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Homotopy classification of higher homotopy commutative loop spaces with finitely generated cohomology

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ABSTRACT. Suppose X is a simply connected mod p loop space such that the mod p cohomology $H^*(X)$ is finitely generated as an algebra. We show that if X is a C_n-space in the sense of Williams then X is the total space of a C_n-fibration over a finite C_n-space. By using this result, we can reduce problems about C_n-spaces with finitely generated cohomology to the case of finite C_n-spaces. In particular, we give classification theorems for C_2-spaces and C_p-spaces with finitely generated cohomology.

1. Introduction

Loop space plays an important role in homotopy theory of Lie groups, and it has been investigated from several points of view (cf. [15], [19], [25], [26]). It is convenient to consider the loop space at a prime by using completion theory due to Bousfield-Kan [3]. Let p be a prime. A loop space which is completed at p is called a mod p loop space. Throughout the paper, homotopy equivalence means mod p homotopy equivalence and cohomology is mod p cohomology unless otherwise specified.

Dwyer-Wilkerson [11] defined the p-compact group and studied its properties. A loop space X is said to be a p-compact group if the classifying space BX is p-completed and the mod p cohomology $H^*(X)$ is finite dimensional. Besides compact Lie groups, other useful examples of p-compact groups are known. In fact, the p-completion of an odd dimensional sphere S^{2n-1} is a p-compact group if $n|(p-1)$. In recent years, many theorems have been proved about p-compact groups (cf. [11], [26]), and those theorems suggest that p-compact groups have similar properties to those of Lie groups.

In this paper, we consider loop spaces which need not be finite, but whose mod p cohomology rings are finitely generated. Recently, Broto and Crespo [4], [8] gave remarkable results for H-spaces with finitely generated cohomology. It follows from their results that a mod p loop space with

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finitely generated mod p cohomology is the total space of a loop fibration over a p-compact group.

The present paper is devoted to study the higher homotopy commutativity of mod p loop spaces with finitely generated cohomology. Such a notion was first introduced by Sugawara [36]. He used it to give a criterion of a homotopy commutative loop space to be the loop space of an H-space. McGibbon [22] proved that a connected finite C_p-space in the sense of Sugawara has the homotopy type of a torus, and Kawamoto-Lin [17] generalized his result to the case of finitely generated cohomology.

Later Williams [38] defined another kind of higher homotopy commutativity which is weaker than the one of Sugawara. If X is a loop space, then by using the Moore loop structure, we can choose a multiplication on X which is associative. C_1-space means a loop space, and C_2-space is just a homotopy commutative loop space. Let X be a C_2-space, and $Q_2 : I \times X^2 \to X$ be a map satisfying that $Q_2(0, x_1, x_2) = x_1 \cdot x_2$ and $Q_2(1, x_1, x_2) = x_2 \cdot x_1$ for $x_1, x_2 \in X$. By using the map Q_2, we can define a map $\tilde{Q}_3 : S^1 \times X^3 \to X$ such that $\tilde{Q}_3(t, x_1, x_2, x_3)$ is a loop connecting $\{x_{\sigma(1)} \cdot x_{\sigma(2)} \cdot x_{\sigma(3)}\}_{\sigma \in \Sigma_3}$ for $x_1, x_2, x_3 \in X$, where Σ_3 denotes the symmetric group on 3 letters. X is said to be a C_3-space if there exists a map $Q_3 : D^2 \times X^3 \to X$ such that $Q_3|_{S^1 \times X^3} = \tilde{Q}_3$. In general, a C_n-space is defined as a loop space together with a C_n-form $\{Q_i : K_i \times X^i \to X\}_{1 \leq i \leq n}$ satisfying certain boundary conditions (see §2), where K_i is homeomorphic to the $(i-1)$-dimensional cell. If there exists a system of maps $\{Q_i : K_i \times X^i \to X\}_{i \geq 1}$ such that $\{Q_i\}_{1 \leq i \leq n}$ is a C_n-form for $n \geq 1$, then X is said to be a C_∞-space. It is known that the loop space of an H-space is a C_∞-space, and in particular, Eilenberg-MacLane spaces are C_∞-spaces. Similarly, a C_n-map is defined as a loop map preserving the C_n-forms (see §2). A C_n-fibration is a loop fibration consisting of C_n-spaces and C_n-maps.

Our main result is stated as follows:

THEOREM A. *Let p be an odd prime. If X is a simply connected C_n-space such that the* mod p *cohomology $H^*(X)$ is finitely generated as an algebra, then there exists a C_n-fibration*

$$K \to X \to F,$$

where F is a simply connected finite C_n-space and K is a finite product of $K(\mathbf{Z}, 2)$.

From Theorem A, we can reduce problems about C_n-spaces with finitely generated cohomology to the case of finite C_n-spaces. In the case of $p = 2$, Slack has shown the following classification theorem about homotopy commutative mod 2 H-spaces:

THEOREM 1.1 ([34; Cor. 0.2]). *If X is a simply connected homotopy commutative* mod 2 *H-space such that the* mod 2 *cohomology $H^*(X)$ is finitely generated as an algebra, then X is homotopy equivalent to a finite product of $K(\mathbf{Z}, 2)$.*

Broto-Crespo [4] reproved Theorem 1.1 by using another method. On the other hand, at odd primes, it is known that any connected mod p H-space admits an H-structure which is homotopy commutative by a result of Iriye-Kono [14]. Furthermore, there are examples of mod p loop spaces which have homotopy commutative loop structures.

Clark-Ewing [7] constructed many p-compact groups. Let $G \subset GL(l, \hat{\mathbf{Z}}_p)$ be a finite pseudo-reflection group contained in the list [7]. We see that G acts on the mod p cohomology $H^*(BT^l)$, where BT^l denotes the classifying space of the l-dimensional torus. Clark-Ewing proved that if the order $|G|$ is coprime to p, then there exists a space $BX(G)$ such that

$$H^*(BX(G)) \cong H^*(BT^l)^G \cong \mathbf{Z}/p[y_1, \dots, y_l],$$

where $\deg y_i = 2t_i$ for $1 \leq i \leq l$ and the order $|G| = t_1 \dots t_l$. Moreover, Dwyer-Miller-Wilkerson [10] has shown that the homotopy type of $BX(G)$ is determined by the cohomology and the action of the mod p Steenrod algebra. If we put $X(G) = \Omega BX(G)$, then by a spectral sequence argument, the mod p cohomology $H^*(X(G))$ is finite dimensional, which implies that $X(G)$ is a p-compact group. The sequence of numbers $(t_1, \dots, t_l)$ is called the type of $X(G)$.

McGibbon [21] studied the homotopy commutativity of compact Lie groups, and Saumell [32] generalized his result to the cases of several p-compact groups. From their results, we have the following theorem:

THEOREM 1.2 ([21; Thm. 2], [32; Thm. 1.1]). *Let p be an odd prime and G be a pseudo-reflection group contained in the Clark-Ewing list. If the pair (G, p) is contained in the following table, then the p-compact group $X(G)$ is a C_2-space.*

G	p	$X(G)$	Types
G	$\geq 2t_l$	$S^{2t_1-1} \times \cdots \times S^{2t_l-1}$	$(t_1, \dots, t_l)$
G_{2b}	≥ 3	$B_1(p)$	$(2, p+1)$
G_9	17	$B_7(17)$	$(8, 24)$
G_{14}	19	$B_5(19)$	$(6, 24)$
G_{17}	41	$B_{19}(41)$	$(20, 60)$
G_{20}	19	$B_{11}(19)$	$(12, 30)$
G_{24}	11	$B_3(11) \times S^{11}$	$(4, 6, 14)$
G_{30}	19	$B_1(19) \times B_{11}(19)$	$(2, 12, 20, 30)$

Here $B_n(p)$ denotes the S^{2n+1}-bundle over $S^{2n+2p-1}$ whose mod p cohomology is given as $H^*(B_n(p)) \cong \Lambda(\sigma, \mathscr{P}^1(\sigma))$ with $\deg \sigma = 2n+1$ (cf. [24]).

Let F be one of the p-compact groups $X(G)$ given in Theorem 1.2. From Theorem 1.2, we see that F is a C_2-space with finitely generated cohomology. If the integral cohomology $H^3(F;\mathbf{Z}) \cong \mathbf{Z}$, then we can define a map $[p^i] : F \to K(\mathbf{Z},3)$ as $[p^i]^*(v) = p^i u$ for $i \geq 0$, where $u \in H^3(F;\mathbf{Z})$ and $v \in H^3(K(\mathbf{Z},3);\mathbf{Z})$ denote the generators. Let $F\langle 3; p^i\rangle$ be the homotopy fiber of the map $[p^i] : F \to K(\mathbf{Z},3)$. In the case of $i = 0$, we see that $F\langle 3;1\rangle = F\langle 3\rangle$ is the three-connected cover of F. From a result of Williams [38; Thm. 21, 23], we see that $F\langle 3; p^i\rangle$ is a C_2-space, and by a spectral sequence argument the cohomology $H^*(F\langle 3; p^i\rangle)$ is not finite but finitely generated as an algebra for $i \geq 0$.

Now from Theorem A, we obtain a classification of C_2-spaces with finitely generated cohomology.

THEOREM B. *Let p be an odd prime and X be a simply connected* mod p *loop space such that the* mod p *cohomology $H^*(X)$ is finitely generated as an algebra. Then X is a C_2-space if and only if there exists a system $\{F_i\}_{1\leq i\leq q}$ consisting of some of the finite C_2-spaces $X(G)$ on the table in Theorem* 1.2 *such that*

$$X \simeq \prod_{i=1}^{s} F_i\langle 3\rangle \times \prod_{i=s+1}^{t} F_i\langle 3; p^{e_i}\rangle \times \prod_{i=t+1}^{q} F_i \times K(\mathbf{Z},2)^r,$$

where $1 \leq e_i \leq e_{i+1}$ *for* $s+1 \leq i \leq t-1$.

The next theorem can be regarded as an odd prime version of Theorem 1.1 since C_2-space is exactly a homotopy commutative loop space.

THEOREM C. *Let p be an odd prime. If X is a simply connected C_p-space such that the* mod p *cohomology $H^*(X)$ is finitely generated as an algebra, then X is homotopy equivalent to a finite product of $K(\mathbf{Z},2)$.*

We remark that Theorem C is extended to the case of connected C_p-spaces (see Corollary 3.12).

Hemmi [13; Thm. 1.1] proved that if X is a simply connected finite quasi C_p-space, then X is contractible, where quasi C_p-space is defined on the category of higher homotopy associative H-spaces. On the category of loop spaces, the quasi C_p-space is exactly the C_p-space of Williams, and Theorem C can be regarded as a generalization of his result in the case of finitely generated cohomology. Theorem C also generalizes results of Kawamoto and Lin [16], [17], [18], in which the same type of problems were treated in the cases of the loop spaces of H-spaces and C_p-spaces in the sense of Sugawara.

It is natural to ask the explicit higher homotopy commutativity of the

p-compact groups $X(G)$ on the table in Theorem 1.2. Saumell [33] gave a partial result for this problem. In §4, we determine the higher homotopy commutativity of almost all p-compact groups (see Theorems 4.2 and 4.5).

This paper is organized as follows: In §2, we give a proof of Theorem A. It is shown that the Dror Farjoun localization functor [9] preserves C_n-forms (see Theorem 2.14), and by combining Theorem 2.14 with results of Broto and Crespo [4], [8], we can complete a proof of Theorem A. §3 is devoted to the proofs of Theorems B and C. By generalizing results of McGibbon [21] and Saumell [32], we give a classification for finite C_2-spaces (see Theorem 3.3). Theorem B is proved by using Theorems A and 3.3. A result of Hemmi [13] about the classification of finite C_p-spaces is used to prove Theorem C. In §4, we study the higher homotopy commutativity of p-compact groups, by which we give a necessary condition for a mod p loop space with finitely generated cohomology to be a C_{p-1}-space (see Corollary 4.9).

