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# Properties of harmonic boundary in nonlinear potential theory

*Dedicated to Professor Masayuki Itô for his sixtieth birthday*

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ABSTRACT. We consider a quasi-linear second order elliptic differential equation on a euclidean domain, and for a compactification of the domain we define the harmonic boundary relative to the structure condition of the equation. Properties of harmonic boundary known in the classical potential theory are extended to our nonlinear case. We show that the comparison principle with respect to harmonic boundary holds for our equation, and give relations between Dirichlet-regular points and the harmonic boundary points.

## Introduction

In an ideal boundary theory for Riemann surfaces, the notion of harmonic boundary has been introduced as a potential theoretically essential part of the given ideal boundary (cf. [CC]). Among others, the minimum principle with respect to harmonic boundary (cf. [CC; Satz 8.4 and Folgesatz 8.1]) and the fact that the harmonic boundary on the Royden boundary coincides with the set of all regular points with respect to the Dirichlet problem (cf. [CC; Folgesatz 9.2]) are typical results showing the importance of this notion. Such results have been also considered on Riemannian manifolds (cf. e.g., [GN]) and behavior of solutions of the equation $\Delta u - Pu = 0$ at the harmonic boundary have been studied (cf. [GKa] and [GN]). Further, these results are extended to the $p$-Royden boundary of a Riemannian manifold $\Omega$, for which the minimum principle (or, rather the comparison principle) and the Dirichlet problem are considered with respect to the $p$-Laplacian ([T1] and [T2]) or more generally, with respect to the quasi-linear elliptic equation

$$-\operatorname{div} \mathscr{A}(x, \nabla u(x)) = 0,$$

where $\mathscr{A}(x, \xi) : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ satisfies structure conditions of $p$-th order with $1 < p < \infty$ (see [N]).

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On the other hand, the authors discussed Dirichlet problems with respect to ideal boundaries for the equation

$$\text{(E)} \qquad -\operatorname{div} \mathscr{A}(x, \nabla u(x)) + \mathscr{B}(x, u(x)) = 0$$

on a euclidean domain $\Omega$, where $\mathscr{A}(x, \xi)$ satisfies weighted structure conditions of $p$-th order with a weight $\mu$ and $\mathscr{B}(x, t) : \Omega \times \mathbf{R} \to \mathbf{R}$ is nondecreasing in $t$ (see [MaO] or §1 below for more details).

In this paper, we consider the $Q$-compactification of $\Omega$ for a family $Q$ of bounded continuous functions with finite $(p, \mu)$-Dirichlet integrals and the associated harmonic boundary. We show that comparison principle with respect to this harmonic boundary still holds for the equation (E) and that the set of regular points for the Dirichlet problem with respect to the $Q$-compactification and the equation (E) coincides with the harmonic boundary, under an additional condition on $Q$. To obtain these results, we first discuss in §2 harmonizability of bounded continuous functions with finite $(p, \mu)$-Dirichlet integrals with respect to (E).

## §1. Preliminaries

In this section, we recall definitions and results in [MaO] which will be used in our later discussions. Throughout this paper, let $\Omega$ be a fixed domain in $\mathbf{R}^N$ and we consider a quasi-linear elliptic differential equation

$$\text{(E)} \qquad -\operatorname{div} \mathscr{A}(x, \nabla u(x)) + \mathscr{B}(x, u(x)) = 0$$

on $\Omega$. Here, $\mathscr{A} : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathscr{B} : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions for $1 < p < \infty$ and a *weight* $w$ which is *p-admissible* in the sense of [HKM]:

(A.1) $x \mapsto \mathscr{A}(x, \xi)$ is measurable on $\Omega$ for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathscr{A}(x, \xi)$ is continuous for a.e. $x \in \Omega$;

(A.2) $\mathscr{A}(x, \xi) \cdot \xi \geq \alpha_1 w(x)|\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathscr{A}(x, \xi)| \leq \alpha_2 w(x)|\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \Omega$ with a constant $\alpha_2 > 0$;

(A.4) $(\mathscr{A}(x, \xi_1) - \mathscr{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$ whenever $\xi_1, \xi_2 \in \mathbf{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in \Omega$;

(B.1) $x \mapsto \mathscr{B}(x, t)$ is measurable on $\Omega$ for every $t \in \mathbf{R}$ and $t \mapsto \mathscr{B}(x, t)$ is continuous for a.e. $x \in \Omega$;

(B.2) For any open set $D \Subset \Omega$, there is a constant $\alpha_3(D) \geq 0$ such that $|\mathscr{B}(x, t)| \leq \alpha_3(D) w(x)(|t|^{p-1} + 1)$ for all $t \in \mathbf{R}$ and a.e. $x \in D$;

(B.3) $t \mapsto \mathscr{B}(x, t)$ is nondecreasing on $\mathbf{R}$ for a.e. $x \in \Omega$.

We remark that if $\mathscr{A}$ and $\mathscr{B}$ satisfy the above conditions, then $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ which are defined by

$$\tilde{\mathscr{A}}(x,\xi) = -\mathscr{A}(x,-\xi) \qquad \text{and} \qquad \tilde{\mathscr{B}}(x,t) = -\mathscr{B}(x,-t)$$

also satisfy these conditions with the same constants $\alpha_1$, $\alpha_2$ and $\alpha_3(D)$.

