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Asymptotic expansions of the null distributions for the Dempster trace criterion

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Abstract. In this paper, we study the Dempster trace criterion. When the number of variables and the dimension of the null hypothesis are large relative to sample size, we derive the asymptotic distribution and Cornish-Fisher expansion of the Dempster trace criterion in the cases such that the covariance matrix is known and that the covariance matrix is unknown. Finally, we study the accuracy of the asymptotic expansion by the numerical simulation.

1. Introduction

We consider a multivariate linear model:

$$Y = XB + \mathscr{E},$$

where Y is an $N \times p$ observation matrix, X is an $N \times k$ design matrix with $\operatorname{rank}(X) = k$, B is a $k \times p$ matrix of regression coefficients, and $\mathscr{E}$ is an $N \times p$ error matrix distributed according to $\mathrm{N}_{N\times p}(O, I_N \otimes \Sigma)$. We test the hypothesis

$$H_0 : CB = O,$$

where C is a $q \times k$ known matrix of rank q. For the null hypothesis, the Dempster trace criterion (See Dempster (1958, 1960)) is defined by $\mathrm{tr}(S_h)/\mathrm{tr}(S_e)$, where

$$S_h = \hat{B}'C'(C(X'X)^{-1}C')^{-1}C\hat{B},$$

$$S_e = (Y - X\hat{B})'(Y - X\hat{B}),$$

with $\hat{B} = (X'X)^{-1}X'Y$ (See Muirhead (1982)). For testing the above hypothesis, the likelihood ratio test statistic, Lawley-Hotelling's generalized T^2 statistic, and Bartlett-Nanda-Pillai's test statistic are often used. However, when $p > N - k$, these statistics can not be defined since S_e is singular. In practice,

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the number of variables is sometimes larger than the sample size, such as microarray data and economic data. So we study the Dempster trace criterion. Under the null hypothesis, the statistic

$$\frac{n\,\mathrm{tr}(S_h)}{q\,\mathrm{tr}(S_e)}$$

is approximated by F distribution with $q\rho$ and $n\rho$ degrees of freedom, where $n = N - k$ and $\rho = \{\mathrm{tr}(\Sigma)\}^2/\mathrm{tr}(\Sigma^2)$ (See Dempster (1958)). Usually ρ is unknown, so it should be estimated by the data. In this case it is difficult to assess how close is the estimated critical point to the true value. So we approximate the critical point by using the asymptotic expansion. Using the traditional asymptotic expansion, if the fixed parameters p and q are large, then the approximation is bad. We want to derive the useful approximation to a variety of size of parameters. In Section 2, we derive the asymptotic expansion under the framework:

$$n, p, q \to \infty, \qquad \frac{p}{n} \to \gamma_1 \in (0, \infty), \qquad \frac{q}{n} \to \gamma_2 \in (0, \infty). \tag{1}$$

In Section 3, under the condition such that the parameter Σ is unknown, we derive the asymptotic expansion. Finally, in Section 4, we check the accuracy of approximation by the numerical simulation.

2. The case such that the covariance matrix Σ is known

In this section, we derive an asymptotic expansion of the distribution of the Dempster trace criterion under the framework (1).

2.1. The stochastic expansion. Under the null hypothesis, we know that S_h and S_e are independently distributed according to the central Wishart distributions $\mathrm{W}_p(q, \Sigma)$ and $\mathrm{W}_p(n, \Sigma)$, respectively. For deriving the asymptotic distribution, we assume that

$$\mathrm{tr}(\Sigma^k) = O(p). \tag{2}$$

In the following, we use the notation $c_k = \mathrm{tr}(\Sigma^k)/p$. Let T be defined by

$$T = p\left(\frac{n\,\mathrm{tr}(S_h)}{q\,\mathrm{tr}(S_e)} - 1\right),$$

and U, V be defined by

$$U = \frac{1}{\sqrt{pq}} (\mathrm{tr}(S_h) - q\,\mathrm{tr}(\Sigma)),$$

$$V = \frac{1}{\sqrt{np}} (\mathrm{tr}(S_e) - n\,\mathrm{tr}(\Sigma)).$$

Then under the assumption (2), both U and V are asymptotically distributed according to the same normal distribution $\mathrm{N}(0, 2c_2)$ (See Appendix A.1). T is expanded as

$$\begin{aligned}
T &= p\left(\frac{n\,\mathrm{tr}(S_h)}{q\,\mathrm{tr}(S_e)} - 1\right) \\
&= p\left\{\left(\frac{1}{\sqrt{pq}}\frac{U}{c_1} + 1\right)\left(1 - \frac{1}{\sqrt{np}}\frac{V}{c_1} + \frac{1}{np}\frac{V^2}{c_1^2} - \frac{1}{(np)^{3/2}}\frac{V^3}{c_1^3} + O_p((np)^{-2})\right) - 1\right\} \\
&= \frac{1}{c_1}(\sqrt{r_1}U - \sqrt{r_2}V) + \frac{1}{n}\frac{V^2}{c_1^2} - \frac{1}{\sqrt{nq}}\frac{UV}{c_1^2} + \frac{1}{n\sqrt{pq}}\frac{UV^2}{c_1^3} - \frac{1}{n\sqrt{np}}\frac{V^3}{c_1^3} + O_p(n^{-3}),
\end{aligned}$$

where $r_1 = p/q$ and $r_2 = p/n$.

2.2. The characteristic function. The characteristic function $C_1(t)$ of T is given by

$$\begin{aligned}
C_1(t) &= \mathrm{E}[\exp(itT)] \\
&= \mathrm{E}\left[\exp\left(\frac{it}{c_1}(\sqrt{r_1}U - \sqrt{r_2}V)\right)\left\{1 + \frac{it}{n}\frac{V^2}{c_1^2} + \frac{(it)^2}{n^2}\frac{V^4}{2c_1^4}\right\}\right. \\
&\qquad \times\left\{1 - \frac{it}{\sqrt{nq}}\frac{UV}{c_1^2} + \frac{(it)^2}{nq}\frac{U^2V^2}{2c_1^4}\right\}\left\{1 + \frac{it}{n\sqrt{pq}}\frac{UV^2}{c_1^3}\right\} \\
&\qquad \left.\times\left\{1 - \frac{it}{n\sqrt{np}}\frac{V^3}{c_1^3}\right\} + O_p(n^{-3})\right] \\
&= \mathrm{E}\left[\exp\left(\frac{it}{c_1}(\sqrt{r_1}U - \sqrt{r_2}V)\right) g_1(U,V)\right],
\end{aligned}$$

where

$$\begin{aligned}
g_1(U,V) &= 1 + \frac{it}{n}\frac{V^2}{c_1^2} - \frac{it}{\sqrt{nq}}\frac{UV}{c_1^2} + \frac{(it)^2}{n^2}\frac{V^4}{2c_1^4} + \frac{(it)^2}{nq}\frac{U^2V^2}{2c_1^4} \\
&\quad + \frac{it}{n\sqrt{pq}}\frac{UV^2}{c_1^3} - \frac{it}{n\sqrt{np}}\frac{V^3}{c_1^3} - \frac{(it)^2}{n\sqrt{nq}}\frac{UV^3}{c_1^4} + O_p(n^{-3}).
\end{aligned}$$

Let Z_1 be a $p \times q$ random matrix distributed according to $\mathrm{N}_{p\times q}(O, I_{pq})$ and Z_2 be a $p \times n$ random matrix distributed according to $\mathrm{N}_{p\times n}(O, I_{pn})$. Then

$$U = \frac{1}{\sqrt{pq}}(\mathrm{tr}(\Sigma Z_1 Z_1') - pqc_1) \qquad \text{and} \qquad V = \frac{1}{\sqrt{np}}(\mathrm{tr}(\Sigma Z_2 Z_2') - npc_1). \tag{3}$$

By using (3), we rewrite the characteristic function as

$$\begin{aligned} C_1(t) &= \mathrm{E}\left[\exp\left\{\frac{it}{c_1}\left(\frac{\sqrt{r_1}}{\sqrt{pq}}\ \mathrm{tr}(\Sigma Z_1 Z_1') - \frac{\sqrt{r_2}}{\sqrt{np}}\ \mathrm{tr}(\Sigma Z_2 Z_2')\right)\right.\right. \\ &\qquad \left.\left. - it\sqrt{r_1}\sqrt{pq} + it\sqrt{r_2}\sqrt{np}\right\} g_1(U,V)\right] \\ &= \int\!\!\int (2\pi)^{-p(n+q)/2}\ \mathrm{etr}\left\{-\frac{1}{2}\left(I_p - \frac{2it}{qc_1}\Sigma\right) Z_1 Z_1'\right\} \\ &\qquad \times \mathrm{etr}\left\{-\frac{1}{2}\left(I_p + \frac{2it}{nc_1}\Sigma\right) Z_2 Z_2'\right\} g_1(U,V)\, dZ_1 dZ_2. \end{aligned}$$

