this implies that q has a harmonic minorant outside a compact set and consequently q is the sum of a potential in R and a harmonic function (not necessarily positive). This means that in the equation $\Delta q = \widehat{R_1^K}$, q can be taken as a potential.

Recall that (Théorème 2.6 [1]) if ω_n is a regular exhaustion of R and if $D_n u$ denotes the Dirichlet solution in ω_n with boundary values u (u being the above mentioned harmonic function outside a compact set), then $D_n u$ converges locally uniformly to a harmonic function v in R so that $|u - v|$ is bounded outside a compact set.

Now q being a potential, $D_n q$ tends locally uniformly to 0 in R. Thus, the equation $q = b + u$ outside a compact set implies that the harmonic function $v = \lim D_n u$ is bounded in R since b is bounded. Consequently, u is also bounded near infinity and hence the potential q in R is bounded near infinity and $\Delta q = \widehat{R_1^K}$. Hence R is tapered (Definition 9).

3) $\Rightarrow$ 1): Since a tapered manifold is a bipotential manifold, the implication 3) $\Rightarrow$ 1) is a consequence of the following theorem.

THEOREM 11. *A bipotential manifold R is tapered if and only if for any nonpolar compact set A in R, there exists a potential u in R bounded near infinity such that $\Delta u = \widehat{R_1^A}$.*

PROOF. Let R be tapered and A be a nonpolar compact set in R. Since R is tapered, there exist potentials p and q, q being bounded near infinity, such that $\Delta q = p > 0$. Now, p being a potential, $\widehat{R_1^A} \leq (\inf_A p)^{-1} p$.

Hence, if u is the potential in R such that $\Delta u = \widehat{R_1^A}$ (Theorem 5), $u \leq (\inf_A p)^{-1} q$. For, if $q_1 = (\inf_A p)^{-1} q$, there exists a subharmonic function s in R such that $u = q_1 + s$ in R. This implies that $s \leq 0$ since the potential $u \geq s$; hence $u \leq q_1$. Since q_1 is bounded near infinity, so is u.

The converse is evident.

LEMMA 12. *Let R be a tapered manifold and $s \in \Im_0$ (see Notation in §3). Then $s = v + h$ where h is harmonic in R and v is a potential bounded near infinity.*

PROOF. Let $\Delta s = p$ where p is a finite continuous potential with compact support A in R. If K is an outerregular compact set, $K^0 \supset A$, $p \leq (\sup_K p)\widehat{R_1^K}$ in R. Since by Theorem 11, there exists a potential u in R, bounded near infinity and $\Delta u = \widehat{R_1^K}$, there also exists a potential v in R, bounded near infinity and $\Delta v = p$; consequently, $s = v + h$ in R, where h is harmonic in R.

THEOREM 13. *Let R be a bipotential Riemannian maniflod. Then the following are equivalent:*

1) *R is tapered.*

2) *For any biharmonic function b defined outside a compact set in R, there exists a biharmonic function B in R such that $B - b$ is regular at infinity (Definition 2) and bounded outside a compact set.*

3) *Any bipotential with compact biharmonic support in R is bounded near infinity.*

PROOF. 1) $\Rightarrow$ 2): Let R be tapered and b be a biharmonic function defined outside a compact set in R.

By Remark 2(a), b can be represented as $(q_1 - q_2) + (p_1 - p_2) + B$ where Δq_i and p_i are in $\wp_0$ and B is biharmonic in R. By Lemma 12, we can take q_1 and q_2 to be bounded near infinity and p_1 and p_2, being potentials with compact support, are also bounded near infinity. Hence $b - B$ is bounded near infinity.

Also, since $|\Delta(b - B)| = |\Delta q_1 - \Delta q_2|$ outside a compact set and since $\Delta q_i \in \wp_0$, $(b - B)$ is regular at infinity (see Remark 2.3 [1]).

2) $\Rightarrow$ 3): Let q be a bipotential with compact biharmonic support in R. Then, by the assumption, there exists a biharmonic function B in R such that $(q - B)$ is regular at infinity and bounded near infinity.

Since $|\Delta(q - B)| \leq p$ near infinity, where p is a potential in R and since Δq is also a potential in R, the harmonic function ΔB is bounded by a potential near infinity and hence $\Delta B \equiv 0$; that is, B is harmonic in R.

