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Boundedness of multilinear oscillatory singular integrals on Hardy type spaces

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ABSTRACT. In this paper, the authors discuss a class of multilinear singular integrals and obtain their boundedness from the weighted Hardy space $H^1_\omega(\mathbf{R}^n)$ to the weighted Lebesgue space $L^1_\omega(\mathbf{R}^n)$ for $\omega \in A_1(\mathbf{R}^n)$ (the class of Muckenhoupt's weights) and from the weighted Herz-type Hardy space $H\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)$ (or $HK_p(\omega_1, \omega_2; \mathbf{R}^n)$) to the weighted Herz space $\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)$ (or $K_p(\omega_1, \omega_2; \mathbf{R}^n)$) for any $p \in (1, \infty)$ and $\omega_1, \omega_2 \in A_1(\mathbf{R}^n)$.

1. Introduction

In recent years, there has been significant progress in the study of oscillatory singular integrals with polynomial phase functions. Let $P(x, y)$ be a real-valued polynomial defined on $\mathbf{R}^n \times \mathbf{R}^n$ and K be a standard Calderón-Zygmund kernel, that is, K is C^1 on $\mathbf{R}^n$ away from the origin and has mean value zero on the unit sphere centered at the origin. Define the oscillatory singular integral operator T by

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(x,y)} K(x-y) f(y) dy. \tag{1.1}$$

A well-known result of Ricci-Stein [13] states that T is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ with the (operator) bound depending only on n, p and $\deg P$ (the total degree of P), and being independent of the coefficients of the polynomial P. Chanillo and Christ [2] proved that T is also bounded from $L^1(\mathbf{R}^n)$ to weak $L^1(\mathbf{R}^n)$ with bound independent of the coefficients of P. Pan [12] considered the behaviour of T on $H^1_{\mathrm{E}}(\mathbf{R}^n)$ (a variant of the Hardy space $H^1(\mathbf{R}^n)$). There are many other works about the operator T; we refer to the references [7], [9] and [11].

The purpose of this paper is to study a class of multilinear operators which

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are closely related to the operator T defined by (1.1). Let $m \in \mathbf{N}$ and $m \geq 2$. Let Ω be homogeneous of degree zero, belong to the space $\mathrm{Lip}_1(S^{n-1})$ and satisfy the moment conditions

$$(1.2) \qquad \int_{S^{n-1}} \Omega(\theta)\theta^{\alpha} d\theta = 0 \qquad \text{for } \alpha \in (\mathbf{N} \cup \{0\})^n \quad \text{and} \quad |\alpha| = m.$$

Let A have derivatives of order m in $BMO(\mathbf{R}^n)$ and let $R_m(A; x, y)$ denote the m-th order Taylor series remainder of A at x about y, that is,

$$R_m(A; x, y) = A(x) - \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} D^{\alpha} A(y)(x-y)^{\alpha}.$$

The operator we will consider here is of the form:

$$(1.3) \qquad T^A f(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} Q_{m+1}(A; x, y) f(y) dy,$$

where $Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^{\alpha} A(x)(x-y)^{\alpha}$. Recall that if b is a $BMO(\mathbf{R}^n)$ function and T is a linear operator, then the commutator $[b, T]$ is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

for suitable functions f. It is obvious that the difference between the operator T^A and the operator $\tilde{T}^A$ defined by

$$(1.4) \qquad \tilde{T}^A f(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} R_{m+1}(A; x, y) f(y) dy$$

is a sum of the commutators of $BMO(\mathbf{R}^n)$ functions $\{D^{\alpha} A\}_{|\alpha|=m}$ and the operators $\{T_{\alpha}\}_{|\alpha|=m}$ of type (1.1) with the kernel K replacing by $K_{\alpha}(x) = \Omega(x)x^{\alpha}/|x|^{n+m}$ which is a Calderón-Zygmund kernel. In fact, we have

$$\tilde{T}^A f(x) - T^A f(x) = \sum_{|\alpha|=m} [D^{\alpha} A, T_{\alpha}] f(x).$$

The boundedness of $\tilde{T}^A$ on $L^p_{\omega}(\mathbf{R}^n)$ for $1 < p < \infty$ and $\omega \in A_p(\mathbf{R}^n)$ (the class of Muckenhoupt's weights), has been disposed in [3]. Here, we will study the behaviour of T^A on the weighted Hardy space $H^1_{\omega}(\mathbf{R}^n)$ and the weighted Herz-type Hardy space $H\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)$. Before stating our results, let us recall some necessary definitions.

Definition 1. Given a non-negative weight $\omega(x)$ on $\mathbf{R}^n$, the *weighted Hardy space* $H^1_{\omega}(\mathbf{R}^n)$ is the space of those $f \in \mathcal{S}'(\mathbf{R}^n)$ for which $G(f)$, the grand maximal function of f (see [14]), belongs to $L^1_\omega(\mathbf{R}^n)$, and define

$$\|f\|_{H^1_\omega(\mathbf{R}^n)} = \|G(f)\|_{L^1_\omega(\mathbf{R}^n)},$$

where $\mathscr{S}'(\mathbf{R}^n)$ is the space of Schwartz distributions on $\mathbf{R}^n$.

Definition 2. Let $1 < p < \infty$ and ω_1, ω_2 be two non-negative weights on $\mathbf{R}^n$.

