Hiroshima Math. J.
30 (2000), 1–8

Dimensions of scattered sets

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(Received October 14, 1996)
(Revised May 8, 1998)

Abstract. Hausdorff and packing dimensions are used to measure the complexity and irregularity of a set, though their calculations are in general not so easy. In this paper we define scattered sets which describe a typical form of explosions and then estimate their Hausdorff and packing dimensions.

1. Introduction

Let (X, d) be a complete metric space and $\mathscr{P}_n(X)$ denote the family of subsets of X having n elements. Then we call a set valued map $\phi : A \subset X \to \mathscr{P}_n(X)$, $n > 1$, an *n-scattered map* on A.

For a sequence $\Phi = \{\phi^k\}$ of n_k-scattered maps, $\phi^k : A^{k-1} \to \mathscr{P}_{n_k}(X)$, in which $\mathbf{A} = \{A^k\}$ is defined inductively by

$$A^0 = \{x_1, x_2, \dots, x_{n_0}\} \qquad \text{and} \qquad A^k = \bigcup\nolimits_{y_{k-1} \in A^{k-1}} \phi^k(y_{k-1})$$

for each $k \in \mathbf{N}$, we call $(\mathbf{A}, \Phi)$ a *scattered system*. Here the image of $y_{k-1} \in A^{k-1}$ may also be denoted by $\phi^k(y_{k-1}) = \{y_{k-1j} : j = 1, 2, \dots, n_k\}$. Moreover, when there exists a function $f : \bigcup_{k=0}^{\infty} A^k \to (0, \infty)$ and $0 < a < 1$ such that

(1) $d(y_k, y_{k+1}) \le f(y_k) - f(y_{k+1})$,

(2) $d(y_k, y_k') > (1 + a)\{f(y_k) + f(y_k')\}$,

(3) $a \le f(y_{kj})/f(y_k) = f(y_{kj}')/f(y_k')$ and $f(y_k) \searrow 0$ as $k \to \infty$

for each y_k, $y_k' \in A^k$ and y_{k+1}, y_{kj}, $y_{kj}' \in A^{k+1}$, we call $(\mathbf{A}, \Phi)$ an *f-bounded scattered system*.

We recall the definition of Hasudorff and packing dimensions [2], [3]. Let F be a given set and $|U|$ denote the diameter of a set U. Then $\{U_i\}_{i=1}^{\infty}$ is called a δ-covering of F if $F \subset \bigcup_i U_i$ and $|U_i| < \delta$, and is called δ-packing of F if $\{U_i\}$ is pairwise disjoint, $\overline{U_i} \cap \bar{F} \neq \varnothing$ and $|U_i| < \delta$. Then s-dimensional Hausdorff measure $H^s(F)$ and Hausdorff dimension $\dim_{\mathrm{H}}(F)$ are defined by

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2000 *Mathematics Subject Classification.* 28A78, 28A80

Key words and phrases. scattered set, Hausdorff dimension, packing dimension.

$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F)$ where $H^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}$ is a δ-covering of $F\}$, and $\dim_{\mathrm{H}}(F) = \sup\{s \geq 0 : H^s(F) = \infty\}$ (or $\inf\{s \geq 0 : H^s(F) = 0\}$).

And let $\mathscr{P}(F) = \lim_{\delta\to 0} \mathscr{P}_\delta(F)$ where $\mathscr{P}_\delta(F) = \sup\{\sum_i |U_i|^s : \{U_i\}$ is a δ-packing of $F\}$. Then s-dimensional packing measure $P^s(F)$ and packing dimension $\dim_{\mathrm{P}}(F)$ are defined by $P^s(F) = \inf\{\sum_i \mathscr{P}(F_i) : F \subset \bigcup_{i=1}^{\infty} F_i\}$ and $\dim_{\mathrm{P}}(F) = \sup\{s \geq 0 : P^s(F) = \infty\}$ (or $\inf\{s \geq 0 : P^s(F) = 0\}$).

In this paper, we define a scattered set from the scattered system and estimate its Hausdorff and packing dimensions. From now on, $B(y_n)$ denotes the closed ball $B(y_n, f(y_n))$.

