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On the action of β_1 in the stable homotopy of spheres at the prime 3

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ABSTRACT. Let S^0 denote the sphere spectrum localized away from 3. The element β_1 is the generator of the homotopy group $\pi_{10}(S^0)$. Toda showed that $\beta_1^5 \neq 0$ and $\beta_1^6 = 0$. In this paper, we generalize his result and show that $\beta_1^4\beta_{9t+1} \neq 0$ and $\beta_1^5\beta_{9t+1} = 0$ for $\beta_{9t+1} \in \pi_{144t+10}(S^0)$ with $t \geq 0$. In particular, $\beta_1^4\beta_{10} \neq 0$ and $\beta_1^5\beta_{10} = 0$, where the existence of β_{10} was shown by Oka. This is proved by determining subgroups of $\pi_*(L_2S^0)$. Here L_2 denotes the Bousfield localization functor with respect to $v_2^{-1}BP$.

1. Introduction

Let p be a prime number and S^0 the sphere spectrum localized away from p. Let $E_r^*(X)$ denote the E_r-term of the Adams-Novikov spectral sequence converging to $\pi_*(X)$ for a spectrum X localized away from p. Miller, Ravenel and Wilson [1] introduced β-elements $\beta_{s/j,i+1}$ in $E_2^2(S^0)$ for $(s,j,i+1) \in \boldsymbol{B}^+$, where

$$\boldsymbol{B}^+ = \{(s,j,i+1) \in \boldsymbol{Z}^3 \mid s = mp^n, n \geq 0, p \nmid m \geq 1, j \geq 1, i \geq 0, \text{ subject to } \\ \text{i) } j \leq p^n \text{ if } m = 1, \text{ ii) } p^i | j \leq a_{n-i}, \text{ and iii) } a_{n-i-1} < j \text{ if } p^{i+1} | j\}$$

for integers a_k defined by $a_0 = 1$ and $a_k = p^k + p^{k-1} - 1$. Here we use the abbreviation $\beta_{s/j,1} = \beta_{s/j}$ and $\beta_{s/1,1} = \beta_s$.

Let $V(1)$ denote the Toda-Smith spectrum, which is a cofiber of the Adams map $\alpha : \Sigma^{2p-2}V(0) \to V(0)$, where $V(0)$ is the mod p Moore spectrum. Since there exists a map $\beta : \Sigma^{2p^2-2}V(1) \to V(1)$ which induces v_2 on BP-homology at a prime $p > 3$ by [9], we have homotopy elements $\beta_t \in \pi_{2t(p^2-1)-2p}(S^0)$ with $t > 0$. On the other hand, there is no such self map at the prime 3. However there are homotopy elements β_i for $i = 1,2,3,5,6,10$ in this case due to Toda and Oka (*cf.* [2]). Besides, assuming the existence of the self map $B : \Sigma^{144}V(1) \to V(1)$ that induces v_2^9 on BP-homology, we see

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that there exists a family $\{\beta_{9t+i} \mid i = 0, 1, 2, 5, 6, t \geq 0\}$ in $\pi_*(S^0)$. The existence of B seems to be shown by Pemmaraju in his thesis. Furthermore, the existence of $\beta_{6/3} \in \pi_{82}(S^0)$ is shown by Ravenel [4].

In this paper, we obtain the following relations among β_{9t+1}, β_2 and $\beta_{6/3}$:

THEOREM A. *Let t, i, j and k be non-negative integers. Then in the homotopy groups $\pi_*(S^0)$ of sphere spectrum localized away from 3,*

$$\beta_{9t+1}\beta_1^i \neq 0 \in \pi_*(S^0) \qquad \text{if and only if } i < 5,$$

$$\beta_{9t+1}\beta_2\beta_1^j \neq 0 \in \pi_*(S^0) \qquad \text{if and only if } j < 2, \qquad \text{and}$$

$$\beta_{9t+1}\beta_{6/3}\beta_1^k \neq 0 \in \pi_*(S^0) \qquad \text{if and only if } k < 4.$$

As is seen in [3, p. 624], we have a relation

$$uv\beta_s\beta_t = st\beta_u\beta_v \qquad \text{for } s + t = u + v$$

in the E_2-term $E_2^4(S^0)$. This implies the following:

COROLLARY B. *In the homotopy groups $\pi_*(S^0)$ localized away from 3,*

$$\prod_{i=1}^k \beta_{9t_i+1} \neq 0 \qquad \text{if and only if } k < 6, \qquad \text{and}$$

$$\left(\prod_{i=1}^k \beta_{9t_i+1}\right)\beta_{9t+2} \neq 0 \qquad \text{if and only if } k < 2,$$

for integers $t, t_i \geq 0$. In particular, $\beta_{9t+1}^k \neq 0$ if and only if $k < 6$.

REMARK. If the self-map B does not exist, the above theorems are valid only for the homotopy elements such as β_1 and β_{10}.

We prove Theorem A by determining subgroups of $\pi_*(L_2S^0)$, where $L_2 : \mathscr{S}_{(3)} \to \mathscr{S}_{(3)}$ denotes the Bousfield localization functor on the category $\mathscr{S}_{(3)}$ of spectra localized away from 3 with respect to the Johnson-Wilson spectrum $E(2)$. In $\pi_*(L_2S^0)$, we have generalized β-elements $\beta_{s/j,i+1} \in E_2^2(L_2S^0)$ for $(s, j, i+1) \in \boldsymbol{B}$, where

$$\boldsymbol{B} = \{(s, j, i+1) \in \boldsymbol{Z}^3 \mid s = mp^n, n \geq 0, 3 \nmid m \in \boldsymbol{Z}, j \geq 1, i \geq 0,$$
$$\text{such that } 3^i \mid j \leq a_{n-i} \text{ and either } 3^{i+1} \nmid j \text{ or } a_{n-i-1} < j\}.$$

Consider the $\boldsymbol{Z}/3[\beta_1]$-modules

$$\begin{aligned}\hat{G} = \sum_{t \in \boldsymbol{Z}} (&B_5\{\beta_{9t+1}\} \oplus B_4\{\beta_{9t+1}\beta_{6/3}\} \\ &\oplus B_3\{\overline{\beta_{9t+7}\alpha_1}\} \\ &\oplus B_2\{\beta_{9t+1}\alpha_1, [\beta_{9t+2}\beta_1'], [\beta_{9t+5}\beta_1']\}),\end{aligned}$$

$$\hat{G}^* = \sum_{t\in \boldsymbol{Z}} (B_5\{g_{16(9t+7)+15}\} \oplus B_4\{g_{16(9t+3)+7}\}$$

$$\oplus B_2\{g_{144t}, g_{16(9t+5)+2}, g_{16(9t+8)+2}\}$$

$$\oplus \sum_{n\geq 1} (B_3\{g_{16(3^{n+2}t+9u+3)} \mid u \in \boldsymbol{Z} - I(n)\}$$

$$\oplus B_2\{g_{16(3^{n+2}t+9u+3)} \mid u \in I(n)\})).$$

Here $B_k = \boldsymbol{Z}/3[\beta_1]/(\beta_1^k)$,

$$I(n) = \{x \in \boldsymbol{Z} \mid x = (3^{n-1}-1)/2 \text{ or } x = 5\cdot 3^{n-2} + (3^{n-2}-1)/2\},$$

$\overline{x}$ denotes a homotopy element detected by x in the E_2-term, $[x]$ is an element of $\pi_*(L_2S^0)$ such that $i_*([x]) = x \in \pi_*(L_2V(0))$ for the inclusion $i: S^0 \to V(0) = S^0 \cup_3 e^1$, and $g_i \in \pi_i(L_2S^0)$ is the generator. Then the direct sum $\hat{G} \oplus \hat{G}^*$ is generated by

$$S = \{\beta_{9t+1}, \beta_{9t+1}\beta_{6/3}, \overline{\beta_{9t+7}\alpha_1}, \beta_{9t+1}\alpha_1, [\beta_{9t+2}\beta_1'], [\beta_{9t+5}\beta_1'],$$

$$g_{16(9t+7)+15}, g_{16(9t+3)+7}, g_{144t}, g_{16(9t+5)+2}, g_{16(9t+8)+2}, g_{16(9t+3)} \mid t \in \boldsymbol{Z}\}$$

as a $\boldsymbol{Z}/3[\beta_1]$-module. Our key lemma is the following:

THEOREM C. *The homotopy groups $\pi_*(L_2S^0)$ contain the subgroups $\hat{G} \oplus \hat{G}^*$.*

Consider the localization map $\iota: S^0 \to L_2S^0$. Then we immediately see the following:

COROLLARY D. *For any element $x \in \pi_*(S^0)$ such that $\iota_*(x) \in S$, we have $x\beta_1 \neq 0 \in \pi_*(S^0)$.*