(i) The *homogeneous weighted Herz space* $\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)$ is the space of those functions $f \in L^p_{loc}(\mathbf{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n)} \equiv \sum_{k=-\infty}^{\infty} \omega_1(B_k)^{1-(1/p)} \|f\chi_k\|_{L^p_{\omega_2}(\mathbf{R}^n)} < \infty$$

with $B_k = B(0, 2^k)$, $C_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{C_k}$.

(ii) The *non-homogeneous weighted Herz space* $K_p(\omega_1, \omega_2; \mathbf{R}^n)$ is the space of those functions $f \in L^p_{loc}(\mathbf{R}^n)$ such that

$$\|f\|_{K_p(\omega_1,\omega_2;\mathbf{R}^n)} \equiv \omega_1(B_0)^{1-(1/p)} \|f\chi_{B_0}\|_{L^p_{\omega_2}(\mathbf{R}^n)} + \sum_{k=1}^{\infty} \omega_1(B_k)^{1-(1/p)} \|f\chi_k\|_{L^p_{\omega_2}(\mathbf{R}^n)} < \infty.$$

(iii) The *homogeneous weighted Herz-type Hardy space* $H\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)$ is defined by

$$H\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n) \equiv \{f \in \mathscr{S}'(\mathbf{R}^n) : G(f) \in \dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)\}$$

with

$$\|f\|_{H\dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n)} \equiv \|G(f)\|_{\dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n)}.$$

(iv) The *non-homogeneous weighted Herz-type Hardy space* $HK_p(\omega_1, \omega_2; \mathbf{R}^n)$ is defined by

$$HK_p(\omega_1, \omega_2; \mathbf{R}^n) \equiv \{f \in \mathscr{S}'(\mathbf{R}^n) : G(f) \in K_p(\omega_1, \omega_2; \mathbf{R}^n)\}$$

with

$$\|f\|_{HK_p(\omega_1,\omega_2;\mathbf{R}^n)} \equiv \|G(f)\|_{K_p(\omega_1,\omega_2;\mathbf{R}^n)}.$$

Theorem 1. *Let $\omega \in A_1(\mathbf{R}^n)$, T^A be defined as in* (1.3) *and $P(x, y)$ be a real-valued polynomial on $\mathbf{R}^n \times \mathbf{R}^n$ with $\nabla_y P(0, y) = 0$. Then T^A is bounded from $H^1_\omega(\mathbf{R}^n)$ to $L^1_\omega(\mathbf{R}^n)$, that is,*

$$\|T^A f\|_{L^1_\omega(\mathbf{R}^n)} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO(\mathbf{R}^n)} \|f\|_{H^1_\omega(\mathbf{R}^n)},$$

where C depends only on n, m, $\deg P$ and $A_1(\omega)$, the $A_1(\mathbf{R}^n)$-constant of ω.

Theorem 2. *Let* $\omega_1,\ \omega_2 \in A_1(\mathbf{R}^n)$, T^A *and* $P(x,y)$ *be the same as in Theorem* 1. *If* $1 < p < \infty$, *then* T^A *is bounded from* $H\dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n)$ *to* $\dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n)$ *and from* $HK_p(\omega_1,\omega_2;\mathbf{R}^n)$ *to* $K_p(\omega_1,\omega_2;\mathbf{R}^n)$, *that is,*

$$\|T^A f\|_{\dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n)} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO(\mathbf{R}^n)} \|f\|_{H\dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n)}$$

and

$$\|T^A f\|_{K_p(\omega_1,\omega_2;\mathbf{R}^n)} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO(\mathbf{R}^n)} \|f\|_{HK_p(\omega_1,\omega_2;\mathbf{R}^n)},$$

where C's depend only on n, m, p, $\deg P$ *and the* $A_1(\mathbf{R}^n)$*-constants of* ω_1 *and* ω_2.

Obviously,

$$\dot{K}_1(\omega_1,\omega_2;\mathbf{R}^n) = K_1(\omega_1,\omega_2;\mathbf{R}^n) = L^1_{\omega_2}(\mathbf{R}^n)$$

and

$$H\dot{K}_1(\omega_1,\omega_2;\mathbf{R}^n) = HK_1(\omega_1,\omega_2;\mathbf{R}^n) = H^1_{\omega_2}(\mathbf{R}^n);$$

while when $1 < p < \infty$ and $\omega_1(x) \equiv \omega_2(x) \equiv 1$,

$$K_p(\omega_1,\omega_2;\mathbf{R}^n) \subsetneqq \dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n) \subsetneqq L^1_{\omega_2}(\mathbf{R}^n)$$

and

$$HK_p(\omega_1,\omega_2;\mathbf{R}^n) \subsetneqq H\dot{K}_p(\omega_1,\omega_2;\mathbf{R}^n) \subsetneqq H^1_{\omega_2}(\mathbf{R}^n).$$

Thus Theorem 2 can be regarded as a local version at the origin of Theorem 1.

We finally remark that Theorem 1 with $P(x,y) = P(x-y)$ and $\omega(x) \equiv 1$ has been obtained by Hu and Yang in [8]. Theorem 2 is new even when $\omega_1(x) \equiv \omega_2(x) \equiv 1$.

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2. Proof of Theorem 1

We begin with some known facts. The following Lemma 1 is the lemma of ([4], p. 448).

Lemma 1. *Let* $b(x)$ *be a function on* $\mathbf{R}^n$ *with m-th order derivatives in* $L^q_{loc}(\mathbf{R}^n)$ *for some* $q > n$. *Then*

$$|R_m(b; x, y)| \le C_{m,n}|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ *is the cube centered at* x *and having diameter* $5\sqrt{n}|x-y|$.