2. Main results

Lemma 1. *Let $(\mathbf{A}, \Phi)$ be an f-bounded scattered system and let $\{y_k\}$ be any sequence such that $y_{k-1} \in A^{k-1}$ and $y_k \in \phi^k(y_{k-1})$ for each $k \in \mathbf{N}$. Then $\{y_k\}$ is a Cauchy sequence in X and so has a limit point.*

Proof. Since $f(y_n) \searrow 0$ as $n \to \infty$, for every $\varepsilon > 0$ there exists an $\mathscr{N} \in \mathbf{N}$ such that $f(y_k) < \varepsilon$ for $k \geq \mathscr{N}$ and $y_k \in A^k$. Then for any $n > m \geq \mathscr{N}$,

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \cdots + d(y_{n-1}, y_n) \\ &\leq \{f(y_m) - f(y_{m+1})\} + \{f(y_{m+1}) - f(y_{m+2})\} + \cdots \\ &\quad + \{f(y_{n+1}) - f(y_n)\} < \varepsilon. \end{aligned}$$

Thus $\{y_n\}$ is a Cauchy sequence. □

From the above Lemma, we can define the following set,

Definition 2. For a given f-bounded scattered system $(\mathbf{A}, \Phi)$, its *scattered set* is defined by

$$\Lambda(\mathbf{A}, \Phi) = \{y \in X : y \text{ is the limit of a sequence } \{y_n\} \text{ satisfying the condition of Lemma 1}\}.$$

From the following Theorem, we can find some topological properties of the scattered set.

Theorem 3. *$\Lambda(\mathbf{A}, \Phi)$ is perfect, totally bounded and compact.*

Proof. Let $y \in \Lambda(\mathbf{A}, \Phi)$. Then there exists $\{y_n\}$ such that $y_n \to y$ as $n \to \infty$. Since $f(y_n) \searrow 0$ as $n \to 0$, for every $\varepsilon > 0$ there exists an $\mathscr{N} \in \mathbf{N}$ such that $f(y_n) < \varepsilon/2$ and $d(y_n, y) < \varepsilon/2$ for each $n \geq \mathscr{N}$. And for this $\mathscr{N}$ any sequence $\{y'_n\}$ with $y'_\ell = y_\ell$ for $\ell < \mathscr{N}$ and $y_{\mathscr{N}} \neq y'_{\mathscr{N}}$ satisfying the condition of Lemma 1 has a limit point y' in $\Lambda(\mathbf{A}, \Phi)$. Since $d(y_k, y'_k) > f(y_k) + f(y'_k)$, $B(y_k) \cap B(y'_k) = \varnothing$ and since $d(y_k, y_{k+1}) \leq f(y_k) - f(y_{k+1})$ for each $y_{k+1} \in \phi^{k+1}(y_k)$, $B(y_{k+1}) \subset B(y_k)$. Then $\bigcap_{n=0}^{\infty} B(y_n) = \{y\}$, $\bigcap_{n=0}^{\infty} B(y_n') = \{y'\}$ and $y' \in B(y, \varepsilon)\backslash\{y\}$. Hence $\Lambda(\mathbf{A}, \Phi)$ is a perfect set.

Now for any $\varepsilon > 0$, take an $\mathcal{N} \in \mathbf{N}$ such that $f(y_n) < \varepsilon$ for any $y_n \in A^n$ with $n > \mathcal{N}$. Then as above, every $y \in \Lambda(\mathbf{A}, \Phi)$ is contained in $B(y_n)$ for some $y_n \in A^n$ and so $\Lambda(\mathbf{A}, \Phi) \subset \bigcup_{y_n \in A^n} B(y_n, \varepsilon)$. Hence $\Lambda(\mathbf{A}, \Phi)$ is totally bounded.

The compactness follows from the above two properties. □

We will estimate the Hausdorff and packing dimensions of the scattered set.

THEOREM 4. *Let*

$$\underline{D} = \sup\left\{ s \geq 0 : \liminf_{n\to\infty} \sum_{y_n \in A^n} f(y_n)^s = \infty \right\}$$

$$= \inf\left\{ s \geq 0 : \liminf_{n\to\infty} \sum_{y_n \in A^n} f(y_n)^s = 0 \right\}.$$

Then $\dim_{\mathrm{H}} \Lambda(\mathbf{A}, \Phi) = \underline{D}$.

