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Stationary solutions to boundary problem for the heat equations

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ABSTRACT. The necessary and sufficient conditions for the existence of a stationary solutions to the boundary value problem for an abstract heat equation with a stationary disturbances and to the stochastic boundary value problem for such equation in the strip are given. The existence of a bounded solutions to the deterministic boundary value problem is also considered.

1. Introduction

In this paper we deal with an abstract stochastic heat equations, for which one of the independent variables represents time. It is supposed that random disturbances on the right-hand side are stationary with respect to the time variable. We are interested in solutions which are stationary with respect to the time variable of a boundary value problem in the strip. Periodic solutions for the deterministic partial differential equations are intensively studied, see, for example, well known book [15]. The problem of the existence of stationary solutions to a stochastic ordinary differential equation is also well understood, see books [8], [5] and a survey [6] for more references. During the past years it has become apparent that it is natural and more adequate in many applications to consider an input source as a random source or random disturbances. Thus investigations of stochastic partial differential equations are important. We consider the stationary solutions to some boundary value problem for a heat equation and will present some approach to obtain the existence theorem of stationary solutions. This approach is based on the results from [3] and [4]. We will demonstrate it in a simple situation relative to the random disturbances.

Let $(B, \|\cdot\|)$ be a complex separable Banach space, $\bar{0}$ the zero element in B, and $\mathscr{L}(B)$ the Banach space of bounded linear operators on B with the operator norm, denoted also by the symbol $\|\cdot\|$. For a B-valued function the continuity and differentiability means correspondingly the continuity and differentiability in the B-norm. For an operator A the sets $\sigma(A)$ and $\rho(A)$ are its spectrum and resolvent sets, respectively. Let I be the identity operator.

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In what follows, we shall consider all random elements on the same complete probability space $(\Omega, \mathscr{F}, P)$. The uniqueness of the random process, satisfying some equation, means its uniqueness up to stochastic equivalence. We consider only B-valued random functions which are continuous with probability one. All equalities with random elements in this article are always the equalities which hold with probability one.

2. Stationary solutions to the boundary value problem with stationary disturbances

Denote by S_1 the set of all B-valued stationary processes $\xi = \{\xi(t) | t \in \mathbf{R}\}$ with continuous sample functions satisfying the following condition

$$E\left[\sup_{0 \le t \le \delta} \|\xi(t)\|\right] < +\infty$$

for some $\delta > 0$.

Define

$$C_0^1 := \{g : [0, \pi] \to \mathbf{C} \,|\, g(0) = g(\pi) = 0\} \cap C^1([0, \pi]); \qquad Q := \mathbf{R} \times [0, \pi].$$

Given $g \in C_0^1$, $A \in \mathscr{L}(B)$ and $\xi \in S_1$ let us consider the problem

$$\text{(1)} \qquad \begin{cases} u_t'(t, x) - u_{xx}''(t, x) = Au(t, x) + \xi(t)g(x), & (t, x) \in Q; \\ u(t, 0) = u(t, \pi) = \bar{0}, \quad t \in \mathbf{R}. \end{cases}$$

DEFINITION 1. *A B-valued random function u defined on Q is a solution of the problem* (1) *if u, u_t' and u_{xx}'' are continuous with probability one on Q and equality* (1) *holds with probability one.*

DEFINITION 2. *A B-valued random function u defined on Q is stationary (with respect to t) if*

$$\forall t \in \mathbf{R} \qquad \forall n \in \mathbf{N} \qquad \forall \{(t_1, x_1), \ldots, (t_n, x_n)\} \subset Q \qquad \forall \{D_1, \ldots, D_n\} \subset \mathscr{B}(B):$$

$$P\left\{\bigcap_{k=1}^{n} \{\omega : u(\omega; t_k + t, x_k) \in D_k\}\right\} = P\left\{\bigcap_{k=1}^{n} \{\omega : u(\omega; t_k, x_k) \in D_k\}\right\},$$

where $\mathscr{B}(B)$ is Borel σ-algebra of B.

We come now to the main result of this section.