LEMMA 2. *Let* T^A *be defined as in* (1.3). *Then for* $1 < p < \infty$ *and* $\omega \in A_p(\mathbf{R}^n)$, T^A *is bounded on* $L^p_\omega(\mathbf{R}^n)$, *that is, for all* $f \in L^p_\omega(\mathbf{R}^n)$,

$$\|T^A f\|_{L^p_\omega(\mathbf{R}^n)} \le C(m, n, p, \deg P, A_p(\omega)) \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO(\mathbf{R}^n)} \|f\|_{L^p_\omega(\mathbf{R}^n)},$$

where and in what follows, $A_p(\omega)$ *denotes the* $A_p(\mathbf{R}^n)$*-constant of* ω.

PROOF. Consider the operator $\tilde{T}^A$ defined in (1.4). The main result in [3] shows that if $\omega \in A_p(\mathbf{R}^n)$, then

$$\|\tilde{T}^A f\|_{L^p_\omega(\mathbf{R}^n)} \le C(m, n, p, \deg P, A_p(\omega)) \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO(\mathbf{R}^n)} \|f\|_{L^p_\omega(\mathbf{R}^n)}.$$

Note that for each fixed α with $|\alpha| = m$, $\Omega(x)x^\alpha/|x|^{n+m}$ is a standard Calderón-Zygmund kernel. Thus, from this and the well-known $L^p_\omega(\mathbf{R}^n)$-boundedness of the commutators (see [6]), it follows that for $\omega \in A_p(\mathbf{R}^n)$,

$$\left\| \int_{\mathbf{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{n+m}} (D^\alpha A(x) - D^\alpha A(y)) f(y) dy \right\|_{L^p_\omega(\mathbf{R}^n)}$$
$$\le C(m, n, p, \deg P, A_p(\omega)) \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO(\mathbf{R}^n)} \|f\|_{L^p_\omega(\mathbf{R}^n)}.$$

Combining these two inequalities, we obtain the desired estimate. This finishes the proof of Lemma 2.

To show Theorem 1, we will need the atomic decomposition of $H^1_\omega(\mathbf{R}^n)$.

DEFINITION 3. Let $\omega \in A_1(\mathbf{R}^n)$. A function $a(x)$ on $\mathbf{R}^n$ is called a $(1,\omega)$-*atom* if

(i) supp $a \subset B(x_0, r) \equiv \{x \in \mathbf{R}^n : |x - x_0| < r\}$ for some $x_0 \in \mathbf{R}^n$ and $r > 0$;
(ii) $\|a\|_{L^\infty(\mathbf{R}^n)} \le \omega(B(x_0, r))^{-1}$;
(iii) $\int_{\mathbf{R}^n} a(x) dx = 0$.

The following atomic decomposition of the Hardy space $H^1_\omega(\mathbf{R}^n)$ is obtained by Bui in ([1], Theorem 5.1).

LEMMA 3. *Let* $\omega \in A_1(\mathbf{R}^n)$. *A distribution* f *on* $\mathbf{R}^n$ *belongs to* $H^1_\omega(\mathbf{R}^n)$ *if and only if* f *can be written as a series* $f = \sum_j \lambda_j a_j$ *convergent in the sense of distributions, where each* a_j *is a* $(1,\omega)$*-atom and the coefficients* λ_j *satisfy*

$\sum_j |\lambda_j| < \infty$. *Moreover, in this case,*

$$\|f\|_{H^1_\omega(\mathbf{R}^n)} \sim \inf\left\{\sum_j |\lambda_j|\right\},$$

where the infimum is taken over all the decompositions of f.

The following lemma has been essentially proved by Pan in ([12], pp. 59–60). In fact, Pan proved the case where $\psi = \chi_{\{1/4 \le |x| \le 4\}}$. However, by minor modification of his proof, we can easily see that the conclusion is still true when $\psi \in C_0^\infty(\mathbf{R}^n)$. We omit the details.

Lemma 4. *Let* $\varphi \in C_0^\infty(\mathbf{R}^n)$ *satisfy that* $\operatorname{supp}\varphi \subset \{x \in \mathbf{R}^n : |x| \le 2\}$ *and* $\varphi(x) = 1$ *for* $|x| \le 1$ *and* $\psi \in C_0^\infty(\mathbf{R}^n)$ *satisfy that* $\operatorname{supp}\psi \subset \{x \in \mathbf{R}^n : 1/4 \le |x| \le 4\}$ *and* $\psi(x) = 1$ *for* $1/2 \le |x| \le 2$. *Define*

$$T_k f(x) = \psi(2^{-k}x) \int_{\mathbf{R}^n} e^{iP(x,y)} \varphi(y) f(y) dy.$$

If the polynomial $P(x, y)$ *has the form*

$$P(x, y) = \sum_{|\mu| \ge 1, |\nu| = l} a_{\mu\nu} x^\mu y^\nu + Q(x, y),$$

where $Q(x, y)$ *is a polynomial with degree in y smaller than l, then for each sufficiently large positive integer N,*

$$\|T_k f\|_{L^2(\mathbf{R}^n)} \le C_N 2^{nk/2} |a_{\mu_0\nu_0}|^{-1/(2Nl)} 2^{-k|\mu_0|/(2Nl)} \|f\|_{L^2(\mathbf{R}^n)},$$

where $|a_{\mu_0\nu_0}|^{1/|\mu_0|} = \max_{|\mu| \ge 1, |\nu| = l} |a_{\mu\nu}|^{1/|\mu|}$.