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2. Proof of Theorem A

Dror Farjoun [9] introduced the localization functor with respect to a space. In Theorem 2.14, we show that the localization functor preserves the higher homotopy commutativity of mod p loop spaces. By combining Theorem 2.14 with results of Broto and Crespo [4], [8], we can complete the proof of Theorem A. First we recall the localization functor.

Let A be a space. A space X is called A-local if the base point evaluation map $e : \mathrm{Map}(A, X) \to X$ is a homotopy equivalence, or equivalently, if $\mathrm{Map}_*(A, X)$ is contractible. Dror Farjoun constructed a localization functor $L_A : \mathscr{S}_* \to \mathscr{S}_*$ with respect to the space A, where $\mathscr{S}_*$ denotes the category of pointed spaces. For any space X, the localization $L_A(X)$ is A-local, and there exists a natural map $\phi_X : X \to L_A(X)$. It is known that the map ϕ_X is homotopically universal, that is, for an A-local space Z and a map $\zeta : X \to Z$, there exists a map $\tilde{\zeta}: L_A(X) \to Z$ unique up to homotopy so that $\tilde{\zeta}\phi_X \simeq \zeta$. Furthermore, the natural map ϕ_X induces a homotopy equivalence

$$(2.1) \qquad (\phi_X)^* : \mathrm{Map}_*(L_A(X), Z) \to \mathrm{Map}_*(X, Z)$$

for any A-local space Z.

The localization functor L_A does not necessarily preserve a fibration. Dror Farjoun has shown the following:

Proposition 2.2 ([9; Thm. 1.H.1, Cor. 3.D.3]). *Let F, E and B be* *connected spaces and $F \to E \to B$ be a fibration. If the base space B is A-local or $L_A(F) \simeq *$, then the localization L_A preserves the fibration.*

Let Y be a space and $Y^{(i)}$ denote the i-fold smash product of Y for $i \geq 1$. In the proof of Theorem 2.14, we need the following:

PROPOSITION 2.3. *Let A be a space and Z be a A-local space. If X is a space and $\phi_X : X \to L_A(X)$ denotes the natural map, then $(\phi_X)^i : X^i \to L_A(X)^i$ and $(\phi_X)^{(i)} : X^{(i)} \to L_A(X)^{(i)}$ induce the following homotopy equivalences:*

$$((\phi_X)^i)^* : \mathrm{Map}_*(L_A(X)^i, Z) \to \mathrm{Map}_*(X^i, Z) \tag{2.4}$$

$$((\phi_X)^{(i)})^* : \mathrm{Map}_*(L_A(X)^{(i)}, Z) \to \mathrm{Map}_*(X^{(i)}, Z) \tag{2.5}$$

for $i \geq 1$.

To prove Proposition 2.3, we need the following lemma:

LEMMA 2.6. *Let X and Y be spaces and $\lambda_{X,Y} = L_A(\phi_X \wedge \phi_Y) : L_A(X \wedge Y) \to L_A(L_A(X) \wedge L_A(Y))$. Then $\lambda_{X,Y}$ is a homotopy equivalence and $\lambda_{X,Y}\phi_{X\wedge Y} \simeq \phi_{L_A(X)\wedge L_A(Y)}(\phi_X \wedge \phi_Y)$.*

PROOF. For a space W, we put $L(W) = L_A(W)$. From the definition of $\lambda_{X,Y}$, we have that $\lambda_{X,Y}\phi_{X\wedge Y} \simeq \phi_{L(X)\wedge L(Y)}(\phi_X \wedge \phi_Y)$.

Let $a : X \to \mathrm{Map}_*(Y, L(X \wedge Y))$ be the adjoint map of $\phi_{X\wedge Y} : X \wedge Y \to L(X \wedge Y)$. From [9; 1.A.8 e.2], the mapping space $\mathrm{Map}_*(Y, L(X \wedge Y))$ is A-local. By the universality of ϕ_X, there exists a map $\tilde{a} : L(X) \to \mathrm{Map}_*(Y, L(X\wedge Y))$ with $\tilde{a}\phi_X \simeq a$. If $\tilde{\kappa}_{X,Y} : L(X)\wedge Y \to L(X\wedge Y)$ is the adjoint map of $\tilde{a}$, then $\tilde{\kappa}_{X,Y}(\phi_X \wedge 1_Y) \simeq \phi_{X\wedge Y}$. By using the same arguments, we have a map $\hat{\kappa}_{X,Y} : L(X) \wedge L(Y) \to L(X \wedge Y)$ with $\hat{\kappa}_{X,Y}(\phi_X \wedge \phi_Y) \simeq \phi_{X\wedge Y}$, and the universality of $\phi_{L(X)\wedge L(Y)}$ gives a map $\kappa_{X,Y} : L(L(X) \wedge L(Y)) \to L(X \wedge Y)$ with $\kappa_{X,Y}\phi_{L(X)\wedge L(Y)} \simeq \hat{\kappa}_{X,Y}$.

Since $\kappa_{X,Y}\lambda_{X,Y}\phi_{X\wedge Y} \simeq \kappa_{X,Y}\phi_{L(X)\wedge L(Y)}(\phi_X \wedge \phi_Y) \simeq \phi_{X\wedge Y}$, we have that $\kappa_{X,Y}\lambda_{X,Y} \simeq 1_{L(X\wedge Y)}$. Similarly, we see that $\lambda_{X,Y}\kappa_{X,Y}\phi_{L(X)\wedge L(Y)}(\phi_X \wedge \phi_Y) \simeq \phi_{L(X)\wedge L(Y)}(\phi_X \wedge \phi_Y)$, and by using the universality of ϕ_X, ϕ_Y and $\phi_{L(X)\wedge L(Y)}$, we have that $\lambda_{X,Y}\kappa_{X,Y} \simeq 1_{L(L(X)\wedge L(Y))}$. This completes the proof. □

Now we prove Proposition 2.3 as follows:

PROOF OF PROPOSITION 2.3. First we show (2.4). By [9; 1.A.8 e.4], there exists a homotopy equivalence $\gamma_{X^i,X} : L_A(X^{i+1}) \to L_A(X^i) \times L_A(X)$ with $\gamma_{X^i,X}\phi_{X^{i+1}} \simeq \phi_{X^i} \times \phi_X$ for $i \geq 1$. If we define a map $\gamma_i : L_A(X^i) \to L_A(X)^i$ as $\gamma_i = (\gamma_{X,X} \times 1_{L_A(X)^{i-2}}) \cdots (\gamma_{X^{i-2},X} \times 1_{L_A(X)})\gamma_{X^{i-1},X}$, then γ_i is a homotopy equivalence and $(\phi_X)^i \simeq \gamma_i\phi_{X^i}$. By (2.1), $(\phi_{X^i})^*$ is a homotopy equivalence, and so $((\phi_X)^i)^* : \mathrm{Map}_*(L_A(X)^i, Z) \to \mathrm{Map}_*(X^i, Z)$ is a homotopy equivalence, which shows (2.4).

Next we show (2.5). By using the induction on $i \geq 1$, we show that there exists a homotopy equivalence $\zeta_i : L_A(X^{(i)}) \to L_A(L_A(X)^{(i)})$ satisfying the following homotopy commutative diagram:

$$\begin{array}{ccc} X^{(i)} & \xrightarrow{(\phi_X)^{(i)}} & L_A(X)^{(i)} \\ \phi_{X^{(i)}} \downarrow & & \downarrow \phi_{L_A(X)^{(i)}} \\ L_A(X^{(i)}) & \xrightarrow{\zeta_i} & L_A(L_A(X)^{(i)}). \end{array} \tag{2.7}$$

For $i = 1$, we can put $\zeta_1 = \phi_{L_A(X)} : L_A(X) \to L_A(L_A(X))$. Now we assume that there exists a homotopy equivalence $\zeta_{i-1} : L_A(X^{(i-1)}) \to L_A(L_A(X)^{(i-1)})$ such that the diagram (2.7) is homotopy commutative. If we define a map $\zeta_i : L_A(X^{(i)}) \to L_A(L_A(X)^{(i)})$ by $\zeta_i = \kappa_{L_A(X)^{(i-1)}, L_A(X)} L_A(\zeta_{i-1} \wedge \phi_{L_A(X)}) \lambda_{X^{(i-1)}, X}$, where $\kappa_{L_A(X)^{(i-1)}, L_A(X)}$ is in the proof of Lemma 2.6, then ζ_i is a homotopy equivalence and the diagram (2.7) is homotopy commutative.

By taking the mapping spaces for the diagram (2.7), we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_*(L_A(L_A(X)^{(i)}), Z) & \xrightarrow[\simeq]{\zeta_i^*} & \mathrm{Map}_*(L_A(X^{(i)}), Z) \\ (\phi_{L_A(X)^{(i)}})^* \downarrow \simeq & & (\phi_{X^{(i)}})^* \downarrow \simeq \\ \mathrm{Map}_*(L_A(X)^{(i)}, Z) & \xrightarrow{((\phi_X)^{(i)})^*} & \mathrm{Map}_*(X^{(i)}, Z), \end{array}$$

which shows (2.5). This completes the proof of Proposition 2.3. □

Now we recall the definition of the C_n-space in the sense of Williams [38]. Let $\mathbf{n} = (1, \ldots, n)$. A subsequence of $\mathbf{n}$ is denoted as $\alpha_l = (a_1, \ldots, a_l)$ for $l \geq 1$ with the inclusion $i_\alpha : \alpha_l \to \mathbf{n}$, and $\alpha : \mathbf{l} \to \mathbf{n}$ denotes the composite $i_\alpha j_\alpha$, where $j_\alpha : \mathbf{l} \to \alpha_l$ is defined as $j_\alpha(i) = a_i$ for $1 \leq i \leq l$. A (l, m)-partition of $\mathbf{n}$ is an ordered pair (α_l, β_m) with $l + m = n$ of disjoint subsequences of $\mathbf{n}$ satisfying that $i_\alpha(\alpha_l) \cup i_\beta(\beta_m) = \mathbf{n}$. If we consider $\mathbf{n} = (1, \ldots, n)$ as a point of $\mathbf{R}^n$, then the symmetric group Σ_n on n letters acts on $\mathbf{R}^n$ by permuting the coordinates. Let K_n denote the convex hull of the orbit of $\mathbf{n}$ of this action. Then from the definition, we see that K_n is homeomorphic to an $(n-1)$-dimensional cell.

If we denote the boundary of K_n as $L_n = \partial K_n$, then it is the union of $(n-2)$-dimensional cells which are corresponding to partitions (α_l, β_m) of $\mathbf{n}$. Let $\varepsilon(\alpha_l, \beta_m) : K_l \times K_m \to L_n$ denote the inclusion.

If X is a loop space, then by using the Moore loop structure, we can assume that X is an associative H-space. A C_n-form on X in the sense of Williams [38] is defined as a sequence of maps $\{Q_i : K_i \times X^i \to X\}_{1 \leq i \leq n}$

satisfying the following conditions:

$$(2.8) \qquad Q_1 = 1_X : K_1 \times X \to X,$$

where $K_1 \times X = \{1\} \times X$ is identified with X. For a partition (α_r, β_s) of $\mathbf{i}$,

$$(2.9) \quad Q_i(\varepsilon(\alpha_r, \beta_s)(\rho, \sigma), x_1, \dots, x_i) = Q_r(\rho, x_{\alpha(1)}, \dots, x_{\alpha(r)}) \cdot Q_s(\sigma, x_{\beta(1)}, \dots, x_{\beta(s)}),$$

where $\rho \in K_r$ and $\sigma \in K_s$.

$$(2.10) \quad Q_i(\tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = Q_{i-1}(s_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i),$$

where $s_j : K_i \to K_{i-1}$ denotes the degeneracy map for $1 \leq j \leq i$ (see [38; Lemma 4]). A mod p loop space X together with a C_n-form is said to be a C_n-space. If X has a sequence of maps $\{Q_i : K_i \times X^i \to X\}_{i \geq 1}$ such that $\{Q_i\}_{1 \leq i \leq n}$ is a C_n-form on X for any $n \geq 1$, then X is called a C_∞-space.