For the nonnegative measure $\mu : d\mu(x) = w(x)dx$ and an open subset $D$ of $\Omega$, we consider the weighted Sobolev spaces $H^{1,p}(D;\mu)$, $H_0^{1,p}(D;\mu)$ and $H_{\mathrm{loc}}^{1,p}(D;\mu)$ (see [HKM] for details). $u \in H_{\mathrm{loc}}^{1,p}(D;\mu)$ is said to be a (weak) *solution* of (E) in $D$ if

$$\int_D \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_D \mathscr{B}(x,u)\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(D)$. $u \in H_{\mathrm{loc}}^{1,p}(D;\mu)$ is said to be a *supersolution* (resp. *subsolution*) of (E) in $D$ if

$$\int_D \mathscr{A}(x,\nabla u)\cdot\nabla\varphi\,dx + \int_D \mathscr{B}(x,u)\varphi\,dx \geq 0 \qquad (\text{resp. } \leq 0)$$

for all nonnegative $\varphi \in C_0^\infty(D)$.

A continuous solution of (E) in an open set $D \subset \Omega$ is called $(\mathscr{A},\mathscr{B})$-*harmonic* in $D$. Note that if $h$ is $(\mathscr{A},\mathscr{B})$-harmonic in $D$, then $-h$ is $(\tilde{\mathscr{A}},\tilde{\mathscr{B}})$-harmonic in $D$.

Proposition 1.1. (Harnack principle) [MaO; Theorem 1.6] *If $\{h_n\}$ is a nondecreasing or nonincreasing sequence of $(\mathscr{A},\mathscr{B})$-harmonic functions in a domain $D$ and if $\{h_n(x_0)\}$ is bounded for some $x_0 \in D$, then $h := \lim_{n\to\infty} h_n$ is $(\mathscr{A},\mathscr{B})$-harmonic in $D$.*

We say that an open set $D$ in $\Omega$ is $(\mathscr{A},\mathscr{B})$-*regular*, if $D \Subset \Omega$ and for any $\theta \in H_{\mathrm{loc}}^{1,p}(\Omega;\mu)$ which is continuous at each point of $\partial D$, there exists a unique $h \in C(\bar{D}) \cap H^{1,p}(D;\mu)$ such that $h = \theta$ on $\partial D$ and $h$ is $(\mathscr{A},\mathscr{B})$-harmonic in $D$.

Proposition 1.2. [MaO; Corollary 1.2] *For any compact set $K$ and an open set $D$ such that $K \subset D \subset \Omega$, there exists an $(\mathscr{A},\mathscr{B})$-regular open set $G$ such that $K \subset G \subset D$.*

A function $u : D \to \mathbf{R} \cup \{\infty\}$ is said to be $(\mathscr{A},\mathscr{B})$-*superharmonic* in $D$ if it is lower semicontinuous, finite on a dense set in $D$ and, for each open set $G \Subset D$ and for $h \in C(\bar{G})$ which is $(\mathscr{A},\mathscr{B})$-harmonic in $G$, $u \geq h$ on $\partial G$ implies $u \geq h$ in $G$. $(\mathscr{A},\mathscr{B})$-*subharmonic* functions are similarly defined. A function $v$ is $(\mathscr{A},\mathscr{B})$-subharmonic in $D$ if and only if $-v$ is $(\tilde{\mathscr{A}},\tilde{\mathscr{B}})$-superharmonic in $D$.

Theorem 1.1. (Comparison principle) [MaO; Theorem 2.1] *Let $u$ be $(\mathscr{A},\mathscr{B})$-superharmonic in $D$ and let $v$ be $(\mathscr{A},\mathscr{B})$-subharmonic in $D$. If*

$$\liminf_{x\to\xi}\{u(x) - v(x)\} \geq 0$$

*for all* $\xi \in \partial^a D$, *then* $u \geq v$ *in* $D$, *where* $\partial^a D$ *is the boundary of* $D$ *in the one point compactification of* $\mathbf{R}^N$.

The following two propositions are given in [MaO; Propositions 2.1, 2.3 and Remark 2.1].

PROPOSITION 1.3. *If* $u$ *and* $v$ *are* $(\mathscr{A}, \mathscr{B})$*-superharmonic in* $D$, *then so is* $\min(u, v)$.

PROPOSITION 1.4. *Let* $D$ *be an open set in* $\Omega$ *and let* $G \Subset D$ *be an* $(\mathscr{A}, \mathscr{B})$*-regular open set. For an* $(\mathscr{A}, \mathscr{B})$*-superharmonic function* $u$ *on* $D$, *we define*

$$u_G = \sup\{h \in C(\bar{G}) : h \leq u \text{ on } \partial G \text{ and } h \text{ is } (\mathscr{A}, \mathscr{B})\text{-harmonic in } G\}.$$

*Then*

$$P(u, G) := \begin{cases} u & \text{in } D \backslash G \\ u_G & \text{in } G \end{cases}$$

*is* $(\mathscr{A}, \mathscr{B})$*-superharmonic in* $D$ *and* $(\mathscr{A}, \mathscr{B})$*-harmonic in* $G$, *and* $P(u, G) \leq u$ *in* $D$. *If* $u \in H_{\mathrm{loc}}^{1,p}(D; \mu)$, *then* $u|_G - u_G \in H_0^{1,p}(G; \mu)$.