Since q is also a potential and since $|q - B| \leq \lambda$ outside a compact set A, $|B| \leq q + \lambda$ in $R \backslash A$. Hence the subharmonic function $|B|$ in R is bounded above by λ. Consequently, q is bounded outside the compact set A.

3) $\Rightarrow$ 1): Let A be a compact nonpolar set in R. Since R is a bipotential manifold, there exists a bipotential q in R such that $\Delta q = \widehat{R_1^A}$ (Theorem 5). Since q has compact biharmonic support A, by the assumption, q is bounded near infinity. Hence R is tapered.

We prove now a result that is useful in the extension of biharmonic functions with singularities.

THEOREM 14. *In a tapered manifold R, let K be a compact set and ω an open set $\supset K$. Suppose b is a biharmonic function in $\omega \backslash K$. Then there exists a biharmonic function v in ω and a biharmonic function u in $R \backslash K$ which is bounded near infinity such that $b = u + v$ in $\omega \backslash K$.*

PROOF. A similar result in $\mathbf{R}^n$ is given in Theorem 3.1 [5]. It is not difficult to modify the proof in the context of any Riemannian manifold; but in this general case we cannot prove that u is bounded near infinity.

So, we need the restriction that the Riemannian manifold R should be tapered in which case Theorem 10 will ensure that u is bounded near infinity, proving the theorem in the given form.

We conclude this section with a characterization of tapered manifolds, using Remark 1(b) and Theorems 10 and 13.

THEOREM 15. *In a hyperbolic manifold R, the following are equivalent:*

1) *R is tapered.*

2) *For any biharmonic function b defined outside a compact set and regular at infinity, there exists a harmonic function h in R such that $b - h$ is bounded near infinity.*

PROOF. 1) $\Rightarrow$ 2): Let b be a biharmonic function regular at infinity. Then by Theorem 13, there exists a biharmonic function B in R such that $B - b$ is regular at infinity and bounded outside a compact set. Since b and $B - b$ are regular at infinity, so is the biharmonic function B defined on R. This implies that B is harmonic. Set $B = h$ to obtain (2).

2) $\Rightarrow$ 1): Let K be a compact nonpolar set and $\Delta b = \widehat{R_1^K}$ in R. Then b is biharmonic in K^c and regular at infinity. Hence, by the assumption there exists a harmonic function h in R such that $b - h$ is bounded near infinity. Set $s = b - h$. Then s is a bounded function outside a compact set such that $\Delta s = \widehat{R_1^K}$ is superharmonic. Hence, by Theorem 10, R is tapered.

5. Biharmonic extension in $\mathbf{R}^n$

In $\mathbf{R}^n$, $n \geq 5, s_n(x) = |x|^{4-n}$ is a bipotential tending to 0 at infinity. Hence these spaces are tapered and the biharmonic extensions mentioned in Theorems 13 and 14 are valid here.

But the spaces $\mathbf{R}^n$, $2 \leq n \leq 4$, are not tapered. For, $\mathbf{R}^2$ is not hyperbolic; and $\mathbf{R}^3$ and $\mathbf{R}^4$, though hyperbolic, are not even bipotential spaces. In this section, we show that if b is a biharmonic function near infinity in $\mathbf{R}^n$, $n = 3$ or 4, then there exists a biharmonic function B in $\mathbf{R}^n$ such that $(b - B)$ is bounded near infinity if and only if the flux at infinity of Δb is 0. This result is contained in the following theorem, where we denote by E_n the fundamental solution $\Delta E_n = \delta$ in $\mathbf{R}^n$, $n \geq 2$.

THEOREM 16. *For a biharmonic function b defined outside a compact set in $\mathbf{R}^n$, $2 \leq n \leq 4$, the following are equivalent:*

1) *Flux at infinity of Δb is 0.*

2) *There exists a biharmonic function B in $\mathbf{R}^n$ and a constant α such that $(b - B - \alpha E_n)$ is bounded near infinity.*

3) *For some r_0, the mean-value $M(r, \Delta b)$ of Δb on $|x| = r > r_0$ is independent of r.*

In particular, if b is harmonic so is B in (2).

PROOF. For the simplicity of writing, we give the proof when $n = 3$; the case $n = 4$ does not differ much from $n = 3$; when $n = 2$, we indicate the relevant changes.