In [7], we showed that the β-elements $\beta_{s/j,i+1}$ for $(s,j,i+1) \in \boldsymbol{B}^c$ do not exist in $\pi_*(L_2S^0)$, where

$$\boldsymbol{B}^c = \{(9t+4,1,1), (9t+7,1,1), (9t+8,1,1), (9t+3,3,1), (9s,3,2), (3^is, 3^i, 1)$$

$$\mid t \in \boldsymbol{Z}, s \in \boldsymbol{Z} - 3\boldsymbol{Z}, i > 1\}.$$

Moreover β-elements $\beta_{s/j,i+1}$ for $(s,j,i+1) \in \boldsymbol{B}^c$ do not exist in $\pi_*(S^0)$ if $t \geq 1$, $3 \not| s \geq 1$ and $i > 1$. We further showed in [7] the existence of β-elements β_{9t}, β_{9t+1} and β_{9t+5} in $\pi_*(L_2S^0)$ for $t \in \boldsymbol{Z}$. Here we extend the existence theorem of β-elements in $\pi_*(L_2S^0)$:

THEOREM E. *$\beta_{s/j,i+1}$ for $(s,j,i+1) \in \boldsymbol{B} - \boldsymbol{B}^c$ survives to a homotopy element of $\pi_*(L_2S^0)$.*

Note that $\beta_{s/j,i+1}$ are homotopy elements for all $(s,j,i+1)\in \boldsymbol{B}$ at a prime >5 ([8]).

The following sections 2 to 6 are devoted to proving Theorem C and to giving subgroups of $\pi_*(M^2)$ for an L_2-local spectrum M^2 with $E(2)_*(M^2)=E(2)_*/(3^\infty, v_1^\infty)$. Theorems A and E are actually corollaries of Theorem C, and proved in §7.

2. Basic properties of $H^*M_0^2$

Let $E(2)$ be the Johnson-Wilson spectrum with coefficient ring $E(2)_*=\boldsymbol{Z}_{(3)}[v_1,v_2^{\pm1}]$. Then $E(2)_*E(2)$ is a Hopf algebroid over $E(2)_*$ with $E(2)_*E(2)=E(2)_*[t_1,t_2,\dots]/(\eta_R(v_i):i>2)$. For an $E(2)_*E(2)$-comodule M, $\mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*,M)$ is the cohomology of the cobar complex $\Omega^*M=\Omega^*_{E(2)_*E(2)}M$, and we will denote it by H^*M.

The chromatic comodules N_j^i and M_j^i are defined inductively by $N_0^0=E(2)_*$, $N_1^0=E(2)_*/(3)$, $N_2^0=E(2)_*/(3,v_1)$, $M_j^i=v_{i+j}^{-1}N_j^i$ and the short exact sequence $0\to N_j^i\to M_j^i\to N_j^{i+1}\to 0$ for $i+j+1\le 2$ [1]. Note that $N_j^i=M_j^i$ if $i+j=2$. These have $E(2)_*E(2)$-comodule structure induced from the right unit $\eta_R: E(2)_*\to E(2)_*E(2)$. Consider the long exact sequences associated to these short ones $0\to N_0^0\to M_0^0\to N_0^1\to 0$ and $0\to N_0^1\to M_0^1\xrightarrow{f} M_0^2\to 0$, whose connecting homomorphisms are $\delta': H^sN_0^1\to H^{s+1}E(2)_*$ and $\delta: H^sM_0^2\to H^sN_0^1$. Then we see that $\delta'\delta: H^sM_0^2\to H^{s+2}E(2)_*$ is an epimorphism if $s\ge 1$, and an isomorphism if $s>1$, since $H^sM_0^0=0$ for $s\ge1$ and $H^sM_0^1=0$ for $s>1$ by [1]. In particular,

LEMMA 2.1. *$H^sE(2)_*$ for $s\ge 3$ consists of torsion elements.*

We have a short exact sequence $0\to M_1^1\xrightarrow{i} M_0^2\xrightarrow{3} M_0^2\to 0$ $(i(x)=x/3)$ which induces a long exact sequence

$$\cdots\to H^{s-1}M_0^2\xrightarrow{\delta} H^sM_1^1\xrightarrow{i_*} H^sM_0^2\xrightarrow{3} H^sM_0^2\xrightarrow{\delta}\cdots.$$

An easy diagram chasing shows the following:

LEMMA 2.2. ([1, Remark 3.11]) *Consider the following commutative diagram of modules with horizontal exact sequences and a 3 torsion module B^s:*

$$\begin{tikzcd}
B^{s-1} \arrow[r, "\delta"] \arrow[d, "f^{s-1}"'] & H^sM_1^1 \arrow[r, "i_*"] \arrow[d, "g^s"'] & B^s \arrow[r, "3"] \arrow[d, "f^s"'] & B^s \arrow[r, "\delta"] \arrow[d, "f^s"'] & H^{s+1}M_1^1 \arrow[d, "g^{s+1}"'] \\
H^{t-1}M_0^2 \arrow[r, "\delta"] & H^tM_1^1 \arrow[r, "i_*"] & H^tM_0^2 \arrow[r, "3"] & H^tM_0^2 \arrow[r, "\delta"] & H^{t+1}M_1^1
\end{tikzcd}$$

If g^s and g^{s+1} are isomorphisms, then f^s is an epimorphism. Moreover, if f^{s-1} is an epimorphism, then f^s is an isomorphism.

Let b_{10} denote the element of $H^2E(2)_*$ represented by the cocycle $-t_1 \otimes t_1^2 - t_1^2 \otimes t_1$. Then b_{10} acts on H^*M for any comodule M. In [7], we show the following:

PROPOSITION 2.3. *The multiplication by b_{10} yields an isomorphism $H^sM_1^1 \to H^{s+2}M_1^1$ if $s > 3$ and an epimorphism if $s = 3$.*

This together with Lemma 2.2 implies

COROLLARY 2.4. *The multiplication by b_{10} yields an isomorphism $H^sM_0^2 \to H^{s+2}M_0^2$ if $s > 3$ and an epimorphism if $s = 3$.*

COROLLARY 2.5. *$H^*M_0^2 \cong (H^4M_0^2 \oplus H^5M_0^2) \otimes \boldsymbol{Z}/3[b_{10}]$ for $* > 3$.*

3. Some formulae in $\Omega^*E(2)_*$

The Adams-Novikov E_2-term for computing $\pi_*(L_2X)$ is the cohomology $H^*E(2)_*(X)$ of the cobar complex $\Omega^*E(2)_*(X)$, and in particular, $H^*E(2)_* = H^*N_0^0$ is the E_2-term for $\pi_*(L_2S^0)$.

Take X to be the Toda-Smith spectrum $V(1)$. Then $E(2)_*(V(1)) = K(2)_* = \boldsymbol{Z}/3[v_2^{\pm 1}]$ and $H^*K(2)_* = K(2)_*[b_{10}] \otimes F \otimes \Lambda(\zeta_2)$. Here F denotes the module $\boldsymbol{Z}/3\{1, h_{10}, h_{11}, b_{11}, \xi, \psi_0, \psi_1, b_{11}\xi\}$ satisfying the following relations (*cf.* [6, Prop. 5.9]):

$$
\begin{gathered}
v_2^2 h_{10} b_{10} = h_{11} b_{11}, \qquad v_2 h_{11} b_{10} = -h_{10} b_{11} \\
b_{11}\xi = v_2 h_{10} \psi_1 = v_2 h_{11} \psi_0, \qquad b_{10}\xi = -h_{10}\psi_0 = v_2^{-1} h_{11}\psi_1, \\
v_2^3 b_{10}^2 = -b_{11}^2, \qquad b_{10}\psi_1 = -v_2^{-1} b_{11}\psi_0, \qquad b_{10}\psi_0 = v_2^{-2} b_{11}\psi_1
\end{gathered}
\tag{3.1}
$$

If X is the mod 3 Moore spectrum $V(0)$, then the E_2-term for $V(0)$ is $H^*E(2)_*/(3)$, which is determined in [7]. In particular, we see the following:

LEMMA 3.2.

$$
\begin{aligned}
&H^{2,0}E(2)_*/(3) = 0 \\
&H^{3,0}E(2)_*/(3) = \{v_2^{-1}\psi_0, v_2^{-1}h_{10}b_{10}\} \\
&H^{4,0}E(2)_*/(3) = \{v_2^{-3}b_{10}b_{11}, v_2^{-1}\psi_0\zeta_2, v_2^{-1}h_{10}b_{10}\zeta_2\} \\
&H^{5,0}E(2)_*/(3) = \{v_2^{-3}b_{10}b_{11}\zeta_2\}.
\end{aligned}
$$

Consider the long exact sequence

$$
\cdots \to H^{*-1}E(2)_*/(3) \xrightarrow{\delta} H^*E(2)_* \xrightarrow{3} H^*E(2)_* \xrightarrow{j_*} H^*E(2)_*/(3) \xrightarrow{\delta} \cdots
$$

associated to the short exact sequence $0 \to E(2)_* \xrightarrow{3} E(2)_* \xrightarrow{j} E(2)_*/(3) \to 0$.