Besides, we consider the following transformations:

$$Z_1 = \left(I_p - \frac{2it}{qc_1}\Sigma\right)^{-1/2} \tilde{Z}_1 \qquad \text{and} \qquad Z_2 = \left(I_p + \frac{2it}{nc_1}\Sigma\right)^{-1/2} \tilde{Z}_2. \tag{4}$$

These transformations imply that

$$\begin{aligned} C_1(t) &= \left|I_p - \frac{2it}{qc_1}\Sigma\right|^{-q/2} \left|I_p + \frac{2it}{nc_1}\Sigma\right|^{-n/2} \\ &\qquad \times \int\!\!\int (2\pi)^{-p(n+q)/2}\ \mathrm{etr}\left(-\frac{1}{2}\tilde{Z}_1\tilde{Z}_1' - \frac{1}{2}\tilde{Z}_2\tilde{Z}_2'\right) g_1(U,V)\, d\tilde{Z}_1 d\tilde{Z}_2 \\ &\qquad (\tilde{Z}_1 \sim \mathrm{N}_{p\times q}(O, I_{pq}) \ \text{ and } \ \tilde{Z}_2 \sim \mathrm{N}_{p\times n}(O, I_{pn})) \\ &= \left|I_p - \frac{2it}{qc_1}\Sigma\right|^{-q/2} \left|I_p + \frac{2it}{nc_1}\Sigma\right|^{-n/2} \mathrm{E}_{\tilde{Z}_1,\tilde{Z}_2}[g_1(U,V)], \end{aligned}$$

where $|A|$ denotes the determinant of A for any square matrix A. For the Jacobian, it holds that

$$\log\left|I_p - \frac{2it}{qc_1}\Sigma\right|^{-q/2} = pit + \frac{r_1 c_2}{c_1^2}(it)^2 + \frac{(it)^3}{q}\frac{4r_1 c_3}{3c_1^3} + \frac{(it)^4}{q^2}\frac{2r_1 c_4}{c_1^4} + O(q^{-3}),$$

$$\log\left|I_p + \frac{2it}{nc_1}\Sigma\right|^{-n/2} = -pit + \frac{r_2 c_2}{c_1^2}(it)^2 - \frac{(it)^3}{n}\frac{4r_2 c_3}{3c_1^3} + \frac{(it)^4}{n^2}\frac{2r_2 c_4}{c_1^4} + O(n^{-3}).$$

So we obtain the following expansions:

$$\left|I_p - \frac{2it}{qc_1}\Sigma\right|^{-q/2} = \exp\left(pit + \frac{r_1c_2}{c_1^2}(it)^2\right) \times \left\{1 + \frac{(it)^3}{q}\frac{4r_1c_3}{3c_1^3} + \frac{(it)^4}{q^2}\frac{2r_1c_4}{c_1^4} + \frac{(it)^6}{q^2}\frac{8r_1^2c_3^2}{9c_1^6} + O(q^{-3})\right\},$$

$$\left|I_p + \frac{2it}{nc_1}\Sigma\right|^{-n/2} = \exp\left(-pit + \frac{r_2c_2}{c_1^2}(it)^2\right) \times \left\{1 - \frac{(it)^3}{n}\frac{4r_2c_3}{3c_1^3} + \frac{(it)^4}{n^2}\frac{2r_2c_4}{c_1^4} + \frac{(it)^6}{n^2}\frac{8r_2^2c_3^2}{9c_1^6} + O(n^{-3})\right\}.$$

Next, in order to calculate the expectation $\mathrm{E}_{\tilde{Z}_1,\tilde{Z}_2}[g_1(U,V)]$, we use the stochastic expansion of U and V given by

$$\begin{aligned}
U &= \frac{1}{\sqrt{pq}}(\mathrm{tr}(\Sigma Z_1Z_1') - pqc_1) \\
&= \frac{1}{\sqrt{pq}}\left(\mathrm{tr}\left[\Sigma\left(I_p - \frac{2it}{qc_1}\Sigma\right)^{-1}\tilde{Z}_1\tilde{Z}_1'\right] - pqc_1\right) \\
&= U_1 + \frac{2it\sqrt{r_1}c_2}{c_1} + \frac{1}{q}\left(\frac{2it}{c_1}U_2 + \frac{4(it)^2\sqrt{r_1}c_3}{c_1^2}\right) + O_p(p^{-2}) \\
&\quad \left(U_1 = \frac{1}{\sqrt{pq}}(\mathrm{tr}(\Sigma\tilde{Z}_1\tilde{Z}_1') - pqc_1) \text{ and } U_2 = \frac{1}{\sqrt{pq}}(\mathrm{tr}(\Sigma^2\tilde{Z}_1\tilde{Z}_1') - pqc_2)\right), \\
V &= \frac{1}{\sqrt{np}}(\mathrm{tr}(\Sigma Z_2Z_2') - npc_1) \\
&= \frac{1}{\sqrt{np}}\left(\mathrm{tr}\left[\Sigma\left(I_p + \frac{2it}{nc_1}\Sigma\right)^{-1}\tilde{Z}_2\tilde{Z}_2'\right] - npc_1\right) \\
&= V_1 - \frac{2it\sqrt{r_2}c_2}{c_1} - \frac{1}{n}\left(\frac{2it}{c_1}V_2 - \frac{4(it)^2\sqrt{r_2}c_3}{c_1^2}\right) + O_p(p^{-2}) \\
&\quad \left(V_1 = \frac{1}{\sqrt{np}}(\mathrm{tr}(\Sigma\tilde{Z}_2\tilde{Z}_2') - npc_1) \text{ and } V_2 = \frac{1}{\sqrt{np}}(\mathrm{tr}(\Sigma^2\tilde{Z}_2\tilde{Z}_2') - npc_2)\right).
\end{aligned}$$

Using Appendix A.2, we obtain the following expectation:

$$\mathrm{E}_{\tilde{Z}_1,\tilde{Z}_2}[g_1(U,V)] = 1 + \frac{1}{n}\left(\frac{2(it)c_2}{c_1^2} + \frac{4(it)^3(r_1+r_2)c_2^2}{c_1^4}\right)$$
$$+ \frac{1}{n^2}\left(-\frac{8(it)^2c_3}{c_1^3} + \frac{18(it)^2c_2^2}{c_1^4} - \frac{16(it)^4r_2c_2c_3}{c_1^5}\right.$$
$$\left. + \frac{(it)^4(24r_1+32r_2)c_2^3}{c_1^6} + \frac{(it)^6(16r_1+8r_2)r_2c_2^4}{c_1^8}\right)$$
$$+ \frac{1}{nq}\left(\frac{6(it)^2c_2^2}{c_1^4} + \frac{8(it)^4(r_1-r_2)c_2c_3}{c_1^5}\right.$$
$$\left. + \frac{(it)^4(4r_1+12r_2)c_2^3}{c_1^6} + \frac{8(it)^6r_1r_2c_2^4}{c_1^8}\right) + O(n^{-3}).$$

Consequently, we see that the characteristic expansion can be expanded as

$$C_1(t) = \exp((it)^2\sigma^2/2)\left[1 + \frac{1}{q}(it)^3a_1 + \frac{1}{n}\{(it)a_2 + (it)^3a_3\}\right.$$
$$+ \frac{1}{q^2}\{(it)^4a_4 + (it)^6a_5\} + \frac{1}{n^2}\{(it)^2a_6 + (it)^4a_7 + (it)^6a_8\}$$
$$\left. + \frac{1}{nq}\{(it)^2a_9 + (it)^4a_{10} + (it)^6a_{11}\}\right] + O(n^{-3}),$$

where

$$\sigma = \frac{\sqrt{2(r_1+r_2)c_2}}{c_1}, \qquad a_1 = \frac{4r_1c_3}{3c_1^3}, \qquad a_2 = \frac{2c_2}{c_1^2},$$
$$a_3 = \frac{4(r_1+r_2)c_2^2}{c_1^4} - \frac{4r_2c_3}{3c_1^3},$$
$$a_4 = \frac{2r_1c_4}{c_1^4}, \qquad a_5 = \frac{8r_1^2c_3^2}{9c_1^6}, \qquad a_6 = \frac{18c_2^2}{c_1^4} - \frac{8c_3}{c_1^3},$$
$$a_7 = \frac{2r_2c_4}{c_1^4} - \frac{8(3r_1+7r_2)c_2c_3}{3c_1^5} + \frac{32(r_1+r_2)c_2^3}{c_1^6},$$
$$a_8 = \frac{8r_2^2c_3^2}{9c_1^6} - \frac{16r_2(r_1+r_2)c_2^2c_3}{3c_1^7} + \frac{8(2r_1+r_2)r_2c_2^4}{c_1^8},$$

$$a_9 = \frac{6c_2^2}{c_1^4}, \qquad a_{10} = \frac{32r_1c_2c_3}{3c_1^5} + \frac{4(r_1+r_2)c_2^3}{c_1^6},$$

$$a_{11} = -\frac{16r_1r_2c_3^2}{9c_1^6} + \frac{8r_1r_2c_2^4}{c_1^8} + \frac{16r_1(r_1+r_2)c_2^2c_3}{3c_1^7}.$$

