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Absolutely continuous invariant measures for expansive diffeomorphisms of the 2-torus

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Abstract. The aim of this paper is to establish an equivalent criterion for certain expansive diffeomorphisms of the 2-torus to admit an invariant Borel probability measure that is absolutely continuous with respect to the Riemannian volume. Our result is closely related to the well known Livšic-Sinai theorem for Anosov diffeomorphisms.

1. Introduction

Let $g : M \to M$ be a transitive C^2 Anosov diffeomorphism of a compact Riemannian manifold M. A celebrated work of Livšic and Sinai [14] says that g admits an invariant Borel probability measure that is absolutely continuous with respect to the Riemannian volume on M if and only if $|\mathrm{Jac}(D_p g^n)| = 1$ holds for every periodic point $p \in \mathrm{Fix}(g^n)$ and $n \in \mathbf{N}$, where Jac stands for the Jacobian and $\mathrm{Fix}(g^n) = \{x \in M : g^n(x) = x\}$. We refer the reader to [2] for more precise definitions. Our aim here is to further the study of relations of this type for certain expansive diffeomorphisms.

Let $f : M \to M$ be a $C^{1+\alpha}$ $(\alpha > 0)$ diffeomorphism of a compact Riemannian manifold M preserving a hyperbolic Borel probability measure μ. In Corollary 5.6 of [10] Ledrappier proved that the following (A) and (B) are equivalent.

- *Property* (A) The measure μ is absolutely continuous with respect to the volume on M.
- *Property* (B) The measure μ is absolutely continuous with respect to both the stable and unstable laminations (see the definition in the next section).

It follows from the Pesin entropy formula ([17]) that (B) is equivalent to the following:

- *Property* (C) the measure μ is absolutely continuous with respect to the unstable lamination and

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$$\int \log|\mathrm{Jac}(D_x f)|\, d\mu(x) = 0 \tag{1}$$

(see Lemma 5.2 below). Moreover we can derive (C) from the following:

- *Property* (D) the measure μ is absolutely continuous with respect to the unstable lamination and $|\mathrm{Jac}(D_p f^n)| = 1$ holds for $p \in \mathrm{Fix}(f^n)$ and $n \in \mathbf{N}$ (see Lemma 5.3 below).

In this context, the Livšic-Sinai theorem for transitive C^2 Anosov diffeomorphisms could be reformulated as the properties (A) and (D) are equivalent. It then asserts that all the properties above are equivalent, particularly that (C) implies (D). This implication seems to be little known in the broader context beyond Anosov. In this paper, we would turn to this problem.

To state the result we recall the following notion. Let $x \in M$ and $\delta > 0$. Define the *local stable* and *local unstable sets* at x by

$$\mathscr{W}_\delta^s(x) = \{y \in M : d(f^n(x), f^n(y)) \le \delta \ \ (n \ge 0)\},$$
$$\mathscr{W}_\delta^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \le \delta \ \ (n \ge 0)\},$$

where d is the distance on M induced by the Riemannian metric.

THEOREM 1.1. *Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be an expansive C^2 diffeomorphism of the 2-torus preserving a hyperbolic Borel probability measure μ. Assume that for all $x \in \mathbf{T}^2$ the local stable and unstable sets at x form C^1 curves and they intersect transversally at x in the sense that $T_x\mathbf{T}^2 = T_x\mathscr{W}_\delta^s(x) \oplus T_x\mathscr{W}_\delta^u(x)$. Then the following two assertions are equivalent:*

(1) *μ is absolutely continuous with respect to the Riemannian volume on $\mathbf{T}^2$.*

(2) *μ is absolutely continuous with respect to the unstable lamination and $|\mathrm{Jac}(D_p f^n)| = 1$ for $p \in \mathrm{Fix}(f^n)$ and $n \in \mathbf{N}$.*

As an immediate corollary of this theorem we have the following.

COROLLARY 1.2. *Under the same assumption as in Theorem 1.1, all the properties (A), (B), (C) and (D) are equivalent for an expansive C^2 diffeomorphism f on the 2-torus preserving a hyperbolic Borel probability measure μ.*

Background material is given in §2. Sections 3, 4 and 5 are devoted to our proof of Theorem 1.1. The implication that (2) follows from (1) is shown in section 4 (Proposition 4.10). Since the proof of Proposition 4.10 makes use of the bounded distortion property of surface diffeomorphisms (§4, Case 2, Lemma 4.8), it seems to be necessary to use an alternative method in order to extend our theorem to higher dimensional dynamical systems.

The reverse implication is given in section 5 with no assumption on local manifolds as in the theorem. More precisely we establish the implication for *every* diffeomorphism on a Riemannian manifold preserving a hyperbolic probability measure (Proposition 5.1).

We emphasize the assumption in Theorem 1.1 is not sufficient to guarantee the existence of hyperbolic absolutely continuous invariant probability measures. Indeed, after the construction of a diffeomorphism of a compact surface with nonzero Lyapunov exponents which is not Anosov due to Katok [8], a diffeomorphism of $\mathbf{T}^2$ admitting no hyperbolic absolutely continuous invariant probability measures is given in §6.

2. Definitions

2.1. Let M be a compact C^∞ manifold with a Riemannian norm $\|\cdot\|$, $f : M \to M$ a $C^{1+\alpha}$ $(\alpha > 0)$ diffeomorphism of M and $Df : TM \to TM$ the derivative of f. Let also μ be a Borel probability measure invariant under f. A point $x \in M$ is said to be *Lyapunov regular* if there exist real numbers $\chi_1(x) > \chi_2(x) > \cdots > \chi_{r(x)}(x)$ and a $D_x f$-invariant decomposition $T_x M = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{r(x)}(x)$ such that for each $i = 1, 2, \dots, r(x)$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|D_x f^n(v)\| = \chi_i(x) \qquad (v \in E_i(x)\backslash\{0\})$$

exists, and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log|\mathrm{Jac}(D_x f^n)| = \sum_{i=1}^{r(x)} \chi_i(x) \dim E_i(x).$$

We denote by Γ the set of Lyapunov regular points. By the multiplicative ergodic theorem ([15]) Γ is a full μ-measure subset. The numbers $\chi_i(x)$ are called the *Lyapunov exponents* of f at the point x. The functions $x \mapsto \chi_i(x)$, $r(x)$ and $\dim E_i(x)$ are Borel measurable and f-invariant. A measure μ is said to be *hyperbolic* if none of the Lyapunov exponents of f for μ vanish and there exist Lyapunov exponents with different signs for μ-almost every $x \in M$.

Let $x \in \Gamma$. We define the *stable* and *unstable manifolds* at x as

$$\mathscr{W}^s(x) = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^n(x), f^n(y)) < 0 \right\},$$

$$\mathscr{W}^u(x) = \left\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0 \right\}.$$

Then $\mathscr{W}^s(x)$ and $\mathscr{W}^u(x)$ are injectively immersed manifolds satisfying

$$T_x\mathscr{W}^s(x) = E^s(x), \qquad T_x\mathscr{W}^u(x) = E^u(x),$$

where $E^s(x) = \bigoplus_{i:\chi_i(x)<0} E_i(x)$ and $E^u(x) = \bigoplus_{i:\chi_i(x)>0} E_i(x)$ ([1]). Both $\mathscr{W}^s(x)$ and $\mathscr{W}^u(x)$ inherit a Riemannian structure from M and hence a Riemannian volume and a distance. We write the volume and the distance on $\mathscr{W}^\tau(x)$ as m_x^τ and d_x^τ, respectively $(\tau = s, u)$.

2.2. We call

$$e_f(\mu) = -\int \log|\mathrm{Jac}(D_x f)| d\mu(x)$$

the *entropy production* for μ (in the sense of Ruelle [20]). It is easy to see that the entropy production is independent of the choice of Riemannian metrics and the multiplicative ergodic theorem asserts

$$e_f(\mu) = -\int \sum_{i=1}^{r(x)} \chi_i(x) \dim E_i(x) d\mu(x).$$

We refer the reader to [20, 21] for more precise definitions and results. Note that the equation (1) says the entropy production for μ vanishes.

2.3. Let $\mathscr{B}$ be the Borel σ-algebra of M completed with respect to μ and ξ a partition of M. We say a subset $A \subset M$ *ξ-set* if it is the union of elements of ξ. A countable system $\{A_i\}_{i\in\mathbf{N}} \subset \mathscr{B}$ of measurable ξ-sets is said to be a *basis* of ξ if for any two distinct elements C_1, C_2 of ξ, there exists A_{i_0} such that, up to sets of measure zero, either $C_1 \subset A_{i_0}$ and $C_2 \not\subset A_{i_0}$ or $C_1 \not\subset A_{i_0}$ and $C_2 \subset A_{i_0}$. A partition with a basis is said to be *measurable*. Denote by $\mathscr{B}_\xi$ the sub σ-algebra of $\mathscr{B}$ whose elements are ξ-sets. We denote by $C_\xi(x)$ the element of ξ containing $x \in M$. We write $\eta \le \xi$ if η is, up to sets of measure zero, a sub-partition of ξ.

For a measurable partition ξ of M, there exists a *canonical system of conditional measures*: for μ-almost every $x \in M$ there is a probability measure μ_x^ξ defined on $C_\xi(x)$ such that the function $x \mapsto \mu_x^\xi(A)$ is $\mathscr{B}_\xi$-measurable and $\mu(A) = \int \mu_x^\xi(A) d\mu(x)$ for every $A \in \mathscr{B}$. See [19] for more details.

Let $\mathscr{W}^u = \{\mathscr{W}^u(x) : x \in \Gamma\}$ be the *unstable lamination* and ξ^u a measurable partition of M. We say that ξ^u is *subordinate* to the $\mathscr{W}^u$-lamination if for μ-almost every $x \in M$, $C_{\xi^u}(x) \subset \mathscr{W}^u(x)$ and $C_{\xi^u}(x)$ contains an open neighborhood of x in $\mathscr{W}^u(x)$. The measure μ is said to be *absolutely continuous with respect to the $\mathscr{W}^u$-lamination* if for every measurable partition ξ^u subordinate to the $\mathscr{W}^u$-lamination, $\mu_x^{\xi^u}$ is absolutely continuous with respect to m_x^u for μ-almost every $x \in M$. The measurable partition subordinate to the $\mathscr{W}^s$-lamination and the absolute continuity with respect to the $\mathscr{W}^s$-lamination are defined similarly.