Proof of Theorem 1. Without loss of generality, we may assume that $\sum_{|\alpha|=m} \|D^\alpha A\|_{BMO(\mathbf{R}^n)} = 1$. By Lemma 3, it is enough to show that for any $(1, \omega)$-atom a,

$$\|T^A a\|_{L^1_\omega(\mathbf{R}^n)} \le C(m, n, \deg P, A_1(\omega)). \tag{2.1}$$

Noting that T^A is translation and dialation invariant, we may assume that $\operatorname{supp} a \subset B_0 = B(0, 1)$. Write

$$\int_{\mathbf{R}^n} |T^A a(x)| \omega(x) dx = \int_{|x| \le 2} |T^A a(x)| \omega(x) dx + \int_{|x| > 2} |T^A a(x)| \omega(x) dx \equiv I_1 + I_2.$$

We have by Lemma 2 that

$$I_1 \le \|T^A a\|_{L^2_\omega(\mathbf{R}^n)} \left(\int_{|x| \le 2} \omega(x) dx\right)^{1/2} \le C\|a\|_{L^2_\omega(\mathbf{R}^n)} \omega(B_0)^{1/2} \le C.$$

To estimate I_2, noting that $\nabla_y P(0,y)=0$, we can write

$$P(x,y)=\sum_{|\mu|\ge 1,|\nu|=l} a_{\mu\nu}x^{\mu}y^{\nu}+Q(x,y),$$

where $Q(x,y)$ has the degree in y less than l and $\nabla_y Q(0,y)=0$. Let

$$b=|a_{\mu_0\nu_0}|^{-1/|\mu_0|}=\left(\max_{|\mu|\ge 1,|\nu|=l}|a_{\mu\nu}|^{1/|\mu|}\right)^{-1}$$

and $r_0=\max(2,|a_{\mu_0\nu_0}|^{-1/|\mu_0|})$. Write

$$I_2=\int_{2<|x|\le r_0}|T^A a(x)|\omega(x)dx+\int_{|x|>r_0}|T^A a(x)|\omega(x)dx\equiv I_{21}+I_{22}.$$

We first estimate I_{22}. Set

$$A_k(x)=A(x)-\sum_{|\alpha|=m}\frac{1}{\alpha!}m_{B_{k+n_0}}(D^{\alpha}A)x^{\alpha},$$

where $m_{B_{k+n_0}}(D^{\alpha}A)$ denotes the mean value of $D^{\alpha}A$ on $B_{k+n_0}=B(0,2^{k+n_0})$, and n_0 is any fixed integer satisfying $2^{n_0}\ge 20\sqrt{n}$. It is easy to see that $Q_{m+1}(A;x,y)=Q_{m+1}(A_k;x,y)$ and for $q\in[1,\infty)$,

$$\left(2^{-kn}\int_{B_{k+n_0}}|D^{\alpha}A_k(x)|^q\,dx\right)^{1/q}\le C\|D^{\alpha}A\|_{BMO(\mathbf{R}^n)}. \tag{2.2}$$

In what follows, we suppose $k\ge 2$. For $2^{k-1}<|x|\le 2^k$, we write

$$\begin{aligned}
|T^A a(x)|&=\left|\int_{\mathbf{R}^n}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^{n+m}}Q_{m+1}(A_k;x,y)a(y)dy\right|\\
&\le\int_{\mathbf{R}^n}\left|\frac{\Omega(x-y)}{|x-y|^{n+m}}R_m(A_k;x,y)-\frac{\Omega(x)}{|x|^{n+m}}R_m(A_k;x,0)\right||a(y)|dy\\
&\quad+\sum_{|\alpha|=m}\frac{1}{\alpha!}|D^{\alpha}A_k(x)|\int_{\mathbf{R}^n}\left|\frac{\Omega(x-y)(x-y)^{\alpha}}{|x-y|^{n+m}}-\frac{\Omega(x)x^{\alpha}}{|x|^{n+m}}\right||a(y)|dy\\
&\quad+\left(\left|\frac{\Omega(x)}{|x|^{n+m}}R_m(A_k;x,0)\right|+\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{|D^{\alpha}A_k(x)|}{|x|^n}\right)\left|\int_{\mathbf{R}^n}e^{iP(x,y)}a(y)dy\right|\\
&\equiv T^{A,1}a(x)+T^{A,2}a(x)+T^{A,3}a(x).
\end{aligned}$$

With the aid of the formula (see (10) in [4])

$$R_m(A_k;x,y)-R_m(A_k;x,0)=\sum_{|\alpha|<m}\frac{1}{\alpha!}R_{m-|\alpha|}(D^{\alpha}A_k;0,y)(x-y)^{\alpha},$$

we can obtain from Lemma 1 and (2.2) that for $2^{k-1} < |x| \le 2^k$ and $y \in B_0$,

$$\begin{aligned}(2.3)\quad &\left|\frac{\Omega(x-y)}{|x-y|^{n+m}}R_m(A_k;x,y)-\frac{\Omega(x)}{|x|^{n+m}}R_m(A_k;x,0)\right| \\ &\le \left|\frac{\Omega(x-y)}{|x-y|^{n+m}}-\frac{\Omega(x)}{|x|^{n+m}}\right||R_m(A_k;x,y)| \\ &\quad+\frac{|\Omega(x)|}{|x|^{n+m}}|R_m(A_k;x,y)-R_m(A_k;x,0)| \\ &\le C\left(|x-y|^{-n-1}+\sum_{l=0}^{m-1}|x|^{-n-m}|x-y|^l\right) \\ &\le C|x|^{-n-1}.\end{aligned}$$