THEOREM 1. *Let $A \in \mathscr{L}(B)$. Then the following two statements are equivalent:*

(i) *For any stationary process* $\xi \in S_1$ *and function* $g \in C_0^1$ *the boundary value problem* (1) *has a unique stationary solution* u *such that*

$$\sup_{0 \le x \le \pi} E\|u(0,x)\| < +\infty.$$

(ii) $\{k^2 + i\alpha \mid k \in \mathbf{N}, \alpha \in \mathbf{R}\} \subset \rho(A)$.

PROOF. (ii) implies (i). Let $\xi \in S_1$ and $g \in C_0^1$ be given. If the condition (ii) holds then we have, for some $k_0 \in \mathbf{N}$

$$\sigma(A) = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_{k_0},$$

where

$$\sigma_1 \subset \{z \in \mathbf{C} \mid \operatorname{Re} z < 1\},$$

$$\sigma_2 \subset \{z \in \mathbf{C} \mid 1 < \operatorname{Re} z < 4\};$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\sigma_{k_0} \subset \{z \in \mathbf{C} \mid (k_0 - 1)^2 < \operatorname{Re} z < k_0^2\}.$$

Now let P_-^k (or P_+^k) be the Riesz projectors corresponding to the part of the spectrum $\sigma_1 \cup \cdots \cup \sigma_k$ (or $\sigma_{k+1} \cup \cdots \cup \sigma_{k_0}$), $k = 1, \dots, k_0 - 1$. Note that if, for example, $\sigma_1 = \sigma(A)$, then $P_-^1 = I$ and P_+^1 is the zero operator. It is known that

$$P_-^k + P_+^k = I,$$

$$\|e^{(A-k^2 I)t} P_-^k\| \le L e^{-at}, \qquad t > 0;$$

$$\|e^{(A-k^2 I)t} P_+^k\| \le L e^{at}, \qquad t < 0$$

for $1 \le k \le k_0 - 1$ and

$$\|e^{(A-k_0^2 I)t}\| \le L e^{-at}, \qquad t \ge 0$$

with some numbers $L \ge 0$, $a > 0$, which are fixed below.

The sequence $\{\sin kx,\ x \in [0,\pi] : k \ge 1\}$ is complete in C_0^1, which means that for any $g \in C_0^1$,

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \quad x \in [0,\pi]; \qquad g_k = \frac{2}{\pi}\int_0^{\pi} g(x) \sin kx\, dx, \quad k \ge 1,$$

where the series on the right-hand side is uniformly convergent.

Consider the random function

$$u(t,x) := \sum_{k=1}^{k_0-1} \left(\int_{-\infty}^{t} G_k(t-s) P_-^k \xi(s) ds - \int_t^{+\infty} G_k(t-s) P_+^k \xi(s) ds \right) g_k \sin kx$$

$$+ \sum_{k=k_0}^{+\infty} \int_{-\infty}^{t} G_k(t-s) \xi(s) ds g_k \sin kx, \qquad (t,x) \in Q,$$

where

$$G_k(t) := e^{(A-k^2 I)t}, \qquad t \in \mathbf{R}.$$

All integrals for u are convergent as a Bochner integral with respect to Lebesgue measure (or improper Riemann integrals for continuous B-valued function) with probability one. To see this, observe that for $0 \le k \le k_0 - 1$

$$\int_{-\infty}^{t} E\|G_k(t-s)P_-^k \xi(s)\|ds \le \int_{-\infty}^{t} Le^{-a(t-s)} E\|\xi(0)\|ds < +\infty;$$

$$\int_{t}^{+\infty} E\|G_k(t-s)P_+^k \xi(s)\|ds \le \int_{t}^{+\infty} Le^{a(t-s)} E\|\xi(0)\|ds < +\infty$$

and

$$(2) \quad \int_{-\infty}^{t} E\|G_k(t-s)\xi(s)\|ds \le \int_{-\infty}^{t} L\exp(-(a+k^2-k_0^2)(t-s))E\|\xi(0)\|ds$$
$$\le \frac{L}{a+k^2-k_0^2} E\|\xi(0)\|, \qquad k \ge k_0.$$