PROOF. Since $f(y_n) \searrow 0$ as $n \to \infty$, $\Phi(s,k) = \sum_{y_k \in A^k} f(y_k)^s$ is continuous and decreasing for s and for sufficiently large k. And since $\Phi(s,k) \to 0$ as $s \to \infty$ and $\Phi(0,k) = n_0 n_1 n_2 \cdots n_k \to \infty$ as $k \to \infty$, $0 < \liminf_{k\to\infty} \Phi(s,k) < \infty$ implies $\liminf_{k\to\infty} \Phi(s_1,k) = \infty$ and $\liminf_{k\to\infty} \Phi(s_2,k) = 0$ for $s_1 < s < s_2$, and so $\underline{D}$ is well-defined. We may suppose $0 < \underline{D} < \infty$.

For $s > \underline{D}$ and for given $\delta > 0$, take an $n \in \mathbf{N}$ such that $f(y_n) < \delta/2$ for all $y_n \in A^n$. Then as in the proof of Theorem 3, we have $\Lambda(\mathbf{A}, \Phi) \subset \bigcup_{y_n \in A^n} B(y_n)$ and

$$H^s_\delta(\Lambda(\mathbf{A}, \Phi)) \leq \sum_{y_n \in A^n} |B(y_n)|^s = 2^s \sum_{y_n \in A^n} f(y_n)^s.$$

Hence $H^s(\Lambda(\mathbf{A}, \Phi)) \leq \liminf_{n\to\infty} 2^s \sum_{y_n \in A^n} f(y_n)^s < \infty$ for $s > \underline{D}$ or $\dim_{\mathrm{H}} \Lambda(\mathbf{A}, \Phi) \leq \underline{D}$.

Now let $s < \underline{D}$. To define a mass distribution on X, let $\mathscr{F}_n$ be the family of arbituary unions of $B(y_n)$'s and $\mathscr{F}$ be the completion of the smallest σ-algebra generated by $\bigcup \mathscr{F}_n$. And define a set function μ on $\{B(y_n) : y_n \in A^n, n \in \mathbf{N}\}$ by

$$\mu(B(y_n)) = f(y_n)^s \Big/ \sum_{y_n' \in A^n} f(y_n')^s.$$

Since $f(y_{kj}')/f(y_k') = f(y_{kj})/f(y_k)$ for $j = 1, 2, \ldots, n_{k+1}$,

$$\sum_{y'_{kj}\in\phi^{k+1}(y'_k)} \{f(y'_{kj})/f(y'_k)\}^s = \sum_{y_{kj}\in\phi^{k+1}(y_k)} \{f(y_{kj})/f(y_k)\}^s.$$

Therefore we have

$$\begin{aligned}
&\mu(B(y_{nj}))\\
&= f(y_{nj})^s \Big/ \sum_{y'_{n+1}\in A^{n+1}} f(y'_{n+1})^s\\
&= f(y_n)^s \cdot \{f(y_{nj})/f(y_n)\}^s \cdot \left[\sum_{y'_n\in A^n} f(y'_n)^s \left\{\sum_{y'_{nj}\in\phi^{n+1}(y'_n)} (f(y'_{nj})/f(y'_n))^s\right\}\right]^{-1}\\
&= f(y_n)^s \cdot \{f(y_{nj}/f(y_n)\}^s \cdot \left[\sum_{y'_n\in A^n} f(y'_n)^s \cdot \sum_{y''_{nj}\in\phi^{n+1}(y_n)} (f(y''_{nj})/f(y_n))^s\right]^{-1}\\
&= \left\{f(y_n)^s \Big/ \sum_{y'_n\in A^n} f(y'_n)^s\right\} \cdot \left\{f(y_{nj})^s \Big/ \sum_{y''_{nj}\in\phi^{n+1}(y_n)} f(y''_{nj})^s\right\}\\
&= \mu(B(y_n)) \cdot \left\{f(y_{nj})^s \Big/ \sum_{y_{n+1}\in\phi^{n+1}(y_n)} f(y_{n+1})^s\right\},
\end{aligned}$$

and so $\sum_{y_{nj}\in\phi^{n+1}(y_n)} \mu(B(y_{nj}) = \mu(B(y_n))$. Moreover

$$\begin{aligned}
&\sum_{y_n\in A^n} \mu(B(y_n))\\
&= \sum_{y_{n-1}\in A^{n-1}} \sum_{y_{n-1i}\in\phi^n(y_{n-1})} \left[\mu(B(y_{n-1}))\cdot\left\{f(y_{n-1i})^s \Big/ \sum_{y_{n-1j}\in\phi^n(y_{n-1})} f(y_{n-1j})^s\right\}\right]\\
&= \sum_{y_{n-1}\in A^{n-1}} \mu(B(y_{n-1}) = \cdots = \sum_{y_0\in A^0} \mu(B(y_0)) = 1.
\end{aligned}$$