This in turn implies that

$$(2.4)\qquad T^{A,1}a(x) \le C|x|^{-n-1}\int_{\mathbf{R}^n}|a(y)|dy \le C2^{-k(n+1)}\omega(B_0)^{-1}.$$

On the other hand, using $\Omega \in \mathrm{Lip}_1(S^{n-1})$, $2^{k-1} < |x| \le 2^k$ and $|y| \le 1$, it is easy to see that for $|\alpha| = m$,

$$\left|\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{n+m}}-\frac{\Omega(x)x^\alpha}{|x|^{n+m}}\right| \le C|x|^{-n-1},$$

where C is independent of x and y. From this, we can easily deduce that for $2^{k-1} < |x| \le 2^k$,

$$(2.5)\qquad T^{A,2}a(x) \le C2^{-k(n+1)}\sum_{|\alpha|=m}|D^\alpha A_k(x)|\omega(B_0)^{-1}.$$

For $T^{A,3}$, let φ, ψ and T_k be the same as in Lemma 4; then another application of Lemma 1 and (2.2) leads to that

$$(2.6)\qquad T^{A,3}a(x) \le C2^{-kn}\left(1+\sum_{|\alpha|=m}|D^\alpha A_k(x)|\right)|T_k a(x)|.$$

Let k_0 be the integer such that $2^{k_0} \le b < 2^{k_0+1}$; then

$$\begin{aligned}I_{22} &\le \sum_{k=2}^{\infty}\int_{2^{k-1}<|x|\le 2^k}T^{A,1}a(x)\omega(x)dx+\sum_{k=2}^{\infty}\int_{2^{k-1}<|x|\le 2^k}T^{A,2}a(x)\omega(x)dx \\ &\quad+\sum_{k=k_0+1}^{\infty}\int_{2^{k-1}<|x|\le 2^k}T^{A,3}a(x)\omega(x)dx \\ &\equiv I_{22}^1+I_{22}^2+I_{22}^3\end{aligned}$$

Recall that $\omega \in A_1(\mathbf{R}^n)$ and so $\omega(B_k)/\omega(B_0) \leq C2^{kn}$. Therefore, (2.4) gives

$$I_{22}^1 \leq C\sum_{k=2}^{\infty} 2^{-k(n+1)}\omega(B_0)^{-1}\int_{2^{k-1}<|x|\leq 2^k} \omega(x)dx$$

$$\leq C\sum_{k=2}^{\infty} 2^{-k(n+1)}\frac{\omega(B_k)}{\omega(B_0)} \leq C.$$

By the reverse Hölder's inequality, it follows that for some $\varepsilon > 0$ small enough,

$$\frac{(\omega^{1+\varepsilon}(B_k))^{1/(1+\varepsilon)}}{\omega(B_k)} \leq C|B_k|^{-\varepsilon/(1+\varepsilon)} \leq C2^{-kn(\varepsilon/(1+\varepsilon))}, \tag{2.7}$$

where $(\omega^{1+\varepsilon}(B_k)) = \int_{B_k} \omega(x)^{1+\varepsilon}\,dx$ and C depends only on n and $A_1(\omega)$. Thus, (2.5) gives

$$I_{22}^2 \leq C\sum_{k=2}^{\infty} 2^{-k(n+1)}\omega(B_0)^{-1}\int_{2^{k-1}<|x|\leq 2^k}\left(\sum_{|\alpha|=m}|D^\alpha A_k(x)|\right)\omega(x)dx$$

$$\leq C\sum_{k=2}^{\infty} 2^{-k(n+1)}\omega(B_0)^{-1}\left(\int_{2^{k-1}<|x|\leq 2^k}\omega(x)^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)}$$

$$\times \sum_{|\alpha|=m}\left(\int_{2^{k-1}<|x|\leq 2^k}|D^\alpha A_k(x)|^{(1+\varepsilon)/\varepsilon}\,dx\right)^{\varepsilon/(1+\varepsilon)}$$

$$\leq C\sum_{k=2}^{\infty} 2^{-k(n+1-n(\varepsilon/(1+\varepsilon)))}\frac{(\omega^{1+\varepsilon}(B_k))^{1/(1+\varepsilon)}}{\omega(B_0)} \quad \text{(by (2.2))}$$

$$\leq C\sum_{k=2}^{\infty} 2^{-k} \leq C.$$

Interpolation between the inequality

$$\|T_k f\|_{L^2(\mathbf{R}^n)} \leq C_N 2^{nk/2}|a_{\mu_0 v_0}|^{-1/(2Nl)}2^{-k|\mu_0|/(2Nl)}\|f\|_{L^2(\mathbf{R}^n)}, \tag{2.8}$$

and the trivial estimate

$$\|T_k f\|_{L^\infty(\mathbf{R}^n)} \leq C\|f\|_{L^\infty(\mathbf{R}^n)}$$

gives that

$$\|T_k f\|_{L^p(\mathbf{R}^n)} \leq C2^{nk/p}|a_{\mu_0 v_0}|^{-1/(pNl)}2^{-k|\mu_0|/(pNl)}\|f\|_{L^p(\mathbf{R}^n)}, \qquad 2 \leq p \leq \infty. \tag{2.9}$$