3. Preliminaries

3.1. *The Pesin invariant manifolds.* Let $f : M \to M$ be a C^2 diffeomorphism of a compact Riemannian manifold M preserving a hyperbolic Borel probability measure μ. Recall that Γ denotes the set of Lyapunov regular points. There exist an increasing family $\{\Lambda_l\}_{l \in \mathbf{N}}$ of closed subsets of M, a family $\{\mathscr{W}^\tau_{loc}(x)\}$ $(\tau = s, u)$ of C^2 disks passing through $x \in \Lambda_l$ and positive numbers r_l, δ_l, A_l and B_l such that

(i) $\Gamma \subset \bigcup_{l \geq 1} \Lambda_l$ and $f^n(\Lambda_l) \subset \Lambda_q$ for some positive integer $q = q(l, n)$;

(ii) $\mathscr{W}^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(\mathscr{W}^s_{loc}(f^n(x)))$ and
$\mathscr{W}^u(x) = \bigcup_{n=0}^{\infty} f^{n}(\mathscr{W}^u_{loc}(f^{-n}(x)))$ for $x \in \Gamma$;

(iii) for each $x \in \Lambda_l$ the disk $\mathscr{W}^\tau_{loc}(x)$ contains the closed ball centered at x of radius δ_l with respect to the induced distance d^τ_x on $\mathscr{W}^\tau(x)$;

(iv) for each $x \in \Lambda_l$ there is $c_l \in (0, 1)$ such that for all $y \in \Lambda_l \cap B(x, rc_l)$ and $r \in (0, r_l]$, $\mathscr{W}^\tau_{loc}(y) \cap B(x, r)$ is connected, and the map

$$\Lambda_l \cap B(x, r_l c_l) \ni y \mapsto \mathscr{W}^\tau_{loc}(y) \cap B(x, r_l)$$

is continuous with respect to the Hausdorff metric on the space of all subsets of $B(x, r_l)$;

(v) if $y \in \mathscr{W}^s_{loc}(x)$ and $x \in \Lambda_l$, then for every $n \geq 0$

$$d^s_{f^n(x)}(f^n(y), f^n(x)) \leq A_l e^{-nB_l} d^s_x(y, x),$$

and if $y \in \mathscr{W}^u_{loc}(x)$, then for every $n \geq 0$

$$d^u_{f^{-n}(x)}(f^{-n}(y), f^{-n}(x)) \leq A_l e^{-nB_l} d^u_x(y, x)$$

(see [11]). We see, in particular, $\mu(\bigcup_{l \geq 1} \Lambda_l) = 1$ by (i).

Fix $l > 1$ so large that $\mu(\Lambda_l) > 1/2$ and a measurable partition η^τ subordinate to the $\mathscr{W}^\tau$-lamination, $\tau = s, u$. We may take the partitions η^s and η^u so that

(a) $\eta^s \leq f\eta^s$ and $\eta^u \leq f^{-1}\eta^u$;

(b) $\bigcup_{i=0}^{\infty} f^{-i}(C_{\eta^s}(f^i(x))) = \mathscr{W}^s(x)$ and
$\bigcup_{i=0}^{\infty} f^{i}(C_{\eta^u}(f^{-i}(x))) = \mathscr{W}^u(x)$ for μ-almost every $x \in M$;

(c) both $\bigvee_{i=0}^{\infty} f^i\eta^s$ and $\bigvee_{i=0}^{\infty} f^{-i}\eta^u$ are partitions into points.

See [11] for complete description. Assertion (c) yields

$$\operatorname{diam} C_{f^i\eta^s}(y) \to 0 \qquad \text{and} \qquad \operatorname{diam} C_{f^{-i}\eta^u}(y) \to 0,$$

as $i \to \infty$, for μ-almost every $y \in M$. Here and below we write $\operatorname{diam} A = \sup\{d(a, b) : a, b \in A\}$. Given $r \in (0, \min\{\delta_l, r_l\}/100)$, we thus let

$$\Lambda^s_{l,r,i} = \{y \in \Lambda_l : \operatorname{diam} C_{f^i\eta^s}(y) < r\};$$

$$\Lambda^u_{l,r,i} = \{y \in \Lambda_l : \operatorname{diam} C_{f^{-i}\eta^u}(y) < r\}.$$

Note that $\Lambda^{\tau}_{l,r,i} \in \mathscr{B}$ (see [22] for example). We see

$$\Lambda^s_{l,r,i} \subset \Lambda^s_{l,r,i+1}, \qquad \Lambda^u_{l,r,i} \subset \Lambda^u_{l,r,i+1}$$

and

$$\mu(\Lambda_l) = \mu\left(\bigcup_{i\geq 0} \Lambda^s_{l,r,i}\right), \qquad \mu(\Lambda_l) = \mu\left(\bigcup_{i\geq 0} \Lambda^u_{l,r,i}\right).$$

Therefore

$$\mu(\Lambda^s_{l,r,i} \cap \Lambda^u_{l,r,i}) \geq \frac{1}{2}\mu(\Lambda_l)$$

holds for some integer $i > 1$ large enough. Fix such an integer i, and below we may write

$$\xi^s = f^i \eta^s, \qquad \xi^u = f^{-i}\eta^u,$$

and

$$\Lambda_{l,r,i} = \Lambda^s_{l,r,i} \cap \Lambda^u_{l,r,i}.$$

That ξ^τ is still a measurable partition subordinate to the $\mathscr{W}^\tau$-lamination, $\tau = s, u$.

For $j \in \mathbf{N}$ we consider

$$\Lambda^s_{l,r,i,j} = \{y \in \Lambda_{l,r,i} : d(\partial^s C_{\xi^s}(y), y) \geq r/j\};$$

$$\Lambda^u_{l,r,i,j} = \{y \in \Lambda_{l,r,i} : d(\partial^u C_{\xi^u}(y), y) \geq r/j\},$$

where $\partial^\tau A$ denotes the boundary of A in $W^\tau(y)$, $\tau = s, u$. Again we see

$$\Lambda^s_{l,r,i,j} \subset \Lambda^s_{l,r,i,j+1}, \qquad \Lambda^u_{l,r,i,j} \subset \Lambda^u_{l,r,i,j+1}$$

and

$$\mu(\Lambda_{l,r,i}) = \mu\left(\bigcup_{j\geq 1} \Lambda^s_{l,r,i,j}\right), \qquad \mu(\Lambda_{l,r,i}) = \mu\left(\bigcup_{j\geq 1} \Lambda^u_{l,r,i,j}\right).$$

Hence

$$\mu(\Lambda^s_{l,r,i,j} \cap \Lambda^u_{l,r,i,j}) \geq \frac{1}{2}\mu(\Lambda_{l,r,i})$$

holds for some integer $j > 1$ large enough. Fix such an integer j, and below we may write

$$\Lambda_{l,r} = \Lambda_{l,r,i,j} = \Lambda^s_{l,r,i,j} \cap \Lambda^u_{l,r,i,j}$$

for notational simplicity. Let $x_0 \in \Lambda_{l,r}$ be a density point of μ and put

$$\Lambda_{l,r}(x_0, q) = \Lambda_{l,r} \cap B(x_0, q),$$

for $q \in (0, r/20j]$. Define

$$Q_l^s(x_0) = \bigcup_{y \in \Lambda_{l,r}(x_0,q)} L^s(y);$$

$$Q_l^u(x_0) = \bigcup_{y \in \Lambda_{l,r}(x_0,q)} L^u(y),$$

where $L^s(y) = \mathscr{W}_{loc}^s(y) \cap B(x_0, r/2j)$ and $L^u(y) = \mathscr{W}_{loc}^u(y) \cap B(x_0, r/2j)$, respectively. Clearly we have $\mu(Q_l^\tau(x_0)) > 0$ since $Q_l^\tau(x_0)$ contains $\Lambda_{l,r}(x_0,q)$, $\tau = s,u$. Notice that $L^\tau(y)$ is connected and $L^\tau(y) \subset C_{\xi^\tau}(y)$ for $y \in \Lambda_{l,r}(x_0,q)$, $\tau = s,u$.

Recall that μ is absolutely continuous with respect to the Riemannian volume on M if and only if μ is absolutely continuous with respect to both the $\mathscr{W}^s$-lamination and the $\mathscr{W}^u$-lamination ([10]). Thus we let the density functions along these laminations be defined as follows:

$$h^s(x) = \frac{d\mu_y^{\xi^s}}{dm_y^s}(x), \qquad x \in C_{\xi^s}(y);$$

$$h^u(x) = \frac{d\mu_y^{\xi^u}}{dm_y^u}(x), \qquad x \in C_{\xi^u}(y),$$

for μ-almost every $y \in M$. Given a constant $C > 1$, we define

$$\Lambda_{l,r,C}^s(x_0,q) = \{y \in \Lambda_{l,r}(x_0,q) : C^{-1} \le h^s(x) \le C \ \ (x \in L^s(y))\};$$

$$\Lambda_{l,r,C}^u(x_0,q) = \{y \in \Lambda_{l,r}(x_0,q) : C^{-1} \le h^u(x) \le C \ \ (x \in L^u(y))\}.$$

Then it is proved in Corollary 6.1.4 of [12] that these densities h^s and h^u are indeed of class C^1 and strictly positive along $\mathscr{W}^s(y)$ and $\mathscr{W}^u(y)$, respectively, for μ-almost every $y \in M$. Thus we might choose $C > 1$ so large that

$$\mu(\Lambda_{l,r,C}^s(x_0,q) \cap \Lambda_{l,r,C}^u(x_0,q)) \ge \frac{1}{2}\mu(\Lambda_{l,r}(x_0,q)).$$

Put

$$\Lambda_{l,r,C}(x_0,q) = \Lambda_{l,r,C}^s(x_0,q) \cap \Lambda_{l,r,C}^u(x_0,q).$$

Without loss of generality we may assume $x_0 \in \Lambda_{l,r,C}(x_0,q)$. Set

$$Q^s = \bigcup_{y \in \Lambda_{l,r,C}(x_0,q)} L^s(y) \qquad \text{and} \qquad Q^u = \bigcup_{y \in \Lambda_{l,r,C}(x_0,q)} L^u(y),$$

respectively. That $\mu(Q^\tau) > 0$ holds since Q^τ contains $\Lambda_{l,r,C}(x_0,q)$, $\tau = s,u$. It follows that Q^τ has positive volume since μ is assumed to be absolutely continuous with respect to the volume ($\tau = s, u$). In what follows we denote by $L^\tau(y)$ the component passing through $y \in Q^\tau$, $\tau = s, u$, whether y belongs to $\Lambda_{l,r,C}(x_0, q)$ or not.