The series for u converges uniformly on $[u,v] \times [0,\pi]$ for any $u < v$, $v - u < \delta$ with probability one. Indeed, we have

$$E\left[\sup_{u \le t \le v} \left\| \int_{-\infty}^{t} G_k(t-s)\xi(s)ds \right\|\right] = E\left[\sup_{u \le t \le v} \left\| \int_{0}^{+\infty} G_k(p)\xi(t-p)dp \right\|\right]$$
$$\le E\left[\sup_{u \le t \le v} \int_{0}^{+\infty} Le^{-(a+k^2-k_0^2)p}\|\xi(t-p)\|dp\right]$$
$$\le \int_{0}^{+\infty} Le^{-(a+k^2-k_0^2)p} E\left[\sup_{u \le t \le v} \|\xi(t-p)\|\right]dp$$
$$\le \frac{L}{a+k^2-k_0^2} E\left[\sup_{0 \le t \le \delta} \|\xi(t)\|\right], \qquad k \ge k_0.$$

Hence the random function u is continuous with probability one. The random function u is stationary with respect to t, see [5].

By a similar reasoning, it is verified that

$$(3) \qquad u_t'(t,x) = \sum_{k=1}^{k_0-1} (A-k^2 I)\left(\int_{-\infty}^{t} G_k(t-s)P_-^k \xi(s)ds\right.$$
$$\left. - \int_{t}^{+\infty} G_k(t-s)P_+^k \xi(s)ds\right) g_k \sin kx$$

$$+\sum_{k=k_0}^{\infty}(A-k^2I)\int_{-\infty}^{t}G_k(t-s)\xi(s)dsg_k\sin kx$$
$$+\sum_{k=1}^{k_0-1}(P_-^k\xi(t)+P_+^k\xi(t))g_k\sin kx+\sum_{k=k_0}^{\infty}\xi(t)g_k\sin kx$$
$$=Au(t,x)+\xi(t)g(x)$$
$$-\sum_{k=1}^{k_0-1}k^2\left(\int_{-\infty}^{t}G_k(t-s)P_-^k\xi(s)ds\right.$$
$$\left.-\int_{t}^{+\infty}G_k(t-s)P_+^k\xi(s)ds\right)g_k\sin kx$$
$$-\sum_{k=k_0}^{+\infty}k^2\int_{-\infty}^{t}G_k(t-s)\xi(s)dsg_k\sin kx,\qquad (t,x)\in Q;$$

and

$$u''_{xx}(t,x)=\sum_{k=1}^{k_0-1}\left(\int_{-\infty}^{t}G_k(t-s)P_-^k\xi(s)ds\right.\tag{4}$$
$$\left.-\int_{t}^{+\infty}G_k(t-s)P_+^k\xi(s)ds\right)g_k(-k^2)\sin kx$$
$$+\sum_{k=k_0}^{\infty}\int_{-\infty}^{t}G_k(t-s)\xi(s)dsg_k(-k^2)\sin kx,\qquad (t,x)\in Q$$

with probability one. The series appearing on the right hand sides of (3) and (4) converge unifofmly on $[u,v]\times[0,\pi]$ for any $u<v$, $v-u<\delta$ with probability one. Indeed, by conditions $\xi\in S_1$ and $g\in C_0^1$ we have

$$\sum_{k=k_0}^{\infty}E\left(\sup_{u\le t\le v,0\le x\le\pi}\left\|(A-k^2I)\int_{-\infty}^{t}G_k(t-s)\xi(s)dsg_k\sin kx\right\|\right)$$
$$\le\sum_{k=k_0}^{\infty}(\|A\|+k^2)E\left(\sup_{u\le t\le v}\left\|\int_0^{+\infty}G_k(p)\xi(t-p)dp\right\|\cdot|g_k|\right)$$
$$\le\sum_{k=k_0}^{\infty}\frac{(\|A\|+k^2)|g_k|}{a+k^2-k_0^2}E\left(\sup_{0\le t\le\delta}\|\xi(t)\|\right)<+\infty;$$
$$\sum_{k=k_0}^{+\infty}E\left(\sup_{u\le t\le v,0\le x\le\pi}\left\|k^2\int_{-\infty}^{t}G_k(t-s)\xi(s)dsg_k\sin kx\right\|\right)$$
$$\le\sum_{k=k_0}^{\infty}\frac{k^2|g_k|}{a+k^2-k_0^2}E\left(\sup_{0\le t\le\delta}\|\xi(t)\|\right)<+\infty.$$