LEMMA 3.3. *The map* $d_1 = j_*\delta : H^*E(2)_*/(3) \to H^{*+1}E(2)_*/(3)$ *sends* $v_2^{-1}h_{10}b_{10}$ *(resp.* $v_2^{-1}h_{10}b_{10}\zeta_2$*) to* $v_2^{-3}b_{10}b_{11}$ *(resp.* $v_2^{-3}b_{10}b_{11}\zeta_2$*).*

PROOF. Note that $v_2^{-1}h_{10}b_{10}$ is represented by a cochain whose leading term is $v_2^{-3}t_1^3 \otimes b_{11}$. Since $d(t_1^{3^{i+1}}) = 3b_{1i}$ by definition, we compute

$$d(v_2^{-3}t_1^3 \otimes b_{11}) = 3(v_2^{-3}b_{10} \otimes b_{11} + \cdots), \tag{3.4}$$

which shows $\delta(v_2^{-1}h_{10}b_{10}) = v_2^{-3}b_{10}b_{11} + \cdots$. For $v_2^{-1}h_{10}b_{10}\zeta_2$, the result follows immediately from (3.4) and Proposition 4.2 in the next section. q.e.d.

Let x denote a cochain that represents ξ. Then it is shown in [7, Lemma 4.4] that $d(x) \equiv v_1^2 f_0 \mod (3)$ for f_0 that represents $-v_2^{-1}\psi_0 \mod (3, v_1)$ (In [7], x is denoted by $X(0)$). So we have a cochain A such that

$$d(x) \equiv v_1^2 f_0 \mod (3) \qquad \text{and} \qquad d(f_0) = 3A \tag{3.5}$$

in the cobar complex $\Omega^*E(2)_*$. Then A is a cocycle of $H^{4,0}E(2)_*$. Furthermore, we have

LEMMA 3.6. $d(f_0) = \pm 3f_0 \otimes z \mod (9)$ *in the cobar complex* $\Omega^4E(2)_*$. *Here* z *denotes a cocycle that represents the generator* ζ_2.

PROOF. The projection $E(2)_* \to E(2)_*/(3)$ sends A in (3.5) to a cocycle, which is also denoted by A. By virtue of Lemmas 3.2 and 3.3, we put

$$[A] = k_1 v_2^{-1}h_{10}b_{10}\zeta_2 + k_2 v_2^{-1}\psi_0\zeta_2,$$

where $[A]$ denotes a cohomology class represented by $A \in \Omega^*E(2)_*/(3)$. In fact, f_0 may be replaced by $f_0 + kv_2^{-3}t_1^3 \otimes b_{11}$ for some $k \in \mathbf{Z}/3$ if necessary. Since $d_1([A]) = 0$ and $d_1(v_2^{-1}\psi_0) = 3[A]$ by definition,

$$0 = k_1 v_2^{-3}b_{10}b_{11}\zeta_2 + k_2(k_1 v_2^{-1}h_{10}b_{10}\zeta_2 + k_2 v_2^{-1}\psi_0\zeta_2)\zeta_2,$$

by Lemma 3.3. Noticing that $\zeta_2^2 = 0$ and $v_2^{-3}b_{10}b_{11}\zeta_2 \neq 0$, we see that $k_1 = 0$. On the other hand, if A represents 0, then we have an element $v_2^{-1}\psi_0$ in $H^{3,0}E(2)_*$. Since $H^{2,0}E(2)_*/(3) = 0$ by Lemma 3.2, $v_2^{-1}\psi_0$ generates a $\mathbf{Z}_{(3)}$-free submodule in $H^{3,0}E(2)_*$, which contradicts Lemma 2.1. Therefore A represents a non-zero element. This means that $k_2 = \pm 1$. q.e.d.

LEMMA 3.7. *Put* $\widetilde{v_2t_1} = v_2t_1 + v_1\tau$ *and* $\widetilde{v_2^2t_1^3} = v_2^2t_1^3 + v_1v_2t_1^6$. *Then* $d(\widetilde{v_2t_1}) = v_1^2b_{10}$ *and* $d(\widetilde{v_2^2t_1^3}) = -v_1^2b_{11}$. *Furthermore, there exists a cochain* $u \in \Omega^2E(2)_*$ *such that*

$$d(u) \equiv \widetilde{v_2t_1} \otimes b_{11} + \widetilde{v_2^2t_1^3} \otimes b_{10} \mod (3, v_1^2).$$

PROOF. The first statement is checked by a routine computation.

Turn to the second. We obtain an element $u' \in \Omega^2 E(2)_*$ such that $d(u') \equiv v_2 t_1 \otimes b_{11} + b_{10} \otimes v_2^2 t_1^3 \bmod (3, v_1)$ from the relation $v_2 h_{11} b_{10} = -h_{10} b_{11}$ in $H^3 E(2)_*/(3, v_1)$ of (3.1). Put $d(u') \equiv \widetilde{v_2 t_1} \otimes b_{11} + b_{10} \otimes \widetilde{v_2^2 t_1^3} + v_1 w \bmod (3, v_1^3)$ for some cochain w. Sending this by d, we have $0 \equiv v_1^2 b_{10} \otimes b_{11} - b_{10} \otimes v_1^2 b_{11} + v_1 d(w) \bmod (3, v_1^3)$. Then $w \in H^{3,52} E(2)_*/(3, v_1^2)$, which is 0 since $H^{3,52} E(2)_*/(3, v_1) = \{v_2 b_{11} \zeta_2\}$ by [6, Th. 5.8] and $d(v_2 b_{11} \zeta_2) \not\equiv 0 \bmod (3, v_1^2)$ by [7, Lemma 3.3]. Therefore, we see that there is a cochain $\overline{w}$ such that $d(\overline{w}) \equiv w$, and put $u'' = u' - v_1 \overline{w}$ to obtain $d(u'') \equiv \widetilde{v_2 t_1} \otimes b_{11} + b_{10} \otimes \widetilde{v_2^2 t_1^3} \bmod (3, v_1^3)$. There is also a cochain a such that $d(a) \equiv \widetilde{v_2^2 t_1^3} \otimes b_{10} - b_{10} \otimes \widetilde{v_2^2 t_1^3} \bmod (3, v_1^2)$, and so we have the lemma by putting $u = u'' + a$. q.e.d.

Lemma 3.8. *There exists a cochain w such that*

$$d(w) \equiv \widetilde{v_2 t_1} \otimes f_0 - x' \otimes b_{10} \quad \bmod (3, v_1^3).$$

Here x' denotes a cocycle that represents ξ $\bmod (3, v_1)$ *and $t_1 \otimes x'$ is homologous to $t_1 \otimes x$* $\bmod (3, v_1^3)$.

Proof. This is shown in the same way as the above lemma. By the equation $h_{10} \psi_0 = -\xi b_{10}$ in (3.1), we have a cochain w' such that $d(w') \equiv v_2 t_1 \otimes f_0 - b_{10} \otimes x \bmod (3, v_1)$. Put $d(w') \equiv \widetilde{v_2 t_1} \otimes f_0 - b_{10} \otimes x + v_1 a$ for a cochain a, and send this by d. Then we see that a is a cocycle of $\Omega^{4,16} E(2)_*/(3, v_1^2)$. Since we see that $H^{4,16} E(2)_*/(3, v_1^2) = \{h_{10} b_{10} \zeta_2\}$ by [7], $a = k t_1 \otimes z \otimes b_{10}$ for some $k \in \boldsymbol{Z}/3$. Furthermore, $b_{10} \otimes x$ is homologous to $x \otimes b_{10}$, which yields a cochain w such that $d(w) \equiv \widetilde{v_2 t_1} \otimes f_0 - (x - k v_1 t_1 \otimes z) \otimes b_{10}$. Now put $x' = x - k v_1 t_1 \otimes z$, and we have the lemma. q.e.d.

Lemma 3.9. *In the cobar complex $\Omega^* E(2)_*$, there exists a cochain y such that*

$$d(y) \equiv t_1 \otimes x - v_1 v_2^{-1} f_1 - v_1 z \otimes x - k v_1 v_2^{-2} t_1 \otimes b_{11} \quad \bmod (3, v_1^2)$$

for some $k \in \boldsymbol{Z}/3$. Here f_1 denotes a cocycle that represents ψ_1.