Consider the families of local manifolds

$$\mathscr{L}^s = \{L^s(y) : y \in Q^s\},$$
$$\mathscr{L}^u = \{L^u(y) : y \in Q^u\}.$$

Given a manifold $T \subset M$, we denote by m_T the induced Riemannian volume on T. A manifold T is said to be a *transversal* to the family $\mathscr{L}^s$ if T intersects each $L \in \mathscr{L}^s$ in a unique point and the intersection is transverse. Let T_1 and T_2 be two transversals to the family $\mathscr{L}^s$. We then define the *holonomy map* on Q^s sliding along $\mathscr{L}^s$

$$\mathfrak{p}^s : Q^s \cap T_1 \to Q^s \cap T_2$$

by setting

$$\mathfrak{p}^s(z) = L^s(w) \cap T_2$$

for $z \in L^s(w) \cap T_1$ and $w \in Q^s$. The holonomy map $\mathfrak{p}^s$ is a homeomorphism onto its image. It is called *absolutely continuous* if m_{T_2} is absolutely continuous with respect to $\mathfrak{p}^s_* m_{T_1}$. Define the *Jacobian* $J_z(\mathfrak{p}^s)$ of $\mathfrak{p}^s$ at $z \in Q^s \cap T_1$ to be the Radon-Nikodym derivative

$$J_z(\mathfrak{p}^s) = \frac{dm_{T_2}}{d(\mathfrak{p}^s_* m_{T_1})}(\mathfrak{p}^s(z)).$$

It is well known that the holonomy map $\mathfrak{p}^s$ is absolutely continuous and has a bounded Jacobian in the sense that there is a constant $J = J(l) > 1$ such that

$$J^{-1} \le J_z(\mathfrak{p}^s) \le J \tag{2}$$

for $z \in Q^s \cap T_1$ ([17]). Below we say $\mathfrak{p}^s$ has the *J-distortion property* on Q^s if it satisfies (2).

The holonomy map $\mathfrak{p}^u : Q^u \cap T_1 \to Q^u \cap T_2$ sliding along $\mathscr{L}^u$ is defined analogously, where T_1 and T_2 are transversals to the family $\mathscr{L}^u$. The holonomy map $\mathfrak{p}^u$ also possesses the absolute continuity property in the sense explained above and has the J-distortion property on Q^u.

3.2. *Local product structure.* Let us recall several known facts from topological dynamics we need later. Let $\mathbf{T}^2$ be the 2-torus. Given $x \in \mathbf{T}^2$ and $\delta > 0$, we define the *local stable* and *local unstable sets* at x by

$$\mathscr{W}^s_\delta(x) = \{y \in \mathbf{T}^2 : d(f^n(x), f^n(y)) \le \delta \ (n \ge 0)\},$$
$$\mathscr{W}^u_\delta(x) = \{y \in \mathbf{T}^2 : d(f^{-n}(x), f^{-n}(y)) \le \delta \ (n \ge 0)\}.$$

Clearly $\mathscr{W}_\delta^\tau(x)$ is a closed subset of $\mathbf{T}^2$, $\tau = s, u$. A homeomorphism $f : \mathbf{T}^2 \to \mathbf{T}^2$ is said to be *expansive* if there exists a constant $\varepsilon_0 > 0$, called an *expansivity constant*, such that $d(f^n(x), f^n(y)) < \varepsilon_0$ for all $n \in \mathbf{Z}$ implies $x = y$. It is well known that every expansive homeomorphism of $\mathbf{T}^2$ possesses the *local product structure*: for any small $\delta > 0$ there exists $\varepsilon_0 > 0$ such that the intersection $\mathscr{W}_\delta^s(z) \cap \mathscr{W}_\delta^u(w)$ consists of one point, denoted by $[z, w]$, whenever $d(z, w) \le \varepsilon_0$. See [5, 6, 13] for more precise. For $Z, W \subset \mathbf{T}^2$ we denote by $[Z, W]$ the subset $\{[z, w] : z \in Z, w \in W\}$ if it makes sense.

Given $\delta \in (0, \varepsilon_0/100)$ and $\varepsilon \in (0, \delta/100)$, the following

$$R^s(x, \varepsilon, \delta) = [\mathscr{W}_\varepsilon^u(x), \mathscr{W}_\delta^s(x)],$$

$$R^u(x, \delta, \varepsilon) = [\mathscr{W}_\delta^u(x), \mathscr{W}_\varepsilon^s(x)]$$

make sense for $x \in \mathbf{T}^2$. We call these sets the *stable* and *unstable rectangles around* x, respectively. Define also $R(x, \delta) = [\mathscr{W}_\delta^u(x), \mathscr{W}_\delta^s(x)]$.

4. Preserving the volume around periodic orbits

Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be an expansive C^2 diffeomorphism of the 2-torus $\mathbf{T}^2$ preserving a hyperbolic Borel probability measure μ and $\varepsilon_0 > 0$ an expansivity constant for f. Notice that the map f is topologically conjugate to a hyperbolic toral automorphism ([5, 6]). Throughout this section we let $\delta \in (0, \varepsilon_0/100)$ and $\varepsilon \in (0, \delta/100)$, and assume that for all $x \in \mathbf{T}^2$ both the local stable set $\mathscr{W}_\delta^s(x)$ and the local unstable set $\mathscr{W}_\delta^u(x)$ form C^1 curves and they intersect transversally at x:

$$T_x\mathbf{T}^2 = T_x\mathscr{W}_\delta^s(x) \oplus T_x\mathscr{W}_\delta^u(x). \tag{3}$$

We then show the invariant measure μ is absolutely continuous with respect to the $\mathscr{W}^u$-lamination and $|\mathrm{Jac}(D_p f^n)| = 1$ for all $p \in \mathrm{Fix}(f^n)$ and $n \in \mathbf{N}$ provided that μ is absolutely continuous with respect to the Riemannian volume on $\mathbf{T}^2$.

Fix $p \in \mathrm{Fix}(f^n)$. Without loss of generality we may assume that $n = 1$. It follows from (3) that the point p is neither attracting nor repelling and its eigenvalues at p are real numbers. We may as well assume that $D_p f$ has positive eigenvalues $\lambda^s \le 1 \le \lambda^u$.

Define the (*global*) *stable* and (*global*) *unstable sets* at p as

$$\mathscr{W}^s(p) = \{y \in \mathbf{T}^2 : d(f^n(p), f^n(y)) \to 0 \text{ as } n \to \infty\},$$

$$\mathscr{W}^u(p) = \{y \in \mathbf{T}^2 : d(f^{-n}(p), f^{-n}(y)) \to 0 \text{ as } n \to \infty\}.$$

It can be shown that for $\delta \in (0, \varepsilon_0/100)$

$$\mathscr{W}^s(p) = \bigcup_{n \geq 0} f^{-n}(\mathscr{W}^s_\delta(f^n(p))), \qquad \mathscr{W}^u(p) = \bigcup_{n \geq 0} f^n(\mathscr{W}^u_\delta(f^{-n}(p)))$$

and that $\mathscr{W}^\tau(p)$ is dense in $\mathbf{T}^2$, $\tau = s, u$. We refer the reader to [5, 6], for instance. It then follows from (3.1)(ii) and (v) that for Lyapunov regular points $x \in \Gamma$ the stable and unstable sets defined here coincide with the stable and unstable manifolds defined in the end of (2.1), respectively. Observe also that the assumption (3) implies $\mathscr{W}^u(p)$ and $\mathscr{W}^s(p)$ intersect $\mathscr{L}^s$ and $\mathscr{L}^u$ transversely, respectively.

Take a point $a \in L^s(x_0) \cap \mathscr{W}^u(p)$ so close to x_0 that $d(a, x_0) < r/100$, and set $V^u = [Q^s, a]$. Note that $V^u \subset \mathscr{W}^u(p)$. Similarly take $b \in L^u(x_0) \cap \mathscr{W}^s(p)$ so that $d(b, x_0) < r/100$, and set $V^s = [b, Q^u]$. Note that $V^s \subset \mathscr{W}^s(p)$. Below we write $a_i = f^{-i}(a)$, $b_i = f^i(b)$ and $V^u_i = f^{-i}(V^u)$, $V^s_i = f^i(V^s)$ for notational simplicity ($i \in \mathbf{N}$).

Suppose on the contrary that $|\mathrm{Jac}(D_p f)| \neq 1$. We split the proof into following two cases: whether the point p is hyperbolic, that is $\lambda^\tau \neq 1$ ($\tau = s, u$), or not.

Case 1: $p \in \mathrm{Fix}(f)$ is hyperbolic. In this case either $\lambda^s < (\lambda^u)^{-1}(<1)$ or $(\lambda^u)^{-1} < \lambda^s(<1)$ might occur. Without loss of generality we may assume $\lambda^s < (\lambda^u)^{-1}$, that is area contracting in a neighborhood of p.

Let us choose an integer $N > 1$ so large that $a_N \in \mathscr{W}^u_{\delta/2}(p)$ and diam $V^u_N < \varepsilon$. Similarly choose an integer $M > 1$ so large that $b_M \in \mathscr{W}^s_{\delta/2}(p)$ and diam $V^s_M < \varepsilon$. Consider

$$\mathscr{L}^s_N = \{L^s_N(z) : z \in V^u_N\} \qquad \text{and} \qquad \mathscr{L}^u_M = \{L^u_M(z) : z \in V^s_M\},$$

where $L^s_N(z) = [z, \mathscr{W}^s_\delta(a_N)]$ for $z \in V^u_N$ and $L^u_M(z) = [\mathscr{W}^u_\delta(b_M), z]$ for $z \in V^s_M$, respectively. Define

$$Q^s_N = \bigcup_{z \in V^u_N} L^s_N(z) \qquad \text{and} \qquad Q^u_M = \bigcup_{z \in V^s_M} L^u_M(z).$$

We see $Q^s_N \subset R^s(a_N, \varepsilon, \delta)$ and $Q^u_M \subset R^u(b_M, \delta, \varepsilon)$ by the choice of N and M, respectively.