The convergence of the series

$$\sum_{k=k_0}^{\infty} \frac{k^2|g_k|}{a+k^2-k_0^2}$$

can be justified by showing $\sum_{k=1}^{\infty} |g_k| < \infty$ for any $g \in C_0^1$. Applying the formula for integration by parts, we have

$$\sum_{k=1}^{\infty} |g_k| = \sum_{k=1}^{\infty} \frac{|h_k|}{k}, \qquad h_k = \frac{2}{\pi} \int_0^{\pi} g'(x) \cos x \, dx, \quad k \geq 1.$$

By the Cauchy-Schwarz inequality and the Parseval's identity the last series converges. Now (1) follows from (3) and (4).

Let us establish the uniqueness of stationary solution for the problem (1) by contradiction. Given a solution u of (1), let

$$v_k(t) := \frac{2}{\pi} \int_0^{\pi} u(t,x) \sin kx \, dx, \qquad t \in \mathbf{R}, \quad k \geq 1$$

be its Fourier coefficients. Then they satisfy the following equation

$$v_k'(t) = (A - k^2 I) v_k(t) + \xi(t), \qquad t \in \mathbf{R}.$$

By the uniqueness of the Fourier expansion, the non-uniqueness of (1) implies the non-uniqueness of the solution of (5) for some $k \geq 1$. But under condition (ii) every equation (5) has a unique stationary solution by Theorem 1 in [3].

(i) implies (ii). Let $k \in \mathbf{N}$ and $\alpha \in \mathbf{R}$ be given. Let u be a unique stationary solution of (1) for

$$g(x) = \sin kx, \quad x \in [0, \pi]; \qquad \xi \in S_1.$$

Define

$$\tilde{v}_k(t) := \int_0^{\pi} u(t,x) \sin kx \, dx, \qquad t \in \mathbf{R},$$

the last integral is a Riemann integral for B-valued continuous function with probability one. It can be easily checked, that $\tilde{v}_k \in S_1$, see [5] for more details. From (1) we have

$$\tilde{v}_k'(t) = (A - k^2 I) \tilde{v}_k(t) + \frac{\pi}{2} \xi(t), \qquad t \in \mathbf{R} \tag{6}$$

with probability one. The stationary process $\tilde{v}_k$ is a unique solution of (6), because if the equation (6) had two different stationary solutions, then one could construct, following the method described in part (ii) $\Rightarrow$ (i) of the proof, two different stationary solutions of the problem (1). By Theorem 1 in [3] we have

$$\sigma(A - k^2 I) \cap i\mathbf{R} = \varnothing.$$

This completes the proof of Theorem 1.

3. Deterministic problem

In this section we shall formulate conditions for the existence of a bounded solution u to the deterministic boundary problem of the form

$$\begin{cases} u_t'(t,x) - u_{xx}''(t,x) = Au(t,x) + v(t,x), & (t,x) \in Q, \\ u(t,x) = u(t,\pi) = \bar{0}, \quad t \in \mathbf{R}, \end{cases} \tag{7}$$

where v is a bounded function on Q.

Set

$$C_{b1} := \left\{ v : Q \to B \,\middle|\, \begin{array}{l} \{v, v_x'\} \subset C(Q,B); \sup\limits_{Q} \|v\| < +\infty, \\ v(t,0) = v(t,\pi) = \bar{0}, \quad t \in \mathbf{R} \end{array} \right\},$$

$$C_{b12} := \left\{ u : Q \to B \,\middle|\, \begin{array}{l} \{u, u_t', u_{xx}''\} \subset C(Q,B); \sup\limits_{Q} \|u\| < +\infty, \\ u(t,0) = u(t,\pi) = \bar{0}, \quad t \in \mathbf{R} \end{array} \right\}.$$

THEOREM 2. *Let $A \in \mathscr{L}(B)$. Then*

$$\forall v \in C_{b1} \quad \exists! u \in C_{b12}, \ u \text{ is a solution of } (7)$$
$$\Leftrightarrow \{k^2 + i\alpha \,|\, k \in \mathbf{N}, \ \alpha \in \mathbf{R}\} \subset \rho(A).$$