Next we consider the following spaces:

$$\mathscr{D}^p(\Omega; \mu) := \{f \in H_{\mathrm{loc}}^{1,p}(\Omega; \mu) : |\nabla f| \in L^p(\Omega; \mu),\ f \text{ is bounded continuous}\},$$

$$\mathscr{D}_0^p(\Omega; \mu) := \left\{ f \in \mathscr{D}^p(\Omega; \mu) : \begin{array}{l} {}^\exists \varphi_n \in C_0^\infty(\Omega) \text{ s.t. } \varphi_n \to f \text{ a.e., } \{\varphi_n\} \text{ is} \\ \text{uniformly bounded, } \nabla \varphi_n \to \nabla f \text{ in } L^p(\Omega; \mu) \end{array} \right\}.$$

We say that $\Omega$ is *$(p, \mu)$-hyperbolic* if $1 \notin \mathscr{D}_0^p(\Omega; \mu)$.

PROPOSITION 1.5. *Let* $h_1$, $h_2 \in \mathscr{D}^p(\Omega; \mu)$ *be* $(\mathscr{A}, \mathscr{B})$*-harmonic functions in* $\Omega$. *If* $h_1 - h_2 \in \mathscr{D}_0^p(\Omega; \mu)$ *and* $\int_\Omega |\mathscr{B}(x, h_1(x)) - \mathscr{B}(x, h_2(x))| dx < \infty$, *then* $h_1 - h_2 \equiv$ constant. *If, in addition,* $\Omega$ *is* $(p, \mu)$*-hyperbolic, then* $h_1 = h_2$.

PROOF. There exist $\varphi_n \in C_0^\infty(\Omega)$ such that $\{\varphi_n\}$ is uniformly bounded and $\varphi_n \to h_1 - h_2$ a.e., $\nabla \varphi_n \to \nabla(h_1 - h_2)$ in $L^p(\Omega; \mu)$ as $n \to \infty$. Since both $h_1$ and $h_2$ are $(\mathscr{A}, \mathscr{B})$-harmonic in $\Omega$, we have

$$\int_\Omega \mathscr{A}(x, \nabla h_1) \cdot \nabla \varphi_n \, dx + \int_\Omega \mathscr{B}(x, h_1) \varphi_n \, dx = 0,$$

$$\int_\Omega \mathscr{A}(x, \nabla h_2) \cdot \nabla \varphi_n \, dx + \int_\Omega \mathscr{B}(x, h_2) \varphi_n \, dx = 0.$$

Subtracting these two equations and letting $n \to \infty$, we have

$$\int_\Omega [\mathscr{A}(x,\nabla h_1) - \mathscr{A}(x,\nabla h_2)]\cdot(\nabla h_1 - \nabla h_2)dx$$
$$+ \int_\Omega [\mathscr{B}(x,h_1) - \mathscr{B}(x,h_2)](h_1 - h_2)dx = 0.$$

It follows from (A.4) and (B.3) that $\nabla h_1 = \nabla h_2$ a.e., so that $h_1 = h_2 + c$. If $\Omega$ is $(p,\mu)$-hyperbolic, we see that $c = 0$, namely $h_1 = h_2$.

In order to prove a resolutivity result, we prepared the following two lemmas in [MaO; Lemmas 5.1, 5.2], which we will use in this paper, too.

LEMMA 1.1. *Let $\{u_n\}$ be a uniformly bounded sequence of functions in $H_0^{1,p}(\Omega;\mu)$ such that $\{\int_\Omega |\nabla u_n|^p\,d\mu\}$ is bounded and $u_n \to u$ a.e. in $\Omega$ as $n \to \infty$. If $u$ is continuous, then $u \in \mathscr{D}_0^p(\Omega;\mu)$.*

LEMMA 1.2. *Let $f \in \mathscr{D}^p(\Omega;\mu)$ and suppose that there is a bounded supersolution $g$ of* (E) *in $\Omega$ such that $g \geq f$ in $\Omega$ and suppose*

$$\int_\Omega \mathscr{B}(x,f)^-\,dx < \infty. \tag{1.1}$$

*Then there exists an $(\mathscr{A},\mathscr{B})$-superharmonic function $u$ in $\Omega$ such that $u \geq f$ in $\Omega$ and $u - f \in \mathscr{D}_0^p(\Omega;\mu)$.*

## § 2. $(\mathscr{A},\mathscr{B})$-harmonizable functions

Let $f$ be a real function in $\Omega$ and, let

$$\mathscr{U}_f^* = \left\{u : \begin{array}{l}(\mathscr{A},\mathscr{B})\text{-superharmonic in } \Omega \text{ and}\\ u \geq f \text{ outside a compact set in } \Omega\end{array}\right\}$$

and

$$\mathscr{L}_f^* = \left\{v : \begin{array}{l}(\mathscr{A},\mathscr{B})\text{-subharmonic in } \Omega \text{ and}\\ v \leq f \text{ outside a compact set in } \Omega\end{array}\right\}.$$