We start with the following remark: In $\mathbf{R}^3$, let μ be a measure with compact support K. Then $u(x) = |x| * \mu$ is well-defined in $\mathbf{R}^3$, biharmonic in K^c. Let $y \in K$ be fixed. Then, for $x_0 \in K^c$, we have $|u(x_0) - |x_0 - y|\mu(\mathbf{R}^3)| \le \mu(\mathbf{R}^3)$ (diameter of K). Consequently, for $i = 1, 2$, if μ_i is a measure with compact support K_i such that $\mu_1(\mathbf{R}^3) = \mu_2(\mathbf{R}^3) = \| \mu \|$ and if $u_i(x) = |x| * \mu_i$ then $|u_1(x_0) - u_2(x_0)| \le 3 \| \mu \| \operatorname{diam}(K_1 \cup K_2)$ for any $x_0 \in (K_1 \cup K_2)^c$.

1) $\Rightarrow$ 2): By Theorem 8, the given biharmonic function b outside a compact set is of the form $b = s_1 - s_2 + p_1 - p_2 + B$. By the construction, there exist two measures μ_i, $i = 1, 2$, with compact support such that $\Delta^2 s_i = \mu_i$.

If the flux Δb at infinity is 0, then $\mu_1(\mathbf{R}^3) = \mu_2(\mathbf{R}^3)$ and by the above remark we can find u_i such that $\Delta^2 u_i = \mu_i$, so that u_1 and u_2 are biharmonic outside a compact set and $|u_1 - u_2|$ is bounded near infinity. Thus, in the above decomposition of b, we can assume that $s_1 - s_2$ is bounded outside a compact set.

Moreover, p_1 and p_2 are potentials (resp. logarithmic potentials if $n = 2$) with compact support in $\mathbf{R}^3$. Hence there are constants α_1 and α_2 such that $p_i - \alpha_i E_n$ $(2 \le n \le 4)$ is bounded near infinity. (Actually, if $n = 3$ or 4 we can take $\alpha_1 = \alpha_2 = 0$). Finally, if $\alpha = \alpha_1 - \alpha_2$, $(b - B - \alpha E_n)$ is bounded near infinity.

2) $\Rightarrow$ 1): Conversely, suppose that for the biharmonic function b near infinity in $\mathbf{R}^n$, $2 \le n \le 4$, there exists a biharmonic function B in $\mathbf{R}^n$ and a constant α such that $(b - B - \alpha E_n)$ is bounded near infinity. We shall show that the flux Δb at infinity is 0.

Whatever be flux Δb, there exists a constant β such that if $u = b - \beta s_n$ where $\Delta^2 s_n = \delta$, then the flux Δu at infinity is 0. Consequently, from the above proof of 1) $\Rightarrow$ 2), there exist a biharmonic function v in $\mathbf{R}^n$ and a constant ν such that $(u - v - \nu E_n)$ is bounded near infinity.

Hence $B - v = (u - v - \nu E_n) - (b - B - \alpha E_n) + (\nu - \alpha)E_n + \beta s_n$, and consequently $B - v$ which is biharmonic in $\mathbf{R}^n$ is bounded on one side. Such a biharmonic function is of the form $c|x|^2 +$ (a harmonic polynomial of degree ≤ 2) (see Nicolesco [10] p. 20; can also be proved as Lemma 17 below).

Now, taking the mean-values on $|x| = r$ of both sides and allowing $r \to \infty$, we obtain $\beta = 0$ (recall that up to a multiplicative constant, $s_2 = |x|^2 \log|x|$, $s_3 = |x|$ and $s_4 = \log|x|$). This means that $u = b$ and the flux Δb at infinity is 0.

3) $\Leftrightarrow$ 1): This is an old result for harmonic functions in $\mathbf{R}^n$, proved in M. Brelot [7] p. 303.

Finally, suppose b is harmonic outside a compact set so that $\Delta b = 0$. Then, using the representation above that $b = s_1 - s_2 + p_1 - p_2 + B$, we have $0 = \Delta b = \Delta s_1 - \Delta s_2 + \Delta B$ near infinity. Since ΔB is harmonic in $\mathbf{R}^n$ and Δs_i is a potential in $\mathbf{R}^n$ for $n = 3$ and 4 (resp. a logarithmic potential if $n = 2$) with compact support, we conclude that $\Delta B = 0$.