2.3. Asymptotic distribution and Cornish-Fisher expansion. Inverting the characteristic function, we obtain the following density function of T/σ:

$$\begin{aligned} f_1(x) = \phi(x)\bigg\{ & 1 + \frac{1}{q}\frac{a_1}{\sigma^3}h_3(x) + \frac{1}{n}\left(\frac{a_2}{\sigma}h_1(x) + \frac{a_3}{\sigma^3}h_3(x)\right) \\ & + \frac{1}{q^2}\left(\frac{a_4}{\sigma^4}h_4(x) + \frac{a_5}{\sigma^6}h_6(x)\right) + \frac{1}{n^2}\left(\frac{a_6}{\sigma^2}h_2(x) + \frac{a_7}{\sigma^4}h_4(x) + \frac{a_8}{\sigma^6}h_6(x)\right) \\ & + \frac{1}{nq}\left(\frac{a_9}{\sigma^2}h_2(x) + \frac{a_{10}}{\sigma^4}h_4(x) + \frac{a_{11}}{\sigma^6}h_6(x)\right)\bigg\} + O(n^{-3}), \end{aligned}$$

where $\phi(x)$ is the density function of the standard normal distribution and $h_j(x)$'s are the Hermite polynomials given by

$$h_1(x) = x, \qquad h_2(x) = x^2 - 1, \qquad h_3(x) = x^3 - 3x, \qquad h_4(x) = x^4 - 6x^2 + 3x,$$
$$h_5(x) = x^5 - 10x^3 + 15x, \qquad h_6(x) = x^6 - 15x^4 + 45x^2 - 15.$$

Integrating the density function, we obtain the following theorem.

THEOREM 1. *Under the framework* (1) *and the condition* (2), *we obtain the asymptotic expansion of the null distribution of* T/σ *as*

$$\begin{aligned} Pr\left(\frac{T}{\sigma} \le z\right) = \Phi(z) - \phi(z)\bigg\{ & \frac{1}{q}\frac{a_1}{\sigma^3}h_2(z) + \frac{1}{n}\left(\frac{a_2}{\sigma} + \frac{a_3}{\sigma^3}h_2(z)\right) \\ & + \frac{1}{q^2}\left(\frac{a_4}{\sigma^4}h_3(z) + \frac{a_5}{\sigma^6}h_5(z)\right) + \frac{1}{n^2}\left(\frac{a_6}{\sigma^2}h_1(z) + \frac{a_7}{\sigma^4}h_3(z) + \frac{a_8}{\sigma^6}h_5(z)\right) \\ & + \frac{1}{nq}\left(\frac{a_9}{\sigma^2}h_1(z) + \frac{a_{10}}{\sigma^4}h_3(z) + \frac{a_{11}}{\sigma^6}h_5(z)\right)\bigg\} + O(n^{-3}), \end{aligned}$$

where $\Phi(z)$ *is the distribution function of the standard normal distribution.*

Using this asymptotic expansion, we obtain the following Cornish-Fisher expansion of the percentile (See Appendix A.3 for the proof).

COROLLARY 1. *Let* $z_1(\alpha)$ *be*

$$z_1(\alpha) = u + \frac{1}{q}p_1(u) + \frac{1}{n}p_2(u) + \frac{1}{q^2}p_3(u) + \frac{1}{n^2}p_4(u) + \frac{1}{nq}p_5(u),$$

where

$$u = z_\alpha,$$

(z_α *is the upper* 100α % *point of the standard normal distribution.*)

$$p_1(u) = \frac{a_1}{\sigma^3}(u^2 - 1),$$

$$p_2(u) = \frac{a_2}{\sigma} + \frac{a_3}{\sigma^3}(u^2 - 1),$$

$$p_3(u) = \frac{u}{2}p_1^2(u) + \frac{a_4}{\sigma^4}h_3(u) + \frac{a_5}{\sigma^6}h_5(u) + \frac{2ua_1}{\sigma^3}p_1(u) - \frac{ua_1}{\sigma^3}h_2(u)p_1(u),$$

$$p_4(u) = -\frac{u}{2}p_2^2(u) + \frac{a_6}{\sigma^2}h_1(u) + \frac{a_7}{\sigma^4}h_3(u) + \frac{a_8}{\sigma^6}h_5(u) + \frac{2ua_3}{\sigma^3}p_2(u),$$

$$p_5(u) = \frac{a_9}{\sigma^2}h_1(u) + \frac{a_{10}}{\sigma^4}h_3(u) + \frac{a_{11}}{\sigma^6}h_5(u) + \frac{2ua_1}{\sigma^3}p_2(u) + \frac{2ua_3}{\sigma^3}p_1(u) - \frac{ua_1}{\sigma^3}h_2(u)p_2(u).$$

Then it holds that

$$Pr\left(\frac{T}{\sigma} \le z_1(\alpha)\right) = 1 - \alpha + O(n^{-3}).$$

3. The case such that the covariance matrix Σ is unknown

Usually the covariance matrix Σ is unknown. So we need to replace the unknown parameter σ with the consistent estimator in the formulas obtained in Section 2.

3.1. The stochastic expansion. In Section 2, we defined that

$$\sigma = \frac{\sqrt{2(r_1 + r_2)c_2}}{c_1},$$

which contains the unknown parameter c_1 and c_2. So we use the unbiased consistent estimators of c_1 and c_2 (See Srivastava (2005)).

$$\hat{c}_1 = \frac{\mathrm{tr}(S_e)}{np},$$

$$\hat{c}_2 = \frac{n^2}{p(n+2)(n-1)}\left(\frac{\mathrm{tr}(S_e^2)}{n^2} - \frac{(\mathrm{tr}(S_e))^2}{n^3}\right).$$

By using these consistent estimators, we obtain the consistent estimator of σ:

$$\hat{\sigma} = \frac{\sqrt{2n^2(r_1+r_2)\{\mathrm{tr}(S_e^2)/n^2 - (\mathrm{tr}(S_e))^2/n^3\}/(p(n+2)(n-1))}}{\mathrm{tr}(S_e)/(np)}.$$

Then it holds that

$$\frac{T}{\hat{\sigma}} = \frac{\mathrm{tr}(S_h)/q - \mathrm{tr}(S_e)/n}{\sqrt{2n^2(r_1+r_2)\{\mathrm{tr}(S_e^2)/n^2 - (\mathrm{tr}(S_e))^2/n^3\}/(p(n+2)(n-1))}}.$$

Let W be defined by

$$W = \frac{n}{p}\left[\frac{n^2}{(n+2)(n-1)}\left(\frac{\mathrm{tr}(S_e^2)}{n^2} - \frac{(\mathrm{tr}(S_e))^2}{n^3}\right) - \mathrm{tr}(\Sigma^2)\right].$$

Then $W = O_p(1)$ (See Srivastava (2005)), and $T/\hat{\sigma}$ is expanded as

$$\begin{aligned}\frac{T}{\hat{\sigma}} &= \frac{\mathrm{tr}(S_h)/q - \mathrm{tr}(S_e)/n}{\sqrt{2(r_1+r_2)(c_2+W/n)}}\\ &= \left(\frac{\mathrm{tr}(S_h)}{q} - \frac{\mathrm{tr}(S_e)}{n}\right)\frac{1}{\sqrt{2(r_1+r_2)c_2}}\left(1 - \frac{W}{2nc_2} + O_p(n^{-2})\right).\end{aligned}$$

3.2. The expansion of the characteristic function. The characteristic function of $T_1/\hat{\sigma}$ is calculated as

$$\begin{aligned}C_2(t) &= \mathrm{E}\left[\exp\left(\frac{itT}{\hat{\sigma}}\right)\right]\\ &= \mathrm{E}\left[\exp\left\{\frac{it}{\sqrt{2(r_1+r_2)c_2}}\left(\frac{\mathrm{tr}(S_h)}{q} - \frac{\mathrm{tr}(S_e)}{n}\right)\right\}g_2(S_h,S_e)\right],\end{aligned}$$

where

$$g_2(S_h,S_e) = 1 - \frac{itW}{2nc_2\sqrt{2(r_1+r_2)c_2}}\left(\frac{\mathrm{tr}(S_h)}{q} - \frac{\mathrm{tr}(S_e)}{n}\right) + O_p(n^{-2}).$$