THEOREM 2.1. *If both $\mathscr{U}_f^*$ and $\mathscr{L}_f^*$ are nonempty, then*

$$\bar{h}_f = \bar{h}_f^{(\mathscr{A},\mathscr{B})} := \inf \mathscr{U}_f^* \qquad \textit{and} \qquad \underline{h}_f = \underline{h}_f^{(\mathscr{A},\mathscr{B})} := \sup \mathscr{L}_f^*$$

*are $(\mathscr{A},\mathscr{B})$-harmonic in $\Omega$ and $\underline{h}_f \leq \bar{h}_f$.*

PROOF. The comparison principle (Theorem 1.1) implies $\underline{h}_f \leq \bar{h}_f$. The rest of the assertion follows from Propositions 1.3, 1.4 and 1.1 in the same way as in [HKM; Theorem 9.2].

We say that $f$ is $(\mathscr{A},\mathscr{B})$-*harmonizable* if both $\mathscr{U}_f^*$ and $\mathscr{L}_f^*$ are nonempty and $\underline{h}_f = \bar{h}_f$. In this case we write $h_f = h_f^{(\mathscr{A},\mathscr{B})}$ for $\underline{h}_f = \bar{h}_f$.

The following proposition is clear.

PROPOSITION 2.1. *If $f$ and $g$ are $(\mathscr{A},\mathscr{B})$-harmonizable and $f \le g$ outside a compact set, then $h_f \le h_g$.*

We recall the following conditions, which have been given in [MaO] for the discussion of resolutivity (see Theorem 2.3).

($C_1$) There exist a bounded supersolution of (E) in $\Omega$ and a bounded subsolution of (E) in $\Omega$.

(B.5) $\int_\Omega |\mathscr{B}(x,t)|dx < \infty$ for any $t \in \mathbf{R}$.

THEOREM 2.2. *Suppose that $\Omega$ is $(p,\mu)$-hyperbolic and suppose that conditions ($C_1$) and (B.5) are satisfied. If $f \in \mathscr{D}^p(\Omega;\mu)$, then $f$ is $(\mathscr{A},\mathscr{B})$-harmonizable and $h_f - f \in \mathscr{D}_0^p(\Omega;\mu)$.*

PROOF. Let $f \in \mathscr{D}^p(\Omega;\mu)$ and let $v_1$ (resp. $v_2$) be a bounded supersolution (resp. subsolution) of (E) in $\Omega$. By the boundedness of $f$ and $v_1$, there is a constant $c_1 \ge 0$ such that $v_1 + c_1 \ge f$ in $\Omega$. Then $g_1 := v_1 + c_1$ is a supersolution of (E) and $g_1 \ge f$. Also, by condition (B.5), (1.1) is satisfied. Hence, by Lemma 1.2, there is an $(\mathscr{A},\mathscr{B})$-superharmonic function $u$ in $\Omega$ such that $u \ge f$ and $u - f \in \mathscr{D}_0^p(\Omega;\mu)$. Let $\{D_n\}$ be an exhaustion of $\Omega$ by $(\mathscr{A},\mathscr{B})$-regular open sets and let $u_n = P(u, D_n)$ in the notation in Proposition 1.4. Then since $D_n \Subset \Omega$ and $u_n = u$ in $\Omega \backslash D_n$, $u_n \in \mathscr{U}_f^*$. By the boundedness of $f$ and $v_2$, there is a constant $c_2$ such that $g_2 := v_2 - c_2 \le f$ in $\Omega$. We may assume that $g_2$ is $(\mathscr{A},\mathscr{B})$-subharmonic in $\Omega$ ([MaO; Corollary 4.1]). Hence $\mathscr{L}_f^* \ne \emptyset$, so that $\bar{h}_f$ exists and $u_n \ge \bar{h}_f$ for each $n$. On the other hand, we obtain from the comparison principle that $u \ge u_n \ge u_{n+1}$. Thus, by Proposition 1.1, $\bar{u} := \lim_{n\to\infty} u_n$ is $(\mathscr{A},\mathscr{B})$-harmonic in $\Omega$ and $u \ge \bar{u} \ge \bar{h}_f$.

Since $u_n$ is $(\mathscr{A},\mathscr{B})$-harmonic in $\Omega$, $u_n = u$ in $\Omega \backslash D_n$ and $u_n - u \in H_0^{1,p}(D_n;\mu)$, we have

$$\int_\Omega \mathscr{A}(x,\nabla u_n)\cdot(\nabla u_n - \nabla u)dx + \int_\Omega \mathscr{B}(x,u_n)(u_n - u)dx = 0,$$

so that, by (A.2) and (A.3), we have

$$\begin{aligned}(2.1)\qquad \alpha_1 \int_\Omega |\nabla u_n|^p\,d\mu &\le \alpha_2 \int_\Omega |\nabla u_n|^{p-1}|\nabla u|d\mu + \int_\Omega |\mathscr{B}(x,u_n)|(u-u_n)dx \\ &\le \alpha_2\left(\int_\Omega |\nabla u_n|^p\,d\mu\right)^{(p-1)/p}\left(\int_\Omega |\nabla u|^p\,d\mu\right)^{1/p} \\ &\quad + \int_\Omega |\mathscr{B}(x,u_n)|(u-u_n)dx,\end{aligned}$$