Let X and Y be C_n-spaces with the C_n-forms $\{Q_i : K_i \times X^i \to X\}_{1 \leq i \leq n}$ and $\{R_i : K_i \times Y^i \to Y\}_{1 \leq i \leq n}$, respectively. A loop map $\psi : X \to Y$ is said to be a C_n-map if there exists a sequence of maps $\{D_i : I \times K_i \times X^i \to Y\}_{1 \leq i \leq n}$ satisfying the following conditions:

$$(2.11) \qquad D_i(t, \tau, x_1, \dots, x_i) = \begin{cases} R_i(\tau, \psi(x_1), \dots, \psi(x_i)) & \text{if } t = 0, \\ \psi(Q_i(\tau, x_1, \dots, x_i)) & \text{if } t = 1. \end{cases}$$

For a partition (α_r, β_s) of $\mathbf{i}$,

$$(2.12) \qquad D_i(t, \varepsilon(\alpha_r, \beta_s)(\rho, \sigma), x_1, \dots, x_i)$$
$$= D_r(t, \rho, x_{\alpha(1)}, \dots, x_{\alpha(r)}) \cdot D_s(t, \sigma, x_{\beta(1)}, \dots, x_{\beta(s)}),$$

where $\rho \in K_r$ and $\sigma \in K_s$.

$$(2.13) \quad D_i(t, \tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = D_{i-1}(s_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i)$$

for $1 \leq j \leq i$.

The following theorem is the key in the proof of Theorem A:

Theorem 2.14. *Let A be a space. If X is a C_n-space, then the localization $L_A(X)$ is a C_n-space and the natural map $\phi_X : X \to L_A(X)$ is a C_n-map.*

Lemma 2.15. *Let A and B be spaces and $\phi : A \to B$ be a homotopy equivalence. If (K, L) denotes a relative CW-complex, then the following statements hold:*

(1) *If there are maps $f : K \to B$ and $g : L \to A$ with $\phi g = f|_L$, then there exists a map $h : K \to A$ such that $h|_L = g$ and $\phi h \simeq f \operatorname{rel} L$.*

(2) *If $h, k : K \to A$ are maps satisfying that $h|_L = k|_L$ and $\phi h \simeq \phi k \operatorname{rel} L$, then $h \simeq k \operatorname{rel} L$.*

Now we prove Theorem 2.14 as follows:

PROOF OF THEOREM 2.14. Since X is a mod p loop space, there exists a space Z such that $X = \Omega Z$. By a result of Dror Farjoun [9; Thm. 3.A.1], there exists a homotopy equivalence $\zeta : L_A(X) \to \Omega L_{\Sigma A}(Z)$ so that $\zeta\phi_X \simeq \Omega(\phi_Z) : X \to \Omega L_{\Sigma A}(Z)$. Then it is sufficient to show that $Y = \Omega L_{\Sigma A}(Z)$ is a C_n-space so that $\psi = \Omega(\phi_Z) : X \to Y$ is a C_n-map. By using the following homotopy commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}_*(Y^i, Y) \arrow[r, "(\psi^i)^*"] \arrow[d, "(\zeta^i)^*"', "\simeq"] & \mathrm{Map}_*(X^i, Y) \arrow[d, equal] \\
\mathrm{Map}_*(L_A(X)^i, Y) \arrow[r, "((\phi_X)^i)^*"] & \mathrm{Map}_*(X^i, Y) \\
\mathrm{Map}_*(L_A(X)^i, L_A(X)) \arrow[u, "\zeta_*", "\simeq"'] \arrow[r, "((\phi_X)^i)^*", "\simeq"'] & \mathrm{Map}_*(X^i, L_A(X)), \arrow[u, "\simeq", "\zeta_*"']
\end{tikzcd}$$

we have a homotopy equivalence

$$(2.16) \qquad (\psi^i)^* : \mathrm{Map}_*(Y^i, Y) \to \mathrm{Map}_*(X^i, Y)$$

since $((\phi_X)^i)^* : \mathrm{Map}_*(L_A(X)^i, L_A(X)) \to \mathrm{Map}_*(X^i, L_A(X))$ is a homotopy equivalence by (2.4). By using (2.5) and the same arguments as above, we have that

$$(2.17) \qquad (\psi^{(i)})^* : \mathrm{Map}_*(Y^{(i)}, Y) \to \mathrm{Map}_*(X^{(i)}, Y)$$

is a homotopy equivalence for $i \geq 1$.

Since X is a C_n-space, there exists a system of maps $\{Q_i : K_i \times X^i \to X\}_{1 \leq i \leq n}$ satisfying the conditions (2.8)–(2.10). By using induction on i, we shall construct C_n-forms $\{R_i : K_i \times Y^i \to Y\}_{1 \leq i \leq n}$ and $\{D_i : I \times K_i \times X^i \to Y\}_{1 \leq i \leq n}$ satisfying the conditions (2.8)–(2.10) and (2.11)–(2.13), which implies that Y is a C_n-space and $\psi : X \to Y$ is a C_n-map. In the case of $i = 1$, we can define maps $R_1 : K_1 \times Y \to Y$ and $D_1 : I \times K_1 \times X \to Y$ as $R_1(1, y) = y$ and $D_1(t, 1, x) = \psi(x)$.

Now we assume that there exist maps $R_j : K_j \times Y^j \to Y$ and $D_j : I \times K_j \times X^j \to Y$ for $1 \leq j \leq i-1$ satisfying the conditions (2.8)–(2.10) and (2.11)–(2.13). Let $S_i = I \times (L_i \times X^i \cup K_i \times X^{[i]}) \cup \{1\} \times K_i \times X^i$. If we define a map $E : S_i \to Y$ as

$$E(t, \varepsilon(\alpha_r, \beta_s)(\rho, \sigma), x_1, \dots, x_i) = D_r(t, \rho, x_{\alpha(1)}, \dots, x_{\alpha(r)}) \cdot D_s(t, \sigma, x_{\beta(1)}, \dots, x_{\beta(s)}),$$

$$E(t, \tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = D_{i-1}(s_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i)$$

and

$$E(1,\tau,x_1,\dots,x_i) = \psi(Q_i(\tau,x_1,\dots,x_i)),$$

then we have the extension $\tilde{E} : I \times K_i \times X^i \to Y$ with $\tilde{E}|_{S_i} = E$ by using the homotopy extension property. Let $F : K_i \times X^i \to Y$ be a map defined as $F(\tau,x_1,\dots,x_i) = \tilde{E}(0,\tau,x_1,\dots,x_i)$ and $\gamma : K_i \to \mathrm{Map}_*(X^i, Y)$ denote the adjoint map of F. If we set a map $\lambda : L_i \to \mathrm{Map}_*(Y^i, Y)$ as

$$\lambda(\varepsilon(\alpha_r,\beta_s)(\rho,\sigma), y_1,\dots,y_i) = R_r(\rho, y_{\alpha(1)},\dots,y_{\alpha(r)}) \cdot R_s(\sigma, y_{\beta(1)},\dots,y_{\beta(s)}),$$

then we see that $(\psi^i)^*\lambda = \gamma|_{L_i}$. By Lemma 2.15 and (2.16), there exists a map $\theta : K_i \to \mathrm{Map}_*(Y^i, Y)$ such that $\theta|_{L_i} = \lambda$ and $(\psi^i)^*\theta \simeq \gamma \operatorname{rel} L_i$.

To construct a map $R_i : K_i \times Y^i \to Y$ satisfying the conditions (2.8)–(2.10), we show that the induced map

$$(\psi^{[i]})^* : \mathrm{Map}_*(Y^{[i]}, Y)_f \to \mathrm{Map}_*(X^{[i]}, Y)_{(\psi^{[i]})^*f} \tag{2.18}$$

is a homotopy equivalence, where $Y^{[i]}$ denotes the i-fold fat wedge of Y defined as

$$Y^{[i]} = \{(y_1,\dots,y_i) \in Y^i \mid y_j = * \text{ for some } 1 \le j \le i\},$$

and $f : Y^{[i]} \to Y$ is a map defined as $f(y_1,\dots,y_i) = (\cdots(y_1 \cdot y_2)\cdots) \cdot y_i$. Since Y is a mod p loop space, if we define a map $\nu_1 : \mathrm{Map}_*(Y^{[i]}, Y)_c \to \mathrm{Map}_*(Y^{[i]}, Y)_f$ and $\nu_2 : \mathrm{Map}_*(X^{[i]}, Y)_c \to \mathrm{Map}_*(X^{[i]}, Y)_{(\psi^{[i]})^*f}$ as $\nu_1(g) = \mu_Y(g \times f)\Delta_{Y^{[i]}}$ and $\nu_2(h) = \mu_Y(h \times (\psi^{[i]})^*f)\Delta_{X^{[i]}}$, then the following diagram is homotopy commutative:

$$\begin{tikzcd}
\mathrm{Map}_*(Y^{[i]}, Y)_c \arrow[r, "(\psi^{[i]})^*"] \arrow[d, "\nu_1"', "\simeq"] & \mathrm{Map}_*(X^{[i]}, Y)_c \arrow[d, "\nu_2"', "\simeq"] \\
\mathrm{Map}_*(Y^{[i]}, Y)_f \arrow[r, "(\psi^{[i]})^*"] & \mathrm{Map}_*(X^{[i]}, Y)_{(\psi^{[i]})^*f},
\end{tikzcd}$$

where c denote the constant maps and the vertical arrows are homotopy equivalences. Hence it is sufficient to show that the top horizontal arrow is a homotopy equivalence. There is the following homotopy commutative diagram of fibrations:

$$\begin{tikzcd}
\mathrm{Map}_*(Y^{(i)}, Y)_{C_1} \arrow[r, "(\psi^{(i)})^*", "\simeq"'] \arrow[d, "(\rho_Y)^*"] & \mathrm{Map}_*(X^{(i)}, Y)_{C_2} \arrow[d, "(\rho_X)^*"] \\
\mathrm{Map}_*(Y^i, Y)_c \arrow[r, "(\psi^i)^*", "\simeq"'] \arrow[d, "(\iota_Y)^*"] & \mathrm{Map}_*(X^i, Y)_c \arrow[d, "(\iota_X)^*"] \\
\mathrm{Map}_*(Y^{[i]}, Y)_c \arrow[r, "(\psi^{[i]})^*"] & \mathrm{Map}_*(X^{[i]}, Y)_c,
\end{tikzcd}$$

where $C_1 = \{g : Y^{(i)} \to Y \mid (\rho_Y)^* g \simeq c\}$ and $C_2 = \{h : X^{(i)} \to Y \mid (\rho_X)^* h \simeq c\}$. By using the homotopy exact sequences for the fibrations and the five lemma, the bottom horizontal arrow is a homotopy equivalence, and so we have (2.18).