Lemma 4.1. *$\mu(Q^s_N) > 0$ and $\mu(Q^u_M) > 0$.*

Proof. We prove only the first statement as the latter can be shown in the same way.

By the invariance of μ we have

$$\begin{aligned}\mu(Q^s_N) &= \int_{\mathbf{T}^2} \mu^{f^{-N}\xi^s}_{f^{-N}(z)}(Q^s_N)\,d\mu(z) \\ &= \int_{\mathbf{T}^2} \mu^{\xi^s}_z(f^N(Q^s_N))\,d\mu(z)\end{aligned}$$

$$= \int_{\mathbf{T}^2} \int_{f^N(Q_N^s)\cap C_{\xi^s}(z)} h^s(w)dm_z^s(w)d\mu(z)$$

$$\geq C^{-1} \int_{\mathbf{T}^2} m_z^s(f^N(Q_N^s) \cap L^s(z))d\mu(z) > 0.$$

This completes the proof. □

It follows from Lemma 4.1 that both Q_N^s and Q_M^u have positive volume since μ is assumed to be absolutely continuous with respect to the volume.

From now on we assume that the ranges of the holonomy maps $\mathfrak{p}^s$ and $\mathfrak{p}^u$ are in $\mathscr{W}^u(p)$ and $\mathscr{W}^s(p)$, respectively. More precisely let $\mathfrak{p}^s : Q^s \cap L \to V^u$ ($L \in \mathscr{L}^u$) be the holonomy map sliding along $\mathscr{L}^s$ and $\mathfrak{p}^u : Q^u \cap L \to V^s$ ($L \in \mathscr{L}^s$) the holonomy map sliding along $\mathscr{L}^u$. The holonomy maps $\mathfrak{p}_N^s$ on Q_N^s sliding along $\mathscr{L}_N^s$ and $\mathfrak{p}_M^u$ on Q_M^u sliding along $\mathscr{L}_M^u$ can be defined similarly as in (3.1).

Lemma 4.2. *The holonomy map $\mathfrak{p}_N^s : Q_N^s \cap L \to V_N^u$ possesses the J-distortion property on Q_N^s for a constant $J = J(l, N) > 1$:*

$$J^{-1} \leq \frac{m_p^u(V_N^u)}{m_L(Q_N^s \cap L)} \leq J$$

for $z \in Q_N^s \cap L$ and $L \in \mathscr{L}_M^u$. Similarly, the holonomy map $\mathfrak{p}_M^u : Q_M^u \cap L \to V_M^s$ has the J-distortion property on Q_M^u for a constant $J = J(l, M) > 1$:

$$J^{-1} \leq \frac{m_p^s(V_M^s)}{m_L(Q_M^u \cap L)} \leq J$$

for $z \in Q_M^u \cap L$ and $L \in \mathscr{L}_N^s$.

Proof. It is enough to prove only the case for the stable holonomy $\mathfrak{p}^s : Q_N^s \cap L \to V_N^u$ as just consider f^{-1} instead of f for the case for the unstable holonomy.

As we noted in (3.1) the holonomy $\mathfrak{p}^s : Q^s \cap L \to V^u$ ($L \in \mathscr{L}^u$) possesses the J-distortion property. Since the map $\mathfrak{p}^s$ is a bijection and $\mathfrak{p}_N^s = f^{-N} \circ \mathfrak{p}^s \circ f^N$ holds on $Q_N^s \cap L$ ($L \in \mathscr{L}_M^u$), the map $\mathfrak{p}_N^s$ is also a bijection and possesses the J-distortion property for a constant $J = J(l, N)$.

Lemma 4.1 asserts that the numerator and the denominator do not vanish. □

Associated to the hyperbolic point $p \in \mathrm{Fix}(f)$ there is a splitting $T_p\mathbf{T}^2 = E^s(p) \oplus E^u(p)$. We thus obtain $T_z\mathbf{T}^2 = E^s(z) \oplus E^u(z)$ by identifying $T_z\mathbf{T}^2 = \mathbf{R}^2$ for $z \in \mathbf{T}^2$ near to p. For $a \in (0, 1)$ define the *stable* and *unstable cones* at $z \in \mathbf{T}^2$ as

$$C_a^s(z) = \{w \in T_z\mathbf{T}^2 : \|w_u\| \le a\|w_s\|\},$$

$$C_a^u(z) = \{w \in T_z\mathbf{T}^2 : \|w_s\| \le a\|w_u\|\},$$

where $w = w_s + w_u$ with respect to the splitting $T_z\mathbf{T}^2 = E^s(z) \oplus E^u(z)$. Set $R = R(p,\delta) = [\mathscr{W}_\delta^u(p), \mathscr{W}_\delta^s(p)]$. A map that associates to every point $z \in R$ a cone $C_a^\tau(z)$ in $T_z\mathbf{T}^2$ is said to be a *cone field* on R $(\tau = s,u)$. Since the number δ is small, for sufficiently small $\beta > 0$ and $a \in (0,1)$ there are continuous cone fields C_a^s and C_a^u on R so that

(i) if $x \in R \cap f(R)$
 (a) $D_x f^{-1}(C_a^s(x)) \subset C_a^s(f^{-1}(x))$;
 (b) $(\lambda^s - \beta)\|v\| \le \|D_x f(v)\| \le (\lambda^s + \beta)\|v\|$ for $v \in C_a^s(x)\backslash\{0\}$;

(ii) $T_p\mathscr{W}^s(p) \subset C_a^s(p)$,

and

(iii) if $x \in R \cap f^{-1}(R)$
 (a) $D_x f(C_a^u(x)) \subset C_a^u(f(x))$;
 (b) $(\lambda^u + \beta)\|v\| \ge \|D_x f(v)\| \ge (\lambda^u - \beta)\|v\|$ for $v \in C_a^u(x)\backslash\{0\}$;

(iv) $T_p\mathscr{W}^u(p) \subset C_a^u(p)$.

Below we write the cone field C^τ instead of C_a^τ for notational simplicity $(\tau = s,u)$.

Proposition 4.3 (the inclination lemma or λ-lemma [16]). *Let p be hyperbolic and $L \subset \mathbf{T}^2$ a C^1 curve having a transversal intersection point q with $\mathscr{W}^s(p)$. Then $f^n(L)$ converges to $\mathscr{W}^u(p)$ as $n \to \infty$ in the sense that for each n there is a disk $D_n \subset f^n(L)$, a neighborhood of $f^n(q)$ in $f^n(L)$, such that $\lim_{n\to\infty} D_n = D$. Here $D \subset \mathscr{W}^u(p)$ is a disk around p, and the convergence means that for each n large enough, D_n and D are C^1 near. Similarly, for a C^1 curve $L \subset \mathbf{T}^2$ having a transversal intersection with $\mathscr{W}^u(p)$, $f^{-n}(L)$ converges to $\mathscr{W}^s(p)$ as $n \to \infty$.*

It follows from the inclination lemma that

$$T_zL \subset C^s(z) \qquad \text{for } z \in L,\ L \in \mathscr{L}_N^s; \tag{4}$$

$$T_zL \subset C^u(z) \qquad \text{for } z \in L,\ L \in \mathscr{L}_M^u. \tag{5}$$

We now construct a transversely laminated set in the unstable rectangle $R^u(b_M,\delta,\varepsilon)$ around b_M as follows. Given $k > 1$ large enough, define

$$Q_{N+k}^s = \bigcup_{z \in V_{N+k}^u} L_{N+k}^s(z),$$

where $L_{N+k}^s(z) = [z, \mathscr{W}_\delta^s(a_{N+k})]$ for $z \in V_{N+k}^u$. Define then

$$Q_k = Q_{N+k}^s \cap Q_M^u.$$

This is the desired. Notice that

$$Q_k = [V^u_{N+k}, V^s_M]$$

holds. Observe that the maps $\mathfrak{p}^s_N : f^k(Q_k \cap L^u_M(z)) \to V^u_N$, $z \in V^s_M$, and $\mathfrak{p}^u_M : Q_k \cap f^{-k}(L^s_N(z)) \to V^s_M$, $z \in V^u_N$, are bijections by construction.

Lemma 4.4. *Let γ be a subset of a leaf, say $L^u_M(z)$, of the family $\mathscr{L}^u_M$. If $f^i(\gamma) \subset R$ for $i = 0, 1, \ldots, k-1$, then*

$$m^u_{f^k(z)}(f^k(\gamma)) \le (\lambda^u + \beta)^k m^u_z(\gamma).$$

Proof. It follows from (5) and (iii-b) that

$$\begin{aligned} m^u_{f(z)}(f(\gamma)) &= \int_\gamma |\mathrm{Jac}(D_x f \mid T_x \mathscr{W}^u_\delta(z))| dm^u_z(x) \\ &= \int_\gamma \|D_x f \mid T_x \mathscr{W}^u_\delta(z)\| dm^u_z(x) \\ &\le (\lambda^u + \beta) m^u_z(\gamma). \end{aligned}$$

Successive use of this inequality proves the lemma. □

Proposition 4.5. *There is a constant $K_1 > 0$ such that $\mu(Q_k) \ge K_1(\lambda^u + \beta)^{-k}$ for large k.*

Proof. Put $h^u_M = d\mu^{f^M\xi^u}_z / dm^u_z$. By the invariance of μ we have

$$\begin{aligned} \mu(Q_k) &= \int_{\mathbf{T}^2} \mu^{f^M\xi^u}_z (Q_k) d\mu(z) \\ &= \int_{\mathbf{T}^2} \int_{Q_k \cap C_{f^M\xi^u}(z)} h^u_M(w) dm^u_z(w) d\mu(z) \\ &\ge C_1^{-1} \int_{\mathbf{T}^2} m^u_z(Q_k \cap L^u_M(z)) d\mu(z) \end{aligned}$$

for a constant $C_1 = C_1(C, M) > 1$ since $C^{-1} \le h^u | L \le C$ $(L \in Q^u)$ holds. By Lemmas 4.4 and 4.2 we have

$$\begin{aligned} \int_{\mathbf{T}^2} m^u_z(Q_k \cap L^u_M(z)) d\mu(z) &\ge (\lambda^u + \beta)^{-k} \int_{\mathbf{T}^2} m^u_{f^k(z)}(f^k(Q_k \cap L^u_M(z))) d\mu(z) \\ &\ge (\lambda^u + \beta)^{-k} J^{-1} \int_{Q^u_M} m^u_p(V^u_N) d\mu(z) \\ &= (\lambda^u + \beta)^{-k} J^{-1} m^u_p(V^u_N) \mu(Q^u_M). \end{aligned}$$