where in the last inequality we have used Hölder's inequality. The comparison principle implies that $g_2 \le u_n$. Since $u_n \le u$ and $u$ is bounded, we see that $\{u-u_n\}$ is uniformly bounded. Hence, by condition (B.5), $\{\int_\Omega \mathscr{B}(x,u_n)(u-u_n)dx\}$ is bounded. Also $u\in\mathscr{D}^p(\Omega;\mu)$ implies $\int_\Omega|\nabla u|^p\,d\mu<\infty$. It follows from (2.1) that $\{\int_\Omega|\nabla u_n|^p\,d\mu\}$ is bounded. Hence, since $u-u_n\in H_0^{1,p}(\Omega;\mu)$, $\{u-u_n\}$ is uniformly bounded and $u_n\to u$, Lemma 1.1 yields $u-\bar{u}\in\mathscr{D}_0^p(\Omega;\mu)$, so that $\bar{u}-f\in\mathscr{D}_0^p(\Omega;\mu)$.

Similarly, applying the above arguments to $(\tilde{\mathscr{A}},\tilde{\mathscr{B}})$ and $-f$, we can find a bounded $(\mathscr{A},\mathscr{B})$-harmonic function $\underline{u}$ in $\Omega$ such that $\underline{u}\le\underline{h}_f$ and $f-\underline{u}\in\mathscr{D}_0^p(\Omega;\mu)$. Therefore, the linearity of $\mathscr{D}_0^p(\Omega;\mu)$ implies $\bar{u}-\underline{u}\in\mathscr{D}_0^p(\Omega;\mu)$. It follows from Proposition 1.5 that $\bar{u}=\underline{u}$, and hence $\bar{h}_f=\underline{h}_f$.

Given a compactification $\Omega^*$ of $\Omega$ and a bounded function $\psi$ on $\partial^*\Omega=\Omega^*\backslash\Omega$, let

$$\mathscr{U}_\psi=\left\{u:\ \begin{array}{l}(\mathscr{A},\mathscr{B})\text{-superharmonic in }\Omega\text{ and}\\ \liminf_{x\to\xi}u(x)\ge\psi(\xi)\text{ for all }\xi\in\partial^*\Omega\end{array}\right\}$$

and

$$\mathscr{L}_\psi=\left\{v:\ \begin{array}{l}(\mathscr{A},\mathscr{B})\text{-subharmonic in }\Omega\text{ and}\\ \limsup_{x\to\xi}v(x)\le\psi(\xi)\text{ for all }\xi\in\partial^*\Omega\end{array}\right\}.$$

If both $\mathscr{U}_\psi$ and $\mathscr{L}_\psi$ are nonempty, then

$$\overline{H}(\psi;\Omega^*)=\overline{H}^{(\mathscr{A},\mathscr{B})}(\psi;\Omega^*):=\inf\mathscr{U}_\psi$$

and

$$\underline{H}(\psi;\Omega^*)=\underline{H}^{(\mathscr{A},\mathscr{B})}(\psi;\Omega^*):=\sup\mathscr{L}_\psi$$

are $(\mathscr{A},\mathscr{B})$-harmonic in $\Omega$ and $\underline{H}(\psi;\Omega^*)\le\overline{H}(\psi;\Omega^*)$ ([MaO; Theorm 3.1]). We say that $\psi$ is $(\mathscr{A},\mathscr{B})$-*resolutive* if both $\mathscr{U}_\psi$ and $\mathscr{L}_\psi$ are nonempty and $\underline{H}(\psi;\Omega^*)=\overline{H}(\psi;\Omega^*)$. In this case we write $H(\psi;\Omega^*)=H^{(\mathscr{A},\mathscr{B})}(\psi;\Omega^*)$ for $\underline{H}(\psi;\Omega^*)=\overline{H}(\psi;\Omega^*)$. $\Omega^*$ is called an $(\mathscr{A},\mathscr{B})$-*resolutive* compactification, if all $\psi\in C(\partial^*\Omega)$ are $(\mathscr{A},\mathscr{B})$-resolutive.

PROPOSITION 2.2. *Let $f\in C(\Omega^*)$ and let $\psi:=f|_{\partial^*\Omega}$. $\psi$ is $(\mathscr{A},\mathscr{B})$-resolutive if and only if $f|_\Omega$ is $(\mathscr{A},\mathscr{B})$-harmonizable, and then $H(\psi;\Omega^*)=h_f$.*

PROOF. If $u\in\mathscr{U}_f^*$, then $u\in\mathscr{U}_\psi$. Hence $\overline{H}(\psi;\Omega^*)\le u$, so that $\overline{H}(\psi;\Omega^*)\le\bar{h}_f$. Similarly, we have $\underline{h}_f\le\underline{H}(\psi;\Omega^*)$. Therefore, $\psi$ is $(\mathscr{A},\mathscr{B})$-resolutive if $f$ is $(\mathscr{A},\mathscr{B})$-harmonizable, and $H(\psi;\Omega^*)=h_f$.