If we define a map $\omega : K_i \to \mathrm{Map}_*(Y^{[i]}, Y)_f$ as

$$\omega(\tau)(y_1, \dots, y_{j-1}, *, y_{j+1}, \dots, y_i) = R_{i-1}(s_j(\tau), y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_i)$$

for $1 \le j \le i$, then we see that $(\psi^{[i]})^*(\iota_Y)^*\theta \simeq (\psi^{[i]})^*\omega \,\mathrm{rel}\, L_i$. By Lemma 2.15 and (2.18), we have that $(\iota_Y)^*\theta \simeq \omega \,\mathrm{rel}\, L_i$, and so there exists a map $\xi : I \times K_i \to \mathrm{Map}_*(Y^{[i]}, Y)_f$ such that

$$\xi(t,\tau) = \begin{cases} (\iota_Y)^*\theta(\tau) & \text{if } (t,\tau) \in \{0\} \times K_i \cup I \times L_i, \\ \omega(\tau) & \text{if } (t,\tau) \in \{1\} \times K_i. \end{cases}$$

Let $T_i = I \times (L_i \times Y^i \cup K_i \times Y^{[i]}) \cup \{0\} \times K_i \times Y^i$. For a map $G : T_i \to Y$ defined as $G(t, \tau, y_1, \dots, y_i)$

$$= \begin{cases} \theta(\tau)(y_1, \dots, y_i) & \text{if } (t, \tau, y_1, \dots, y_i) \in I \times L_i \times Y^i \cup \{0\} \times K_i \times Y^i, \\ \xi(t,\tau)(y_1, \dots, y_i) & \text{if } (t, \tau, y_1, \dots, y_i) \in I \times K_i \times Y^{[i]}, \end{cases}$$

we have the extension $\tilde{G} : I \times K_i \times Y^i \to Y$ with $\tilde{G}|_{T_i} = G$ by using the homotopy extension property. If we define a map $R_i : K_i \times Y^i \to Y$ as $R_i(\tau, y_1, \dots, y_i) = \tilde{G}(1, \tau, y_1, \dots, y_i)$, then R_i satisfies the conditions (2.8)–(2.10). Since $R_i(1_{K_i} \times \psi^i) \simeq F \,\mathrm{rel}\, L_i \times X^i$, there exists a map $H : I \times K_i \times X^i \to Y$ satisfying that

$$H(t, \tau, x_1, \dots, x_i) = \begin{cases} R_i(\tau, \psi(x_1), \dots, \psi(x_i)) & \text{if } t = 0, \\ \psi(Q_i(\tau, x_1, \dots, x_i)) & \text{if } t = 1 \end{cases}$$

and

$$H(t, \varepsilon(\alpha_r, \beta_s)(\rho, \sigma), x_1, \dots, x_i) = D_r(t, \rho, x_{\alpha(1)}, \dots, x_{\alpha(r)}) \cdot D_s(t, \sigma, x_{\beta(1)}, \dots, x_{\beta(s)}).$$

If we set a map $M : I \times K_i \times X^{[i]} \to Y$ as

$$M(t, \tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) = D_{i-1}(s_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i)$$

for $1 \le j \le i$, then $H|_{(\partial I \times K_i \cup I \times L_i) \times X^{[i]}} = M|_{(\partial I \times K_i \cup I \times L_i) \times X^{[i]}}$. By [38; Remark 10], there exists a map $D_i : I \times K_i \times X^i \to Y$ such that $D_i|_{(\partial I \times K_i \cup I \times L_i) \times X^i} = H|_{(\partial I \times K_i \cup I \times L_i) \times X^i}$ and $D_i|_{I \times K_i \times X^{[i]}} = M$. Hence we see that the map D_i satisfies the conditions (2.11)–(2.13), which implies that Y is a C_i-space and $\psi : X \to Y$ is a C_i-map. This completes the proof of Theorem 2.14. $\square$

Now we give a proof of Theorem A as follows:

Proof of Theorem A. Let X be a simply connected C_n-space with finitely generated cohomology. From Theorem 2.14, the localization $L_{B\mathbf{Z}/p}(X)$ is a

C_n-space and the natural map $\phi_X : X \to L_{B\mathbf{Z}/p}(X)$ is a C_n-map. By using results of Broto and Crespo [4], [8], there exists an H-fibration

$$K \to X \to F, \tag{2.19}$$

where F is a simply connected mod p finite H-space and $K = K(\mathbf{Z}, 2)^m$ for some $m \geq 0$. Since the mod p cohomology $H^*(F)$ is finite dimensional, by Miller [23; Thm. A] F is $B\mathbf{Z}/p$-local. By Proposition 2.2 the localization functor $L_{B\mathbf{Z}/p}$ preserves the H-fibration (2.19), and so we have the following homotopy commutative diagram of fibrations:

$$\begin{tikzcd}
K \arrow[r, "\simeq"'] \arrow[d, equal] & \tilde{K} \arrow[r] \arrow[d] & * \arrow[d] \\
K \arrow[r] \arrow[d, "\phi_K"'] & X \arrow[r] \arrow[d, "\phi_X"'] & F \arrow[d, "\phi_F"', "\simeq"] \\
L_{B\mathbf{Z}/p}(K) \arrow[r] & L_{B\mathbf{Z}/p}(X) \arrow[r, "\simeq"'] & L_{B\mathbf{Z}/p}(F),
\end{tikzcd}$$

where $\tilde{K}$ denotes the homotopy fiber of $\phi_X : X \to L_{B\mathbf{Z}/p}(X)$. It is shown in [2; Remark 9.5] that $L_{B\mathbf{Z}/p}(K) \simeq *$, which implies that $L_{B\mathbf{Z}/p}(X) \simeq F$ and $K \simeq \tilde{K}$. Since $\phi_X : X \to L_{B\mathbf{Z}/p}(X)$ is a C_n-map, by Williams [38; Thm. 21], the homotopy fiber $\tilde{K}$ is a C_n-space and the fiber inclusion $\tilde{K} \to X$ is a C_n-map. This completes the proof of Theorem A. $\square$

3. Proofs of Theorems B and C

Let X be a simply connected C_n-space with finitely generated cohomology. Then from Theorem A, X is the total space of a C_n-fibration over a simply connected finite C_n-space, and so we can reduce the problems to the cases of finite C_n-spaces.

If F is a simply connected finite C_2-space, then by a result of Browder [6; Thm. 8.6] the mod p cohomology $H^*(F)$ is an exterior algebra generated by the odd dimensional elements. By a spectral sequence argument $H^*(BF)$ is a polynomial algebra, where BF denotes the classifying space of F. We can put

$$H^*(BF) = \mathbf{Z}/p[x_1, \ldots, x_l],$$

where $\deg x_i = 2t_i$ for $1 \leq i \leq l$. From results of Adams-Wilkerson [1] and Dwyer-Miller-Wilkerson [10], there exists a pseudo-reflection group $G(F) \subset GL(l, \hat{\mathbf{Z}}_p)$ such that $H^*(BF) \cong H^*(BT^l)^{G(F)}$, where BT^l denotes the classifying space of the l-dimensional torus. This implies that the type $(t_1, \ldots, t_l)$ of F is a union of irreducible types on the Clark-Ewing list [7].

The topological realizations of the invariant rings $H^*(BT^l)^G$ for pseudo-reflection groups $G \subset GL(l, \hat{\mathbf{Z}}_p)$ have been investigated by Clark-Ewing [7] and many others (cf. [26; § 3]). Notbohm [27] studied the topological realization of a family of pseudo-reflection groups. Let p be an odd prime, $q > 1, r \geq 1$ and $l > 1$ such that $q|(p-1)$ and $r|q$. A subgroup $G(q,r,l) \subset GL(l, \hat{\mathbf{Z}}_p)$ is defined as the semidirect product $G(q,r,l) = A(q,r,l) \rtimes \Sigma_l$, where

$$A(q,r,l) = \{(z_1, \dots, z_l) \in (\hat{\mathbf{Z}}_p)^l \,|\, z_i^q = 1 \text{ for } 1 \leq i \leq l \text{ and } (z_1 \dots z_l)^r = 1\},$$

and Σ_l denotes the symmetric group on l letters. Here an element of $A(q,r,l)$ is considered as a diagonal matrix and Σ_l acts on $(\hat{\mathbf{Z}}_p)^l$ by the permutation of the coordinates. It is known that $G(q,r,l)$ is generated by the pseudo-reflections (cf. [27; Remark 1.3]), and $G(q,r,l)$ acts on the mod p cohomology $H^*(BT^l)$. From [27; Prop. 1.4], the invariant ring is given as

$$H^*(BT^l)^{G(q,r,l)} = \mathbf{Z}/p[s_1, \dots, s_l]^{G(q,r,l)} \cong \mathbf{Z}/p[y_1, \dots, y_{l-1}, e],$$

where $y_i = \sigma_i(s_1^q, \dots, s_l^q)$ for $1 \leq i \leq l-1$ and $e = (s_1 \dots s_l)^r$. We see that the invariant ring has the type of no. 2a on the Clark-Ewing list [7]. The following result is due to Notbohm:

Theorem 3.1 ([27; Thm. 1.5, 1.6]). *Let p be an odd prime. If $q > 1$, $r \geq 1$ and $l > 1$ such that $q|(p-1)$ and $r|q$, then the following statements hold:*

(1) *There exists a space $BX(q,r,l)$ such that the* mod p *cohomology*

$$H^*(BX(q,r,l)) \cong \mathbf{Z}/p[y_1, \dots, y_{l-1}, e].$$

(2) *If Y is a space such that the* mod p *cohomology $H^*(Y) \cong H^*(BX(q,r,l))$ as algebras over the* mod p *Steenrod algebra, then $Y \simeq BX(q,r,l)$.*

By generalizing results of Saumell [32; Prop. 3.3, 4.1], we have the following:

Lemma 3.2. *Let p be an odd prime. If $q > 1$, $r \geq 1$ and $l > 1$ such that $q|(p-1)$ and $r|q$, then the p-compact group $X(q,r,l) = \Omega BX(q,r,l)$ is not a C_2-space.*

Proof. If $p > l$ or $r = q$, then we have the required result by results of Saumell [32; Prop. 3.3, 4.1], and so we assume that $p \leq l$ and $r < q$. Let $s = (p-1)/q$. Then $s < l$ since $s|(p-1)$ and $p \leq l$. If D^3 denotes the 3-fold decomposable module of $H^*(BX(q,r,l))$ defined as $D^3 = \tilde{H}^*(BX(q,r,l)) \cdot \tilde{H}^*(BX(q,r,l)) \cdot \tilde{H}^*(BX(q,r,l))$, then D^3 is closed under the action of the mod p Steenrod algebra, and by [32; Lemma 3.2], the action of $\mathscr{P}^1$ on $H^*(BX(q,r,l))/D^3$ is given as

$$\mathscr{P}^1(y_i) = \begin{cases} (-1)^s(s+i)q\tilde{y}_{s+i} + \sum_{j=1}^{s}(-1)^{s+1}jq\tilde{y}_j\tilde{y}_{s+i-j} & \text{if } 1 \le i \le l-s, \\ \sum_{j=i+s-l}^{s}(-1)^{s+1}jq\tilde{y}_j\tilde{y}_{s+i-j} & \text{if } l-s+1 \le i \le l-1, \end{cases}$$

$$\mathscr{P}^1(e) = (-1)^{s+1}sr\tilde{y}_s e,$$

where $\tilde{y}_i = y_i$ for $1 \le i \le l-1$ and $\tilde{y}_l = e^{q/r}$. By applying a result of McGibbon [21; Lemma 3.2] (see also Lemma 4.6 in §4) to $e \in H^*(BX(q,r,l))$ and $\mathscr{P}^1$, we see that $X(q,r,l) = \Omega BX(q,r,l)$ is not a C_2-space. This completes the proof. □

THEOREM 3.3. *If F is a simply connected finite C_2-space, then there exists a system $\{F_i\}_{1 \le i \le k}$ consisting of some of the p-compact groups $X(G)$ on the following table such that the classifying space $BF \simeq \prod_{i=1}^{k} BF_i$.*

G	p	$X(G)$	Types
G	$\ge 2t_l$	$S^{2t_1-1} \times \cdots \times S^{2t_l-1}$	$(t_1, \dots, t_l)$
G_{2b}	≥ 3	$B_1(p)$	$(2, p+1)$
G_9	17	$B_7(17)$	$(8, 24)$
G_{14}	19	$B_5(19)$	$(6, 24)$
G_{17}	41	$B_{19}(41)$	$(20, 60)$
G_{20}	19	$B_{11}(19)$	$(12, 30)$
G_{24}	11	$B_3(11) \times S^{11}$	$(4, 6, 14)$
G_{30}	19	$B_1(19) \times B_{11}(19)$	$(2, 12, 20, 30)$

PROOF. By results of Dwyer-Wilkerson [12] and Notbohm [28], there exists a decomposition $BF \simeq \prod_{i=1}^{k} BF_i$ with simple p-compact groups F_i for $1 \le i \le k$ (cf. [11], [26]). From a result of Dwyer-Miller-Wilkerson [10], $H^*(BF_i) \cong H^*(BT)^G$ for some pseudo-reflection group G since the cohomology $H^*(BF_i)$ is a polynomial algebra. This implies that the type of F_i is obtained from the Clark-Ewing list [7] for $1 \le i \le k$. Since $F \simeq \prod_{i=1}^{k} F_i$ as p-compact groups, we can show that F_i is a C_2-space for $1 \le i \le k$. Hence it is sufficient to consider the homotopy commutativity of p-compact groups on the Clark-Ewing list.