It follows from the above consideration that

$$\mu(Q_k) \geq (\lambda^u + \beta)^{-k}(C_1 J)^{-1}\, m_p^u(V_N^u)\mu(Q_M^u),$$

thereby the desired estimate follows for $K_1 = (C_1 J)^{-1}\, m_p^u(V_N^u)\mu(Q_M^u)$, which is positive by Lemmas 4.1 and 4.2. □

LEMMA 4.6. *Let γ be a subset of a leaf, say $L_N^s(z)$, of the family $\mathscr{L}_N^s$. If $f^{-i}(\gamma) \subset R$ for $i = 0, 1, \dots, k-1$, then*

$$m_{f^{-k}(z)}^s(f^{-k}(\gamma)) \geq (\lambda^s + \beta)^{-k} m_z^s(\gamma).$$

PROOF. It follows from (4) and (i-b) that

$$\begin{aligned} m_{f^{-1}(z)}^s(f^{-1}(\gamma)) &= \int_\gamma |\mathrm{Jac}(D_x f^{-1} \mid T_x \mathscr{W}_\delta^s(z))| dm_z^s(x) \\ &= \int_\gamma \|D_x f^{-1} \mid T_x \mathscr{W}_\delta^s(z)\| dm_z^s(x) \\ &\geq (\lambda^s + \beta)^{-1} m_z^s(\gamma). \end{aligned}$$

Successive use of this inequality proves the lemma. □

Notice that

$$f^k(Q_k) = Q_N^s \cap f^k(Q_M^u).$$

PROPOSITION 4.7. *There is a constant $K_2 > 0$ such that $\mu(f^k(Q_k)) \leq K_2(\lambda^s + \beta)^k$ for large k.*

PROOF. Put $h_N^s = d\mu_z^{f^{-N}\xi^s}/dm_z^s$. By the invariance of μ we have

$$\begin{aligned} \mu(f^k(Q_k)) &= \int_{\mathbf{T}^2} \mu_z^{f^{-N}\xi^s}(f^k(Q_k)) d\mu(z) \\ &= \int_{\mathbf{T}^2} \int_{f^k(Q_k) \cap C_{f^{-N}\xi^s}(z)} h_N^s(w) dm_z^s(w) d\mu(z) \\ &\leq C_2 \int_{\mathbf{T}^2} m_z^s(f^k(Q_k) \cap L_N^s(z)) d\mu(z) \end{aligned}$$

for a constant $C_2 = C_2(C, N) > 1$ since $C^{-1} \leq h^s | L \leq C$ $(L \in Q^s)$ holds. By Lemmas 4.6 and 4.2 we have

$$\begin{aligned} &\int_{\mathbf{T}^2} m_z^s(f^k(Q_k) \cap L_N^s(z)) d\mu(z) \\ &\qquad \leq (\lambda^s + \beta)^k \int_{\mathbf{T}^2} m_{f^{-k}(z)}^s(Q_k \cap f^{-k}(L_N^s(z))) d\mu(z) \end{aligned}$$

$$= (\lambda^s + \beta)^k \int_{\mathbf{T}^2} m^s_{f^{-k}(z)}(Q_k \cap L^s_{N+k}(f^{-k}(z)))d\mu(z)$$

$$= (\lambda^s + \beta)^k \int_{f^k(Q_k)} m^s_{f^{-k}(z)}(Q^u_M \cap L^s_{N+k}(f^{-k}(z)))d\mu(z)$$

$$\leq (\lambda^s + \beta)^k J \int_{Q^s_N} m^s_p(V^s_M)d\mu(z)$$

$$= (\lambda^s + \beta)^k J\, m^s_p(V^s_M)\mu(Q^s_N).$$

It follows that

$$\mu(f^k(Q_k)) \leq (\lambda^s + \beta)^k C_2 J\, m^s_p(V^s_M)\mu(Q^s_N),$$

thereby the desired estimate follows for $K_2 = C_2 J\, m^s_p(V^s_M)\mu(Q^s_N)$, which is positive by Lemmas 4.1 and 4.2. □

Combining Propositions 4.5 and 4.7 we obtain

$$K_1(\lambda^u + \beta)^{-k} \leq \mu(Q_k) \leq K_2(\lambda^s + \beta)^k$$

as μ is invariant under f, where neither K_1 nor K_2 depend on k. It follows that

$$K_1 K_2^{-1} \leq \{(\lambda^s + \beta)(\lambda^u + \beta)\}^k$$

and the second term above tends to zero as k goes infinity since $\lambda^s\lambda^u < 1$. This yields a contradiction. Our proof for Case 1 is completed.

Case 2: $p \in \mathrm{Fix}(f)$ is non-hyperbolic. In this case either $\lambda^s < 1 = \lambda^u$ or $\lambda^s = 1 < \lambda^u$ might occur. Without loss of generality we may assume $\lambda^s < 1 = \lambda^u$. The argument below is a simple adaptation to our case of an argument in the proof of Theorem A in [7] by Hu and Young and is presented for the sake of completeness. Given a positive number $\rho > 0$, we denote by $O_\rho(A)$ the ρ-neighborhood of $A \subset \mathbf{T}^2$, that is $O_\rho(A) = \{v \in \mathbf{T}^2 : d(v, A) < \rho\}$.

Let $E^s(p)$ and $E^u(p)$ be the eigenspaces of $D_p f$ corresponding to λ^s and λ^u, respectively. Consider again $R(p,\delta) = [\mathscr{W}^u_\delta(p), \mathscr{W}^s_\delta(p)]$, where $\delta \in (0, \varepsilon_0/100)$ is chosen so small that there is a continuous (stable) cone field C^s, which can be defined similarly as in Case 1, on an open neighborhood U of $R(p,\delta) \cup f(R(p,\delta))$. In the rest of this section we will write R instead of $R(p,\delta)$ for notational simplicity.

We first construct an f-invariant Lipschitz stable foliation of smooth leaves as follows. Given $\rho \in (0, \delta/100)$, we consider the ρ-neighborhood $O_\rho(f(R)\backslash R)$ of the set $f(R)\backslash R$, and denote it by V_ρ for notational simplicity, that is $V_\rho = O_\rho(f(R)\backslash R)$. Without loss of generality we may assume the positive number ρ is so small that $V_\rho \subset U$ and $V_\rho \cap f^{-2}(V_\rho) = \varnothing$.

There is a C^1 vector field $X_0 : V_\rho \to T\mathbf{T}^2$ such that $X_0(z) \in C^s(z)$ since $V_\rho \subset U$. Thus pushing forward the vector field X_0 by f^{-1} gives a C^1 vector field $(f^{-1})_* X_0$ on $f^{-1}(V_\rho)$:

$$((f^{-1})_* X_0)(z) = D_z f^{-1}(X_0(f(z))) \qquad (z \in f^{-1}(V_\rho)).$$

Given an arbitrarily small positive number $\eta \in (0, \rho/100)$ so that

$$O_\eta(V_\rho \backslash f^{-1} V_\rho) \subset U \qquad \text{and} \qquad O_\eta(V_\rho \backslash f^{-1} V_\rho) \cap R = \varnothing,$$

we consider a C^∞ function $\psi = \psi_{\rho,\eta} : V_\rho \to [0,1]$ such that

$$\psi(z) = \begin{cases} 0 & \text{if } z \in V_\rho \backslash f^{-1}(V_\rho); \\ 1 & \text{if } z \in V_\rho \cap f^{-1}(V_\rho) \backslash O_\eta(V_\rho \backslash f^{-1}(V_\rho)). \end{cases}$$

We then define a vector field $X : V_\rho \to C^s$ as

$$X = \psi \cdot (f^{-1})_* X_0 + (1 - \psi) \cdot X_0.$$

The resulting vector field X is of class C^1 and Df-invariant on $V_\rho \cap f^{-1}(V_\rho) \backslash O_\eta(V_\rho \backslash f^{-1}(V_\rho))$ in the sense that $(f^{-1})_* X = X$. Indeed we have, by definition, for $z \in V_\rho \cap f^{-1}(V_\rho) \backslash O_\eta(V_\rho \backslash f^{-1}(V_\rho))$

$$X(z) = ((f^{-1})_* X_0)(z) = D_{f(z)} f^{-1} X_0(f(z)),$$

and

$$((f^{-1})_* X)(z) = D_{f(z)} f^{-1}(X(f(z))) = D_{f(z)} f^{-1}(X_0(f(z)))$$

as $V_\rho \cap f^{-2}(V_\rho) = \varnothing$.

Integrating the vector field X gives a Lipschitz (stable) foliation $\mathscr{F}_0^s$. Notice that the associated holonomy map $\mathfrak{h}_0^s$ on V_ρ sliding along $\mathscr{F}_0^s$ has a bounded distortion in the sense that for all pair of transversals T_1 and T_2 to the family $\mathscr{F}_0^s$, there is some constant $J_0 > 1$ such that

$$J_0^{-1} \leq \frac{l(T_1)}{l(T_2)} \leq J_0 \tag{6}$$

holds, where l denotes the induced leaf volume.

Pushing forward the vector field X by f^{-1} defines a C^1 vector field of which Df-invariant on (a neighborhood of) $R \backslash \mathscr{W}_\delta^s(p)$ and then, integrating the resulting vector field shall give an f-invariant Lipschitz stable foliation. This is the desired. We denote by $\mathscr{F}^s$ the resulting foliation and by $F^s(z)$ the leaf containing z.

Since the intersection $F^s(z) \cap \mathscr{W}_\delta^u(w)$ consists of one point, we now define

$$\lfloor z, w \rfloor = F^s(z) \cap \mathscr{W}_\delta^u(w),$$

whenever $z \in R \backslash \mathscr{W}_\delta^s(p)$ and $w \in R$. For $Z, W \subset R$ we denote by $\lfloor Z, W \rfloor$ the subset $\{\lfloor z, w \rfloor : z \in Z \backslash \mathscr{W}_\delta^s(p), w \in W\}$ if it makes sense.