Proof. It is shown in [7, Lemma 6.4] that there exists a cochain Y_0 such that $d(Y_0) \equiv t_1 \otimes X + v_1 v_2^{-3} \tau^3 \otimes X + v_1^2 v_2^{-1} t_1^3 \otimes X \bmod (3, v_1^3)$. It is also shown that $x \equiv X + v_1 v_2^{-1} Y_1 + k v_1 v_2^{-2} b_{11}$ for some $k \in \boldsymbol{Z}/3$ in [7, Proof of Lemma 4.4.]. Take now y to be Y_0, and we obtain

$$d(y) \equiv t_1 \otimes (x - v_1 v_2^{-1} Y_1 - k v_1 v_2^{-2} b_{11}) - v_1 z \otimes X + v_1 v_2^{-1} t_2 \otimes X$$

$\bmod (3, v_1^2)$. Since $f_1 = t_1 \otimes Y_1 - t_2 \otimes X$ by the proof of [7, Lemma 4.4], we have the result. q.e.d.

4. The E_2-terms $H^sM_0^2$ for $s > 3$.

Let $E(2,1)_*$ denote $\boldsymbol{Z}/3[v_1, v_2^{\pm 3}]$. In [7], $H^*M_1^1$ is given as the direct sum of three $E(2,1)_*$-modules A_i:

$$H^*M_1^1 = A_0 \oplus A_1 \oplus A_2.$$

In order to describe the modules A_i, we use the following notation:

$$\begin{aligned}
k(1)_* &= \boldsymbol{Z}/3[v_1] \\
K(1)_* &= \boldsymbol{Z}/3[v_1^{\pm 1}] \\
PE &= \boldsymbol{Z}/3[b_{10}] \otimes \Lambda(\zeta_2) \\
E(2,n)_* &= \boldsymbol{Z}/3[v_1, v_2^{\pm 3^n}] \\
F_{(h)} &= \boldsymbol{Z}/3[v_2^{\pm 3}]\{v_2/v_1, v_2h_{10}/v_1, v_2^2h_{11}/v_1, v_2b_{11}/v_1\} \\
F_{(t)} &= \boldsymbol{Z}/3[v_2^{\pm 3}]\{v_2^{-1}/v_1, v_2h_{10}/v_1^2, v_2^2h_{11}/v_1^2, v_2^{-1}b_{11}/v_1\} \\
F_{(h)}^* &= \boldsymbol{Z}/3[v_2^{\pm 3}]\{\xi/v_1, \psi_0/v_1, v_2\psi_1/v_1, b_{11}\xi/v_1\} \\
F_{(t)}^* &= \boldsymbol{Z}/3[v_2^{\pm 3}]\{\xi/v_1^2, v_2^{-2}\psi_0/v_1, v_2^{-1}\psi_1/v_1, b_{11}\xi/v_1^2\} \\
F_n &= E(2,n+2)_*\{v_2^{\pm 3^{n+1}}/v_1^{4\cdot 3^n-1}, v_2^{3^{n+1}}h_{10}/v_1^{6\cdot 3^n+1}, \\
&\qquad v_2^{8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n+1}, v_2^{3^n(5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}\}.
\end{aligned}$$

Then the modules A_i are given as follows:

$$\begin{aligned}
A_0 &= (K(1)_*/k(1)_*) \oplus \Lambda(h_{10}, \zeta_2) \\
A_1 &= \sum_{n\geq 0} F_n \oplus \Lambda(\zeta_2) \\
A_2 &= (F_{(h)} \oplus F_{(t)} \oplus F_{(h)}^* \oplus F_{(t)}^*) \oplus PE,
\end{aligned}$$

Consider the exact sequence $H^{1,0}M_1^1 \xrightarrow{\delta} H^{2,0}E(2)_*/(3, v_1^{3^i}) \to H^{2,-4\cdot 3^i}M_1^1$ associated to the short exact sequence $0 \to E(2)_*/(3, v_1^{3^i}) \xrightarrow{1/v_1^{3^i}} M_1^1 \xrightarrow{v_1^{3^i}} M_1^1 \to 0$. Then the structure of $H^*M_1^1$ shows immediately the following:

LEMMA 4.1. *For each $i > 0$, each element of $H^{2,0}E(2)_*/(3, v_1^{3^i})$ is divisible by $v_1^{3^i-3^{i-1}}$.*

Now we obtain the next proposition corresponding to [5, Lemma 2.6]. In fact, the proof of [5, Lemma 2.6] also works at the prime 3 by the above lemma.

Proposition 4.2. *For each integer $i > 0$, there exists a cocycle z_i of $\Omega^{1,0}E(2)_*/(3^{i+1}, v_1^{3^i})$ such that $z_i = z \in \Omega^1 K(2)_*$.*

By virtue of this proposition, we abuse the notation z for a cocycle that represents ζ_2 as we did in the previous papers for a prime >3.

Consider the connecting homomorphism $\delta : H^s M_0^2 \to H^{s+1} M_1^1$ associated to the short exact sequence $0 \to M_1^1 \xrightarrow{i} M_0^2 \xrightarrow{3} M_0^2 \to 0$.

Lemma 4.3. $\delta(v_2^2\xi/3v_1^3) = v_2\psi_0/v_1$.

Proof. In [7, Lemma 4.4], it is shown that there exists a cochain $X(2)$ such that $d(X(2)) \equiv v_1^4 z^3 \otimes X^3 - v_1^4 v_2^{-3} f_1^3 \mod (3, v_1^5)$. Since $H^{3,20}M_2^0 = 0$, the congruence holds mod $(3, v_1^6)$ if we replace $X(2)$ by a suitable cochain x''. Put now $d(x'') \equiv v_1^4 z^3 \otimes X^3 - v_1^4 v_2^{-3} f_1^3 + 3A \mod (9, v_1^6)$. Then $0 \equiv 3v_1^3 t_1 \otimes z^3 \otimes X^3 - 3v_1^3 t_1 \otimes v_2^{-3} f_1^3 + 3d(A) \mod (9, v_1^4)$. This implies that $A \in H^{3,40}E(2)_*/(3, v_1^3)$, which is $\boldsymbol{Z}/3\{v_1^2 v_2 \psi_0\}$ by [7]. Note that $3v_1^3 t_1 \otimes z^3 \otimes X^3$ and $v_1^3 t_1 \otimes v_2^{-3} f_1^3$ are homologous to zero and $v_1^3 b_{11} \otimes X \mod (9, v_1^4)$, respectively, by Lemma 3.9 and (3.1), and that $d(-v_1^2 v_2(v_2 f_0)) \equiv v_1^3 v_2 t_1^3 \otimes f_0 + v_1^3 b_{11} \otimes X \mod (3, v_1^4)$ by [7, Lemma 4.3] whose right hand side is homologous to $-v_1^3 b_{11} \otimes X$ by (3.1). Thus $d(A) = d(-v_1^2 v_2^2 f_0 + \cdots)$, and we see that A is homologous to $-v_1^2 v_2^2 f_0$, and that $d(x'') \equiv -3v_1^2 v_2^2 f_0 \mod (9, v_1^3)$. Since x'' and f_0 represent $v_2^2\xi$ and $-v_2^{-1}\psi_0$, respectively, we have the lemma. q.e.d.

Put

$$G^s = (i_*(F_{(h)} \oplus F_{(h)}^*) \otimes E(2,1)_*[b_{10}] \otimes \Lambda(\zeta_2))^s \subset H^s M_0^2$$

for $i_* : H^* M_1^1 \to H^* M_0^2$ given by $i_*(x) = x/3$.