Recently, Notbohm [30] determined the condition for which a type on the Clark-Ewing list is realizable as the cohomology of a space. From results of Notbohm [29; Thm. 1.4], [30; Thm. 1.4, 1.5], McGibbon [21; Thm. 2], Saumell [32; Thm. 1.1] and Lemma 3.2, we see that each F_i must be contained in the table given in the theorem for $1 \le i \le k$. This completes the proof of Theorem 3.3. □

Now we proceed to the proof of Theorem B. Let F be a finite C_2-space. Then by Theorem 3.3, there exists a homotopy equivalence $BF \simeq \prod_{i=1}^{k} BF_i$, where F_i is one of the p-compact groups $X(G)$ on the table of Theorem 3.3 for $1 \le i \le k$. By permuting F_i if necessary, we can assume that $H^3(F_i; \mathbf{Z}_{(p)}) \cong \mathbf{Z}_{(p)}$ for $1 \le i \le r$ and F_i is 3-connected for $r+1 \le i \le k$. Thus the p-localized cohomology of F is given as

$$H^3(F; \mathbf{Z}_{(p)}) \cong \mathbf{Z}_{(p)}\{u_1, \dots, u_r\},$$

where $u_i \in H^3(F_i; \mathbf{Z}_{(p)})$ is the generator for $1 \le i \le r$. Since $H^3(K(\mathbf{Z}, 3); \mathbf{Z}_{(p)}) \cong \mathbf{Z}_{(p)}$, we put the p-localized cohomology of $K(\mathbf{Z}, 3)^m$ as

$$H^3(K(\mathbf{Z}, 3)^m; \mathbf{Z}_{(p)}) \cong \mathbf{Z}_{(p)}\{v_1, \dots, v_m\},$$

where v_j denotes the generator of $H^3(K(\mathbf{Z}, 3); \mathbf{Z}_{(p)})$ for $1 \le j \le m$.

Let $\phi : F \to K(\mathbf{Z}, 3)^m$ be a map and $\phi^* : H^3(K(\mathbf{Z}, 3)^m; \mathbf{Z}_{(p)}) \to H^3(F; \mathbf{Z}_{(p)})$ be the induced homomorphism. Then it is easy to see that there exist systems of generators $\{\tilde{v}_j \in H^3(K(\mathbf{Z}, 3)^m; \mathbf{Z}_{(p)})\}_{1 \le j \le m}$ and $\{\tilde{u}_i \in H^3(F; \mathbf{Z}_{(p)})\}_{1 \le i \le r}$ satisfying that

$$\phi^*(\tilde{v}_j) = \begin{cases} \tilde{u}_j & \text{if } 1 \le j \le s_1, \\ p^{e_j}\tilde{u}_j & \text{if } s_1 + 1 \le j \le s_2, \\ 0 & \text{if } s_2 + 1 \le j \le m, \end{cases} \tag{3.4}$$

where $s_1 \le s_2 \le r$ and $1 \le e_j \le e_{j+1}$ for $s_1 + 1 \le j \le s_2 - 1$.

In the proof of Theorem B, we need the following fact:

Proposition 3.5. *If F is a finite C_2-space and $\phi : F \to K(\mathbf{Z}, 3)^m$ is a map, then there exist systems of generators $\{\tilde{v}_j \in H^3(K(\mathbf{Z}, 3)^m; \mathbf{Z}_{(p)})\}_{1 \le j \le m}$ and $\{\tilde{u}_i \in H^3(F; \mathbf{Z}_{(p)})\}_{1 \le i \le r}$ satisfying* (3.4), *and moreover, there is a homotopy equivalence $\omega : F \to F$ such that $\omega^*(\tilde{u}_i) = u_i$ for $1 \le i \le r$ on the p-localized cohomology $H^3(F; \mathbf{Z}_{(p)})$.*

By permuting F_i for $1 \le i \le r$, we assume that $F_i = B_1(p) \times \tilde{F}_i$ for $1 \le i \le q$ and $F_i = S^3 \times \tilde{F}_i$ for $q+1 \le i \le r$. Let $\rho^* : H^3(F; \mathbf{Z}_{(p)}) \to H^3(F)$ denote the mod p reduction map and $\sigma_i = \rho^*(u_i) \in H^3(F)$ be the mod p reduction of u_i for $1 \le i \le r$. Since the mod p cohomology of $B_1(p)$ is

$$H^*(B_1(p)) = \Lambda(\sigma, \mathscr{P}^1(\sigma))$$

with $\deg \sigma = 3$, we can assume that $\mathscr{P}^1(\sigma_i) \ne 0$ in $H^{2p+1}(F)$ for $1 \le i \le q$ and $\mathscr{P}^1(\sigma_i) = 0$ for $q+1 \le i \le r$.

The following lemma is used to prove Proposition 3.5:

Lemma 3.6. *Let F and $\phi : F \to K(\mathbf{Z}, 3)^m$ be as in Proposition* 3.5. *Then we can choose systems of generators $\{\hat{v}_j \in H^3(K(\mathbf{Z}, 3)^m; \mathbf{Z}_{(p)})\}_{1 \le j \le m}$ and*

$\{\hat{u}_i \in H^3(F; \mathbf{Z}_{(p)})\}_{1 \le i \le r}$ *such that the following conditions hold:*

(1)

$$\phi^*(\hat{v}_j) = \begin{cases} p^{f_j}\hat{u}_j & \text{if } 1 \le j \le t_1, \\ p^{g_j}\hat{u}_{q-t_1+j} & \text{if } t_1 + 1 \le j \le t_2, \\ 0 & \text{if } t_2 + 1 \le j \le m, \end{cases}$$

where $t_1 \le q$, $t_2 \le r - q + t_1$, $0 \le f_j \le f_{j+1}$ *for* $1 \le j \le t_1 - 1$ *and* $0 \le g_j \le g_{j+1}$ *for* $t_1 + 1 \le j \le t_2 - 1$.

(2) *If we put* $\hat{\sigma}_i = \rho^*(\hat{u}_i) \in H^3(F)$ *for* $1 \le j \le r$, *then* $\mathscr{P}^1(\hat{\sigma}_i) \ne 0$ *in* $H^{2p+1}(F)$ *for* $1 \le i \le q$ *and* $\mathscr{P}^1(\hat{\sigma}_i) = 0$ *for* $q + 1 \le i \le r$.

Proof. By using induction on $r \ge 1$ and $m \ge 1$, we show the lemma. If $\phi^* = 0$, then we can put that $\hat{v}_j = v_j$ for $1 \le j \le m$ and $\hat{u}_i = u_i$ for $1 \le i \le r$, and the result follows. Thus we assume that $\phi^* \ne 0$.

In the case of $r = 1$, we can set $\phi^*(v_j) = p^{a_j}c_j u$, where $a_j \ge 0$ and $(c_j, p) = 1$ for $1 \le j \le m$. By permuting v_j, we can assume that $a_1 \le a_j$ for $2 \le j \le m$. If we put that $e = a_1$, $\hat{v}_1 = v_1$, $\hat{v}_j = v_j - p^{a_j - e}(c_j/c_1)v_1$ for $2 \le j \le m$ and $\hat{u} = c_1 u$, then the result follows.

In the case of $m = 1$, we can set

$$\phi^*(v) = \sum_{i=1}^{q} p^{a_i}c_i u_i + \sum_{i=q+1}^{r} p^{b_i}d_i u_i,$$

where $a_i, b_i \ge 0$ and $(c_i, p) = 1, (d_i, p) = 1$ for $1 \le i \le r$. By permuting u_i, we can assume that $a_1 \le a_i$ for $2 \le i \le q$ and $b_{q+1} \le b_i$ for $q + 2 \le i \le r$. If $a_1 \le b_{q+1}$, then we put $f = a_1$ and choose generators $\hat{u}_i \in H^3(F; \mathbf{Z}_{(p)})$ as $\hat{u}_1 = \sum_{i=1}^{q} p^{a_i - f}c_i u_i + \sum_{i=q+1}^{r} p^{b_i - f}d_i u_i$ and $\hat{u}_i = u_i$ for $2 \le i \le r$. In this case, we see that $\mathscr{P}^1(\hat{\sigma}_1) = \mathscr{P}^1(\rho^*(\hat{u}_1)) = \sum_{i=1}^{q} p^{a_i - f}c_i\mathscr{P}^1(\sigma_i) \ne 0$, and the result follows. If $a_1 > b_{q+1}$, then we put $g = b_{q+1}$, $\hat{u}_{q+1} = \sum_{i=1}^{q} p^{a_i - g}c_i u_i + \sum_{i=q+1}^{r} p^{b_i - g}d_i u_i$ and $\hat{u}_i = u_i$ for $1 \le i \le q$, $q+2 \le i \le r$. In this case, $\mathscr{P}^1(\hat{\sigma}_{q+1}) = \sum_{i=q+1}^{r} p^{b_i - g}d_i\mathscr{P}^1(\sigma_i) = 0$ since $a_i - g \ge a_1 - g > 0$ for $1 \le i \le q$, and so we have the required conclusion.