So μ can be extended to a mass distribution on $\mathscr{F}$ with support in $\Lambda(\mathbf{A},\Phi)$ [2]. Now consider $y\in\Lambda(\mathbf{A},\Phi)$ and $\{y_n\}$ with $y_n\to y$ as $n\to\infty$. As in the proof of Theorem 2, $\{y\} = \bigcap B(y_n)$. For every small $r>0$, take an n such that $f(y_{n+1})\le r < f(y_n)$. For $a>0$ in the definition of f-bounded scattered system, $ar < af(y_n) < d(y_n, y'_n) - \{f(y_n)+f(y'_n)\}$ and so $B(y,ar)\subset \{\bigcup_{y_n\ne y_{n'}\in A^n} B(y'_n)\}^c$ and $\mu(B(y,ar))\le\mu(B(y_n))$. For $0<t<s$,

$$\begin{aligned}\mu(B(y,ar))/(ar)^t &\le \mu(B(y_n))/\{a^t\cdot f(y_{n+1})^t\}\\ &\le \mu(B(y_n))/\{a^{2t}f(y_n)^t\}\\ &= f(y_n)^{s-t}\bigg/\left\{a^{2t}\sum_{y_n'\in A^n} f(y_n')^s\right\},\end{aligned}$$

so $\sup_{r\to 0}\mu(B(y,r)/r^t \le \limsup_{n\to\infty} f(y_n)^{s-t}/\{a^{2t}\cdot\sum_{y_n'\in A^n} f(y_n')^s\} = 0$. Hence $H^t(\Lambda(\mathbf{A},\Phi)) = \infty$ by the density theorem [2], and $\dim_{\mathrm{H}}\Lambda(\mathbf{A},\Phi)\ge \underline{D}$. □

THEOREM 5. *Let*

$$\begin{aligned}\bar{D} &= \sup\left\{s\ge 0 : \limsup_{n\to\infty}\sum_{y_n\in A^n} f(y_n)^s = \infty\right\}\\ &= \inf\left\{s\ge 0 : \limsup_{n\to\infty}\sum_{y_n\in A^n} f(y_n)^s = 0\right\}.\end{aligned}$$

Then $\dim_{\mathrm{P}}\Lambda(\mathbf{A},\Phi) = \bar{D}$.

PROOF. As in Theorem 4, $\bar{D}$ is well defined. Let $0 < s < \bar{D}$. Take an $\mathcal{N}\in\mathbf{N}$ such that $f(y_k) < \varepsilon/2$ for each $y_k\in A^k$ with $k\ge\mathcal{N}$. Then for each $y_n\in A^n$ with $n\ge\mathcal{N}$,

$$\begin{aligned}P_\varepsilon^s(B(y_n)\cap\Lambda(\mathbf{A},\Phi)) &\ge \sup_{k\ge n}\sum_{\substack{y_k\in A^k\\ B(y_k)\subset B(y_n)}} |B(y_k)|^s\\ &\ge 2^s\cdot f(y_n)^s\cdot\left\{\limsup_{k\to\infty}\sum_{y_k\in A^k} f(y_k)^s\bigg/\sum_{y_n\in A^n} f(y_n)^s\right\} = \infty.\end{aligned}$$