Let Z_3 be a $p \times q$ random matrix distributed according to $\mathrm{N}_{p\times q}(O, I_{pq})$ and Z_4 be a $p \times n$ random matrix distributed according to $\mathrm{N}_{p\times n}(O, I_{pn})$. Then we can represent the characteristic function as

$$C_2(t) = \mathrm{E}\left[\exp\left\{\frac{it}{\sqrt{2(r_1+r_2)c_2}}\left(\frac{\mathrm{tr}(\Sigma Z_3 Z_3')}{q} - \frac{\mathrm{tr}(\Sigma Z_4 Z_4')}{n}\right)\right\}g_2(S_h, S_e)\right]$$

$$= \int\int (2\pi)^{-p(n+q)/2}\,\mathrm{etr}\left\{-\frac{1}{2}\left(I_p - \frac{\sqrt{2}it}{q\sqrt{(r_1+r_2)c_2}}\Sigma\right)Z_3Z_3'\right.$$

$$\left. - \frac{1}{2}\left(I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma\right)Z_4Z_4'\right\}g_2(S_h, S_e)dZ_3dZ_4.$$

Next, we transform Z_3 and Z_4 as

$$Z_3 = \left(I_p - \frac{\sqrt{2}it}{q\sqrt{(r_1+r_2)c_2}}\Sigma\right)^{-1/2}\tilde{Z}_3,$$

$$Z_4 = \left(I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma\right)^{-1/2}\tilde{Z}_4,$$

respectively. Then the characteristic function is expressed as

$$C_2(t) = \left|I_p - \frac{\sqrt{2}it}{q\sqrt{(r_1+r_2)c_2}}\Sigma\right|^{-q/2}\left|I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma\right|^{-n/2}$$

$$\times \int\int (2\pi)^{-p(n+q)/2}\,\mathrm{etr}\left\{-\frac{1}{2}\tilde{Z}_3\tilde{Z}_3' - \frac{1}{2}\tilde{Z}_4\tilde{Z}_4'\right\}g_2(S_h, S_e)d\tilde{Z}_3d\tilde{Z}_4$$

$$= \left|I_p - \frac{\sqrt{2}it}{q\sqrt{(r_1+r_2)c_2}}\Sigma\right|^{-q/2}\left|I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma\right|^{-n/2}$$

$$\times \mathrm{E}_{\tilde{Z}_3, \tilde{Z}_4}[g_2(S_h, S_e)].$$

For the Jacobian, it holds that

$$\left|I_p - \frac{\sqrt{2}it}{q\sqrt{(r_1+r_2)c_2}}\Sigma\right|^{-q/2} = \exp\left(\frac{pitc_1}{\sqrt{2(r_1+r_2)c_2}} + \frac{(it)^2 r_1}{2(r_1+r_2)}\right)$$

$$\times\left\{1 + \frac{\sqrt{2}(it)^3 r_1 c_3}{3q\sqrt{(r_1+r_2)^3 c_2^3}} + O(q^{-2})\right\},$$

$$\left| I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma \right|^{-n/2} = \exp\left(-\frac{pitc_1}{\sqrt{2(r_1+r_2)c_2}} + \frac{(it)^2 r_2}{2(r_1+r_2)} \right) \times \left\{ 1 - \frac{\sqrt{2}(it)^3 r_2 c_3}{3n\sqrt{(r_1+r_2)^3 c_2^3}} + O(n^{-2}) \right\}.$$

To calculate the $\mathrm{E}_{\tilde{Z}_3,\tilde{Z}_4}[g_2(S_h,S_e)]$, we expand $g_2(S_h,S_e)$ as

$$\begin{aligned}
g_2(S_h,S_e) &= 1 - \frac{itW}{2nc_2\sqrt{2(r_1+r_2)c_2}}\left(\frac{\mathrm{tr}(S_h)}{q} - \frac{\mathrm{tr}(S_e)}{n}\right) + O_p(n^{-2}) \\
&= 1 - \frac{it}{2pc_2\sqrt{2(r_1+r_2)c_2}} \\
&\quad \times \left\{ \left(\frac{\mathrm{tr}(\Sigma Z_4 Z_4' \Sigma Z_4 Z_4')}{n^2} - \frac{(\mathrm{tr}(\Sigma Z_4 Z_4'))^2}{n^3} \right) - \mathrm{tr}(\Sigma^2) \right\} \\
&\quad \times \left(\frac{\mathrm{tr}(\Sigma Z_3 Z_3')}{q} - \frac{\mathrm{tr}(\Sigma Z_4 Z_4')}{n} \right) + O_p(n^{-2}) \\
&= 1 - \frac{it}{2pc_2\sqrt{2(r_1+r_2)c_2}} \\
&\quad \times \left\{ \frac{1}{n^2}\,\mathrm{tr}\left[\Sigma \left(I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma \right)^{-1} \tilde{Z}_4 \tilde{Z}_4' \right.\right. \\
&\quad \left. \times\, \Sigma \left(I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma \right)^{-1} \tilde{Z}_4 \tilde{Z}_4' \right] \\
&\quad \left. - \frac{1}{n^3}\left(\mathrm{tr}\left[\Sigma \left(I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma \right)^{-1} \tilde{Z}_4 \tilde{Z}_4' \right] \right)^2 - \mathrm{tr}(\Sigma^2) \right\} \\
&\quad \times \left\{ \frac{1}{q}\,\mathrm{tr}\left[\Sigma \left(I_p - \frac{\sqrt{2}it}{q\sqrt{(r_1+r_2)c_2}}\Sigma \right)^{-1} \tilde{Z}_3 \tilde{Z}_3' \right] \right. \\
&\quad \left. - \frac{1}{n}\,\mathrm{tr}\left[\Sigma \left(I_p + \frac{\sqrt{2}it}{n\sqrt{(r_1+r_2)c_2}}\Sigma \right)^{-1} \tilde{Z}_4 \tilde{Z}_4' \right] \right\} + O_p(n^{-2}) \\
&= 1 - \frac{it}{2pc_2\sqrt{2(r_1+r_2)c_2}}
\end{aligned}$$

$$\times\left(\frac{1}{n^2}\ \mathrm{tr}((\Sigma\tilde{Z}_4\tilde{Z}_4')^2)-\frac{2\sqrt{2}it}{n^3\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4'\Sigma^2\tilde{Z}_4\tilde{Z}_4')\right.$$

$$-\frac{1}{n^3}(\mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4'))^2+\frac{2\sqrt{2}it}{n^4\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4')\ \mathrm{tr}(\Sigma^2\tilde{Z}_4\tilde{Z}_4')$$

$$\left.-\ \mathrm{tr}(\Sigma^2)\right)\times\left(\frac{1}{q}\ \mathrm{tr}(\Sigma\tilde{Z}_3\tilde{Z}_3')+\frac{\sqrt{2}it}{q^2\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma^2\tilde{Z}_3\tilde{Z}_3')\right.$$

$$\left.-\frac{1}{n}\ \mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4')+\frac{\sqrt{2}it}{n^2\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma^2\tilde{Z}_4\tilde{Z}_4')\right)+O_p(n^{-2}).$$

By using the above expansion, we obtain the following expression:

$$\mathrm{E}_{\tilde{Z}_3,\tilde{Z}_4}[g_2(S_h,S_e)]=1-\frac{it}{2pc_2\sqrt{2(r_1+r_2)c_2}}$$

$$\times\left\{\left(\mathrm{tr}(\Sigma)+\frac{\sqrt{2}it}{q\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma^2)\right)\right.$$

$$\times\left(\frac{1}{n^2}\mathrm{E}[\mathrm{tr}((\Sigma\tilde{Z}_4\tilde{Z}_4')^2)]-\frac{1}{n^3}\mathrm{E}[(\mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4'))^2]\right.$$

$$-\frac{2\sqrt{2}it}{n^3\sqrt{(r_1+r_2)c_2}}\mathrm{E}[\mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4'\Sigma^2\tilde{Z}_4\tilde{Z}_4')]$$

$$\left.+\frac{2\sqrt{2}it}{n^4\sqrt{(r_1+r_2)c_2}}\mathrm{E}[\mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4')\ \mathrm{tr}(\Sigma^2\tilde{Z}_4\tilde{Z}_4')]\right)$$

$$-\ \mathrm{E}\left[\left(\frac{1}{n}\ \mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4')-\frac{\sqrt{2}it}{n^2\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma^2\tilde{Z}_4\tilde{Z}_4')\right)\right.$$

$$\times\left(\frac{1}{n^2}\ \mathrm{tr}((\Sigma\tilde{Z}_4\tilde{Z}_4')^2)-\frac{1}{n^3}(\mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4'))^2\right.$$

$$-\frac{2\sqrt{2}it}{n^3\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4'\Sigma^2\tilde{Z}_4\tilde{Z}_4')$$

$$\left.\left.+\frac{2\sqrt{2}it}{n^4\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma\tilde{Z}_4\tilde{Z}_4')\ \mathrm{tr}(\Sigma^2\tilde{Z}_4\tilde{Z}_4')\right)\right]$$

$$\left.-\frac{\sqrt{2}it}{\sqrt{(r_1+r_2)c_2}}\left(\frac{1}{q}+\frac{1}{n}\right)(\mathrm{tr}(\Sigma^2))^2\right\}+O(n^{-2}).$$