To show the converse, we suppose $u\in\mathscr{U}_\psi$. Then, because $u+\varepsilon\in\mathscr{U}_f^*$ for any $\varepsilon>0$, $\bar{h}_f\le u+\varepsilon$, so that $\bar{h}_f\le\overline{H}(\psi;\Omega^*)+\varepsilon$. Since $\varepsilon$ is arbitrary, $\bar{h}_f\le\overline{H}(\psi;\Omega^*)$. Similarly, we have $\underline{h}_f\ge\underline{H}(\psi;\Omega^*)$. Therefore, $f$ is $(\mathscr{A},\mathscr{B})$-harmonizable if $\psi$ is $(\mathscr{A},\mathscr{B})$-resolutive.

The following resolutivity result, which is the main theorem in [MaO], can be also shown by using Theorem 2.2 and Proposition 2.2.

THEOREM 2.1. [MaO; Theorem 3.2] *Suppose that $\Omega$ is $(p,\mu)$-hyperbolic and suppose that conditions* $(C_1)$ *and* (B.5) *are satisfied. If $Q \subset \mathscr{D}^p(\Omega;\mu)$, then the $Q$-compactification $\Omega_Q^*$ of $\Omega$* (see [CC]) *is an $(\mathscr{A},\mathscr{B})$-resolutive compactification.*

## §3. Properties of harmonic boundary

Let $\Omega^*$ be a compactification of $\Omega$ and $\partial^*\Omega = \Omega^*\setminus\Omega$. Setting

$$\varDelta^{(p,\mu)} := \left\{\xi \in \partial^*\Omega : \liminf_{x\to\xi} |f(x)| = 0 \text{ for any } f \in \mathscr{D}_0^p(\Omega;\mu)\right\},$$

we call $\varDelta^{(p,\mu)}$ the *$(p,\mu)$-harmonic boundary* of $\Omega^*$. It is a compact subset of $\partial^*\Omega$.

PROPOSITION 3.1. *$\varDelta^{(p,\mu)} \neq \emptyset$ if and only if $\Omega$ is $(p,\mu)$-hyperbolic.*

PROOF. If $\Omega$ is not $(p,\mu)$-hyperbolic, $1 \in \mathscr{D}_0^p(\Omega;\mu)$, and hence $\varDelta^{(p,\mu)} = \emptyset$. To show the converse, we suppose $\varDelta^{(p,\mu)} = \emptyset$. It follows from the definition of $\varDelta^{(p,\mu)}$ that, for each $\xi \in \partial^*\Omega$, there is $f_\xi \in \mathscr{D}_0^p(\Omega;\mu)$ such that $\liminf_{x\to\xi}|f_\xi(x)| > 0$. Since $\mathscr{D}_0^p(\Omega;\mu)$ is closed under max-operation, we may assume that $f_\xi \geq 0$. Thus since $\partial^*\Omega$ is compact, we can choose $f_{\xi_1},\dots,f_{\xi_k}$ such that

$$\liminf_{x\to\xi}\{f_{\xi_1}(x) + \cdots + f_{\xi_k}(x)\} \geq \alpha > 0$$

for any $\xi \in \partial^*\Omega$. Then we can find $g \in C_0^\infty(\Omega)$ such that

$$f_0 := (2/\alpha)(f_{\xi_1} + \cdots + f_{\xi_k}) + g \geq 1$$

on $\Omega$. The linearity of $\mathscr{D}_0^p(\Omega;\mu)$ implies $f_0 \in \mathscr{D}_0^p(\Omega;\mu)$. Hence there exist $g_n \in C_0^\infty(\Omega)$ such that $\{g_n\}$ is uniformly bounded and $g_n \to f_0$ a.e., $\nabla g_n \to \nabla f_0$ in $L^p(\Omega;\mu)$ as $n\to\infty$. Set $\varphi_n := \min\{g_n, 1\}$. Thus $\{\varphi_n\}$ is uniformly bounded and $\varphi_n \to 1$ a.e., $\nabla\varphi_n \to 0$ in $L^p(\Omega;\mu)$ as $n\to\infty$. Since $\varphi_n \in H_0^{1,p}(\Omega;\mu)$, it follows that $1 \in \mathscr{D}_0^p(\Omega;\mu)$, so that $\Omega$ is not $(p,\mu)$-hyperbolic.

Let $Q$ be a family of bounded continuous functions on $\Omega$. Denote by $\varDelta_Q^{(p,\mu)}$ the $(p,\mu)$-harmonic boundary of $\Omega_Q^*$. Also denote by $[Q]$ the smallest linear space containing $Q \cup C_0^\infty(\Omega)$ and constant functions, and closed under max- and min-operations and the uniform convergence. Note that $\Omega_{[Q]}^* = \Omega_Q^*$ and $[Q] = C(\Omega_Q^*)|_\Omega$. If $\mathscr{D}_0^p(\Omega;\mu) \subset [Q]$, then we can write the definition of $\varDelta_Q^{(p,\mu)}$ as

$$\varDelta_Q^{(p,\mu)} = \left\{\xi \in \partial_Q^*\Omega : \lim_{x\to\xi} f(x) = 0 \text{ for any } f \in \mathscr{D}_0^p(\Omega;\mu)\right\},$$

where $\partial_Q^*\Omega = \Omega_Q^*\setminus\Omega$.