Thus $P^s(B(y_n)\cap\Lambda(\mathbf{A},\Phi)) = \infty$. Now consider any $\{\Lambda_n\}$ which satisfies $\bigcup_{n=1}^\infty \Lambda_n = \Lambda(\mathbf{A},\Phi)$. Since $\Lambda(\mathbf{A},\Phi)$ is compact, $\bigcup_{n=1}^\infty \overline{\Lambda_n} = \Lambda(\mathbf{A},\Phi)$. By the Baire category theorem, there exists a $\overline{\Lambda_{n_0}}$ whose interior in $\Lambda(\mathbf{A},\Phi)$ is not empty, so there exists a large n such that $B(y_n)\cap\Lambda(\mathbf{A},\Phi)\subset\overline{\Lambda_{n_0}}$. Since $P^s(\Lambda_{n_0}) = P^s(\overline{\Lambda_{n_0}}) \ge P^s(B(y_n)\cap\Lambda(\mathbf{A},\Phi)) = \infty$, we have $P^s(\Lambda(\mathbf{A},\Phi)) = \inf\{\sum_{i=1}^\infty P^s(\Lambda_n) : \Lambda(\mathbf{A},\Phi) = \bigcup\Lambda_n\} = \infty$. Hence $\dim_{\mathrm{P}}\Lambda(\mathbf{A},\Phi)\ge\bar{D}$.

Now take an $s > \bar{D}$ and let μ be the mass distribution defined as in Theorem 3. Consider $y\in\Lambda(\mathbf{A},\Phi)$ and $\langle y_n\rangle$ satisfying $y_n\to y$ as $n\to\infty$. For given $r > 0$, take an n such that $f(y_{n+1})\le r/2 < f(y_n)$. For $t > s$,

$$\begin{aligned}\mu(B(y,r))/r^t &\geq \mu(B(y_{n+1}))/(2f(y_n))^t \\ &\geq (a/2)^t \cdot \mu(B(y_{n+1}))/f(y_{n+1})^t \\ &= (a/2)^t \cdot f(y_{n+1})^{s-t} \Big/ \sum_{y'_{n+1} \in A^{n+1}} f(y'_{n+1})^s,\end{aligned}$$

and so

$$\liminf_{r\to 0} \mu(B(y,r))/r^t \geq \liminf_{n\to\infty} (a/2)^t \cdot f(y_{n+1})^{s-t} \Big/ \sum_{y'_{n+1} \in A^{n+1}} f(y'_{n+1})^s = \infty.$$

By the packing density theorem [3], $P^t(\Lambda(\mathbf{A},\Phi)) = 0$ and $\dim_{\mathrm{P}} \Lambda(\mathbf{A},\Phi) \leq \bar{D}$. □

THEOREM 6. *Let s_n be the number satisfying*

$$\sum_{y_n \in \phi^n(y_{n-1})} (f(y_n)/f(y_{n-1}))^{s_n} = 1.$$

And put $\underline{s} = \liminf_{n\to\infty} s_n$ and $\bar{s} = \limsup_{n\to\infty} s_n$. Then

$$\underline{s} \leq \dim_{\mathrm{H}} \Lambda(\mathbf{A},\Phi) \leq \dim_{\mathrm{P}} \Lambda(\mathbf{A},\Phi) \leq \bar{s}.$$

PROOF. We may suppose $0 < \underline{s} \leq \bar{s} < \infty$. Take an s with $0 < s < \underline{s}$, then there exists an $\mathcal{N} \in \mathbf{N}$ such that $s < s_k$ for all $k > \mathcal{N}$. Since

$$\begin{aligned}(1+a)f(y_{n+1}) &< d(y_{n+1}, y'_{n+1}) \\ &\leq d(y_{n+1}, y_n) + d(y_n, y'_{n+1}) < 2f(y_n) - f(y_{n+1})\end{aligned}$$

for each $y_{n+1}, y'_{n+1} \in \phi^{n+1}(y_n)$, we have $f(y_{n+1})/f(y_n) < 2/(2+a) < 1$ for each $y_{n+1} \in \phi^{n+1}(y_n)$. Then

$$\begin{aligned}\sum_{y_n \in A^n} f(y_n)^s &= \sum_{y_0 \in A^0} f(y_0)^s \cdot \prod_{k=0}^{n} \sum_{y_{k+1} \in \phi^{k+1}(y_k)} \{f(y_{k+1})/f(y_k)\}^s \\ &\geq c \cdot \prod_{k=N}^{n} \left[\sum_{y_k \in \phi^k(y_{k-1})} \{(f(y_k)/f(y_{k-1}))^{s_k} \cdot (2/2+a)^{s-s_k}\} \right] \\ &\geq c \cdot (2/2+a)^{(s-s_k)(n-\mathcal{N}+1)},\end{aligned}$$

where

$$c = \sum_{y_0 \in A^0} f(y_0)^s \cdot \prod_{k=0}^{\mathcal{N}-1} \sum_{y_{k+1} \in \phi^{k+1}(y_k)} \{f(y_{k+1})/f(y_k)\}^s.$$

So $\liminf_{n\to\infty} \sum_{y_n \in A^n} f(y_n)^s = \infty$ and $\underline{s} \leq \underline{D}$.