Taking $1/(1+\varepsilon) + 1/p + 1/q = 1$ with $p \geq 2$, by Hölder's inequality, (2.2), (ii) of Definition 3, (2.6), (2.7) and (2.9), we have

$$I_{22}^3 \le C \sum_{k=k_0+1}^{\infty} 2^{-kn} \int_{2^{k-1}<|x|\le 2^k} \left(1+\sum_{|\alpha|=m} |D^\alpha A_k(x)|\right) |T_k a(x)| \omega(x) dx\,.$$

$$\le C \sum_{k=k_0+1}^{\infty} 2^{-kn} \left(\int_{2^{k-1}<|x|\le 2^k} \omega(x)^{1+\varepsilon} dx\right)^{1/(1+\varepsilon)} \|T_k a\|_{L^p(\mathbf{R}^n)}$$

$$\times \left[\int_{2^{k-1}<|x|\le 2^k} \left(1+\sum_{|\alpha|=m} |D^\alpha A_k(x)|\right)^q dx\right]^{1/q}$$

$$\le C \sum_{k=k_0+1}^{\infty} 2^{-kn+kn/q+kn/p} |a_{\mu_0\nu_0}|^{-1/(pNl)} 2^{-k|\mu_0|/(pNl)} \frac{(\omega^{1+\varepsilon}(B_k))^{1/(1+\varepsilon)}}{\omega(B_0)}$$

$$\le C \sum_{k=k_0+1}^{\infty} 2^{-kn} |a_{\mu_0\nu_0}|^{-1/(pNl)} 2^{-k|\mu_0|/(pNl)} \frac{\omega(B_k)}{\omega(B_0)}$$

$$\le C \sum_{k=k_0+1}^{\infty} |a_{\mu_0\nu_0}|^{-1/(pNl)} 2^{-k|\mu_0|/(pNl)}$$

$$\le C |a_{\mu_0\nu_0}|^{-1/(pNl)} b^{-|\mu_0|/(pNl)} \le C.$$

Obviously, we can assume that $r_0 = b > 2$, for otherwise $\{x : 2 < |x| \le r_0\}$ is empty. To estimate I_{21}, we first consider the case that l, the degree in y of the polynomial $P(x, y)$, is zero. In this case, by using the moment condition of a, I_2 can be estimated just as $I_{22}^1 + I_{22}^2$. Thus (2.1) holds. Then we can estimate I_{21} by induction on l. Suppose that (2.1) is true when the degree in y of the polynomial $P(x, y)$ is less than l. We need to show that (2.1) is still true when the degree in y of the polynomial $P(x, y)$ equals l. To do so, by the induction hypothesis on l, Lemma 1, (ii) of Definition 3, Hölder's inequality, (2.2) and (2.7), we have

$$I_{21} \le \int_{2<|x|\le r_0} \left| \int_{\mathbf{R}^n} e^{iQ(x,y)} \left[e^{i\sum_{|\mu|\ge 1, |\nu|=l} a_{\mu\nu} x^\mu y^\nu} - 1\right] \frac{\Omega(x-y)}{|x-y|^{n+m}} \right.$$

$$\left. \times Q_{m+1}(A; x, y) a(y) dy \right| \omega(x) dx$$

$$+ \int_{2<|x|\le r_0} \left| \int_{\mathbf{R}^n} e^{iQ(x,y)} \frac{\Omega(x-y)}{|x-y|^{n+m}} Q_{m+1}(A; x, y) a(y) dy \right| \omega(x) dx$$

$$\le C\sum_{k=2}^{k_0+1}\int_{2^{k-1}<|x|\le 2^k}\left\{\int_{\mathbf{R}^n}\left(\sum_{|\mu|\ge 1,|\nu|=l}|a_{\mu\nu}||x|^{|\mu|}\right)|x-y|^{-n-m}\right.$$

$$\left.\times|Q_{m+1}(A_k;x,y)a(y)|dy\right\}\omega(x)dx+C$$

$$\le C\sum_{k=2}^{k_0+1}\sum_{|\mu|\ge 1,|\nu|=l}|a_{\mu\nu}|2^{k(|\mu|-n)}\omega(B_0)^{-1}$$

$$\times\int_{2^{k-1}<|x|\le 2^k}\left(1+\sum_{|\alpha|=m}|D^\alpha A_k(x)|\right)\omega(x)dx+C$$

$$\le C\sum_{k=2}^{k_0+1}\sum_{|\mu|\ge 1,|\nu|=l}|a_{\mu\nu}|2^{k|\mu|}\omega(B_0)^{-1}\left(\frac{1}{|B_k|}\int_{B_k}\omega^{1+\varepsilon}(x)dx\right)^{1/(1+\varepsilon)}$$

$$\times\left\{\frac{1}{2^{kn}}\int_{2^{k-1}<|x|\le 2^k}\left(1+\sum_{|\alpha|=m}|D^\alpha A_k(x)|\right)^{(1+\varepsilon)/\varepsilon}dx\right\}^{\varepsilon/(1+\varepsilon)}+C$$

$$\le C\sum_{k=2}^{k_0+1}\sum_{|\mu|\ge 1,|\nu|=l}|a_{\mu\nu}|2^{k|\mu|}2^{-kn}\frac{\omega(B_k)}{\omega(B_0)}+C$$

$$\le C\sum_{|\mu|\ge 1,|\nu|=l}|a_{\mu\nu}|\sum_{k=2}^{k_0+1}2^{k|\mu|}+C$$

$$\le C\sum_{|\mu|\ge 1,|\nu|=l}|a_{\mu\nu}|b^{|\mu|}+C\le C,$$

where $Q(x,y)$ is a polynomial with its degree in y less than l. This finishes the proof.