THEOREM 3.1. (Comparison principle with respect to $(p,\mu)$-harmonic boundary) *Suppose that $\Omega$ is $(p,\mu)$-hyperbolic and suppose that conditions* $(C_1)$ *and* (B.5) *are satisfied. Let $Q \subset \mathscr{D}^p(\Omega;\mu)$. If $u$ is an $(\mathscr{A},\mathscr{B})$-superharmonic function in $\Omega$ which is bounded below and $v$ is an $(\mathscr{A},\mathscr{B})$-subharmonic function in $\Omega$ which is bounded above, and*

$$\limsup_{x\to\xi} v(x) \le \liminf_{x\to\xi} u(x)$$

*for any $\xi \in \Delta_Q^{(p,\mu)}$, then $v \le u$ in $\Omega$.*

PROOF. Set $\tilde{u}(\xi) = \liminf_{x\to\xi} u(x)$ and $\tilde{v}(\xi) = \limsup_{x\to\xi} v(x)$ for $\xi \in \partial_Q^*\Omega$. Then $\tilde{u}$ (resp. $\tilde{v}$) is lower (resp. upper) semicontinuous and bounded below (resp. above) on $\partial_Q^*\Omega$ and $\tilde{u} \ge \tilde{v}$ on $\Delta_Q^{(p,\mu)}$. Let $\varepsilon > 0$. Then we can choose $\varphi \in C(\partial_Q^*\Omega)$ such that $\tilde{v} - \varepsilon \le \varphi \le \tilde{u} + \varepsilon$ on $\Delta_Q^{(p,\mu)}$. Since $\mathscr{D}^p(\Omega;\mu)$ is a linear space containing constant functions and closed under max- and min-operations, $[Q] \cap \mathscr{D}^p(\Omega;\mu)$ is dense in $[Q]$ with respect to the uniform convergence by the Stone-Weierstrass theorem. Hence, there is $f \in C(\Omega_Q^*) \cap \mathscr{D}^p(\Omega;\mu)$ such that

$$\tilde{v} - 2\varepsilon < f < \tilde{u} + 2\varepsilon$$

on $\Delta_Q^{(p,\mu)}$. Put

$$\Lambda := \{\xi \in \partial_Q^*\Omega : \tilde{u}(\xi) + 2\varepsilon \le f(\xi) \text{ or } \tilde{v}(\xi) - 2\varepsilon \ge f(\xi)\}.$$

Since $\Lambda$ is a compact subset of $\partial_Q^*\Omega \backslash \Delta_Q^{(p,\mu)}$, as in the proof of Proposition 3.1 we can find $g \in \mathscr{D}_0^p(\Omega;\mu)$ such that $g \ge 0$ on $\Omega$ and $\liminf_{x\to\xi} g(x) \ge \delta > 0$ for any $\xi \in \Lambda$. Since $f$ is bounded and $\tilde{u}$ is bounded below, there exists $c_1 > 0$ such that $\tilde{u}(\xi) + c_1 \liminf_{x\to\xi} g(x) \ge f(\xi)$ for any $\xi \in \Lambda$. Thus

$$\liminf_{x\to\xi}\{u(x) + c_1 g(x)\} + 3\varepsilon \ge f(\xi) + \varepsilon$$

for any $\xi \in \partial_Q^*\Omega$. Hence $u + 3\varepsilon \in \mathscr{U}_{f-c_1g}^*$, so that $h_{f-c_1g} \le u + 3\varepsilon$. Also $h_{f-c_1g} - (f - c_1g) \in \mathscr{D}_0^p(\Omega;\mu)$ by Theorem 2.2. Similarly there exists $c_2 > 0$ such that $v - 3\varepsilon \le h_{f+c_2g}$ and $h_{f+c_2g} - (f + c_2g) \in \mathscr{D}_0^p(\Omega;\mu)$. The linearity of $\mathscr{D}_0^p(\Omega;\mu)$ and $g \in \mathscr{D}_0^p(\Omega;\mu)$ yield $h_{f-c_1g} - h_{f+c_2g} \in \mathscr{D}_0^p(\Omega;\mu)$. By $(C_1)$, we see that $h_{f-c_1g}$, $h_{f+c_2g}$ are bounded. Hence, by (B.5), we can apply Proposition 1.5 and obtain $h_{f-c_1g} = h_{f+c_2g}$. Thus we have

$$v - 3\varepsilon \le h_{f+c_2g} = h_{f-c_1g} \le u + 3\varepsilon$$

in $\Omega$. Since $\varepsilon > 0$ is arbitrary, $v \le u$ in $\Omega$.

In case $\Omega^*$ is an $(\mathscr{A},\mathscr{B})$-resolutive compactification, a point $\xi \in \partial^*\Omega$ is said to be *$(\mathscr{A},\mathscr{B})$-regular* if

$$\lim_{x\to\xi} H(\psi;\Omega^*) = \psi(\xi)$$

for any $\psi \in C(\partial^*\Omega)$.