By Appendix A.4, we obtain the following expectation:

$$
\begin{aligned}
\mathrm{E}_{\tilde{Z}_3,\tilde{Z}_4}[g_2(S_h,S_e)] = 1 &- \frac{it}{2pc_2\sqrt{2(r_1+r_2)c_2}} \\
&\times\left\{\left(\mathrm{tr}(\Sigma)+\frac{\sqrt{2}it}{q\sqrt{(r_1+r_2)c_2}}\ \mathrm{tr}(\Sigma^2)\right)\right. \\
&\times\left(\left(1+\frac{1}{n}-\frac{2}{n^2}\right)\mathrm{tr}(\Sigma^2)-\frac{2\sqrt{2}it}{\sqrt{(r_1+r_2)c_2}}\left(\frac{1}{n}+\frac{1}{n^2}\right)\mathrm{tr}(\Sigma^3)\right) \\
&-\left(\frac{1}{n^3}\{4n^2\ \mathrm{tr}(\Sigma^3)+n(n^2+n-2)\ \mathrm{tr}(\Sigma)\ \mathrm{tr}(\Sigma^2))\}\right. \\
&\left.-\frac{2\sqrt{2}it}{n^4\sqrt{(r_1+r_2)c_2}}n(n+2)(n-1)\ \mathrm{tr}(\Sigma)\ \mathrm{tr}(\Sigma^3)\right) \\
&+\frac{\sqrt{2}it}{n^2\sqrt{(r_1+r_2)c_2}}\times\left(\frac{1}{n^2}n^2(n+1)(\mathrm{tr}(\Sigma^2))^2\right. \\
&\left.\left.-\frac{2\sqrt{2}it}{n^3\sqrt{(r_1+r_2)c_2}}n(n^2+n-2)\ \mathrm{tr}(\Sigma^2)\ \mathrm{tr}(\Sigma^3)\right)\right] \\
&\left.-\frac{\sqrt{2}it}{\sqrt{(r_1+r_2)c_2}}\left(\frac{1}{q}+\frac{1}{n}\right)(\mathrm{tr}(\Sigma^2))^2\right\}+O(n^{-2}) \\
= 1 &+ \frac{1}{n}\left\{\frac{\sqrt{2}itc_3}{c_2\sqrt{(r_1+r_2)c_2}}-\frac{(it)^2}{2}+\frac{\sqrt{2}(it)^3c_3}{c_2\sqrt{(r_1+r_2)c_2}}\right\}+O(n^{-2}).
\end{aligned}
$$

Hence, we obtain the expansion of the characteristic function:

$$
\begin{aligned}
C_2(t) = \exp((it)^2/2)\times\left\{1+\frac{\sqrt{2}(it)^3(r_1-r_2)c_3}{3p\sqrt{(r_1+r_2)c_2^3}}+\frac{\sqrt{2}itc_3}{nc_2\sqrt{(r_1+r_2)c_2}}\right. \\
\left.-\frac{(it)^2}{2n}+\frac{\sqrt{2}(it)^3c_3}{nc_2\sqrt{(r_1+r_2)c_2}}\right\}+O(n^{-2}).
\end{aligned}
$$

3.3. The asymptotic expansion and the Cornish-Fisher expansion. By using the inversion formula, we obtain the density function of $T/\hat{\sigma}$:

$$f_2(x) = \phi(x)\left\{1 + \frac{1}{n}\left(\frac{2c_3}{c_2\sqrt{2(r_1+r_2)c_2}}h_1(x) - \frac{1}{2}h_2(x) + \frac{\sqrt{2}c_3}{\sqrt{(r_1+r_2)c_2^3}}\left(\frac{2}{3}+\frac{r_1}{3r_2}\right)h_3(x)\right) + O(n^{-2})\right\}.$$

By integrating the density function and using the consistent estimator, we obtain the following theorem.

THEOREM 2. *We assume the framework* (1) *and the condition* (2). *Then we obtain the asymptotic expansion of the null distribution of* $T/\hat{\sigma}$ *as*

$$Pr\left(\frac{T}{\hat{\sigma}} \le z\right) = \Phi(z) - \phi(z)\frac{1}{n} \times \left(\frac{2\hat{c}_3}{\hat{c}_2\sqrt{2(r_1+r_2)\hat{c}_2}} - \frac{1}{2}h_1(x) + \frac{\sqrt{2}\hat{c}_3}{\sqrt{(r_1+r_2)\hat{c}_2^3}}\left(\frac{2}{3}+\frac{r_1}{3r_2}\right)h_2(x)\right) + O_p(n^{-2}),$$

where

$$\hat{\sigma} = \frac{\sqrt{2(r_1+r_2)\{\mathrm{tr}(S_e^2)/n^2 - (\mathrm{tr}(S_e))^2/n^3\}/p}}{\mathrm{tr}(S_e)/(np)},$$

$$\hat{c}_2 = \frac{n^2}{p(n+2)(n-1)}\left(\frac{\mathrm{tr}(S_e^2)}{n^2} - \frac{(\mathrm{tr}(S_e))^2}{n^3}\right),$$

$$\hat{c}_3 = \frac{1}{pn^3}\left(\mathrm{tr}(S_e^3) - \frac{3}{n}\ \mathrm{tr}(S_e^2)\ \mathrm{tr}(S_e) + \frac{2}{n^2}(\mathrm{tr}(S_e))^3\right).$$

By using the above theorem, we obtain the Cornish-Fisher expansion of the upper percent point of the null distribution as in the following corollary.

COROLLARY 2. *Let* z_α *be the upper* $100\alpha\%$ *point of the standard normal distribution and let*

$$z_2(\alpha) = z_\alpha + \frac{1}{n}\left(\frac{2\hat{c}_3}{\hat{c}_2\sqrt{2(r_1+r_2)\hat{c}_2}} - \frac{1}{2}z_\alpha + \frac{\sqrt{2}\hat{c}_3}{\sqrt{(r_1+r_2)\hat{c}_2^3}}\left(\frac{2}{3}+\frac{r_1}{3r_2}\right)(z_\alpha^2-1)\right).$$

Then

$$Pr\left(\frac{T}{\hat{\sigma}} \le z_2(\alpha)\right) = 1 - \alpha + O(n^{-2}).$$

4. Numerical simulation

In this section, we examine the accuracy of the asymptotic expansion by the numerical simulation.

We study the accuracy of the approximated upper 5 percent points given by Corollary 1 and Corollary 2. The values of p, n, q and Σ were chosen as follows:

$$q : 5, 10, 20, \quad n : 40, 80, \quad p : 40, 80, \quad \Sigma = I_p (=: \Sigma_1), \quad \begin{pmatrix} I_{p/2} & O \\ O & 2I_{p/2} \end{pmatrix} (=: \Sigma_2).$$

Table 1 shows the actual error probabilities of the first kind by using the approximated percent points given by Corollary 1. Here the actual error

Σ	q	n	p	limit	first	second
Σ_1	5	40	40	0.058	0.050	0.050
Σ_1	5	40	80	0.055	0.049	0.049
Σ_1	5	80	40	0.057	0.050	0.050
Σ_1	5	80	80	0.055	0.050	0.050
Σ_1	10	40	40	0.057	0.050	0.049
Σ_1	10	40	80	0.055	0.050	0.050
Σ_1	10	80	40	0.055	0.050	0.050
Σ_1	10	80	80	0.055	0.051	0.051
Σ_1	20	40	40	0.057	0.051	0.050
Σ_1	20	40	80	0.056	0.051	0.050
Σ_1	20	80	40	0.056	0.051	0.051
Σ_1	20	80	80	0.054	0.051	0.051
Σ_2	5	40	40	0.058	0.050	0.050
Σ_2	5	40	80	0.055	0.049	0.049
Σ_2	5	80	40	0.058	0.051	0.051
Σ_2	5	80	80	0.054	0.049	0.049
Σ_2	10	40	40	0.057	0.050	0.050
Σ_2	10	40	80	0.055	0.050	0.049
Σ_2	10	80	40	0.058	0.051	0.051
Σ_2	10	80	80	0.053	0.049	0.049
Σ_2	20	40	40	0.057	0.049	0.048
Σ_2	20	40	80	0.054	0.049	0.049
Σ_2	20	80	40	0.056	0.050	0.050
Σ_2	20	80	80	0.054	0.050	0.050

Table 1. The case such that Σ is known

probabilities of the first kind are estimated by using the 100,000 samples generated by Monte Carlo simulation. We compare the limiting percent point with the expansion up to the first order using the terms of n^{-1} and q^{-1}. Since the difference of the expansions up to the first order and up to the second order using the terms of n^{-2}, q^{-2} and $n^{-1}q^{-1}$ is a range of an error, it is enough even if we do not use the second order. From the Table 1, we see that the approximation is good even if q is small. Table 2 shows the actual error probabilities of the first kind by using the approximated percent point given by Corollary 2. The actual error probabilities of the first kind are estimated by using the 100,000 samples generated by Monte Carlo simulation. This result is good, too. But compared to the Table 1, the approximation is a little worse. We think that the gap is caused by the estimations of the unknown parameters c_1, c_2 and c_3. For the improvement of the approximation, we need to derive the estimation that is better. Since it is difficult to estimate the degrees of freedom of F approximation, the comparison with F approximation is a future problem.