Let $\mathfrak{h}^s$ be the holonomy map on $R \backslash \mathscr{W}_\delta^s(p)$ sliding along $\mathscr{F}^s$, that is

$$\mathfrak{h}^s : (R \backslash \mathscr{W}_\delta^s(p)) \cap L \to \mathscr{W}_\delta^u(p)$$

where L is a transversal to the family $\mathscr{F}^s$. We show the map $\mathfrak{h}^s : (R \backslash \mathscr{W}_\delta^s(p)) \cap L \to \mathscr{W}_\delta^u(p)$ sliding along the foliation $\mathscr{F}^s$ possesses the same property as in (6) for every transversal L to the family $\mathscr{F}^s$. To see this, let $I \subset \mathscr{W}_\delta^u(p) \backslash f^{-1}(\mathscr{W}_\delta^u(p))$ be an arbitrary (nontrivial) curve, $B(n) = \lfloor f^{-n}(I), \mathscr{W}_\delta^s(p) \rfloor$ and $F^s(w,n)$ a leaf of $\mathscr{F}^s$ containing $w \in B(n)$ for $n \in \mathbf{N} \cup \{0\}$. Then we observe that

(1) $f^n(B(n)) \subset V_\rho$;
(2) $\operatorname{diam} f^i(B(n)) \leq \operatorname{diam}(R \cup f(R))$ for $i = 0, 1, \ldots, n$;
(3) there is $\kappa > 1$ so that for all $w \in B(n)$,

$$\sum_{i=0}^{n} l(f^i(F^s(w,n))) \leq \kappa$$

for all $n \in \mathbf{N} \cup \{0\}$.

We therefore have the following by Lemma 3.4.1 in [18].

Lemma 4.8 (Lemma 3.4.1 in [18]). *There is a constant $J = J(J_0, \kappa) > 1$, independent of n, such that the map $\mathfrak{h}^s$ has the bounded distortion property as in* (6) *for every transversal to the family $\mathscr{F}^s$.*

Choose an integer $M > 1$ so large that V_M^s meets $\mathscr{W}_\delta^s(p) \backslash f(\mathscr{W}_\delta^s(p))$ and put $B_M^s = V_M^s \cap (\mathscr{W}_\delta^s(p) \backslash f(\mathscr{W}_\delta^s(p)))$. Define

$$Q_M^u = \bigcup_{z \in B_M^s} L_M^u(z),$$

where $L_M^u(z) = \lfloor \mathscr{W}_\delta^u(p), z \rfloor$ for $z \in B_M^s$. By Lemma 4.1 we have $\mu(Q_M^u) > 0$.

For each $n \in \mathbf{N} \cup \{0\}$ define

$$Q_n = \lfloor \mathscr{W}_\delta^u(p), f^n(B_M^s) \rfloor.$$

Notice that $Q_n \cap Q_m = \varnothing$ for $n \neq m$ and that

$$f^{-n}(Q_n) = \lfloor f^{-n}(\mathscr{W}_\delta^u(p)), B_M^s \rfloor.$$

An argument analogous to Proposition 4.5 shows the following.

Proposition 4.9. *There is a constant $K > 0$ such that $\mu(f^{-n}(Q_n)) \geq K m_p^u(f^{-n}(\mathscr{W}_\delta^u(p)))$ for all n.*

Proof. Put $h_M^u = d\mu_z^{f^M \xi^u} / dm_z^u$. By the invariance of μ we have

$$\begin{aligned}\mu(f^{-n}(Q_n)) &= \int_{\mathbf{T}^2} \mu_z^{f^M\xi^u}(f^{-n}(Q_n))d\mu(z) \\ &= \int_{\mathbf{T}^2}\int_{f^{-n}(Q_n)\cap C_{f^M\xi^u}(z)} h_M^u(w)dm_z^u(w)d\mu(z) \\ &\geq C_3^{-1}\int_{\mathbf{T}^2} m_z^u(f^{-n}(Q_n)\cap L_M^u(z))d\mu(z)\end{aligned}$$

for a constant $C_3 = C_3(C, M) > 1$ since $C^{-1} \leq h^u|L \leq C$ $(L \in Q^u)$ holds. By Lemma 4.8 we have

$$\begin{aligned}\int_{\mathbf{T}^2} m_z^u(f^{-n}(Q_n)\cap L_M^u(z))d\mu(z) &\geq J^{-1}\int_{Q_M^u} m_p^u(f^{-n}(\mathscr{W}_\delta^u(p)))d\mu(z) \\ &= J^{-1}\, m_p^u(f^{-n}(\mathscr{W}_\delta^u(p)))\,\mu(Q_M^u).\end{aligned}$$

It follows from the above consideration that

$$\mu(f^{-n}(Q_n)) \geq (C_3J)^{-1}\,\mu(Q_M^u)\,m_p^u(f^{-n}(\mathscr{W}_\delta^u(p))),$$

thereby the desired estimate follows for $K = (C_3J)^{-1}\,\mu(Q_M^u)$. □

Note that Q_n, $n \in \mathbf{N}\cup\{0\}$, are pairwise disjoint subsets in R. By using Proposition 4.9 we have

$$1 \geq \mu(R) \geq \sum_{n=0}^{\infty}\mu(Q_n) \geq \sum_{n=0}^{\infty}\mu(f^{-n}(Q_n)) \geq K\sum_{n=0}^{\infty} m_p^u(f^{-n}(\mathscr{W}_\delta^u(p))).$$

Lemma 4.1 in [7] would show that the sum $\sum_{n=0}^{\infty} m_p^u(f^{-n}(\mathscr{W}_\delta^u(p)))$ diverges, which gives a contradiction.

The result we have proved in this section is summarized in the following proposition.

Proposition 4.10. *Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be an expansive C^2 diffeomorphism of the 2-torus preserving a hyperbolic Borel probability measure μ. Assume that for all $x \in \mathbf{T}^2$ the local stable and unstable sets at x form C^1 curves and they intersect transversally at x in the sense that $T_xM = T_x\mathscr{W}_\delta^s(x) \oplus T_x\mathscr{W}_\delta^u(x)$. If μ is absolutely continuous with respect to the Riemannian volume on $\mathbf{T}^2$, then μ is absolutely continuous with respect to the unstable lamination and $|\mathrm{Jac}(D_pf^n)| = 1$ for $p \in \mathrm{Fix}(f^n)$ and $n \in \mathbf{N}$.*

5. Vanishing entropy production

As we noted at the end of §1, we show the following.

Proposition 5.1. *Let $f : M \to M$ be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact Riemannian manifold M preserving a hyperbolic Borel probability measure μ. If the measure μ is absolutely continuous with respect to the $\mathscr{W}^u$-lamination and $|\mathrm{Jac}(D_p f^n)| = 1$ for $p \in \mathrm{Fix}(f^n)$ and $n \in \mathbf{N}$, then it is absolutely continuous with respect to the Riemannian volume on M.*

Combining Proposition 4.10 and this proposition shall complete the proof of the main theorem (Theorem 1.1).

Lemma 5.2. *Let f be as in Proposition 5.1. If the entropy production for μ vanishes and μ is absolutely continuous with respect to the $\mathscr{W}^u$-lamination, then μ is absolutely continuous with respect to the Riemannian volume on M.*

Proof. Recall that the measure μ is absolutely continuous with respect to the Riemannian volume on M if and only if μ is absolutely continuous with respect to both the $\mathscr{W}^u$-lamination and the $\mathscr{W}^s$-lamination ([10]). Thus it is enough to show the measure μ is absolutely continuous with respect to the $\mathscr{W}^s$-lamination, equivalently

$$h_\mu(f^{-1}) = -\int \sum_{i:\chi_i(x)<0} \chi_i(x) \dim E_i(x) d\mu(x).$$

We have

$$\begin{aligned}
0 = e_f(\mu) &= -\int \log|\mathrm{Jac}(D_x f)| d\mu(x) \\
&= -\int \sum_i \chi_i(x) \dim E_i(x) d\mu(x) \\
&= -\int \sum_{i:\chi_i(x)>0} \chi_i(x) \dim E_i(x) + \sum_{i:\chi_i(x)<0} \chi_i(x) \dim E_i(x) d\mu(x) \\
&= -h_\mu(f) - \int \sum_{i:\chi_i(x)<0} \chi_i(x) \dim E_i(x) d\mu(x),
\end{aligned}$$

thereby

$$h_\mu(f^{-1}) = h_\mu(f) = -\int \sum_{i:\chi_i(x)<0} \chi_i(x) \dim E_i(x) d\mu(x).$$

This is the desired. □

In the following lemma we assume the invariant measure μ to be ergodic.

Lemma 5.3. *Let $f : M \to M$ be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact Riemannian manifold preserving an ergodic hyperbolic Borel probability* *measure* μ. *Assume* $|\mathrm{Jac}(D_p f^n)| = 1$ *for* $p \in \mathrm{Fix}(f^n)$ *and* $n \in \mathbf{N}$. *Then we have* $e_f(\nu) = 0$ *for every ergodic hyperbolic probability measure* ν *invariant under* f.

PROOF. We recall the following proposition given in [9] and present it in the form that suits to our purpose. The *support* of μ is the smallest closed set F with $\mu(F) = 1$. We denote it by $\mathrm{Supp}\,\mu$.

PROPOSITION 5.4 (Theorem S.5.5 in [9]). *Let* f *and* μ *be as in Lemma* 5.3 *and* $x \in \mathrm{Supp}\,\mu$. *Then for any* $\varepsilon > 0$, *neighborhood* V *of* x *and finite family* Φ *of continuous functions on* M *there exist* $n \in \mathbf{N}$ *and a hyperbolic periodic point* $z \in V \cap \mathrm{Fix}(f^n)$ *such that*

$$\left| \int \varphi \, d\mu - \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)) \right| < \varepsilon$$

for all $\varphi \in \Phi$.

Fix an arbitrarily ergodic hyperbolic Borel probability measure ν invariant under f. Take $\varphi = -\log|\mathrm{Jac}(Df)|$. We have, by the chain rule,

$$\sum_{i=0}^{n-1} \varphi(f^i(z)) = -\sum_{i=0}^{n-1} \log|\mathrm{Jac}(D_{f^i(z)} f)| = -\log|\mathrm{Jac}(D_z f^n)| = 0$$

for the periodic point z of period n for Proposition 5.4, where for the third equality the assumption on the Jacobian is used. On the other hand we obtain

$$\int \varphi \, d\nu = \int -\log|\mathrm{Jac}(Df)| d\nu = e_f(\nu)$$

by definition. Applying Proposition 5.4 shall imply $|e_f(\nu)| < \varepsilon$. Thus Lemma 5.3 follows from the arbitrariness of the positive number ε. □

PROOF OF PROPOSITION 5.1. Combining Lemmas 5.2 and 5.3 yields the proof for the case when the measure μ is ergodic. Reducing the non-ergodic measure to the ergodic one *via* the ergodic decomposition theorem completes the proof.