The proof of Theorem 2 is similar to that of Theorem 1 except for one difference. Instead of Theorem 1 from [3] we must use the M. G. Krein Theorem [11], p. 54:

For $D \in \mathscr{L}(B)$ the equation

$$x'(t) = Dx(t) + f(t), \qquad t \in \mathbf{R}$$

has a unique bounded solution x in B for every bounded function f if and only if

$$\sigma(D) \cap i\mathbf{R} = \varnothing.$$

4. Stationary solutions to the stochastic boundary value problem

We now consider the boundary problem (1), where ξ is a process of the "white noise" type. More precisely, such a problem can be formulated in the following way. Suppose that $B = H$ is a complex separable Hilbert space and consider an H-valued Wiener process $\{w(t) : t \in \mathbf{R}\}$, see, for example, [1] or [5]. In particular, we have

$$Ew(t) = \bar{0}, \qquad E\|w(t) - w(s)\|^2 = |t - s| \operatorname{tr} W, \qquad \{t, s\} \subset \mathbf{R},$$

where W is a nuclear operator. Define $\mathscr{F}_t := \sigma(w(s) : s \leq t),\ t \in \mathbf{R}$.

We would remind the reader that a $\mathscr{L}(H)$-valued random function h on $\mathbf{R}$ is a *nonanticipating* function if for every $t \in \mathbf{R}$ the random element $h(t)$ is $\mathscr{F}_t$-measurable. For such a function h the stochastic integral

$$\int_s^t h(r)dw(r)$$

is defined in the usual way, see, for example, [1], [7], [5]. The definition of a nonanticipating H-valued function is given in a similar manner.

DEFINITION 3. *An H-valued random function u is a solution to the boundary value problem*

$$\begin{aligned} u_t'(t,x) - u_{xx}''(t,x) &= Au(t,x) + g(x)w'(t), \qquad (t,x) \in Q, \\ u(t,0) = u(t,\pi) &= \bar{0}, \qquad t \in \mathbf{R} \end{aligned} \tag{8}$$

with $g \in C_0^1$ if the function u is nonanticipating, the functions u, u_{xx}'' are continuous with probability one and, for every $s < t$, $x \in [0,\pi]$ we have

$$\begin{aligned} u(t,x) - u(s,x) - \int_s^t u_{xx}''(r,x)dr &= \int_s^t Au(r,x)dr + g(x)(w(t) - w(s)), \\ u(t,0) = u(t,\pi) &= \bar{0} \end{aligned} \tag{9}$$

with probability one.

Let us define now the class

$$C_0^3 := \{g : [0,\pi] \to \mathbf{C} \mid g^{(k)}(0) = g^{(k)}(\pi) = 0,\ k = 0,1,2\} \cap C^3([0,\pi]).$$

Now we can formulate the main result of this section.

THEOREM 3. *Let $A \in \mathscr{L}(H)$. Then the following two statements are equivalent:*

(i) *For any Wiener process w and a function $g \in C_0^3$ the boundary problem* (8) *has a unique stationary, nonanticipating solution u such that*

$$\sup_{0 \leq x \leq \pi} E\|u(0,x)\|^2 < +\infty.$$

(ii) $\sigma(A - I) \subset \{z \in \mathbf{C} : \operatorname{Re} z < 0\}$.

PROOF. (ii) implies (i). First note that

$$\|G_1(t)\| = \|e^{(A-I)t}\| \le Le^{-at}, \qquad t \ge 0$$

for some $L \ge 0$ and $a > 0$.

Let a Wiener process w and a function $g \in C_0^3$ be ginen. Then

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \qquad x \in [0, \pi]$$

with $\{g_k : k \ge 1\} \subset \mathbf{C}$. The last series converges uniformly on $[0, \pi]$.