Lemma 4.4. *For the connecting homomorphism $\delta : H^s M_0^2 \to H^{s+1} M_1^1$, $\delta(x)$ for a generator x of G^s is obtained by the following equations:*

$$\delta(v_2/3v_1) = -v_2 h_{10}/v_1^2,$$

$$\delta(v_2 h_{10}/3v_1) = v_2^{-1} b_{11}/v_1 + v_2 h_{10}\zeta_2/v_1,$$

$$\delta(v_2^2 h_{11}/3v_1) = v_2^2 b_{10}/v_1 + v_2^2 h_{11}\zeta_2/v_1,$$

$$\delta(v_2 b_{11}/3v_1) = v_2^2 h_{11} b_{10}/v_1^2;$$

$$\delta(v_2(v_2^{-1}\psi_0)/3v_1) = \xi b_{10}/v_1^2 \pm v_2(v_2^{-1}\psi_0)\zeta_2/v_1,$$

$$\delta(\xi/3v_1) = v_2^{-1}\psi_1/v_1 + (1 \pm 1)\xi\zeta_2/v_1 + kv_2^{-1} h_{11} b_{10}/v_1$$

$$\delta(b_{11}\xi/3v_1) = v_2^2(v_2^{-1}\psi_0) b_{10}/v_1 + (1 \pm 1) b_{11}\xi\zeta_2/v_1 + kv_2 h_{10} b_{10}^2/v_1,$$

$$\delta(v_2\psi_1/3v_1) = b_{11}\xi/v_1^2 \pm v_2\psi_1\zeta_2/v_1.$$

PROOF. Note that v_2h_{10} is represented by a cochain $\widetilde{v_2t_1}=\eta_R(v_2)t_1-v_1t_2$ of Lemma 3.7. In the cobar complex $\Omega^*E(2)_*/(9,v_1^3)$, we compute

$$
\begin{aligned}
d(v_1^2v_2) &= 6v_1t_1\eta_R(v_2)+3v_1^2t_2=6v_1\widetilde{v_2t_1};\\
d(v_1^2v_2t_1) &= 6v_1t_1\otimes v_2t_1+3v_1^2t_2\otimes t_1\\
&= \underline{6v_1v_2t_1\otimes t_1}_1+\underline{6v_1^2\tau\otimes t_1}_2\\
d(3v_1v_2t_1^2) &= \underline{3v_1^2t_1^3\otimes t_1^2}_3-\underline{6v_1v_2t_1\otimes t_1}_1\\
d(3v_1^2t_1\tau) &= -3v_1^2(\underline{t_1^4\otimes t_1}_2+\underline{t_1^3\otimes t_1^2}_3+t_1\otimes\tau+\underline{\tau\otimes t_1}_2)\\
d(-3v_1^2v_2^{-2}t_3) &= 3v_1^2v_2^{-2}(t_1\otimes t_2^3+\underline{t_2\otimes t_1^9}_2+v_2b_{11});\\
d(v_1^2v_2^2t_1^3) &= 6v_1t_1\otimes v_2^2t_1^3+6v_1^2v_2t_2\otimes t_1^3+3v_1^2v_2^2b_{10}\\
&= \underline{6v_1v_2^2t_1\otimes t_1^3}_4+\underline{12v_1^2v_2t_1^4\otimes t_1^3}_7\\
&\quad+\underline{6v_1^2v_2t_2\otimes t_1^3}_7+\underline{3v_1^2v_2^2b_{10}}_6\\
d(6v_1v_2^2t_2) &= \underline{12v_1^2v_2t_1^3\otimes t_2}_5-\underline{6v_1v_2^2t_1\otimes t_1^3}_4-\underline{6v_1^2v_2^2b_{10}}_6\\
d(3v_1^2v_2^{-7}t_3^3) &= -3v_1^2v_2^{-7}(t_1^3\otimes t_2^9+t_2^3\otimes t_1^{27}+v_2^3b_{11}^3)\\
&= -3v_1^2v_2^{-7}(\underline{v_2^8t_1^3\otimes t_2}_5+\underline{v_2^{-6}t_2^3\otimes t_1^3}_7+\underline{v_2^9b_{10})}_6.
\end{aligned}
\tag{4.5}
$$

The underlined terms with the same number sum up to zero except for the terms numbered 6 and 7. The terms numbered 6 and 7 sum up to $3v_1^2v_2^2b_{10}$ and $6v_1^2v_2^2\zeta_2\otimes t_1^3$, respectively. These imply the first three equations. In fact, $\delta([a/3v_1])=[(i_*)^{-1}d(v_1^2a)/9v_1^3]$, where $[a]$ denotes a homology class represented by a. Since $\delta(b_{11}a)=b_{11}\delta(a)$, the first equation gives $\delta(v_2b_{11}/3v_1)=-v_2h_{10}b_{11}/v_1^2$, which equals $v_2^2h_{11}b_{10}/v_1^2$ by Lemma 3.7. Thus we have the fourth equation.

By Lemma 3.6 and the first equation in (4.5), we compute

$$d(v_1^2v_2f_0)\equiv 6v_1\widetilde{v_2t_1}\otimes f_0\pm 3v_1^2v_2f_0\otimes z \mod (9,v_1^3). \tag{4.6}$$

Now by Lemma 3.8, we have the fifth equation. Multiplying h_{10} to the fifth equation yields $\delta(h_{10}\psi_0/3v_1)=h_{10}\xi b_{10}/v_1^2+h_{10}\psi_0\zeta_2/v_1$. Lemma 3.9 says that $h_{10}\xi=v_1v_2^{-1}\psi_1+v_1\xi\zeta_2+kv_1v_2^{-2}h_{10}b_{11}$ for some $k\in \boldsymbol{Z}/3$. Since $h_{10}\psi_0=-\xi b_{10}$, we have the sixth equation. The seventh follows immediately from the product of b_{11} and the sixth equation. The multiplication of b_{10} and the fifth equation gives the last one by the relations (3.1). Here note that $b_{10}^2=-v_2^{-3}b_{11}^2$ holds in $H^4E(2)_*/(3,v_1^2)$. q.e.d.

Proposition 4.7. *$H^sM_0^2 = G^s$ for $s = 4, 5$. In other words, $H^4M_0^2$ and $H^5M_0^2$ are $\boldsymbol{Z}/3[v_2^{\pm 3}]$-modules generated by*

$$G^4: \begin{array}{l} v_2b_{10}^2/3v_1, v_2h_{10}b_{10}\zeta_2/3v_1, v_2^2h_{11}b_{10}\zeta_2/3v_1, v_2b_{11}b_{10}/3v_1; \\ \qquad \psi_0\zeta_2/3v_1, \xi b_{10}/3v_1, b_{11}\xi/3v_1, v_2\psi_1\zeta_2/3v_1, \end{array}$$

and

$$G^5: \begin{array}{l} v_2b_{10}^2\zeta_1/3v_1, v_2h_{10}b_{10}^2/3v_1, v_2^2h_{11}b_{10}^2/3v_1, v_2b_{11}b_{10}\zeta_2/3v_1; \\ \qquad \psi_0b_{10}/3v_1, \xi b_{10}\zeta_2/3v_1, b_{11}\xi\zeta_2/3v_1, v_2\psi_1b_{10}/3v_1, \end{array}$$

respectively.

Proof. Put $B^s = G^s$. Then there is a canonical map $f^s: B^s \to H^sM_0^2$ sitting in the commutative diagram

$$\begin{array}{ccccccccc} H^{s-1}M_0^2 & \xrightarrow{\delta} & H^sM_1^1 & \xrightarrow{i_*} & B^s & \xrightarrow{3} & B^s & \xrightarrow{\delta} & H^{s+1}M_1^1 \\ \| & & \| & & \downarrow{\scriptstyle f^s} & & \downarrow{\scriptstyle f^s} & & \| \\ H^{s-1}M_0^2 & \xrightarrow{\delta} & H^sM_1^1 & \xrightarrow{i_*} & H^sM_0^2 & \xrightarrow{3} & H^sM_0^2 & \xrightarrow{\delta} & H^{s+1}M_1^1. \end{array}$$

Lemma 4.4 implies that the δ-images of the generators of B^s are linearly independent. Therefore we see that the above sequence is exact, and Lemma 2.2 shows that f^s is an isomorphism. q.e.d.

5. On the E_2-terms $H^sM_0^2$ for $s \le 3$

We write down the submodules $A_2^s \subset H^sM_1^1$:

$$A_2^0 = \boldsymbol{Z}/3\{v_2/v_1, v_2^{-1}/v_1\}$$

$$A_2^1 = \boldsymbol{Z}/3\{v_2h_{10}/v_1^2, v_2h_{10}/v_1, v_2^2h_{11}/v_1^2, v_2^2h_{11}/v_1, v_2\zeta_2/v_1, v_2^{-1}\zeta_2/v_1\}$$

$$\begin{aligned} A_2^2 = \boldsymbol{Z}/3\{&v_2b_{11}/v_1, v_2^{-1}b_{11}/v_1, v_2h_{10}\zeta_2/v_1^2, v_2h_{10}\zeta_2/v_1, \\ &v_2^2h_{11}\zeta_2/v_1^2, v_2^2h_{11}\zeta_2/v_1, v_2b_{10}/v_1, v_2^{-1}b_{10}/v_1, \xi/v_1^2, \xi/v_1\} \end{aligned}$$

$$\begin{aligned} A_2^3 = \boldsymbol{Z}/3\{&v_2b_{11}\zeta_2/v_1, v_2^{-1}b_{11}\zeta_2/v_1, v_2h_{10}b_{10}/v_1^2, v_2h_{10}b_{10}/v_1, \\ &v_2^2h_{11}b_{10}/v_1^2, v_2^2h_{11}b_{10}/v_1, v_2b_{10}\zeta_2/v_1, v_2^{-1}b_{10}\zeta_2/v_1, \\ &\psi_0/v_1, v_2\psi_0/v_1, v_2\psi_1/v_1, v_2^{-1}\psi_1/v_1, \xi\zeta_2/v_1^2, \xi\zeta_2/v_1\}. \end{aligned}$$