Σ	q	n	p	limit	first
Σ_1	5	40	40	0.057	0.051
Σ_1	5	40	80	0.056	0.052
Σ_1	5	80	40	0.056	0.050
Σ_1	5	80	80	0.055	0.051
Σ_1	10	40	40	0.058	0.053
Σ_1	10	40	80	0.056	0.052
Σ_1	10	80	40	0.056	0.051
Σ_1	10	80	80	0.054	0.051
Σ_1	20	40	40	0.053	0.053
Σ_1	20	40	80	0.056	0.053
Σ_1	20	80	40	0.055	0.050
Σ_1	20	80	80	0.054	0.052
Σ_2	5	40	40	0.060	0.053
Σ_2	5	40	80	0.056	0.052
Σ_2	5	80	40	0.059	0.052
Σ_2	5	80	80	0.055	0.051
Σ_2	10	40	40	0.059	0.053
Σ_2	10	40	80	0.056	0.052
Σ_2	10	80	40	0.056	0.051
Σ_2	10	80	80	0.055	0.051
Σ_2	20	40	40	0.059	0.053
Σ_2	20	40	80	0.056	0.052
Σ_2	20	80	40	0.056	0.051
Σ_2	20	80	80	0.055	0.052

Table 2. The case such that Σ is unknown

Appendix A

A.1. The limiting distribution of U and V.

Lemma 1. *Let* $S_1 \sim \mathrm{W}_p(q, \Sigma)$ *and* $S_2 \sim \mathrm{W}_p(n, \Sigma)$. *Let* U *and* V *be defined by*

$$U = \frac{1}{\sqrt{pq}}(\mathrm{tr}(S_h) - q\,\mathrm{tr}(\Sigma)),$$

$$V = \frac{1}{\sqrt{np}}(\mathrm{tr}(S_e) - n\,\mathrm{tr}(\Sigma)).$$

Then under the assumption $\mathrm{tr}(\Sigma^k) = O(p)$, *both* U *and* V *are asymptotically distributed according to the same normal distribution* $\mathrm{N}(0, 2c_2)$.

Proof. If $X \sim \chi^2(q)$, then the characteristic function of X is

$$\varphi_X(t) = (1 - 2it)^{-q/2}.$$

Let $\lambda_1, \dots, \lambda_p$ be the eigenvalues of Σ and let $X_1, \dots, X_p \sim i.i.d.\chi^2(q)$. Then U is expressed as

$$U = \frac{1}{\sqrt{pq}}\left(\sum_{j=1}^{p} \lambda_j X_j - q\,\mathrm{tr}(\Sigma)\right).$$

So the characteristic function of U is given by

$$\begin{aligned}
\varphi_U(t) &= \mathrm{E}[\exp(itU)] \\
&= \mathrm{E}\left[\exp\left(\frac{it}{\sqrt{pq}}\sum_{j=1}^{p}\lambda_j X_j - it\sqrt{\frac{q}{p}}\,\mathrm{tr}(\Sigma)\right)\right] \\
&= \left(\prod_{j=1}^{p}\mathrm{E}\left[\exp\left(\frac{it\lambda_j}{\sqrt{pq}}X_j\right)\right]\right)\exp\left(-it\sqrt{\frac{q}{p}}\,\mathrm{tr}(\Sigma)\right) \\
&= \left(\prod_{j=1}^{p}\left(1 - 2i\frac{t\lambda_j}{\sqrt{pq}}\right)^{-q/2}\right)\exp\left(-it\sqrt{\frac{q}{p}}\,\mathrm{tr}(\Sigma)\right).
\end{aligned}$$

The logarithm of the characteristic function is expressed as

$$\log \varphi_U(t) = -\frac{q}{2}\sum_{j=1}^{p}\log\left(1 - 2i\frac{t\lambda_j}{\sqrt{pq}}\right) - it\sqrt{\frac{q}{p}}\,\mathrm{tr}(\Sigma).$$

Generally, for any complex number x, it holds that

$$-\log(1-x) = x + \frac{1}{2}x^2 + O(|x|^3) \qquad (\text{for } |x| < 1).$$

By using this expansion, $\log \varphi_U(t)$ is calculated as

$$\begin{aligned}\log \varphi_U(t) &= \frac{q}{2}\sum_{j=1}^{p} 2i\frac{t\lambda_j}{\sqrt{pq}} + \frac{q}{2}\sum_{j=1}^{p}\frac{1}{2}\left(2i\frac{t\lambda_j}{\sqrt{pq}}\right)^2 + \frac{q}{2}\sum_{j=1}^{p} O\left(\left|\frac{t\lambda_j}{\sqrt{pq}}\right|^3\right) - it\sqrt{\frac{q}{p}}\,\mathrm{tr}(\Sigma) \\ &= \frac{i^2t^2}{p}\,\mathrm{tr}(\Sigma^2) + O\left(\frac{|t|^3\,\mathrm{tr}(\Sigma^3)}{p\sqrt{pq}}\right) \\ &= \frac{1}{2}i^2t^2(2c_2) + O\left(\frac{|t|^3 c_3}{\sqrt{pq}}\right).\end{aligned}$$

Therefore, $\varphi_U(t)$ converges to the characteristic function of $\mathrm{N}(0, 2c_2)$. Similarly, $\varphi_V(t)$ converges to the characteristic function of $\mathrm{N}(0, 2c_2)$. □

A.2. The calculation of the expectation when Σ is known.

LEMMA 2. *Let Z_1 and Z_2 be a $p \times q$ random matrix distributed according to $\mathrm{N}_{p\times q}(O, I_{pq})$ and be a $p \times n$ random matrix distributed according to $\mathrm{N}_{p\times n}(O, I_{pn})$, respectively and let*

$$U = U_1 + \frac{2it\sqrt{r_1}c_2}{c_1} + \frac{1}{q}\left(\frac{2it}{c_1}U_2 + \frac{4(it)^2\sqrt{r_1}c_3}{c_1^2}\right) + O_p(p^{-2})$$

$$\left(U_1 = \frac{1}{\sqrt{pq}}(\mathrm{tr}(\Sigma\tilde{Z}_1\tilde{Z}_1') - pqc_1) \text{ and } U_2 = \frac{1}{\sqrt{pq}}(\mathrm{tr}(\Sigma^2\tilde{Z}_1\tilde{Z}_1') - pqc_2)\right),$$

$$V = V_1 - \frac{2it\sqrt{r_2}c_2}{c_1} - \frac{1}{n}\left(\frac{2it}{c_1}V_2 - \frac{4(it)^2\sqrt{r_2}c_3}{c_1^2}\right) + O_p(p^{-2})$$

$$\left(V_1 = \frac{1}{\sqrt{np}}(\mathrm{tr}(\Sigma\tilde{Z}_2\tilde{Z}_2') - npc_1) \text{ and } V_2 = \frac{1}{\sqrt{np}}(\mathrm{tr}(\Sigma^2\tilde{Z}_2\tilde{Z}_2') - npc_2)\right).$$

Then

$$\mathrm{E}[V^2] = 2c_2 + \frac{4(it)^2 r_2 c_2^2}{c_1^2} - \frac{2}{n}\left\{\frac{4itc_3}{c_1} + \frac{8(it)^3 r_2 c_2 c_3}{c_1^3}\right\} + O(n^{-2}),$$

$$\mathrm{E}[UV] = -\frac{4(it)^2\sqrt{r_1 r_2}c_2^2}{c_1^2} + \frac{1}{p}\frac{8(it)^3\sqrt{r_1 r_2}(r_2 - r_1)c_2 c_3}{c_1^3} + O(n^{-2}),$$

$$\mathrm{E}[V^4] = 12c_2^2 + \frac{48(it)^2 r_2 c_2^3}{c_1^2} + \frac{16(it)^4 r_2^2 c_2^4}{c_1^4} + O(n^{-1}),$$

$$\mathrm{E}[U^2V^2] = 4c_2^2 + \frac{8(it)^2(r_1 + r_2)c_2^3}{c_1^2} + \frac{16(it)^4 r_1 r_2 c_2^4}{c_1^4} + O(n^{-1}),$$

$$\mathrm{E}[UV^2] = \frac{4it\sqrt{r_1}c_2^2}{c_1} + \frac{8(it)^3\sqrt{r_1}r_2c_2^3}{c_1^3} + O(n^{-1}),$$

$$\mathrm{E}[V^3] = -\frac{12it\sqrt{r_2}c_2^2}{c_1} - \frac{8(it)^3 r_2\sqrt{r_2}c_2^3}{c_1^3} + O(n^{-1}),$$

$$\mathrm{E}[UV^3] = -\frac{24(it)^2\sqrt{r_1r_2}c_2^3}{c_1^2} - \frac{16(it)^4 r_2\sqrt{r_1r_2}c_2^4}{c_1^4} + O(n^{-1}).$$