Now we assume that the result holds if r and m are replaced by $r - 1$ and $m - 1$. In the case of $r, m > 1$, the induced homomorphism $\phi^* : H^3(K(\mathbf{Z}, 3)^m; \mathbf{Z}_{(p)}) \to H^3(F; \mathbf{Z}_{(p)})$ is give as

$$\phi^*(v_j) = \sum_{i=1}^{q} p^{a_{i,j}}c_{i,j}u_i + \sum_{i=q+1}^{r} p^{b_{i,j}}d_{i,j}u_i$$

for $1 \le j \le m$, where $a_{i,j}, b_{i,j} \ge 0$ and $(c_{i,j}, p) = 1$, $(d_{i,j}, p) = 1$ for $1 \le i \le r$ and $1 \le j \le m$. By using the similar arguments to the cases of $r = 1$ and $m = 1$, there exist systems of generators $\{\bar{v}_j \in H^3(K(\mathbf{Z}, 3)^m; \mathbf{Z}_{(p)})\}_{1 \le j \le m}$ and

$\{\bar{u}_i \in H^3(F; \mathbf{Z}_{(p)})\}_{1 \le i \le r}$ satisfying (2) and either of the following conditions:

$$\phi^*(\bar{v}_1) = p^{f_1}\bar{u}_1,$$

$$\phi^*(\bar{v}_j) = \sum_{i=2}^{q} p^{\bar{a}_{i,j}}\bar{c}_{i,j}\bar{u}_i + \sum_{i=q+1}^{r} p^{\bar{b}_{i,j}}\bar{d}_{i,j}\bar{u}_i$$

for $2 \le j \le m$ or

$$\phi^*(\bar{v}_1) = p^{g_1}\bar{u}_{q+1},$$

$$\phi^*(\bar{v}_j) = \sum_{i=1}^{q} p^{\bar{a}_{i,j}}\bar{c}_{i,j}\bar{u}_i + \sum_{i=q+2}^{r} p^{\bar{b}_{i,j}}\bar{d}_{i,j}\bar{u}_i$$

for $2 \le j \le m$, where $0 \le f_1 \le \bar{a}_{i,j}, \bar{b}_{i,j}$, $0 \le g_1 \le \bar{a}_{i,j}, \bar{b}_{i,j}$ and $(\bar{c}_{i,j}, p) = (\bar{d}_{i,j}, p) = 1$ for $1 \le i \le r$ and $2 \le j \le m$. From the inductive hypothesis, we can choose systems of generators $\{\hat{v}_j \in H^3(K(\mathbf{Z},3)^m; \mathbf{Z}_{(p)})\}_{1 \le j \le m}$ and $\{\hat{u}_i \in H^3(F; \mathbf{Z}_{(p)})\}_{1 \le i \le r}$ satisfying the required conditions. $\square$

Proof of Proposition 3.5. Let $\{\hat{v}_j \in H^3(K(\mathbf{Z},3)^m; \mathbf{Z}_{(p)})\}_{1 \le j \le m}$ and $\{\hat{u}_i \in H^3(F; \mathbf{Z}_{(p)})\}_{1 \le i \le r}$ be the systems of generators of Lemma 3.6. We construct maps $f_i : B_1(p) \to F$ and $g_i : S^3 \to F$ such that $f_i^*(\hat{u}_i) = u_i$ for $1 \le i \le q$ and $g_i^*(\hat{u}_i) = u_i$ for $q+1 \le i \le r$.

By the universal coefficient theorem, we have that $\pi_3(F) \cong H_3(F; \mathbf{Z}_{(p)}) \cong \mathbf{Z}_{(p)}\{\hat{u}_1, \dots, \hat{u}_r\}$, and so there exists a map $g_i : S^3 \to F$ such that $g_i^*(\hat{u}_i) = u$ in the p-localized cohomology for $1 \le i \le r$. If $\sigma \in H^3(S^3)$ denotes the mod p reduction of $u \in H^3(S^3; \mathbf{Z}_{(p)})$, then we have that $g_i^*(\hat{\sigma}_i) = \sigma$ for $1 \le i \le r$.

From the mod p cohomology $H^*(B_1(p))$, the cell structure of $B_1(p)$ is given as

$$B_1(p) \simeq S^3 \cup_\alpha e^{2p+1} \cup_\beta e^{2p+4},$$

where $\alpha \in \pi_{2p}(S^3) \cong \mathbf{Z}/p$ is the generator and $\beta : S^{2p+3} \to S^3 \cup_\alpha e^{2p+1}$ denotes some attaching map. In the case of $1 \le i \le q$, $\mathscr{P}^1(\hat{\sigma}_i) \ne 0$ in $H^{2p+1}(F)$ by Lemma 3.6. If we consider a cofibration

$$S^{2p} \xrightarrow{\alpha} S^3 \xrightarrow{\gamma} S^3 \cup_\alpha e^{2p+1},$$

then $g_i\alpha \simeq *$ since $\mathscr{P}^1(\hat{\sigma}_i) \ne 0$, and so there exists a map $h_i : S^3 \cup_\alpha e^{2p+1} \to F$ such that $h_i\gamma \simeq g_i$. By Theorem 3.3, [24; Thm. 3.2] and [37; Thm. 3], we see that $\pi_{2p+3}(F) = 0$. If we consider the following cofibration:

$$S^{2p+3} \xrightarrow{\beta} S^3 \cup_\alpha e^{2p+1} \xrightarrow{\delta} B_1(p),$$

then $h_i\beta \simeq *$, and so there exists a map $f_i : B_1(p) \to F$ such that $f_i\delta \simeq h_i$ for $1 \le i \le q$.

If we put $\tilde{F} = \prod_{i=1}^{r} \tilde{F}_i \times \prod_{i=r+1}^{k} F_i$, where $F_i = B_1(p) \times \tilde{F}_i$ for $1 \le i \le q$ and $F_i = S^3 \times \tilde{F}_i$ for $q+1 \le i \le r$, then $F = B_1(p)^q \times (S^3)^{r-q} \times \tilde{F}$. Let $\varepsilon : \tilde{F} \to F$ denote the inclusion map. Since F is a loop space, by using the maps $f_i : B_1(p) \to F$ for $1 \le i \le q$, $g_i : S^3 \to F$ for $q+1 \le i \le r$ constructed above and $\varepsilon : \tilde{F} \to F$, we can construct a map $\omega : F \to F$ which induces an isomorphism on the mod p cohomology.

By permuting the generators obtained in Lemma 3.6, we have systems of generators $\{\tilde{u}_i \in H^3(F; \mathbf{Z}_{(p)})\}_{1 \le i \le r}$ and $\{\tilde{v}_j \in H^3(K(\mathbf{Z},3)^m; \mathbf{Z}_{(p)})\}_{1 \le j \le m}$ satisfying the required conditions. This completes the proof. $\square$

Now we can prove Theorem B as follows:

PROOF OF THEOREM B. If X is a simply connected C_2-space with finitely generated mod p cohomology, then by Theorem A, there exists a C_2-fibration

$$K \to X \to F, \tag{3.7}$$

where F is a simply connected finite C_2-space and $K = K(\mathbf{Z},2)^m$ for some $m \ge 0$. By extending the fibration (3.7), we have an H-fibration

$$X \to F \xrightarrow{\phi} K(\mathbf{Z},3)^m,$$

and we may assume that the p-localized cohomology of $K(\mathbf{Z},3)^m$ and F are given as

$$H^3(K(\mathbf{Z},3)^m; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}\{v_1, \dots, v_m\}$$

and

$$H^3(F; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}\{u_1, \dots, u_r\},$$

where $v_j \in H^3(K(\mathbf{Z},3); \mathbf{Z}_{(p)})$ for $1 \le j \le m$ and $u_i \in H^3(F_i; \mathbf{Z}_{(p)})$ for $1 \le i \le r$ denote the generators. By Proposition 3.5, we can choose systems of generators $\{\tilde{v}_j \in H^3(K(\mathbf{Z},3)^m; \mathbf{Z}_{(p)})\}_{1 \le j \le m}$ and $\{\tilde{u}_i \in H^3(F; \mathbf{Z}_{(p)})\}_{1 \le i \le r}$ such that the induced homomorphism $\phi^* : H^3(K(\mathbf{Z},3)^m; \mathbf{Z}_{(p)}) \to H^3(F; \mathbf{Z}_{(p)})$ is given as

$$\phi^*(\tilde{v}_j) = \begin{cases} \tilde{u}_j & \text{if } 1 \le j \le s_1, \\ p^{e_j} \tilde{u}_j & \text{if } s_1 + 1 \le j \le s_2, \\ 0 & \text{if } s_2 + 1 \le j \le m, \end{cases}$$

where $1 \le e_j \le e_{j+1}$ for $s_1 + 1 \le j \le s_2 - 1$. Furthermore, there exists a homotopy equivalence $\omega : F \to F$ satisfying that $\omega^*(\tilde{u}_i) = u_i$ for $1 \le i \le r$. If $\zeta : K(\mathbf{Z},3)^m \to K(\mathbf{Z},3)^m$ is a map defined as $\zeta^*(\tilde{v}_j) = v_j$ for $1 \le j \le m$, then ζ is a homotopy equivalence, and there exists the following homotopy commutative diagram of fibrations:

$$(3.8)\qquad \begin{array}{ccccc} Y & \longrightarrow & F & \xrightarrow{\psi} & K(\mathbf{Z},3)^m \\ {\scriptstyle\kappa}\downarrow & & {\scriptstyle\omega}\downarrow\simeq & & {\scriptstyle\zeta}\downarrow\simeq \\ X & \longrightarrow & F & \xrightarrow{\phi} & K(\mathbf{Z},3)^m, \end{array}$$

where

$$Y = \prod_{i=1}^{s_1} F_i\langle 3\rangle \times \prod_{i=s_1+1}^{s_2} F_i\langle 3; p^{e_i}\rangle \times \prod_{i=s_2+1}^{k} F_i \times K(\mathbf{Z},2)^{m-s_2}$$

and $\psi : F \to K(\mathbf{Z},3)^m$ is a map defined as

$$\psi^*(v_j) = \begin{cases} u_j & \text{if } 1 \le j \le s_1, \\ p^{e_j}u_j & \text{if } s_1+1 \le j \le s_2, \\ 0 & \text{if } s_2+1 \le j \le m. \end{cases}$$

By using the homotopy exact sequences for the fibrations (3.8) and the five lemma, we see that $\kappa : Y \to X$ is also a homotopy equivalence. This completes the proof of Theorem B. □

Remark 3.9. In the proof of Theorem B, if X is a C_n-space, then F is a finite C_n-space by Theorem A. Since $BF \simeq \prod_{i=1}^{k} BF_i$, by using the next lemma, we have that F_i is a C_n-space for $1 \le i \le k$.

Lemma 3.10. *Let X, Y and Z be loop spaces. If Z is a C_n-space and $Z \simeq X \times Y$ as loop spaces, then X and Y are C_n-spaces.*

Proof. Let $\iota_X : X \to Z$ be the inclusion map and $\rho_X : Z \to X$ be the projection map. Since Z is a C_n-space, there exists a C_n-form $\{Q_i : K_i \times Z^i \to Z\}_{1\le i\le n}$ satisfying the conditions (2.8)–(2.10). If we define a map $Q_i^X : K_i \times X^i \to X$ as $Q_i^X = \rho_X Q_i(1_{K_i} \times (\iota_X)^i)$ for $1 \le i \le n$, then the system $\{Q_i^X : K_i \times X^i \to X\}_{1\le i\le n}$ satisfies the conditions (2.8)–(2.10), which implies that X is a C_n-space. By using similar arguments, we see that Y is a C_n-space. This completes the proof. □

Next we proceed to the proof of Theorem C. By using Theorem A, we can reduce the problem to the case of finite C_p-spaces.