3. Proof of Theorem 2

We begin with the atomic decomposition of the Herz-type Hardy space.

Definition 4. Let $\omega_1,\ \omega_2\in A_1(\mathbf{R}^n)$, $1<p<\infty$. A function $a(x)$ on $\mathbf{R}^n$ is called a *central* $\left(n\left(1-\dfrac{1}{p}\right),p;\omega_1,\omega_2\right)$-*atom*, if it satisfies

(i) $\operatorname{supp} a\subset B(0,r)\equiv\{x\in\mathbf{R}^n:|x|<r\}$ for some $r>0$;

(ii) $\|a\|_{L^p_{\omega_2}(\mathbf{R}^n)}\le[\omega_1(B(0,r))]^{-(1-(1/p))}$;

(iii) $\int_{\mathbf{R}^n}a(x)dx=0$.

The following Lemma 5 is a special case of ([10], Theorem 1).

LEMMA 5. *Let* ω_1, ω_2, p *be the same as in Definition* 4. *Then* $f \in H\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)$ *(or* $HK_p(\omega_1, \omega_2; \mathbf{R}^n)$*) if and only if* $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$, *where each* a_k *is a central* $\left(n\left(1-\frac{1}{p}\right), p; \omega_1, \omega_2\right)$*-atom (or a central* $\left(n\left(1-\frac{1}{p}\right), p; \omega_1, \omega_2\right)$*-atom with the radius of the support* ≥ 1*) and* $\sum_{k=-\infty}^{\infty} |\lambda_k| < \infty$. *Moreover,*

$$\|f\|_{H\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)} \sim \inf\left\{\sum_{k=-\infty}^{\infty} |\lambda_k|\right\}$$

$$\left(or\ \|f\|_{HK_p(\omega_1, \omega_2; \mathbf{R}^n)} \sim \inf\left\{\sum_{k=-\infty}^{\infty} |\lambda_k|\right\}\right),$$

where the infimum is taken over all the above decompositions of f.

PROOF OF THEOREM 2. We only show the theorem in homogeneous case. The non-homogeneous case is similar and we omit the details. As in the proof of Theorem 1, we only prove that for any central $\left(n\left(1-\frac{1}{p}\right), p; \omega_1, \omega_2\right)$-atom a with support $B_0 = B(0,1)$,

$$\|T^A a\|_{\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)} \leq C \tag{3.1}$$

with C independent of a. Write

$$\begin{aligned}\|T^A a\|_{\dot{K}_p(\omega_1, \omega_2; \mathbf{R}^n)} &= \sum_{k=-\infty}^{1} \omega_1(B_k)^{1-(1/p)} \|\chi_k T^A a\|_{L^p_{\omega_2}(\mathbf{R}^n)} \\ &\quad + \sum_{k=2}^{\infty} \omega_1(B_k)^{1-(1/p)} \|\chi_k T^A a\|_{L^p_{\omega_2}(\mathbf{R}^n)} \\ &\equiv J_1 + J_2.\end{aligned}$$

By ([5], Theorem 2.9 in page 401), there is a $\delta > 0$ depending only on n and the $A_1(\mathbf{R}^n)$-constant of ω_1 such that for $k \leq 1$,

$$\omega_1(B_k)/\omega_1(B_0) \leq C 2^{kn\delta},$$

where C is independent of k. By this and Lemma 2, we obtain

$$\begin{aligned}J_1 &\leq C \sum_{k=-\infty}^{1} \omega_1(B_k)^{1-(1/p)} \|a\|_{L^p_{\omega_2}(\mathbf{R}^n)} \leq C \sum_{k=-\infty}^{1} \left(\frac{\omega_1(B_k)}{\omega_1(B_0)}\right)^{1-(1/p)} \\ &\leq C \sum_{k=-\infty}^{1} 2^{kn\delta(1-(1/p))} \leq C.\end{aligned}$$

Let b, r_0 and k_0 be the same as in the proof of Theorem 1. Write

$$J_2 = \sum_{2<2^k\le r_0} \omega_1(B_k)^{1-(1/p)}\|\chi_k T^A a\|_{L^p_{\omega_2}(\mathbf{R}^n)} + \sum_{2^k>r_0} \omega_1(B_k)^{1-(1/p)}\|\chi_k T^A a\|_{L^p_{\omega_2}(\mathbf{R}^n)}$$

$$\equiv J_{21} + J_{22}.$$

We first estimate J_{22}. To do this, we write

$$J_{22} \le \sum_{k=2}^{\infty}\omega_1(B_k)^{1-(1/p)}\|\chi_k T^{A,1}a\|_{L^p_{\omega_2}(\mathbf{R}^n)} + \sum_{k=2}^{\infty}\omega_1(B_k)^{1-(1/p)}\|\chi_k T^{A,2}a\|_{L^p_{\omega_2}(\mathbf{R}^n)}$$

$$+ \sum_{k=k_0+1}^{\infty} \omega_1(B_k)^{1-(1/p)}\|\chi_k T^{A,3}a\|_{L^p_{\omega_2}(\mathbf{R}^n)}$$

$$\equiv J_{22}^1 + J_{22}^2 + J_{22}^3.$$

It follows from (2.4) that for $2^{k-1} < |x| \le 2^k$,

$$|T^{A,1}a(x)| \le C|x|^{-n-1}\int_{\mathbf{R}^n}|a(y)|dy$$
$$\le C2^{-k(n+1)}\omega_1(B_0)^{-(1-(1/p))}\omega_2(B_0)^{-1/p}.$$