Now define the random function

$$u(t,x) := \sum_{k=1}^{\infty} \int_{-\infty}^{t} G_k(t-s)dw(s)g_k \sin kx, \qquad (t,x) \in Q. \tag{10}$$

The integrals on the right-hand side are to be understood in the sence of stochastic integrals with respect to the Wiener process w. These integrals exist, since, for any $t \in \mathbf{R}$ and $k \in \mathbf{N}$, we have

$$\int_{-\infty}^{t} \|G_k(t-s)\|^2 ds \le \frac{L^2}{2(a+k^2-1)}.$$

Moreover, these integrals as functions of t are continuous on $\mathbf{R}$ in H-norm with probability one, the proof of this assertion being derived analogously to [1], see also [5] and [2], [8], [10], [14] for general results. It can be verified, that these integrals are *stationary connected* H-valued processes, defined as follows [5]. The processes $\xi_1, \xi_2, \dots, \xi_m$ are called the stationary connected processes if a vector process $(\xi_1, \xi_2, \dots, \xi_m)$ is a stationary process. Each of these integrals is nonanticipating with respect to w. Let $-\infty < b < c < +\infty$ be given. To establish the uniform convergence of the series (10) on $Q(b,c) := [b,c] \times [0,\pi]$ with probability one, we prove that the series

$$\sum_{k=1}^{\infty} E\left(\sup_{t \in Q(b,c)} \left\| \int_{-\infty}^{t} G_k(t-s)dw(s)g_k \sin kx \right\| \right) \tag{11}$$

is convergent. Using the properties of stochastic integral, we find

$$\begin{aligned}
&E\left(\sup_{t \in Q(b,c)} \left\| \int_{-\infty}^{t} G_k(t-s)dw(s)g_k \sin kx \right\| \right) \\
&\quad \le E\left(\sup_{b \le t \le c} \left(\|G_k(t-b)\| \cdot \left\| \int_{-\infty}^{b} G_k(b-s)dw(s) \right\| \right) \right) |g_k| \\
&\quad + E\left(\sup_{b \le t \le c} \left\| \int_{b}^{t} G_k(t-s)dw(s) \right\| \right) |g_k|
\end{aligned}$$

$$\leq L|g_k|E\left\|\int_{-\infty}^{b} G_k(b-s)dw(s)\right\|$$
$$+|g_k|E\left(\sup_{b\leq t\leq c}\left\|w(t)-G_k(t-b)w(b)+\int_b^t(A-k^2I)G_k(t-s)w(s)ds\right\|\right).$$

This relation implies

$$(12)\quad E\left(\sup_{t\in Q(b,c)}\left\|\int_{-\infty}^{t} G_k(t-s)dw(s)g_k\sin kx\right\|\right)$$
$$\leq L|g_k|\left(\int_{-\infty}^{t}\|G_k(b-s)\|^2\operatorname{tr}Wds\right)^{1/2}$$
$$+|g_k|\left(1+L+\sup_{b\leq t\leq c}\int_b^t\|A-k^2I\|Le^{-(a+k^2-1)(t-s)}ds\right)E\left(\sup_{b\leq t\leq c}\|w(t)\|\right)$$
$$\leq\frac{L^2|g_k|\sqrt{\operatorname{tr}W}}{\sqrt{2(a+k^2-1)}}+\frac{L|g_k|(\|A\|+k^2)}{a+k^2-1}E\left(\sup_{b\leq t\leq c}\|w(t)\|\right).$$

Consider the random process $\{w(t):t\in[b,c]\}$ as a random element in $C([b,c],H)$. This element is Gaussian. As a consequence, we have

$$E\left(\sup_{b\leq t\leq c}\|w(t)\|\right)<+\infty,$$

see, for example, [13] or [12].

The uniform convergence of the series (10) on $Q(b,c)$ now follows from the estimate (12) and the convergence of the series (11). Hence, the random function $\{u(t,x):(t,x)\in Q\}$ is continuous with probability one.

The continuity of the random functions

$$\{u_x'(t,x):(t,x)\in Q\},\qquad\{u_{xx}''(t,x):(t,x)\in Q\}$$

and the equalities

$$u_x'(t,x)=\sum_{k=1}^{\infty}\int_{-\infty}^{t}G_k(t-s)dw(s)kg_k\cos kx,\qquad(t,x)\in Q$$

$$u_{xx}''(t,x)=-\sum_{k=1}^{\infty}\int_{-\infty}^{t}G_k(t-s)dw(s)k^2g_k\sin kx,\qquad(t,x)\in Q$$

can be established by use of a similar argument, except for the following. For a proof of convergence of a series for u_{xx}'' it is enough to check that