Thus, the theorem is proved.

In the context of the final part of the proof of the above theorem, we remark that it is known that if b is harmonic outside a compact set in $\mathbf{R}^n$, $n \geq 2$, then there exists a harmonic function $h(x)$ in $\mathbf{R}^n$ such that $b(x) - h(x) - \beta E_n(x)$ is bounded near infinity, where β is a constant (which can be taken as 0 if $n \geq 3$). Using this fact, we have another proof of the final remark of Theorem 16 when we prove the following lemma.

Lemma 17. *Let B be a biharmonic function and h be a harmonic function in $\mathbf{R}^n$, $n \geq 2$, such that $B(x) - h(x) = o(|x|^2)$ when $|x| \to \infty$. Then B is harmonic.*

Proof. By Almansi representation, $B(x) = |x|^2 h_1(x) + h_2(x)$ in $\mathbf{R}^n$ where $h_1(x)$ and $h_2(x)$ are harmonic. Thus, $|x|^2 h_1(x) + h_2(x) = h(x) + o(|x|^2)$ when $|x|$ is large.

For a fixed $z \in \mathbf{R}^n$, let $\rho_z^r(x)$ denote the harmonic measure on $|x| = r > |z|$ with large r. Then integrating the above equality with respect to $\rho_z^r(x)$ we have

$$r^2 h_1(z) + h_2(z) = h(z) + o(r^2).$$

Dividing by r^2 and allowing $r \to \infty$, we find $h_1(z) = 0$. Since z is arbitrary, this implies that $h_1 \equiv 0$ and hence B is harmonic in $\mathbf{R}^n$.

Remark 3. (a) *If b is a bounded biharmonic function defined outside a compact set in $\mathbf{R}^n$, $2 \leq n \leq 4$, then by Theorem* 16 *the flux Δb at infinity is* 0.

(b) *By considering the function $|x|^{4-n}$, we see that the above remark(a) is not valid if $n \geq 5$.*

Biharmonic functions outside a compact set in $\mathbf{R}^n$:

The above discussion leads to the following more precise representation of a biharmonic function defined near infinity in $\mathbf{R}^n$, $n \geq 2$ (see Theorem 8). If $b(x)$ is a biharmonic function defined outside a compact set in $\mathbf{R}^n$, $n \geq 2$, there exist uniquely determined constants α and β and a biharmonic function $B(x)$ in $\mathbf{R}^n$ unique up to an additive constant such that outside a compact set

$$b(x) = \begin{cases} \alpha \log|x| + \beta|x|^2 \log|x| + B(x) + u(x), & \text{if } n = 2 \\ \beta|x| + B(x) + u(x), & \text{if } n = 3 \\ \beta \log|x| + B(x) + u(x), & \text{if } n = 4 \\ B(x) + u(x), & \text{if } n \geq 5. \end{cases}$$

Here $u(x)$ is a bounded biharmonic function near infinity; also $u(x)$ is regular at infinity if $n \geq 5$; and flux at infinity of Δu is 0 if $2 \leq n \leq 4$.

Consequently, in view of Proposition 3, if $b(x)$ is a biharmonic function near infinity in $\mathbf{R}^n$, $n \geq 5$, then there exists a biharmonic function $B_1(x)$ in $\mathbf{R}^n$ such that $\lim_{|x|\to\infty}[b(x) - B_1(x)] = 0$. In particular, if $b(x)$ is bounded near infinity in $\mathbf{R}^n$, $n \geq 5$, $\lim_{|x|\to\infty} b(x)$ exists.

This last statement concerning the limit of a bounded biharmonic function at infinity is true even when $n = 4$ (can be proved by using the extended Kelvin transform as given in Nicolesco [10] p. 14) but not when $n = 2$ or 3; for example, $\sin 2\theta$ in $\mathbf{R}^2$ and $\dfrac{x_1}{|x|}$ in $\mathbf{R}^3$ where $x = (x_1, x_2, x_3)$.

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Department of Mathematics College of Science
King Saud University
P.O. Box 2455
Riyadh 11451
Saudi Arabia
E-mail: bajunaid@ksu.edu.sa
E-mail: vanandam@ksu.edu.sa