Now consider the map $d_1 = \delta i_*: H^sM_1^1 \to H^{s+1}M_1^1$. Then [1, Prop. 6.9] shows

$$d_1(v_2^3/v_1^3) = v_2^2h_{11}/v_1^2. \tag{5.1}$$

Here we compute:

LEMMA 5.2. *The Bockstein differential $d_1 = \delta i_*$ acts up to sign as follows:*

$$\begin{aligned}
d_1(v_2/v_1) &= v_2h_{10}/v_1^2,\\
d_1(v_2^{-1}/v_1) &= v_2^{-1}\zeta_2/v_1;\\
d_1(v_2h_{10}/v_1) &= v_2^{-1}b_{11}/v_1 + v_2h_{10}\zeta_2/v_1,\\
d_1(v_2^2h_{11}/v_1) &= v_2^{-1}b_{10}/v_1 + v_2^{-1}h_{11}\zeta_2/v_1,\\
d_1(v_2\zeta_2/v_1) &= v_2h_{10}\zeta_2/v_1^2;\\
d_1(v_2b_{11}/v_1) &= v_2^2h_{11}b_{10}/v_1^2,\\
d_1(v_2h_{10}\zeta_2/v_1) &= v_2^{-1}b_{11}\zeta_2/v_1^2,\\
d_1(v_2^2h_{11}\zeta_2/v_1) &= v_2^2b_{10}\zeta_2/v_1,\\
d_1(v_2b_{10}/v_1) &= v_2h_{10}b_{10}/v_1^2,\\
d_1(\xi/v_1^2) &= \xi\zeta_2/v_1^2,\\
d_1(\xi/v_1) &= v_2^{-1}\psi_1/v_1 + \xi\zeta_2/v_1;\\
d_1(v_2b_{11}\zeta_2/v_1) &= v_2^2h_{11}b_{10}\zeta_2/v_1^2,\\
d_1(v_2h_{10}b_{10}/v_1) &= v_2^{-1}b_{11}b_{10}/v_1^2 + v_2h_{10}b_{10}\zeta_2/v_1,\\
d_1(v_2^2h_{11}b_{10}/v_1) &= v_2^{-1}b_{10}^2/v_1 + v_2^{-1}h_{11}b_{10}\zeta_2/v_1,\\
d_1(v_2b_{10}\zeta_2/v_1) &= v_2h_{10}b_{10}\zeta_2/v_1^2;\\
d_1(\xi\zeta_2/v_1) &= v_2^{-1}\psi_1\zeta_2/v_1,\\
d_1(\psi_0/v_1) &= \xi b_{10}/v_1^2 + \psi_0\zeta_2/v_1,\\
d_1(v_2\psi_1/v_1) &= b_{11}\xi/v_1^2.
\end{aligned}$$

The other elements of A_2 missing in the left hand sides are in the image of d_1.

PROOF. Lemma 4.3 and (5.1) show that $v_2^{-2}\psi_0/v_1$ and $v_2^2h_{11}/v_1^2$ are in the image of d_1. The other parts follow from Lemma 4.4, except for d_1 on v_2^{-1}/v_1 and ξ/v_1^2.

For the exceptional cases, consider the diagram

$$\begin{tikzcd}
H^{s-1}M_0^2 \arrow[r, "\delta"] \arrow[d, "b_{10}"] & H^sM_1^1 \arrow[r, "i_*"] \arrow[d, "b_{10}"] & H^sM_0^2 \arrow[r, "3"] \arrow[d, "b_{10}"] & H^sM_0^2 \arrow[r, "\delta"] \arrow[d, "b_{10}"] & H^{s+1}M_1^1 \arrow[d, "b_{10}"] \\
H^{s+1}M_0^2 \arrow[r, "\delta"] & H^{s+2}M_1^1 \arrow[r, "i_*"] & H^{s+2}M_0^2 \arrow[r, "3"] & H^{s+2}M_0^2 \arrow[r, "\delta"] & H^{s+3}M_1^1.
\end{tikzcd}$$

If we have a relation $\delta(\alpha/3) = \beta b_{10} + \alpha\zeta_2$ in Lemma 4.4, then we see that $\beta b_{10}/3 = -\alpha\zeta_2/3$ in $H^*M_0^2$, since $i_*(x) = x/3$. Therefore, we compute

$$b_{10}\delta(\beta/3) = \delta(\beta b_{10}/3) = -\delta(\alpha\zeta_2/3) = -\delta(\alpha/3)\zeta_2 = -\beta b_{10}\zeta_2,$$

and so we obtain

$$\delta(\beta/3) = -\beta\zeta_2$$

up to Ker b_{10}. Note that b_{10} acts monomorphically on A_2. Now take β to be the exceptional cases, and we have all d_1. q.e.d.

Hence, we have

PROPOSITION 5.3. *$H^sM_0^2$ contains $E(2,1)_*/(3, v_1)$-module as follows:*

$$H^0M_0^2 \supset E(2,1)_*\{v_2^{\pm 1}/3v_1\}$$

$$H^1M_0^2 \supset E(2,1)_*\{v_2h_{10}/3v_1, v_2^2h_{11}/3v_1, v_2\zeta_2/3v_1\}$$

$$H^2M_0^2 \supset E(2,1)_*\{v_2b_{11}/3v_1, v_2h_{10}\zeta_2/3v_1, v_2^2h_{11}\zeta_2/3v_1, v_2b_{10}/3v_1, \xi/3v_1^2, \xi/3v_1\}$$

$$H^3M_0^2 \supset E(2,1)_*\{v_2b_{11}\zeta_2/3v_1, v_2h_{10}b_{10}/3v_1, v_2^2h_{11}b_{10}/3v_1, v_2b_{10}\zeta_2/3v_1$$
$$\xi\zeta_2/3v_1, \psi_0/3v_1, v_2\psi_1/3v_1\}.$$

6. The Adams-Novikov differentials

Now consider spectra defined by cofiber sequences:

(6.1) $S^0 \to p^{-1}S^0 \to N^1, \qquad N^1 \to L_1N^1 \to N^2, \qquad V(0) \to v_1^{-1}V(0) \to W,$

and $M^2 = L_2N^2$. The Adams-Novikov differentials on $\pi_*(L_2W)$ is determined in [7]. Let $\imath : L_2W \to M^2$ denote the canonical map that induces $i : M_1^1 \to M_0^2$. Suppose that $d_r(x) = y$ in the E_r-term for L_2W. Then $d_r(x/3) = d_r(i_*x) = i_*y = y/3$. In this way, we determine the differentials except for $d_9(v_2^{-1}/3v_1)$ and $d_5(v_2^{3t}\xi/3v_1^2)$.

LEMMA 6.2. *The Adams-Novikov differential d_r is given (up to sign) by*

$$d_r(v_2/3v_1) = 0, \qquad d_5(v_2^4/3v_1) = v_2^2h_{11}b_{10}^2/3v_1, \qquad d_5(v_2^7/3v_1) = v_2^5h_{11}b_{10}^2/3v_1,$$

$$d_r(v_2^2/3v_1) = 0, \qquad d_r(v_2^5/3v_1) = 0, \qquad d_9(v_2^{-1}/3v_1) = v_2^{-5}b_{11}b_{10}^3\zeta_2/3v_1,$$

$$d_r(v_2h_{10}/3v_1) = 0, \qquad d_9(v_2^4h_{10}/3v_1) = v_2b_{10}^5/3v_1, \qquad d_r(v_2^7h_{10}/3v_1) = 0,$$

$$d_r(v_2^2h_{11}/3v_1) = 0, \qquad d_r(v_2^5h_{11}/3v_1) = 0, \qquad d_9(v_2^8h_{11}/3v_1) = v_2^4b_{11}b_{10}^4/3v_1,$$

$$d_5(v_2b_{11}/3v_1) = v_2h_{10}b_{10}^3/3v_1, \qquad d_r(v_2^4b_{11}/3v_1) = 0,$$

$$d_5(v_2^7b_{11}/3v_1) = v_2^7h_{10}b_{10}^3/3v_1, \qquad d_5(\xi/3v_1^2) = v_2^{-3}b_{11}\xi b_{10}\zeta_2/3v_1,$$

$$d_r(v_2^3\xi/3v_1^2) = 0, \qquad d_5(v_2^6\xi/3v_1^2) = v_2^3b_{11}\xi b_{10}\zeta_2/3v_1,$$

$$d_r(\xi/3v_1) = 0, \qquad d_r(v_2^3\xi/3v_1) = 0, \qquad d_9(v_2^6\xi/3v_1) = v_2^3\psi_0b_{10}^4/3v_1,$$

$$d_5(\psi_0/3v_1) = v_2^{-3}b_{11}\xi b_{10}^2/3v_1, \quad d_r(v_2^3\psi_0) = 0, \quad d_5(v_2^6\psi_0/3v_1) = v_2^3b_{11}\xi b_{10}^2/3v_1,$$

$$d_5(v_2\psi_1/3v_1) = \xi b_{10}^3/3v_1, \qquad d_5(v_2^4\psi_1/3v_1) = v_2^3\xi b_{10}^3/3v_1, \qquad d_r(v_2^7\psi_1/3v_1) = 0.$$

$$d_9(b_{11}\xi/3v_1) = v_2^{-2}\psi_1b_{10}^5/3v_1, \qquad d_r(v_2^3b_{11}\xi/3v_1) = 0, \qquad d_r(v_2^6b_{11}\xi/3v_1) = 0.$$