Hemmi [13] introduced the concept of the quasi C_n-space. Let X be an A_n-space in the sense of Stasheff [35] and $P_i(X)$ denote the i-th projective space of X for $1 \le i \le n$. From the construction of $P_i(X)$, we have the inclusion map $\iota_i : P_i(X) \to P_{i+1}(X)$ for $1 \le i \le n-1$ and the projection map $\rho_i : P_i(X) \to (\Sigma X)^{(i)}$ for $1 \le i \le n$. Let $J_i(\Sigma X)$ denote the i-th James reduced product space of ΣX and $\pi_i : J_i(\Sigma X) \to (\Sigma X)^{(i)}$ be the projection map for $i \ge 1$. A quasi C_n-form on X is a system of maps $\{\psi_i : J_i(\Sigma X) \to P_i(X)\}_{1\le i\le n}$ satisfying the following conditions:

$$\psi_1 = 1_{\Sigma X} : \Sigma X \to \Sigma X,$$

$$\psi_i|_{J_{i-1}(\Sigma X)} = \iota_{i-1}\psi_{i-1} \qquad \text{for } 2 \leq i \leq n,$$

$$\rho_i\psi_i \simeq \left(\sum_{\sigma \in \Sigma_i} \sigma\right)\pi_i \qquad \text{for } 1 \leq i \leq n,$$

where the action of the symmetric group Σ_i on $(\Sigma X)^{(i)}$ is given as the permutation of the coordinates, and the summation on the right hand side is defined by using a co-H-structure on $(\Sigma X)^{(i)}$. An A_n-space X together with a quasi C_n-form is said to be a quasi C_n-space. Hemmi has shown the following result:

THEOREM 3.11 ([13; Thm. 1.1]). *Let p be an odd prime. If X is a simply connected finite quasi C_p-space, then X is contractible.*

By using the above theorem, we can prove Theorem C as follows:

PROOF OF THEOREM C. Let X be a simply connected C_p-space with finitely generated mod p cohomology. By Theorem B and Remark 3.9, there exists a homotopy equivalence

$$X \simeq \prod_{i=1}^{s} F_i\langle 3\rangle \times \prod_{i=s+1}^{t} F_i\langle 3; p^{e_i}\rangle \times \prod_{i=t+1}^{q} F_i \times K(\mathbf{Z}, 2)^r,$$

where F_i is a simply connected finite C_p-space for $1 \leq i \leq q$. On the category of loop spaces, the quasi C_p-space is exactly the C_p-space in the sense of Williams by Hemmi [13; Thm. 2.2]. Then by Theorem 3.11, we see that $F_i \simeq *$ for $1 \leq i \leq q$, and so $X \simeq K(\mathbf{Z}, 2)^r$. This completes the proof of Theorem C. □

In the case of connected C_p-spaces, we have the following:

COROLLARY 3.12. *Let p be an odd prime. If X is a connected C_p-space such that the* mod p *cohomology $H^*(X)$ is finitely generated as an algebra, then X is homotopy equivalent to a finite product of $K(\mathbf{Z}, 1)$, $K(\mathbf{Z}, 2)$ and $K(\mathbf{Z}/p^i, 1)$ for $i \geq 1$.*

To prove Corollary 3.12, we need the following lemma:

LEMMA 3.13. *If X is a connected C_n-space, then the universal cover $\tilde{X}$ is a C_n-space and the covering projection map $\omega : \tilde{X} \to X$ is a C_n-map.*

PROOF. We give an outline of the proof. Let $\mu : X \times X \to X$ denote the associative H-structure which makes X a C_n-space. By [25; Thm. II.4.2, 4.3], there exists an associative H-structure $\tilde{\mu}: \tilde{X} \times \tilde{X} \to \tilde{X}$ on $\tilde{X}$ such that $\omega\tilde{\mu} = \mu(\omega \times \omega)$.

Since X is a C_n-space, there exists a C_n-form $\{Q_i : K_i \times X^i \to X\}_{1 \le i \le n}$ satisfying the conditions (2.8)–(2.10). By the covering lifting property (cf. [25; Lemma II.1.7]), there exists a map $\tilde{Q}_i : K_i \times \tilde{X}^i \to \tilde{X}$ such that $\omega\tilde{Q}_i = Q_i(1_{K_i} \times \omega^i)$ for $1 \le i \le n$. By using the uniqueness of the lifting, we see that the system $\{\tilde{Q}_i : K_i \times \tilde{X}^i \to \tilde{X}\}_{1 \le i \le n}$ satisfies the conditions (2.8)–(2.10), and thus $\tilde{X}$ is a C_n-space. This completes the proof. □

The proof of Corollary 3.12 is given as follows:

PROOF OF COROLLARY 3.12. If $\tilde{X}$ denotes the universal cover of X, then there exists an H-fibration

$$\tilde{X} \to X \to K(\pi_1(X), 1), \tag{3.14}$$

where $K(\pi_1(X), 1)$ is a finite product of $K(\mathbf{Z}, 1)$ and $K(\mathbf{Z}/p^i, 1)$ for $i \ge 1$. By Lemma 3.13, $\tilde{X}$ is a simply connected C_p-space, and by using a spectral sequence argument, we see that $H^*(\tilde{X})$ is finitely generated as an algebra. By applying Theorem C to $\tilde{X}$, we have that $\tilde{X} \simeq K(\mathbf{Z}, 2)^r$ for some $r \ge 0$. By a result of Browder [5], we can use a spectral sequence for the H-fibration (3.14). Then the E_2-term is given as

$$E_2^{*,*} \cong H^*(K(\pi_1(X), 1)) \otimes H^*(\tilde{X}),$$

and by the DHA lemma [15; p. 14], the spectral sequence collapses, and so we have that

$$H^*(X) \cong H^*(K(\pi_1(X), 1)) \otimes H^*(\tilde{X}).$$

Since $\tilde{X} \simeq K(\mathbf{Z}, 2)^r$, there exists a map $\zeta : X \to K(\pi_1(X), 1) \times K(\mathbf{Z}, 2)^r$ which induces an isomorphism on the mod p cohomology, which implies the required conclusion. This completes the proof of Corollary 3.12. □

4. Higher homotopy commutativity of p-compact groups

In § 3, we proved the classification theorems for C_2-spaces and C_p-spaces with finitely generated cohomology. In this section, we are interested in C_n-spaces for $2 < n < p$. If X is a simply connected C_n-space such that the mod p cohomology $H^*(X)$ is finitely generated as an algebra, then by Theorem B and Remark 3.9, there exists a homotopy equivalence

$$X \simeq \prod_{i=1}^{s} F_i\langle 3\rangle \times \prod_{i=s+1}^{t} F_i\langle 3; p^{e_i}\rangle \times \prod_{i=t+1}^{q} F_i \times K(\mathbf{Z}, 2)^r,$$

where F_i is a simply connected finite C_n-space for $1 \leq i \leq q$. So we can reduce the classification of C_n-spaces with finitely generated cohomology to the higher homotopy commutativity of the p-compact groups $X(G)$ given in Theorem 1.2. We can give another interpretation of the higher homotopy commutativity of loop spaces by using the generalized higher Whitehead product due to Porter [31].

Let X be a loop space and $X^{(n)}$ denote the n-fold smash product of X. By Porter [31; Thm. 1.2], there exists a map $\omega_n : \Sigma^{n-1}X^{(n)} \to (\Sigma X)^{[n]}$ such that

$$\Sigma^{n-1}X^{(n)} \xrightarrow{\omega_n} (\Sigma X)^{[n]} \xrightarrow{\varepsilon_n} (\Sigma X)^n$$

is a cofibration sequence, where $(\Sigma X)^{[n]}$ is the n-fold fat wedge of ΣX and $\varepsilon_n : (\Sigma X)^{[n]} \to (\Sigma X)^n$ denotes the inclusion. Let $\iota : \Sigma X \to BX$ be the adjoint map of the homotopy equivalence $X \to \Omega BX$. Then the n-fold generalized Whitehead product of ι is defined as

$$[\iota, \ldots, \iota] = \{W(\psi) = \psi\omega_n \,|\, \psi : (\Sigma X)^{[n]} \to BX \text{ with } \psi|_{\Sigma X} = \iota \text{ for each factors}\}.$$

Williams [39] studied the connection of the C_n-space and the higher Whitehead product, and by using his result, Saumell [33] determined the higher homotopy commutativity in the case that a p-compact group X is a finite product of odd dimensional spheres. The following lemma is due to Saumell:

LEMMA 4.1 ([39; Cor. 1.5], [33; Thm. 3.2]). *Let $n \geq 2$ and X be a loop space. Then X is a C_n-space if and only if the k-fold generalized Whitehead product $[\iota, \ldots, \iota]$ contains zero for $2 \leq k \leq n$.*

Now we can prove the following result:

THEOREM 4.2. *Let p be an odd prime. Then we have the following:*
(1) *$S^{2t_1-1} \times \cdots \times S^{2t_l-1}$ is a C_n-space if $p \geq nt_l$.*
(2) *$B_1(p)$ is a $C_{(p-1)/2}$-space.*
(3) *$B_5(19)$ is a C_3-space.*
(4) *$B_{11}(19)$ is a C_3-space.*
(5) *$B_1(19) \times B_{11}(19)$ is a C_3-space.*

PROOF. In the case of $X = S^{2t_1-1} \times \cdots \times S^{2t_l-1}$, the result follows from [33; Thm. B].

Let $X = B_1(p)$ and $G = S^3 \cup_\alpha e^{2p+1}$, where $\alpha \in \pi_{2p}(S^3) \cong \mathbf{Z}/p$ denotes the generator. By the cell structure of X, we see that $G \subset X$. A result of McGibbon [20] implies that ΣG is a retract of ΣX, and so there exists a map $r : \Sigma X \to \Sigma G$ with $r(\Sigma i) \simeq 1_{\Sigma G}$, where $i : G \to X$ denotes the inclusion map. Let $\kappa = \iota(\Sigma i) : \Sigma G \to BX$. As in the proof of [21; Thm. 4], we can assume that $\kappa r \simeq \iota$, and so it is sufficient to show that the k-fold generalized Whitehead product $[\kappa, \ldots, \kappa]$ contains zero for $2 \leq k \leq (p-1)/2$.

If $k = 2$, then $[\kappa, \kappa]$ is the ordinary generalized Whitehead product, and so $[\kappa, \kappa] = 0$ since X is a C_2-space. Now we assume that the $(k-1)$-fold generalized Whitehead product contains zero. Then there exists a map $\psi : (\Sigma G)^{[k]} \to BX$ with $\psi|_{\Sigma G} = \iota$ for each factors. To show $W(\psi) = \psi\omega_k \simeq *$, it is sufficient to construct a map $\tilde{\psi} : (\Sigma G)^k \to BX$ with $\tilde{\psi}|_{(\Sigma G)^{[k]}} = \psi$, and the obstructions for the existence of $\tilde{\psi}$ belong to the following cohomology groups:

$$H^{i+1}((\Sigma G)^k, (\Sigma G)^{[k]}; \pi_i(BX)) \cong \tilde{H}^{i+1}((\Sigma G)^{(k)}; \pi_i(BX)) \tag{4.3}$$

for $i \geq 1$. By [37; Thm. 3], the homotopy groups of BX are given as

$$\pi_i(BX) \cong \begin{cases} \mathbf{Z}/p & \text{if } i = 2p + 2j(p-1) + 1 \text{ for } 1 \leq j < p, \\ 0 & \text{otherwise for } i < 2p^2 + 1. \end{cases} \tag{4.4}$$

By the cell structure of $(\Sigma G)^{(k)}$, the obstruction groups (4.3) are zero unless $i = 4k + 2j(p-1) - 1$ for $0 \leq j \leq k$. Since $2 \leq k \leq (p-1)/2$, we have that $2p + 1 + 2(j-1)(p-1) < 4k + 2j(p-1) - 1 < 2p + 1 + 2j(p-1)$ for $0 \leq j \leq k$, which implies that the obstruction groups are zero by (4.4). By using similar arguments, we can obtain the required results for (3) and (4).