$$\sum_{k=1}^{\infty} k^2 |g_k| < +\infty.$$

We use the assumption $g \in C_0^3$. Let $a_k(f)$, $b_k(f)$, $k \geq 1$ are the Fourier coefficients of $f \in C_0^3$. Then

$$|a_k(f)| + |b_k(f)| \leq \frac{a_k(f''')| + |b_k(f''')|}{k^3}, \qquad k \geq 1$$

and the series

$$\sum_{k=1}^{\infty} (|a_k(f''')|^2 + |b_k(f''')|^2)$$

converges by the Parseval's identity. By the Cauchy-Schwarz inequality the series

$$\sum_{k=1}^{\infty} \frac{|a_k(f''')| + |b_k(f''')|}{k}$$

is convergent.

The random functions u and u''_{xx} are stationary random functions.

With the aid of the above expressions for u and u''_{xx}, we obtain for the right-hand side integral of (9) the following representation

$$\begin{aligned}
&\int_s^t Au(r,x)dr \\
&\quad = \sum_{k=1}^{\infty} \int_s^t \left(\int_{-\infty}^r AG_k(r-p)dw(p) \right) dr g_k \sin kx \\
&\quad = \sum_{k=1}^{\infty} \int_s^t \left(\int_{-\infty}^r (A - k^2 I)G_k(r-p)dw(p) \right) dr g_k \sin kx - \int_s^t u''_{xx}(r,x)dr \\
&\quad = \sum_{k=1}^{\infty} \int_{-\infty}^s \left(\int_s^t \frac{dG_k(r-p)}{dr} dr \right) dw(p) g_k \sin kx \\
&\qquad + \sum_{k=1}^{\infty} \int_s^t \left(\int_p^t \frac{dG_k(r-p)}{dr} dr \right) dw(p) g_k \sin kx - \int_s^t u''_{xx}(r,x)dr \\
&\quad = u(t,x) - u(s,x) - g(x)(w(t) - w(s)) - \int_s^t u''_{xx}(r,x)dr
\end{aligned}$$

with probability 1. Therefore u is a solution of the problem (8).

To show the uniqueness, assume that $\tilde{u}$ is also a solution of the problem (8). Then $p := u - \tilde{u}$ satisfies the following conditions

$$p'_t(t,x) - p''_{xx} = Ap(t,x), \qquad (t,x) \in Q;$$
$$p(t,0) = p(t,\pi) = \bar{0}, \qquad t \in \mathbf{R}$$

with probability 1 and for

$$p_k(t) := \int_0^\pi p(t,x) \sin kx \, dx, \qquad t \in \mathbf{R},$$

we have the equation

$$p'_k(t) = (A - k^2 I) p_k(t)$$

which has under condition (ii) only one and hence trivial stationary solution by Theorem 4.1 from [4], $k \geq 1$. The rest of the proof of the uniqueness is the same as in the proof Theorem 1.

(i) implies (ii). Let a Wiener process w and a function $g \in C_0^3$ be given. Suppose that u is a unique stationary nonanticipating solution of (6) corresponding to w, g and define, for $k \in \mathbf{N}$,

$$v_k(t) := \int_0^\pi u(t,x) \sin kx \, dx, \qquad t \in \mathbf{R}.$$

The function v_k is a stationary nonanticipating H-valued process such that $E\|v_k(0)\|^2 < +\infty$. From (9) we have

$$v_k(t) - v_k(s) + k^2 \int_s^t v_k(r) dr = \int_s^t A v_k(r) dr + g_k(w(t) - w(s)) \tag{13}$$

for $s < t$ with probability one. Note that the equation (13) has unique nonanticipating stationary solution, the proof is similar to the one of Theorem 1. Now by Theorem 4.1 in [4] we have

$$\sigma(A - k^2 I) \subset \{z \in \mathbf{C} : \operatorname{Re} z < 0\}.$$

Since this inclusion holds for all $k \in \mathbf{N}$, we obtain

$$\sigma(A - I) \subset \{z \in \mathbf{C} : \operatorname{Re} z < 0\}.$$

The proof Theorem 3 is complete.

REMARK. We note, that second statements of Theorem 1 and 3 are different. Theorem 3 suggests that the existence of the nonanticipating stationary solution is a stronger condition than the existence of a stationary solution.

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