PROOF. Here we show the exceptional cases. Lemma 4.4 shows

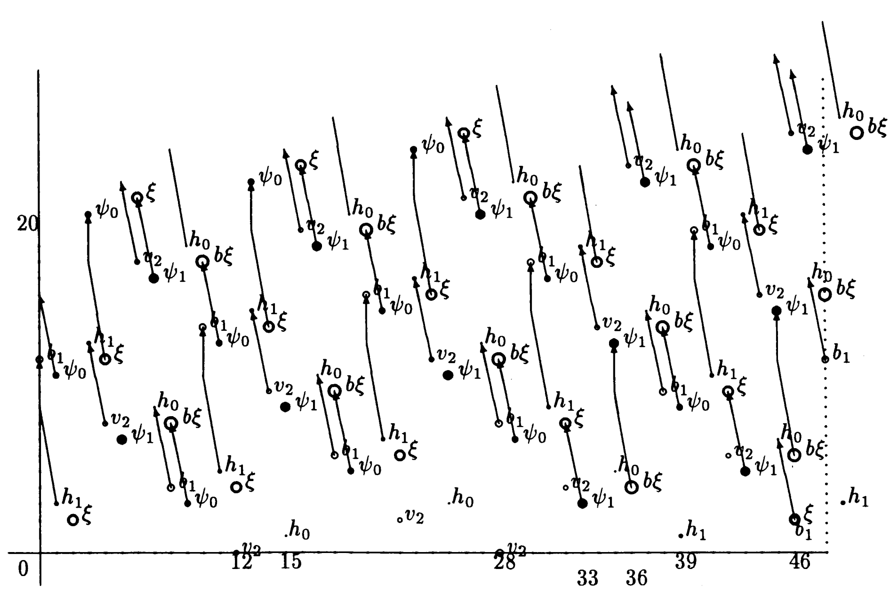
$$v_2^{-1}b_{10}/3v_1 = -v_2^{-1}h_{11}\zeta_2/3v_1 \qquad \text{and} \qquad v_2^{3t}\xi b_{10}/3v_1^2 = \pm v_2^{3t}\psi_0\zeta_2/3v_1.$$

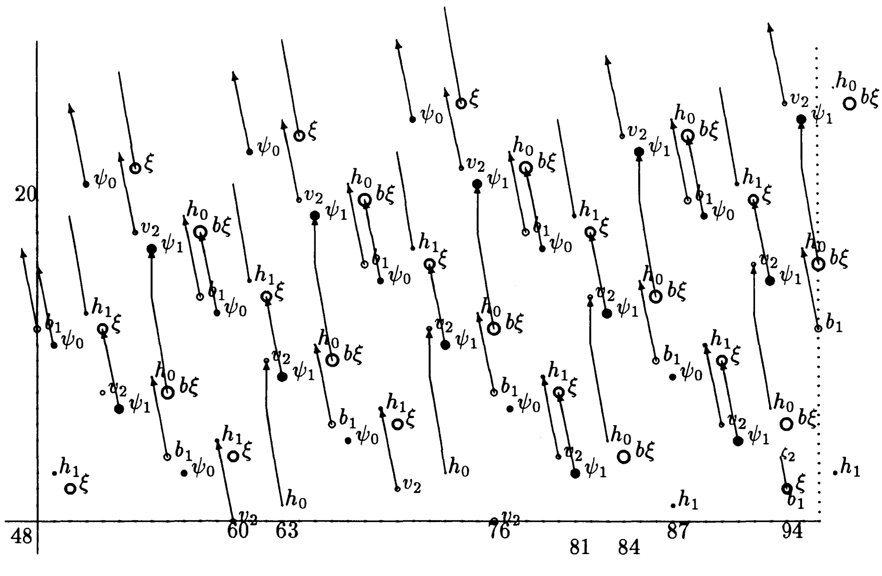
Now we compute

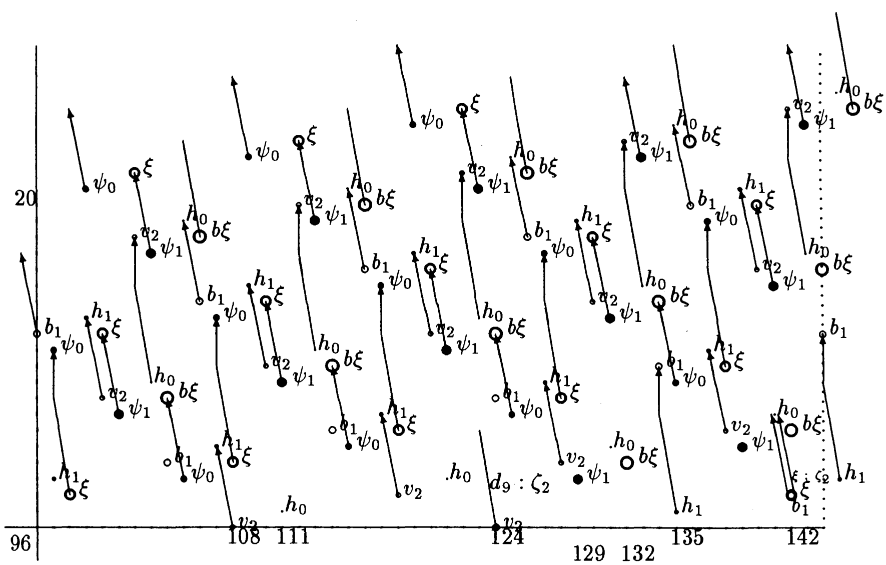
$$b_{10}d_9(v_2^{-1}/3v_1) = d_9(v_2^{-1}b_{10}/3v_1) = -d_9(v_2^{-1}h_{11}\zeta_2/3v_1) = v_2^{-5}b_{11}b_{10}^4\zeta_2/3v_1,$$

and we have $d_9(v_2^{-1}/3v_1) = v_2^{-5}b_{11}b_{10}^3/3v_1$ as desired. In the same way, we have the other case. q.e.d.

Now we display the chart of the Adams-Novikov spectral sequence:







Some of survivors are killed by other differentials derived from [7]:

$$d_5(v_2^{9t+3}/3v_1^3) = v_2^{9t+1}h_{10}b_{10}^2/3v_1,$$

$$d_5(v_2^{9t-1}h_{11}/9v_1^2) = v_2^{9t-2}h_{10}b_{10}^2\zeta_2/3v_1,$$

$$(6.3)\quad d_5(v_2^{3^{n+2}t+3^{n+1}}h_{10}/3v_1^{2\cdot 3^{n+1}+1}) = \pm v_2^{3^{n+2}t+3(3^n-1)/2}\xi b_{10}^2/3v_1 \quad (n \geq 0),$$

$$d_5(v_2^{3^{n+2}t+8\cdot 3^n}h_{10}/3v_1^{10\cdot 3^n+1}) = -v_2^{3^{n+2}t+5\cdot 3^n+3(3^{n-1}-1)/2}\xi b_{10}^2/3v_1 \quad (n > 1).$$