In the case of $X = B_1(19) \times B_{11}(19)$, we put that $G = (S^3 \cup_\alpha e^{39}) \vee (S^{23} \cup_\beta e^{59})$, where $\alpha \in \pi_{38}(S^3) \cong \mathbf{Z}/19$ and $\beta \in \pi_{58}(S^{23}) \cong \mathbf{Z}/19$ denote the generators. Since ΣG is a retract of ΣX, we can show the required result as in the case of (2). This completes the proof of Theorem 4.2. □

Next we show the following result:

THEOREM 4.5. *Let p be an odd prime. Then we have the following:*
(1) $S^{2t_1-1} \times \cdots \times S^{2t_l-1}$ *is not a C_n-space if $p < nt_l$.*
(2) $B_1(p)$ *is not a C_p-space.*
(3) $B_7(17)$ *is not a C_3-space.*
(4) $B_5(19)$ *is not a C_4-space.*
(5) $B_{19}(41)$ *is not a C_3-space.*
(6) $B_{11}(19)$ *is not a C_4-space.*
(7) $B_3(11) \times S^{11}$ *is not a C_3-space.*
(8) $B_1(19) \times B_{11}(19)$ *is not a C_4-space.*

Let X be a simply connected p-compact group. If the mod p cohomology $H^*(X)$ is an exterior algebra, then the mod p cohomology of BX is a polynomial algebra. For $n \geq 1$, we define the n-fold decomposable module of $H^*(BX)$ as $D^1 = \tilde{H}^*(BX)$ and $D^n = D^{n-1} \cdot \tilde{H}^*(BX)$. Then we see that $D^{n+1} \subset D^n$, and D^n is closed under the action of the mod p Steenrod algebra for $n \geq 1$. In the proof of Theorem 4.5, we need the following lemma:

LEMMA 4.6 ([13; Lemma 4.8], [33; Prop. 4.1]). *Let $2 \leq n \leq p-1$ and X be a simply connected p-compact group. If there exists an element $x \in H^*(BX)$*

satisfying that $\theta(x)=0 \mod D^n$ *and* $\theta(x)\neq 0 \mod D^{n+1}+\theta(D^n)$ *for some Steenrod operation* $\theta \in \mathscr{A}_p$, *then* X *is not a* C_n*-space.*

By using Lemma 4.6, we can prove Theorem 4.5 as follows:

PROOF OF THEOREM 4.5. In the cases of $X = S^{2t_1-1} \times \cdots \times S^{2t_l-1}$ and $X = B_1(p)$, we have the required results from [33; Thm. B] and Theorem 3.10, respectively.

Let $X = B_7(17)$. Then the mod 17 cohomology of BX is given as $H^*(BX) \cong \mathbf{Z}/17[x, y]$ with $\mathscr{P}^1(x) = y$, where $\deg x = 16$ and $\deg y = 48$, and we can set that $\mathscr{P}^1(y) = a_1x^5 + a_2x^2y$ for $a_1, a_2 \in \mathbf{Z}/17$. If we assume that $a_2 = 0$, then

$$(\mathscr{P}^1)^7(y) = 7a_1^2x^5y^4 + 3a_1^3x^{11}y^2 + 3a_1^4x^{17},$$

which contradicts the fact that $\mathscr{P}^8(x) = x^{17}$, and so $a_2 \neq 0$. Since $\mathscr{P}^2(x) = 0 \mod D^3$ and $\mathscr{P}^2(x) \neq 0 \mod D^4 + \mathscr{P}^2(D^3)$, by applying Lemma 4.6 to $x \in H^*(BX)$ and $\mathscr{P}^2$, X is not a C_3-space.

If $X = B_5(19)$, then $H^*(BX) \cong \mathbf{Z}/19[x, y]$ with $\mathscr{P}^1(x) = y$, where $\deg x = 12$ and $\deg y = 48$. For the dimensional reason, we can set that $\mathscr{P}^1(y) = a_1x^7 + a_2x^3y$ for $a_1, a_2 \in \mathbf{Z}/19$. If $a_2 = 0$, then $\mathscr{P}^1(y) = a_1x^7$, and by using a routine calculation, we have that $(\mathscr{P}^1)^5(y) = 4a_1x^3y^4 + 13a_1^2x^{11}y^2 + 4a_1^3x^{19}$. Since $\mathscr{P}^6(x) = x^{19}$, this causes a contradiction, and so $a_2 \neq 0$. Hence $\mathscr{P}^2(x) = 0 \mod D^4$ and $\mathscr{P}^2(x) \neq 0 \mod D^5 + \mathscr{P}^2(D^4)$, which implies that X is not a C_4-space.

In the case of $X = B_{19}(41)$, the mod 41 cohomology of BX is given as $H^*(BX) \cong \mathbf{Z}/41[x, y]$ with $\mathscr{P}^1(x) = y$, where $\deg x = 40$ and $\deg y = 120$. For the dimensional reason, we can set that $\mathscr{P}^1(y) = a_1x^5 + a_2x^2y$ for $a_1, a_2 \in \mathbf{Z}/41$. If we assume that $a_2 = 0$, then

$$\begin{aligned}(\mathscr{P}^1)^{19}(y) &= 5a_1^4x^5y^{12} + 35a_1^5x^{11}y^{10} + 9a_1^6x^{17}y^8 \\ &\quad + 7a_1^7x^{23}y^6 + 29a_1^8x^{29}y^4 + 25a_1^9x^{35}y^2 + 22a_1^{10}x^{41},\end{aligned}$$

which contradicts the fact that $\mathscr{P}^{20}(x) = x^{41}$, and so we have that $a_2 \neq 0$. Hence $\mathscr{P}^2(x) = 0 \mod D^3$ and $\mathscr{P}^2(x) \neq 0 \mod D^4 + \mathscr{P}^2(D^3)$, which implies that X is not a C_3-space.

Let $X = B_{11}(19)$. Then the mod 19 cohomology of BX is given as $H^*(BX) \cong \mathbf{Z}/19[x, y]$ with $\mathscr{P}^1(x) = y$, where $\deg x = 24$ and $\deg y = 60$. For the dimensional reason, we can set that $\mathscr{P}^1(y) = ax^4$ for $a \in \mathbf{Z}/19$. If we assume that $a = 0$, then $(\mathscr{P}^1)^{11}(y) = 0$, which contradicts the fact that $\mathscr{P}^{12}(x) = x^{19}$, which implies that $a \neq 0$. Hence $\mathscr{P}^1(y) = 0 \mod D^4$ and $\mathscr{P}^1(y) \neq 0 \mod D^5 + \mathscr{P}^1(D^4)$, and so by Lemma 4.6, X is not a C_4-space.

In the case of $X = B_3(11) \times S^{11}$, the mod 11 cohomology of BX is given as $H^*(BX) \cong \mathbf{Z}/11[x, y, z]$ with $\mathscr{P}^1(x) = y$, where $\deg x = 8$, $\deg y = 28$ and $\deg z = 12$. For the dimensional reasons, we can set that

$$\mathscr{P}^1(z) = a_1x^4 + a_2xz^2, \tag{4.8}$$

$$\mathscr{P}^1(y) = b_1xyz + b_2z^4 + b_3x^3z^2 + b_4x^6,$$

where $a_i, b_j \in \mathbf{Z}/11$ for $1 \le i \le 2$, $1 \le j \le 4$. If we assume that $a_2 = b_1 = 0$, then by using (4.8), we have that

$$\begin{aligned}(\mathscr{P}^1)^3(y) &= (2a_1^2b_3 + 6b_4^2)x^{11} + (a_1^2b_2 + 9b_3b_4)z^2x^8 + 9a_1b_3zx^6y \\ &\quad + (3b_3^2 + 6b_2b_4)z^4x^5 + 5a_1b_2z^3x^3y + 6b_3z^2xy^2 \\ &\quad + 3b_2b_3z^6x^2 + 8b_4x^4y^2,\end{aligned}$$

which contradicts the fact that $\mathscr{P}^4(x) = x^{11}$, and so $a_2 \ne 0$ or $b_1 \ne 0$.

If $a_2 \ne 0$, then $\mathscr{P}^1(z) = 0 \bmod D^3$ and $\mathscr{P}^1(z) \ne 0 \bmod D^4 + \mathscr{P}^1(D^3)$, and so by Lemma 4.6, X is not a C_3-space. In the case of $b_1 \ne 0$, we see that $\mathscr{P}^2(x) = 0 \bmod D^3$ and $\mathscr{P}^2(x) \ne 0 \bmod D^4 + \mathscr{P}^2(D^3)$, and so X is not a C_3-space.

Finally, we consider the case of $X = B_1(19) \times B_{11}(19)$. In this case, the mod 19 cohomology of BX is given as

$$H^*(BX) \cong \mathbf{Z}/19[x, y, z, w],$$

where $\deg x = 4$, $\deg y = 24$, $\deg z = 40$, $\deg w = 60$, $\mathscr{P}^1(x) = z$ and $\mathscr{P}^1(y) = w$. For the dimensional reason, we can set that

$$\begin{aligned}\mathscr{P}^1(w) &= a_1x^{24} + a_2x^{18}y + a_3x^{12}y^2 + a_4x^6y^3 + a_5y^4 + a_6x^{14}z \\ &\quad + a_7x^8yz + a_8x^2y^2z + a_9x^4z^2 + a_{10}x^9w + a_{11}x^3yw,\end{aligned}$$

where $a_i \in \mathbf{Z}/19$ for $1 \le i \le 11$. If (x, z) denotes the ideal of $H^*(BX)$ generated by x and z, then (x, z) is closed under the action of $\mathscr{P}^1$ since $\mathscr{P}^1(z) = 2x^{19} \in (x, z)$. If we assume that $a_5 = 0$, then $\mathscr{P}^1(w) \in (x, z)$, which implies that $(\mathscr{P}^1)^{11}(w) \in (x, z)$. This contradicts the fact that $\mathscr{P}^{12}(y) = y^{19} \notin (x, z)$, and so we have that $a_5 \ne 0$. Hence $\mathscr{P}^1(w) = 0 \bmod D^4$ and $\mathscr{P}^1(w) \ne 0 \bmod D^5 + \mathscr{P}^1(D^4)$, which shows that X is not a C_4-space. This completes the proof of Theorem 4.5. $\square$

From Theorems 4.2 and 4.5, we can determine the higher homotopy commutativity of almost all the p-compact groups on the table in Theorem 1.2. In the following table, n denotes the maximal number for which X is a C_n-space but not a C_{n+1}-space.

X	p	Types	n
$S^{2t_1-1} \times \cdots \times S^{2t_l-1}$	$\geq 2t_l$	$(t_1, \ldots, t_l)$	$[p/t_l]$
$B_1(p)$	3	$(2,4)$	2
——	≥ 5	$(2, p+1)$	$(p-1)/2 \leq \ \leq p-1$
$B_7(17)$	17	$(8,24)$	2
$B_5(19)$	19	$(6,24)$	3
$B_{19}(41)$	41	$(20,60)$	2
$B_{11}(19)$	19	$(12,30)$	3
$B_3(11) \times S^{11}$	11	$(4,6,14)$	2
$B_1(19) \times B_{11}(19)$	19	$(2,12,20,30)$	3

By using Theorem B, Remark 3.9 and Theorem 4.5, we have the following corollary:

COROLLARY 4.9. *Let p be an odd prime. If X is a simply connected C_{p-1}-space such that the* mod p *cohomology $H^*(X)$ is finitely generated as an algebra, then X is homotopy equivalent to a finite product of $K(\mathbf{Z},2)$, $B_1(p)$, $B_1(p)\langle 3\rangle$ and $B_1(p)\langle 3; p^i\rangle$ for $i \geq 1$.*

Kawamoto-Lin [17; Thm. C] has shown the same result under the assumption that X is a C_{p-1}-space in the sense of Sugawara. Corollary 4.9 implies [17; Thm. C] since it is shown in [22; Prop. 6] that the higher homotopy commutativity of Williams is weaker than the one of Sugawara.

It is natural to ask if the converse of Corollary 4.9 holds. In the case of $p = 3$, it is known that $B_1(3)$ has the homotopy type of $Sp(2)$. McGibbon [21; Thm. 2] proved that $Sp(2)$ is a C_2-space, and moreover, $Sp(2)\langle 3\rangle$ and $Sp(2)\langle 3; 3^i\rangle$ are C_2-spaces for $i \geq 1$. At present, the author does not know the corresponding result for $p > 3$. However, it seems to be reasonable to conjecture that $B_1(p)$ is a C_{p-1}-space for $p > 3$.

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