THEOREM 3.2. *Suppose that $\Omega$ is $(p,\mu)$-hyperbolic and suppose that conditions* ($C_1$) *and* (B.5) *are satisfied. Let $Q \subset \mathscr{D}^p(\Omega;\mu)$. Then any $(\mathscr{A},\mathscr{B})$-regular point $\xi \in \partial_Q^*\Omega$ belongs to $\Delta_Q^{(p,\mu)}$.*

PROOF. Let $\xi \notin \Delta_Q^{(p,\mu)}$. Then there is $g \in \mathscr{D}_0^p(\Omega;\mu)$ such that $g \geq 0$ on $\Omega$ and $\liminf_{x\to\xi} g(x) > 0$. We can find $f \in C(\Omega_Q^*)$ such that $0 \leq f \leq g$ on $\Omega$ and $f(\xi) > 0$. Let $\psi = f|_{\partial_Q^*\Omega}$. By Proposition 2.2, the comparison principle (Theorem 1.1) and Proposition 2.1,

$$h_0 = H(0;\Omega_Q^*) \leq H(\psi;\Omega_Q^*) = h_f \leq h_g,$$

where the subscript 0 in $h_0$ signifies the constant function 0 in $\Omega$. Since $g \in \mathscr{D}_0^p(\Omega;\mu)$, $h_g - h_0 \in \mathscr{D}_0^p(\Omega;\mu)$ by Theorem 2.2. Thus by Proposition 1.5, $h_g = h_0$. Hence the above inequalities imply $H(0;\Omega_Q^*) = H(\psi;\Omega_Q^*)$. Thus, if $\xi$ is $(\mathscr{A},\mathscr{B})$-regular, then

$$0 < \psi(\xi) = \lim_{x\to\xi} H(\psi;\Omega_Q^*)(x) = \lim_{x\to\xi} H(0;\Omega_Q^*)(x) = 0,$$

which is impossible. Thus we obtain the conclusion of the theorem.

The converse of the above theorem is valid under an additional condition.

THEOREM 3.3. *Suppose that $\Omega$ is $(p,\mu)$-hyperbolic and suppose that conditions* ($C_1$) *and* (B.5) *are satisfied. If $Q \subset \mathscr{D}^p(\Omega;\mu)$ satisfies $\mathscr{D}_0^p(\Omega;\mu) \subset [Q]$, then any $\xi \in \Delta_Q^{(p,\mu)}$ is an $(\mathscr{A},\mathscr{B})$-regular point.*

PROOF. Let $\xi \in \Delta_Q^{(p,\mu)}$ and $\psi \in C(\partial_Q^*\Omega)$. By the Stone-Weierstrass theorem, for any $\varepsilon > 0$ there is $f \in C(\Omega_Q^*) \cap \mathscr{D}^p(\Omega;\mu)$ such that $f - \varepsilon \leq \psi \leq f + \varepsilon$ on $\partial_Q^*\Omega$ (cf. the proof of Theorem 3.1). By Lemma 1.2 there exists an $(\mathscr{A},\mathscr{B})$-superharmonic function $u$ in $\Omega$ such that $u \geq f$ in $\Omega$ and $u - f \in \mathscr{D}_0^p(\Omega;\mu)$. Then $u + \varepsilon \in \mathscr{U}_\psi$, so that $H(\psi;\Omega_Q^*) \leq u + \varepsilon$ in $\Omega$. Since $\xi \in \Delta_Q^{(p,\mu)}$ and $\mathscr{D}_0^p(\Omega;\mu) \subset [Q]$, we have $\lim_{x\to\xi} u(x) = f(\xi)$. Thus we obtain

$$\limsup_{x\to\xi} H(\psi;\Omega_Q^*)(x) \leq f(\xi) + \varepsilon \leq \psi(\xi) + 2\varepsilon. \tag{3.1}$$

Similarily we have

$$\psi(\xi) - 2\varepsilon \leq \liminf_{x\to\xi} H(\psi;\Omega_Q^*)(x). \tag{3.2}$$

Since $\varepsilon$ is arbitrary, (3.1) and (3.2) yield the regularity of $\xi \in \partial_Q^*\Omega$.

REMARK 3.1. Condition $\mathscr{D}_0^p(\Omega;\mu) \subset [Q]$ cannot be suppressed in Theorem 3.3 even in the linear case. For example, let $A = \{x \in \mathbf{R}^N : 1 < |x| < 2\}$, $x_0 \in A$ and $\Omega = A\backslash\{x_0\}$. Let $f \in C^\infty(\Omega)$ be equal to 1 near $\{|x| = 1\} \cup \{x_0\}$ and equal to 0 near $\{|x| = 2\}$. For $Q = \{f\}$, $\Omega_Q^* = \Omega \cup \{\xi_1, \xi_2\}$, where $\xi_1$

corresponds to $\{|x|=1\} \cup \{x_0\}$ and $\xi_2$ corresponds to $\{|x|=2\}$. Then $\Delta_Q^{(2,dx)} = \{\xi_1, \xi_2\}$, while $\xi_1$ is not regular with respect to the Laplacian, i.e., not $(\mathscr{A}, \mathscr{B})$-regular for $\mathscr{A}(x,\eta) = \eta$ and $\mathscr{B}(x,t) = 0$.

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