This shows that,

THEOREM 6.4. *The E_∞-term of $\pi_*(M^2)$ contains the module $\tilde{G} \oplus \tilde{G}^* \oplus \widetilde{GZ} \oplus \widetilde{GZ}^*$. Here $E(2,1)_*$-modules are given as follows:*

$$\begin{aligned}
\tilde{G} &= B_5(2,2)_*\{v_2/3v_1\} \oplus B_4(2,2)_*\{v_2^4 b_{11}/3v_1\} \\
&\quad \oplus B_3(2,2)_*\{v_2^7 h_{10}/3v_1\} \\
&\quad \oplus B_2(2,2)_*\{v_2 h_{10}/3v_1, v_2^2 h_{11}/3v_1, v_2^5 h_{11}/3v_1\}, \\
\tilde{G}^* &= B_5(2,2)_*\{v_2^7\psi_1/3v_1\} \oplus B_4(2,2)_*\{v_2^3\psi_0/3v_1\} \\
&\quad \oplus B_2(2,2)_*\{\xi/3v_1, v_2^3 b_{11}\xi/3v_1, v_2^6 b_{11}\xi/3v_1\} \\
&\quad \oplus \sum_{n\geq 1}(B_3(2,n+2)_*\{v_2^{9u+3}\xi/3v_1 \mid u \in \boldsymbol{Z} - I(n)\} \\
&\qquad\qquad \oplus B_2(2,n+2)_*\{v_2^{9u+3}\xi/3v_1 \mid u \in I(n)\}), \\
\widetilde{GZ} &= B_5(2,2)_*\{v_2\zeta_2/3v_1\} \\
&\quad \oplus B_3(2,2)_*\{v_2^4 b_{11}\zeta_2/3v_1\} \\
&\quad \oplus B_2(2,2)_*\{v_2 h_{10}\zeta_2/3v_1, v_2^2 h_{11}\zeta_2/3v_1, v_2^5 h_{11}\zeta_2/3v_1, v_2^7 h_{10}\zeta_2/3v_1\}, \\
\widetilde{GZ}^* &= B_5(2,2)_*\{v_2^7\psi_1\zeta_2/3v_1\} \oplus B_4(2,2)_*\{v_2^3\psi_0\zeta_2/3v_1\} \\
&\quad \oplus B_2(2,2)_*\{\xi\zeta_2/3v_1\} \\
&\quad \oplus B_1(2,2)_*\{v_2^3 b_{11}\xi\zeta_2/3v_1, v_2^6 b_{11}\xi\zeta_2/3v_1\} \\
&\quad \oplus \sum_{n\geq 1}(B_3(2,n+2)_*\{v_2^{9u+3}\xi\zeta_2/3v_1 \mid u \in \boldsymbol{Z} - I(n)\} \\
&\qquad\qquad \oplus B_2(2,n+2)_*\{v_2^{9u+3}\xi\zeta_2/3v_1 \mid u \in I(n)\}),
\end{aligned}$$

where $B_k(2,n)_* = (\boldsymbol{Z}/3)[v_2^{\pm 3^n}, b_{10}]/(b_{10}^k)$ and $I(n)$ are given in the introduction.

PROOF. Suppose that $d_r(x) = y \neq 0$ in the Adams-Novikov spectral sequence for $\pi_*(M^2)$. Then y is in the image of $i_*: H^*M_1^1 \to H^*M_0^2$, since y has filtration ≥ 5. Lemma 5.2 shows that $\delta(y) \neq 0$ for the connecting homomorphism $\delta: H^*M_0^2 \to H^*M_1^1$, and so we have $\delta(x) \neq 0$ and $d_r(\delta(x)) = \delta(y)$ in the Adams-Novikov spectral sequence for $\pi_*(L_2W)$. Observing the differentials given in [7] with Lemma 4.4, we see that there is no more new differentials, and obtain the theorem. q.e.d.

7. Application to β-elements

In [1], $H^0M_0^2$ is determined and we see that

$$v_2^s/3^{i+1}v_1^j \in H^0M_0^2 \qquad \text{if and only if } (s, j, i+1) \in \boldsymbol{B}.$$

Consider the universal Greek letter map $\eta = \delta'\delta : H^0M_0^2 \to H^2E(2)_*$, where $\delta : H^0M_0^2 \to H^1N_0^1$ and $\delta' : H^1N_0^1 \to H^2E(2)_*$ are the connecting homomorphisms associated to the short exact sequences $0 \to N_0^1 \to M_0^1 \to M_0^2 \to 0$ and $0 \to E(2)_* \to M_0^0 \to N_0^1 \to 0$, respectively. Then the β-elements are defined by

$$\beta_{s/j,i+1} = \eta(v_2^s/3^{i+1}v_1^j).$$

We obtain the following immediately.

Lemma 7.1. *Mod* $(3, v_1)$, $\beta_1 \equiv b_{10}$, $\beta_2 \equiv v_2h_{11}\zeta_2$ *and* $\beta_{6/3} \equiv v_2^3b_{11}$ *in the* E_2*-term* $E_2(S^0)$.

Furthermore, note that $\beta_1' = h_{11} \in E_2(V(0))$ and $\alpha_1 = h_{10} \in E_2(S^0)$. The generators of $\tilde{G}$ then yields the following elements:

$$\begin{aligned}
\eta(v_2^{9t+1}/3v_1) &= \beta_{9t+1}, & \eta(v_2^{9t+4}b_{11}/3v_1) &= \beta_{9t+1}\beta_{6/3},\\
\eta(v_2^{9t+7}h_{10}/3v_1) &= \beta_{9t+7}\alpha_1, & \eta(v_2^{9t+1}h_{10}/3v_1) &= \beta_{9t+1}\alpha_1,\\
\eta(v_2^{9t+2}h_{11}/3v_1) &= [\beta_{9t+2}\beta_1'], & \eta(v_2^{9t+5}h_{11}/3v_1) &= [\beta_{9t+5}\beta_1'].
\end{aligned}$$

Now we prove the theorems in the introduction.

Proof of Theorem C. Consider the long exact sequences

$$\cdots \to \pi_*(L_0S^0) \to \pi_*(L_2N^1) \to \pi_{*+1}(L_2S^0) \to \cdots$$

and

$$\cdots \to \pi_*(L_1N^1) \to \pi_*(M^2) \to \pi_{*+1}(L_2N^1) \to \cdots$$

associated to the cofiber sequences of (6.1). Note that $\pi_*(L_0S^0) = \boldsymbol{Q}$ and

$$\pi_*(L_1N^1) = \boldsymbol{Q}/\boldsymbol{Z}_{(3)} \otimes \Lambda(y) \oplus A$$

shown in [1], where A is the $\boldsymbol{Z}_{(3)}$-module generated by $v_1^{sp^i}/3^{i+1}$ for $i \geq 0$ and $3 \nmid s \in \boldsymbol{Z}$. Therefore, the module $\tilde{G} \oplus \tilde{G}^* \oplus \widetilde{GZ} \oplus \widetilde{GZ}^*$ given in Theorem 6.4 is isomorphically sent to $\pi_*(L_2S^0)$. Theorem C now follows. q.e.d.

Proof of Theorem A. Consider the localization map $\iota : S^0 \to L_2S^0$. Since the induced map $\iota_* : \pi_*(S^0) \to \pi_*(L_2S^0)$ sends a β-element to the corresponding β-element, the non-triviality of products of β-elements in $\pi_*(S^0)$ is deduced from the one in $\pi_*(L_2S^0)$. The necessity follows immediately from

Theorem C except for β_2. For β_2, note that $\beta_{9t+1}\beta_2 = [\beta_{9t+2}\beta_1']\zeta_2 \in \eta(\widetilde{GZ})$ for the universal Greek letter map η. Thus the necessity for β_2 is shown.

In Lemma 6.2, we have $d_9(v_2^{9t+4}h_{10}/3v_1) = v_2^{9t+1}b_{10}^5/3v_1$ and $d_9(v_2^{9t+8}h_{11}/3v_1) = v_2^{9t+4}b_{11}b_{10}^4/3v_1$, which yield

$$(7.2) \qquad d_9(\beta_{9t+4}\alpha_1) = \beta_1^5\beta_{9t+1} \qquad \text{and} \qquad d_9(\beta_{9t+8}h_{11}) = \beta_{9t+1}\beta_{6/3}\beta_1^4$$

in the E_9-term $E_9^*(L_2S^0)$ as the image of the universal Greek letter map. In the same manner, the equation $d_5(v_2^{9t+4}\zeta_2/3v_1) = v_2^{9t+2}h_{11}\zeta_2 b_{10}^2/3v_1$ in Lemma 6.2 yields

$$(7.3) \qquad d_5(\beta_{9t+4}\zeta_2) = \beta_{9t+1}\beta_2\beta_1^2$$

in the E_9-term $E_5^*(L_2S^0)$. If $t \geq 0$, then the equations (7.2) and (7.3) also hold in the Adams-Novikov spectral sequence for $\pi_*(S^0)$, since the elements appeared in (7.2) and (7.3) are also defined in $E_2(S^0)$. q.e.d.

PROOF OF THEOREM E. In the proof of Theorem 6.4, we read off that the elements on the 0-th line hit nothing except for the β-elements given by $\boldsymbol{B}^c$. Therefore, we obtain Theorem E. q.e.d.

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