6. An example

We present here an example of a diffeomorphism of the 2-torus which satisfies all the assumptions of Theorem 1.1 but does not admit any hyperbolic absolutely continuous invariant probability measures.

Starting with a hyperbolic linear automorphism g of $\mathbf{T}^2$ having positive eigenvalues $\alpha^{-1} < 1 < \alpha$, we let $p \in \mathbf{T}^2$ be a fixed point of g.

Proposition 6.1. *There is a one-parameter family $\{g_a\}_{a\in[0,1]}$ of expansive C^∞ diffeomorphisms of $\mathbf{T}^2$ with $g_0 = g$ satisfying the following:*

(1) *for each $a \in [0, 1)$ the diffeomorphism g_a is Anosov and admits an invariant probability measure which is absolutely continuous with respect to the Riemannian volume on $\mathbf{T}^2$;*

(2) *the diffeomorphism g_1 admits no hyperbolic absolutely continuous invariant probability measures while it has the properties that for all $x \in \mathbf{T}^2$ the local stable and unstable sets at x form C^1 curves and they intersect transversally at x in the sense that $T_xM = T_x\mathscr{W}^s_\delta(x) \oplus T_x\mathscr{W}^u_\delta(x)$, and $|\mathrm{Jac}(D_q g_1^n)| = 1$ for $q \in \mathrm{Fix}(g_1^n)$ and $n \in \mathbf{N}$.*

We shall denote by (x, y) a coordinate system such that p is the origin and in which the map g has the diagonal linear form $g(x, y) = (\alpha x, \alpha^{-1}y)$. Let us denote the neighborhood of p which is given in this coordinate as $\{(x, y) : x^2 + y^2 \le r^2\}$ by D_r. Observe that the automorphism $g(= g_0)$ is the time-one map of the flow generated by the following system of vector fields:

$$\dot{x} = x \log \alpha,$$

$$\dot{y} = -y \log \alpha.$$

Here and below $\dot{x}$ denotes the differentiation dx/dt.

Given sufficiently small positive numbers r_0, r_1 and r_2 so that $0 < r_2 < r_1 < r_0/100\alpha$, choose a real-valued C^∞ function ψ on the unit interval $[0, 1]$ such that

(1) $\psi'(u) \ge 0$;

(2) $\psi(u) = 1$ for $u \ge (r_1)^2$ and $\psi(u) = u^2$ for $0 \le u \le (r_2)^2$.

We let

$$\psi_a(u) = \begin{cases} 1 - a + \dfrac{a}{1-a}\displaystyle\int_0^{1-a} \psi(u+s)ds & \text{if } a \in [0,1); \\ \psi & \text{if } a = 1. \end{cases}$$

Then it is easy to verify that $\psi_0 = 1$ and $\lim_{a\to 1} \psi_a = \psi_1$.

Consider the time-one map $\tilde{g}_a$, defined on D_{r_1}, generated by the following system of vector fields:

$$\dot{x} = x\psi_a(x^2 + y^2) \log \alpha,$$

$$\dot{y} = -y\,\psi_a(x^2 + y^2) \log \alpha.$$

Let g_a be a toral diffeomorphism which coincides with $\tilde{g}_a$ inside D_{r_1} and is extended to g outside D_{r_1}:

$$g_a(z) = \begin{cases} \tilde{g}_a(z) & \text{if } z \in D_{r_1}; \\ g(z) & \text{if } z \in \mathbf{T}^2 \backslash D_{r_1}. \end{cases}$$

Define

$$\rho_a(z) = \begin{cases} \dfrac{1}{\psi_a(x(z)^2 + y(z)^2)} & \text{if } z \in D_{r_1}; \\ 1 & \text{if } z \in \mathbf{T}^2 \backslash D_{r_1}. \end{cases}$$

Observe that the function ρ_a is of class C^∞ and positive everywhere except the case when $a = 1$ and $z = p$. It can be shown that the map g_a, $a \in [0, 1)$, preserves the probability measure

$$dv_a = \Delta_a^{-1} \rho_a \, dm, \tag{7}$$

where $\Delta_a = \int \rho_a \, dm$ and m denotes the Riemannian volume on $\mathbf{T}^2$.

Let $E^s(z)$ and $E^u(z)$ be the eigenspaces corresponding to the eigenvalues α^{-1} and α, respectively. Define the cones at $z \in \mathbf{T}^2$ as

$$C^s(z) = \{w \in T_z\mathbf{T}^2 : \|w_u\| \le \|w_s\|\},$$

$$C^u(z) = \{w \in T_z\mathbf{T}^2 : \|w_s\| \le \|w_u\|\},$$

where $w = w_s + w_u$ with respect to the splitting $T_z\mathbf{T}^2 = E^s(z) \oplus E^u(z)$. We now recall several results from [3, 4, 8].

LEMMA 6.2 (Proposition 4.1 in [8]). *Under the above assumptions we have the following.*

(1) *For every $a \in [0, 1]$ and $z \in \mathbf{T}^2$ the families of cones $C^s(z)$ and $C^u(z)$ are invariant:*

$$D_z g_a^{-1}(C^s(z)) \subset C^s(g_a^{-1}(z)) \qquad \text{and} \qquad D_z g_a(C^u(z)) \subset C^u(g_a(z)).$$

(2) *For every $a \in [0, 1]$ and $z \in \mathbf{T}^2$, except the case when $a = 1$ and $z = p$, the intersections*

$$E_a^s(z) = \bigcap_{n \ge 0} D_{g_a^n(z)} g_a^{-n}(C^s(g_a^n(z)));$$

$$E_a^u(z) = \bigcap_{n \ge 0} D_{g_a^{-n}(z)} g_a^n(C^u(g_a^{-n}(z)))$$

are one-dimensional subspaces of $T_z\mathbf{T}^2$.

LEMMA 6.3 (Corollaries 4.1 and 4.2 in [8]). *Under the same assumptions as in Lemma 6.2 we have for $a \in [0, 1)$ the subspaces $E_a^s(z)$ and $E_a^u(z)$ vary continuously on $z \in \mathbf{T}^2$, and the map g_a is an Anosov diffeomorphism.*

To see, next, the map g_1 admits no hyperbolic absolutely continuous invariant probability measures on $\mathbf{T}^2$ we need several lemmas below. In what follows we will use f instead of g_1 and use the same notations as in sections 3 and 4. We set $\Omega = \mathbf{T}^2 \backslash D_{r_1}$. Then the first return map $F = f^{\omega} : \Omega \to \Omega$ is well-defined, up to sets of measure zero, where $\omega : \Omega \to \mathbf{N}$ is the first return time of f to Ω.

Lemma 6.4. *Let $a = 1$. Under the same assumptions as in Lemma* 6.2 *we have the following.*

(1) *The subspaces $E_1^s(z)$ and $E_1^u(z)$ vary continuously on $z \in \mathbf{T}^2$ except, maybe, the point p* (Corollary 4.2 in [8])*;*

(2) *There exist constants $C > 1$ and $\gamma \in (0,1)$ such that*

$$\|D_z F^n(w_s)\| \le C\gamma^n \|w_s\| \qquad (w_s \in E_1^s(z));$$

$$\|D_z F^{-n}(w_u)\| \le C\gamma^n \|w_u\| \qquad (w_u \in E_1^u(z))$$

holds for $n \in \mathbf{N}$ and $z \in \Omega$ whenever F is defined (Proposition 2.3 in [3])*;*

(3) *The sets $\mathscr{W}_\delta^s(z)$ and $\mathscr{W}_\delta^u(z)$, with respect to f, are smooth curves so that $T_z \mathscr{W}_\delta^s(z) = E_1^s(z)$ and $T_z \mathscr{W}_\delta^u(z) = E_1^u(z)$ for $z \in \mathbf{T}^2$, where $E_1^\tau(p) = E^\tau(p)$, $\tau = s, u$. In particular, f is expansive* (Lemma 4.1 in [8], Corollary 6.1 and Proposition 6.1 in [4]).

Choose $\delta \in (0, \min\{\varepsilon_0, r_2\}/100)$, where ε_0 is an expansivity constant for f. Then we can set $R = R(p,\delta) = [\mathscr{W}_\delta^u(p), \mathscr{W}_\delta^s(p)]$ by the local product structure (see (3.2)). Notice that $R \subset D_{r_2}$. Suppose that f admits a hyperbolic absolutely continuous invariant probability measure μ. Let $V_M^s \subset \mathscr{W}_{\delta/2}^s(p)$ be the same notation defined before Lemma 4.1. Consider

$$\mathscr{L}_M^u = \{L_M^u(z) : z \in V_M^s\},$$

where $L_M^u(z) = [\mathscr{W}_\delta^u(p), z]$ for $z \in V_M^s$. Define

$$Q_M^u = \bigcup_{z \in V_M^s} L_M^u(z).$$

Lemma 6.5. *We have $\mu(Q_M^u) > 0$.*

Proof. Since the measure μ is supposed to be hyperbolic and absolutely continuous with respect to the volume, just the same argument as in the proof of Lemma 4.1 yields Lemma 6.5. □

Define a level set for each $n \in \mathbf{N}$ as

$$J_n = \left\{ (x,y) \in R : \frac{\delta}{n+1} < xy \le \frac{\delta}{n} \right\}.$$

Note that $J_n \cap J_m = \emptyset$ for $n \ne m$ and that J_n, for n sufficiently large, transverses to Q_M^u. Denote by $P_n = J_n \cap Q_M^u$ for such n.

For functions ξ and η that $\xi \sim \eta$ $(x \to a)$ means $\xi(x)/\eta(x) - 1 = o(1)$ $(x \to a)$. Note that $\psi_1 = \psi$ and $(\psi|R)(u) = u^2$.