Proof. By the definition of U and V, and by the property of normal distribution, we obtain

$$\begin{aligned}
\mathrm{E}[V^2] &= \mathrm{E}\left[\left(V_1 - \frac{2it\sqrt{r_2}c_2}{c_1}\right)^2\right] \\
&\quad - \frac{2}{n}\mathrm{E}\left[\left(V_1 - \frac{2it\sqrt{r_2}c_2}{c_1}\right)\left(\frac{2it}{c_1}V_2 - \frac{4(it)^2\sqrt{r_2}c_3}{c_1^2}\right)\right] + O(n^{-2}) \\
&= E[V_1^2] + \frac{4(it)^2 r_2 c_2^2}{c_1^2} - \frac{2}{n}\left(\frac{2it}{c_1}\mathrm{E}[V_1V_2] + \frac{8(it)^3 r_2 c_2 c_3}{c_1^3}\right) + O(n^{-2}) \\
&= 2c_2 + \frac{4(it)^2 r_2 c_2^2}{c_1^2} - \frac{2}{n}\left\{\frac{2it}{npc_1}\mathrm{E}[(\mathrm{tr}(\Sigma\tilde{Z}_2\tilde{Z}_2') - npc_1)(\mathrm{tr}(\Sigma^2\tilde{Z}_2\tilde{Z}_2') - npc_2)]\right. \\
&\quad \left. + \frac{8(it)^3 r_2 c_2 c_3}{c_1^3}\right\} + O(n^{-2}) \\
&= 2c_2 + \frac{4(it)^2 r_2 c_2^2}{c_1^2} - \frac{2}{n}\left\{\frac{2it}{npc_1}(2n\,\mathrm{tr}(\Sigma^3) + n^2\,\mathrm{tr}(\Sigma)\,\mathrm{tr}(\Sigma^2) - n^2p^2c_1c_2)\right. \\
&\quad \left. + \frac{8(it)^3 r_2 c_2 c_3}{c_1^3}\right\} + O(n^{-2}) \\
&= 2c_2 + \frac{4(it)^2 r_2 c_2^2}{c_1^2} - \frac{2}{n}\left\{\frac{4itc_3}{c_1} + \frac{8(it)^3 r_2 c_2 c_3}{c_1^3}\right\} + O(n^{-2}).
\end{aligned}$$

Similarly, we obtain the following expansion.

$$\mathrm{E}[UV] = -\frac{4(it)^2\sqrt{r_1r_2}c_2^2}{c_1^2} + \frac{1}{p}\frac{8(it)^3\sqrt{r_1r_2}(r_2-r_1)c_2c_3}{c_1^3} + O(n^{-2}).$$

Further, by the property such that U_1 and V_1 are asymptotically distributed according to $\mathrm{N}(0, 2c_2)$, we obtain the other expansions. □

A.3. The Cornish-Fisher expansion.

LEMMA 3. *When the asymptotic distribution of T/σ is expressed as*

$$\begin{aligned} Pr\left(\frac{T}{\sigma} \le z\right) = \Phi(z) - \phi(z)\Bigg\{ & \frac{1}{q}\frac{a_1}{\sigma^3}h_2(z) + \frac{1}{n}\left(\frac{a_2}{\sigma} + \frac{a_3}{\sigma^3}h_2(z)\right) \\ & + \frac{1}{q^2}\left(\frac{a_4}{\sigma^4}h_3(z) + \frac{a_5}{\sigma^6}h_5(z)\right) \\ & + \frac{1}{n^2}\left(\frac{a_6}{\sigma^2}h_1(z) + \frac{a_7}{\sigma^4}h_3(z) + \frac{a_8}{\sigma^6}h_5(z)\right) \\ & + \frac{1}{nq}\left(\frac{a_9}{\sigma^2}h_1(z) + \frac{a_{10}}{\sigma^4}h_3(z) + \frac{a_{11}}{\sigma^6}h_5(z)\right)\Bigg\} + O(n^{-3}), \end{aligned}$$

let $z_1(\alpha)$ be

$$z_1(\alpha) = z_\alpha + \frac{1}{q}p_1(z_\alpha) + \frac{1}{n}p_2(z_\alpha) + \frac{1}{q^2}p_3(z_\alpha) + \frac{1}{n^2}p_4(z_\alpha) + \frac{1}{nq}p_5(z_\alpha),$$

where z_α is the upper $100\alpha\%$ point of the standard normal distribution and

$$p_1(u) = \frac{a_1}{\sigma^3}(u^2-1),$$

$$p_2(u) = \frac{a_2}{\sigma} + \frac{a_3}{\sigma^3}(u^2-1),$$

$$p_3(u) = \frac{u}{2}p_1^2(u) + \frac{a_4}{\sigma^4}h_3(u) + \frac{a_5}{\sigma^6}h_5(u) + \frac{2ua_1}{\sigma^3}p_1(u) - \frac{ua_1}{\sigma^3}h_2(u)p_1(u),$$

$$p_4(u) = -\frac{u}{2}p_2^2(u) + \frac{a_6}{\sigma^2}h_1(u) + \frac{a_7}{\sigma^4}h_3(u) + \frac{a_8}{\sigma^6}h_5(u) + \frac{2ua_3}{\sigma^3}p_2(u),$$

$$\begin{aligned} p_5(u) = & \frac{a_9}{\sigma^2}h_1(u) + \frac{a_{10}}{\sigma^4}h_3(u) + \frac{a_{11}}{\sigma^6}h_5(u) + \frac{2ua_1}{\sigma^3}p_2(u) \\ & + \frac{2ua_3}{\sigma^3}p_1(u) - \frac{ua_1}{\sigma^3}h_2(u)p_2(u). \end{aligned}$$

Then it holds that

$$Pr\left(\frac{T}{\sigma} \le z_1(\alpha)\right) = 1 - \alpha + O(n^{-3}).$$

PROOF. Let $z_1(\alpha)$ be the upper $100\alpha\%$ point of T/σ. We assume that $z_1(\alpha)$ is expanded as

$$z_1(\alpha) = u + \frac{1}{q}p_1(u) + \frac{1}{n}p_2(u) + \frac{1}{q^2}p_3(u) + \frac{1}{n^2}p_4(u) + \frac{1}{nq}p_5(u) + O(n^{-3}).$$

By the asymptotic distribution of T/σ and the definition of $z_1(\alpha)$, we derive the following expansion:

$$\begin{aligned}
1 - \alpha &= Pr\left(\frac{T}{\sigma} \le z_1(\alpha)\right) \\
&= \Phi(z_1(\alpha)) - \phi(z_1(\alpha))\left\{\frac{1}{q}\frac{a_1}{\sigma^3}h_2(z_1(\alpha)) + \frac{1}{n}\left(\frac{a_2}{\sigma} + \frac{a_3}{\sigma^3}h_2(z_1(\alpha))\right)\right. \\
&\quad + \frac{1}{q^2}\left(\frac{a_4}{\sigma^4}h_3(z_1(\alpha)) + \frac{a_5}{\sigma^6}h_5(z_1(\alpha))\right) \\
&\quad + \frac{1}{n^2}\left(\frac{a_6}{\sigma^2}h_1(z_1(\alpha)) + \frac{a_7}{\sigma^4}h_3(z_1(\alpha)) + \frac{a_8}{\sigma^6}h_5(z_1(\alpha))\right) \\
&\quad \left. + \frac{1}{nq}\left(\frac{a_9}{\sigma^2}h_1(z_1(\alpha)) + \frac{a_{10}}{\sigma^4}h_3(z_1(\alpha)) + \frac{a_{11}}{\sigma^6}h_5(z_1(\alpha))\right)\right\} + O(n^{-3}) \\
&= \Phi(u) + \phi(u)(z_1(\alpha) - u) + \frac{1}{2}\phi'(u)(z_1(\alpha) - u)^2 \\
&\quad - \{\phi(u) + \phi'(u)(z_1(\alpha) - u)\}\left\{\frac{1}{q}\frac{a_1}{\sigma^3}\{h_2(u) + h_2'(u)(z_1(\alpha) - u)\}\right. \\
&\quad + \frac{1}{n}\left(\frac{a_2}{\sigma} + \frac{a_3}{\sigma^3}\{h_2(u) + h_2'(u)(z_1(\alpha) - u)\}\right) \\
&\quad + \frac{1}{q^2}\left(\frac{a_4}{\sigma^4}h_3(u) + \frac{a_5}{\sigma^6}h_5(u)\right) + \frac{1}{n^2}\left(\frac{a_6}{\sigma^2}h_1(u) + \frac{a_7}{\sigma^4}h_3(u) + \frac{a_8}{\sigma^6}h_5(u)\right) \\
&\quad \left. + \frac{1}{nq}\left(\frac{a_9}{\sigma^2}h_1(u) + \frac{a_{10}}{\sigma^4}h_3(u) + \frac{a_{11}}{\sigma^6}h_5(u)\right)\right\} + O(n^{-3}) \\
&= \Phi(u) + \phi(u)\left\{\frac{1}{q}\left(p_1(u) - \frac{a_1}{\sigma^3}h_2(u)\right) + \frac{1}{n}\left(p_2(u) - \frac{a_2}{\sigma} - \frac{a_3}{\sigma^3}h_2(u)\right)\right. \\
&\quad + \frac{1}{q^2}\left(p_3(u) - \frac{u}{2}p_1^2(u) - \frac{a_4}{\sigma^4}h_3(u) - \frac{a_5}{\sigma^6}h_5(u) - \frac{2ua_1}{\sigma^3}p_1(u) + \frac{ua_1}{\sigma^3}h_2(u)p_1(u)\right)
\end{aligned}$$