Thus,

$$J_{22}^1 \le C\sum_{k=2}^{\infty}2^{-k(n+1)}\left(\frac{\omega_1(B_k)}{\omega_1(B_0)}\right)^{1-(1/p)}\left(\frac{\omega_2(B_k)}{\omega_2(B_0)}\right)^{1/p} \le C.$$

Similarly to (2.5), we can prove

$$|T^{A,2}a(x)| \le C2^{-k(n+1)}\omega_1(B_0)^{-(1-(1/p))}\omega_2(B_0)^{-1/p}\sum_{|\alpha|=m}|D^\alpha A_k(x)|;$$

and then Hölder's inequality together with (2.2) and (2.7) gives

$$J_{22}^2 \le C\sum_{k=2}^{\infty}2^{-k(n+1)}\left(\frac{\omega_1(B_k)}{\omega_1(B_0)}\right)^{1-(1/p)}\omega_2(B_0)^{-1/p}\left(\int_{2^{k-1}<|x|\le 2^k}\omega_2(x)^{1+\varepsilon}\,dx\right)^{1/((1+\varepsilon)p)}$$

$$\times\sum_{|\alpha|=m}\left(\int_{2^{k-1}<|x|\le 2^k}|D^\alpha A_k(x)|^{(1+\varepsilon)p/\varepsilon}\right)^{\varepsilon/((1+\varepsilon)p)}$$

$$\le C.$$

Now we turn our attention to J_{22}^3. We first consider the case $1 < p \le 2$.

Interpolation between (2.8) and the trivial estimate

$$\|T_k f\|_{L^1(\mathbf{R}^n)} \le C2^{nk}\|f\|_{L^1(\mathbf{R}^n)}$$

shows that

$$(3.2)\qquad \|T_k f\|_{L^p(\mathbf{R}^n)} \le C2^{nk/p}|a_{\mu_0 v_0}|^{-1/(p'Nl)}2^{-k|\mu_0|/(p'Nl)}\|f\|_{L^p(\mathbf{R}^n)},\quad 1 < p \le 2,$$

here and in what follows, p' is such that $1/p + 1/p' = 1$. Combining (3.2) and the estimate

$$(3.3)\qquad \|T_k f\|_{L^\infty(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)}$$

gives that

(3.4)
$$\|T_k f\|_{L^{p_0}(\mathbf{R}^n)} \le C2^{nk/p_0}|a_{\mu_0 v_0}|^{-p/(p'p_0Nl)}2^{-kp|\mu_0|/(p'p_0Nl)}\|f\|_{L^p(\mathbf{R}^n)},\quad p \le p_0 < \infty.$$

Thus, by taking $p_0 \in [p, \infty)$ and $q \in (1, \infty)$ such that $1/((1+\varepsilon)p) + 1/p_0 + 1/q = 1/p$, (2.6), Hölder's inequality, (2.2), (2.7) and (3.4), we obtain

$$\begin{aligned}
J_{22}^3 &\le C\sum_{k=k_0+1}^{\infty}\omega_1(B_k)^{1-1/p}2^{-kn}\\
&\quad\times\left\{\int_{2^{k-1}<|x|\le 2^k}\left(1+\sum_{|\alpha|=m}|D^\alpha A_k(x)|\right)^p|T_k a(x)|^p\omega_2(x)dx\right\}^{1/p}\\
&\le C\sum_{k=k_0+1}^{\infty}\omega_1(B_k)^{1-1/p}2^{-kn}\left(\int_{2^{k-1}<|x|\le 2^k}\omega_2(x)^{1+\varepsilon}dx\right)^{1/((1+\varepsilon)p)}\|T_k a\|_{L^{p_0}(\mathbf{R}^n)}\\
&\quad\times\left[\int_{2^{k-1}<|x|\le 2^k}\left(1+\sum_{|\alpha|=m}|D^\alpha A_k(x)|\right)^q dx\right]^{1/q}\\
&\le C\sum_{k=k_0+1}^{\infty}\left(\frac{\omega_1(B_k)}{\omega_1(B_0)}\right)^{1-1/p}2^{-kn}|a_{\mu_0 v_0}|^{-p/(p'p_0Nl)}2^{-kp|\mu_0|/(p'p_0Nl)}\left(\frac{\omega_2(B_k)}{\omega_2(B_0)}\right)^{1/p}\\
&\le C\sum_{k=k_0+1}^{\infty}|a_{\mu_0 v_0}|^{-p/(p'p_0Nl)}2^{-kp|\mu_0|/(p'p_0Nl)}\\
&\le |a_{\mu_0 v_0}|^{-p/(p'p_0Nl)}b^{-p|\mu_0|/(p'p_0Nl)} \le C.
\end{aligned}$$

If $2 < p < \infty$, interpolation between (2.9) and (3.3) gives that

$$(3.5)\qquad \|T_k f\|_{L^{p_0}(\mathbf{R}^n)} \le C2^{nk/p_0}|a_{\mu_0 v_0}|^{-1/(p_0Nl)}2^{-k|\mu_0|/(p_0Nl)}\|f\|_{L^p(\mathbf{R}^n)},\quad 2 < p < p_0.$$

Now if we replace (3.4) by (3.5), similarly to the computation above, we can also obtain a desirable estimate for J_{22}^3 in this case.

Finally, it is easy to see that J_{21} can be estimated similarly to I_{21} by using some techniques as above. This finishes the proof of Theorem 2.

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