LEMMA 6.6. *There is a constant $K_1 > 0$ such that we have $\mu(J_n) \geq K_1 n^4 \mu(P_n)$ for large n.*

PROOF. Given arbitrarily small $c \in (0, \delta)$ so that $c/\delta < \delta/2$, we first estimate the "escape" time T of when the solution of the vector fields along the segment of the hyperbola $\{(x, y) : xy = c, x \in [0, \delta], y \in [0, \delta]\}$ goes out the region R under the initial condition $x = c/\delta$. Since $y = c/x$, it follows that

$$\dot{x} = x\psi(x^2 + y^2) \log \alpha = x\psi(x^2 + (c/x)^2) \log \alpha.$$

Thus we have

$$\begin{aligned}
\int_0^T dt = \int_{c/\delta}^{\delta} \frac{1}{\dot{x}}\, dx &= \int_{c/\delta}^{\delta} \frac{x^3}{(x^4 + c^2)^2 \log \alpha}\, dx \\
&\geq \frac{\kappa}{\log \alpha} \int_{c/\delta}^{\delta} \frac{x^3}{c^4}\, dx \\
&= \frac{\kappa}{4c^4 \log \alpha} \{\delta^4 - (c/\delta)^4\} \\
&\geq \frac{\kappa}{4c^4 \log \alpha} (\delta/2)^4 = \frac{\kappa}{2^6 \log \alpha} (\delta/c)^4
\end{aligned}$$

for a constant $\kappa > 0$ which comes from the similarity $1/(x^4 + c^2) \sim 1/c^2$ $(x \to 0)$. Substituting $c = \delta/n$ shall imply

$$T \geq \frac{\kappa}{2^6 \log \alpha} n^4$$

for $n > 2/\delta$.

It then follows from the invariance of μ that

$$\mu(J_n) \geq \mu(P_n) \cdot T \geq \frac{\kappa}{2^6 \log \alpha} \cdot n^4 \mu(P_n),$$

thereby the estimate holds for $K_1 = \kappa/2^6 \log \alpha$. □

LEMMA 6.7. *There is a constant $K_2 > 0$ such that we have $\mu(P_n) \geq K_2/n^2$ for large n.*

PROOF. Put $h_M^u = d\mu_z^{f^M \xi^u}/dm_z^u$. By the invariance of the measure μ we have

$$\begin{aligned}\mu(P_n) &= \int_{\mathbf{T}^2} \mu_z^{f^M\xi^u}(P_n)d\mu(z)\\ &= \int_{\mathbf{T}^2}\int_{P_n\cap C_{f^M\xi^u}(z)} h_M^u(w)dm_z^u(w)d\mu(z)\\ &\geq C_4^{-1}\int_{\mathbf{T}^2} m_z^u(J_n\cap L_M^u(z))d\mu(z)\\ &= C_4^{-1}\int_{Q_M^u} m_z^u(J_n\cap L_M^u(z))d\mu(z)\end{aligned}$$

for a constant $C_4 = C_4(C, M) > 1$ since $C^{-1} \leq h^u|L \leq C$ $(L \in Q^u)$ holds. There is a constant $\sigma > 1$ independent of z and n such that $m_z^u(J_n \cap L_M^u(z)) \geq \sigma^{-1}/n^2$ since the local unstable manifolds vary uniformly continuously on Λ_l with respect to the C^1 topology ([1]). It follows that

$$\mu(P_n) \geq (C_4\sigma)^{-1}\mu(Q_M^u)/n^2,$$

thereby the desired estimate follows for $K_2 = (C_4\sigma)^{-1}\mu(Q_M^u)$, which is positive by Lemma 6.5. □

Proof of Proposition 6.1. It follows from Lemma 6.3 that for each $a \in [0, 1)$ the map g_a is an Anosov diffeomorphism and as we have seen above the map g_a preserves the absolutely continuous probability measure ν_a defined by (7).

We next show the assertion (2). It can be verified readily that g_1 is expansive and possesses the transverse intersections $T_z\mathbf{T}^2 = T_z\mathscr{W}_\delta^s(z) \oplus T_z\mathscr{W}_\delta^u(z)$ for all $z \in \mathbf{T}^2$ by Lemma 6.4 (3). Put $\mathrm{Fix}(g_1^n)^* = \mathrm{Fix}(g_1^n)\backslash\{p\}$ for each $n \in \mathbf{N}$. It follows from Lemma 6.4 (2) that $\mathrm{Fix}(g_1^n)^*$ consists of hyperbolic periodic points and hence, there is a map (so-called a *continuation*, see [16] for instance)

$$\Psi = \Psi_n : \mathrm{Fix}(g_1^n)^* \times [0, 1] \to \mathbf{T}^2$$

that associates every point $(q, a) \in \mathrm{Fix}(g_1^n)^* \times [0, 1]$ to a periodic point in $\mathrm{Fix}(g_a^n)^*$ such that

- $\Psi(q, 1) = q$, that is $\mathrm{Fix}(g_1^n)^*$ can be identified with $\mathrm{Fix}(g_1^n)^* \times \{1\}$ *via* Ψ;
- for each $q \in \mathrm{Fix}(g_1^n)^*$ the map

  $$\Psi_q = \Psi_n(q) : [0, 1] \to \mathbf{T}^2$$

  defined as $\Psi_q(a) = \Psi(q, a)$ is continuous.

It follows that for $q \in \mathrm{Fix}(g_1^n)^*$

$$|\mathrm{Jac}(D_q g_1^n)| = \lim_{a\to 1} |\mathrm{Jac}(D_{\Psi_q(a)} g_a^n)| = 1$$

since $|\mathrm{Jac}(D_{\Psi_q(a)} g_a^n)| = 1$ for all $a \in [0, 1)$. The equality $|\mathrm{Jac}(D_p g_1^n)| = 1$ follows *a priori*, so does $|\mathrm{Jac}(D_q g_1^n)| = 1$ for all $q \in \mathrm{Fix}(g_1^n)$.

It remains to show g_1 admits no hyperbolic absolutely continuous invariant probability measures. Suppose, to derive a contradiction, g_1 admits a hyperbolic absolutely continuous invariant probability measure μ. Note that J_n, $n \in \mathbf{N} \cup \{0\}$, are pairwise disjoint subsets in R. By using Lemmas 6.6 and 6.7 we obtain

$$1 \geq \mu(R) \geq \sum_{n=1}^{\infty} \mu(J_n) \geq K_1 K_2 \sum_{n=1}^{\infty} n^2.$$

This gives a contradiction.

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References

[1] L. Barreira and Ya. B. Pesin, Lyapunov Exponents and Smooth Ergodic Theory, Univ. Lect. Ser. **23**, Amer. Math. Soc., 2002.

[2] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Math., vol. **470**, Springer-Verlag, 1975.

[3] M. Brin, J. Feldman and A. B. Katok, Bernoulli diffeomorphisms and group extensions of dynamical systems with non-zero characteristic exponents, Ann. of Math. **113** (1981), 159–179.

[4] M. Gerber and A. B. Katok, Smooth models of Thurston's pseudo-Anosov maps, Ann. scient. Éc. Norm. Sup. **15** (1982), 173–204.

[5] K. Hiraide, Expansive homeomorphisms of compact surfaces are pseudo-Anosov, Osaka J. Math. **27** (1990), 117–162.

[6] K. Hiraide, Dynamical systems of expansive maps on compact manifolds, Sugaku Expositions. Amer. Math. Soc. **5** (1992), 133–154.

[7] H. Hu and L.-S. Young, Nonexistence of SBR measures for some diffeomorphisms that are 'almost Anosov', Ergod. Th. & Dynam. Sys. **15** (1995), 67–76.

[8] A. B. Katok, Bernoulli diffeomorphisms on surfaces, Ann. of Math. **110** (1979), 529–547.

[9] A. B. Katok and L. Mendoza, Dynamical systems with nonuniformly hyperbolic behavior, Supplement to Introduction to the Modern Theory of Dynamical Systems, C.U.P., Cambridge, 1995.

[10] F. Ledrappier, Propriétés ergodiques des mesures de Sinaï, Inst. Hautes Ètudes Sci. Publ. Math. **59** (1984), 163–188.

[11] F. Ledrappier and J. M. Strelcyn, A proof of estimation from below in Pesin's entropy formula, Ergod. Th. & Dynam. Sys. **2** (1982), 203–219.

[12] F. Ledrappier and L.-S. Young, The metric entropy of diffeomorphisms, Part I: Characterization of measures satisfying Pesin's entropy formula, Ann. of Math. **122** (1985), 509–539.

[13] J. Lewowicz, Expansive homeomorphisms of surfaces, Bol. Soc. Brasil. Mat. (N. S.) **20** (1989), 113–133.

[14] A. N. Livšic and Ya. G. Sinai, Invariant measures that are compatible with smoothness for transitive $\mathscr{Y}$-systems, Dokl. Akad. Nauk SSSR **207** (1972), 1039–1041; English transl., Soviet Math. Dokl. **13** (1972), 1656–1659.

[15] V. I. Oseledec, A multiplicative ergodic theorem, Lyapunov characteristic number for dynamical systems, Trudy Moskov. Mat. Obšč. **19** (1968), 179–210; English transl., Trans. Mosc. Math. Soc. **19** (1968), 197–231.

[16] J. Palis and F. Takens, Hyperbolicity & Sensitive chaotic dynamics at homoclinic bifurcation, Cambridge University Press, Cambridge, 1993.

[17] Ya. B. Pesin, Characteristic exponents and smooth ergodic theory, Uspehi Mat. Nauk **32** (4) (1977), 55–112; English transl., Russian Math. Surveys. **32** (4) (1977), 55–114.

[18] E. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, Ann. of Math. **151** (2000), 961–1023.

[19] V. A. Rokhlin, Lectures on the theory of entropy of transformation with invariant measure, Uspehi Mat. Nauk **22** (5) (1967), 3–56; English transl., Russian Math. Surveys. **22** (5) (1967), 1–52.

[20] D. Ruelle, Positivity of entropy production in nonequilibrium statistical mechanics, J. Stat. Phys. **85** (1996), 1–23.

[21] D. Ruelle, Entropy production in nonequilibrium statistical mechanics, Commun. Math. Phys. **189** (1997), 365–371.

[22] N. Sumi, Diffeomorphisms with positive entropy and chaos in the sense of Li-Yorke, Ergod. Th. & Dynam. Sys. **23** (2003), 621–635.

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