$$+\frac{1}{n^2}\left(p_4(u)-\frac{u}{2}p_2^2(u)-\frac{a_6}{\sigma^2}h_1(u)-\frac{a_7}{\sigma^4}h_3(u)-\frac{a_8}{\sigma^6}h_5(u)-\frac{2ua_3}{\sigma^3}p_2(u)\right.$$
$$\left.+\frac{ua_2}{\sigma}p_2(u)+\frac{ua_3}{\sigma^3}h_2(u)p_2(u)\right)$$
$$+\frac{1}{nq}\left(p_5(u)-up_1(u)p_2(u)-\frac{a_9}{\sigma^2}h_1(u)-\frac{a_{10}}{\sigma^4}h_3(u)-\frac{a_{11}}{\sigma^6}h_5(u)\right.$$
$$-\frac{2ua_1}{\sigma^3}p_2(u)-\frac{2ua_3}{\sigma^3}p_1(u)+\frac{ua_1}{\sigma^3}h_2(u)p_2(u)+\frac{ua_2}{\sigma}p_1(u)$$
$$\left.\left.+\frac{ua_3}{\sigma^3}h_2(u)p_1(u)\right)\right\}+O(n^{-3}).$$

By comparing the coefficients as a polynomial of $1/q$ and $1/n$, we obtain p_j's. □

A.4. The expectation about the Wishart distribution.

LEMMA 4. *Let $S \sim \mathrm{W}_p(I_p, n)$. Then the following expectations are calculated as*

$$\mathrm{E}[\mathrm{tr}((\Sigma S)^2)] = (n^2+n)\,\mathrm{tr}(\Sigma^2) + n(\mathrm{tr}(\Sigma))^2,$$
$$\mathrm{E}[\mathrm{tr}(\Sigma S \Sigma^2 S)] = (n^2+n)\,\mathrm{tr}(\Sigma^3) + n\,\mathrm{tr}(\Sigma)\,\mathrm{tr}(\Sigma^2),$$
$$\mathrm{E}[(\mathrm{tr}(\Sigma S))^2] = 2n\,\mathrm{tr}(\Sigma^2) + n^2(\mathrm{tr}(\Sigma))^2,$$
$$\mathrm{E}[\mathrm{tr}(\Sigma S)\,\mathrm{tr}(\Sigma^2 S)] = 2n\,\mathrm{tr}(\Sigma^3) + n^2\,\mathrm{tr}(\Sigma)\,\mathrm{tr}(\Sigma^2),$$
$$\mathrm{E}[\mathrm{tr}(\Sigma S)\,\mathrm{tr}((\Sigma S)^2)] = 4n(n+1)\,\mathrm{tr}(\Sigma^3) + n(n^2+n+4)\,\mathrm{tr}(\Sigma)\,\mathrm{tr}(\Sigma^2) + n^2(\mathrm{tr}(\Sigma))^3,$$
$$\mathrm{E}[\mathrm{tr}(\Sigma S)\,\mathrm{tr}(\Sigma S \Sigma^2 S)] = 4n(n+1)\,\mathrm{tr}(\Sigma^4) + n(n^2+n+2)\,\mathrm{tr}(\Sigma)\,\mathrm{tr}(\Sigma^3) + 2n(\mathrm{tr}(\Sigma^2))^2 + n^2(\mathrm{tr}(\Sigma))^2\,\mathrm{tr}(\Sigma^2),$$
$$\mathrm{E}[(\mathrm{tr}(\Sigma S))^3] = 8n\,\mathrm{tr}(\Sigma^3) + 6n^2\,\mathrm{tr}(\Sigma)\,\mathrm{tr}(\Sigma^2) + n^3(\mathrm{tr}(\Sigma))^3,$$
$$\mathrm{E}[(\mathrm{tr}(\Sigma S))^2\,\mathrm{tr}(\Sigma^2 S)] = 8n\,\mathrm{tr}(\Sigma^4) + 2n^2(\mathrm{tr}(\Sigma^2))^2 + 4n^2\,\mathrm{tr}(\Sigma)\,\mathrm{tr}(\Sigma^3) + n^3(\mathrm{tr}(\Sigma))^2\,\mathrm{tr}(\Sigma^2),$$
$$\mathrm{E}[\mathrm{tr}(\Sigma^2 S)\,\mathrm{tr}((\Sigma S)^2)] = 4n(n+1)\,\mathrm{tr}(\Sigma^4) + n^2(n+1)(\mathrm{tr}(\Sigma^2))^2 + 4n\,\mathrm{tr}(\Sigma^3)\,\mathrm{tr}(\Sigma) + n^2(\mathrm{tr}(\Sigma))^2\,\mathrm{tr}(\Sigma^2),$$

$$\mathrm{E}[\mathrm{tr}(\Sigma^2 S)\ \mathrm{tr}(\Sigma S \Sigma^2 S)] = 4n(n+1)\ \mathrm{tr}(\Sigma^5) + n(n^2+n+2)\ \mathrm{tr}(\Sigma^2)\ \mathrm{tr}(\Sigma^3) + 2n\ \mathrm{tr}(\Sigma)\ \mathrm{tr}(\Sigma^4) + n^2(\mathrm{tr}(\Sigma^2))^2\ \mathrm{tr}(\Sigma),$$

$$\mathrm{E}[\mathrm{tr}(\Sigma^2 S)(\mathrm{tr}(\Sigma S))^2] = 8n\ \mathrm{tr}(\Sigma^4) + 2n^2(\mathrm{tr}(\Sigma^2))^2 + 4n^2\ \mathrm{tr}(\Sigma)\ \mathrm{tr}(\Sigma^3) + n^3(\mathrm{tr}(\Sigma))^2\ \mathrm{tr}(\Sigma^2),$$

$$\mathrm{E}[(\mathrm{tr}(\Sigma^2 S))^2\ \mathrm{tr}(\Sigma S)] = 8n\ \mathrm{tr}(\Sigma^5) + 2n^2\ \mathrm{tr}(\Sigma)\ \mathrm{tr}(\Sigma^4) + 4n^2\ \mathrm{tr}(\Sigma^2)\ \mathrm{tr}(\Sigma^3) + n^3(\mathrm{tr}(\Sigma^2))^2\ \mathrm{tr}(\Sigma).$$

Proof. Let $Z \sim \mathrm{N}_{p\times n}(O, I_{pn})$. By the definition of Wishart distribution, we express as $S = ZZ'$. Σ is diagonalized as $\Sigma = P(\mathrm{diag}(\lambda_1,\dots,\lambda_p))P'$ where P is an orthogonal matrix. Then by the property of the normal distribution, we see $Z'P = (z_1,\dots,z_p) \sim \mathrm{N}_{n\times p}(O, I_{pn})$. Hence,

$$\begin{aligned}
\mathrm{E}[\mathrm{tr}((\Sigma S)^2)] &= \mathrm{E}[\mathrm{tr}((\tilde{Z}'\Sigma\tilde{Z})^2)] \\
&= \mathrm{E}\left[\mathrm{tr}\left(\left(\sum_{i=1}^{p}\lambda_i z_i z_i'\right)^2\right)\right] \\
&= \mathrm{E}\left[\mathrm{tr}\left(\sum_{i=1}^{p}\lambda_i^2 (z_i z_i')^2 + \sum_{i\neq j}\lambda_i\lambda_j z_i z_i' z_j z_j'\right)\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{p}\lambda_i^2 (z_i' z_i)^2 + \sum_{i\neq j}\lambda_i\lambda_j (z_i' z_j)^2\right] \\
&= (n^2+2n)\sum_{i=1}^{p}\lambda_i^2 + n\sum_{i\neq j}\lambda_i\lambda_j \\
&= (n^2+n)\ \mathrm{tr}(\Sigma^2) + n(\mathrm{tr}(\Sigma))^2.
\end{aligned}$$

Similarly, expressing by the eigenvalues, we obtain the above formulas. □

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