In a similar way, we have $\limsup \sum_{y_n \in A^n} f(y_n)^s = 0$ for $s > \bar{s}$ and $\bar{D} \leq \bar{s}$. Therefore from the above two theorems, $\underline{s} \leq \dim_{\mathrm{H}} \Lambda(\mathbf{A}, \Phi) \leq \dim_{\mathrm{P}} \Lambda(\mathbf{A}, \Phi) \leq \bar{s}$. □

COROLLARY 7. *If the sequence* $\{s_n\}$ *in* $\Lambda(\mathbf{A}, \Phi)$ *satisfying*

$$\sum_{y_n \in \phi^n(y_{n-1})} (f(y_n)/f(y_{n-1}))^{s_n} = 1$$

has a limit s, then

$$\dim_{\mathrm{H}} \Lambda(\mathbf{A}, \Phi) = \dim_{\mathrm{P}} \Lambda(\mathbf{A}, \Phi) = s.$$

3. Examples

Now we will give two examples.

EXAMPLE 1. Let $X = [0, 1]$, $A^0 = \{x/3 + 1/6 : x = 0, 2\}$ and let $\mathcal{N}^*$ be a fixed positive integer. For each positive integer k represented by $k = m\mathcal{N}^* + l$ for non-negative integers m and l with $0 \leq l < \mathcal{N}^*$, put $a_k = 1/2 \cdot \{m\mathcal{N}^*(\mathcal{N}^* + 1) + (l + 1)(l + 2)\}$. Define $\phi^k : A^{k-1} \to \mathscr{P}_{2^{l+1}}(X)$ by

$$\phi^k(y_{k-1}) = \left\{ y_{k-1} + \sum_{i=a_k-l}^{a_k} x_i/3^i + 1/2 \cdot (1/3^{a_k} - 1/3^{a_{k-1}}) : x_i = 0, 2 \right\}$$

where $A^k = \phi^k(A^{k-1})$. and define $f : \bigcup_{k=0}^{\infty} A^k \to (0, \infty)$ by $f(y_k) = 1/(2 \cdot 3^{a_k})$. Then $\Lambda(\mathbf{A}, \Phi)$ is an f-bounded scattered system. Since $\sum_{y_k \in \phi^k(y_{k-1})} \{f(y_k)/f(y_{k-1})\}^{s_k} = 1$ for each $s_k = \log 2/\log 3$,

$$\dim_{\mathrm{H}} \Lambda(\mathbf{A}, \Phi) = \dim_{\mathrm{P}} \Lambda(\mathbf{A}, \Phi) = \log 2/\log 3.$$

EXAMPLE 2. For the same X and A^0 as in above, we define another f-bounded scattered system by

$$\phi^k(y_{k-1}) = \begin{cases} \{y_{k-1} + \sum_{i=3n-1}^{3n} x_i/3^i - 4/27^n : x_{3n-1} = 0, 2,\ x_{3n} = 2\}, & (k = 2n-1) \\ \{y_{k-1} + x_{3n+1}/3^{3n+1} - 1/(3 \cdot 27^n) : x_{3n+1} = 0, 2\}, & (k = 2n), \end{cases}$$

$$f(y_k) = \begin{cases} 1/(2 \cdot 27^n), & (k = 2n - 1), \\ 1/(6 \cdot 27^n), & (k = 2n), \end{cases}$$

for $y_{k-1} \in A^{k-1}$ and for $y_k \in A^k$. Then for this f-bounded scattered system $\Lambda(\mathbf{A}, \Phi)$, $s_k = \log 2/(2 \log 3)$ for $k = 2n - 1$ and $s_k = \log 2/\log 3$ for $k = 2n$. Hence

$$\log 2/(2 \log 3) \leq \dim_{\mathrm{H}} \Lambda(\mathbf{A}, \Phi) \leq \dim_{\mathrm{P}} \Lambda(\mathbf{A}, \Phi) \leq \log 2/\log 3.$$

Acknowledgement

I would to thank support from the KOSEF through the Global Analysis Research Center at Seoul National University.

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