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Some aspects of the classical potential theory on trees

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Abstract. Potential theory on a Cartier tree T is developed on the lines of the classical and the axiomatic theories on harmonic spaces. The harmonic classifications of such trees are considered; the notion of a subordinate structure on T is introduced to consider more generally the potential theory on T associated with the Schrödinger equation $\Delta u(x) = Q(x)u(x)$, $Q(x) \geq 0$ on T; polysuperharmonic functions and polypotentials on T are defined and a Riesz-Martin representation for positive polysuperharmonic functions is obtained.

1. Introduction

In this note, we study some classical potential-theoretic concepts like balayage, domination principle etc. in the context of a tree T and introduce the notions of polysuperharmonic functions and polypotentials on T and obtain some of their properties. The tree T is taken in the sense of Cartier's [4], a graph with infinite vertices, connected, locally finite and no circuits, provided with a transition probability structure. Bajunaid et al. [1] show that the harmonic functions on the vertices of T can be linearly extended to the edges, so that the extended functions verify the axioms 1, 2, 3 of Brelot. Consequently, some of the properties of harmonic functions and potentials on T can be immediately deduced from the axiomatic potential theory.

However, on many occasions, direct proofs of theorems about harmonic functions on T are simpler and give more informations in comparison to those deduced from the axiomatic theory. Secondly, some theorems in the axiomatic theory require more assumptions than the axioms 1, 2, 3 only. One such is the converse to the Riesz representation theorem in a harmonic space Ω which states that given a positive Radon measure μ on an open set ω in Ω, there exists a superharmonic function s on ω with associated measure μ in a local Riesz representation. To prove this, we need the axiom of analyticity (de La Pradelle [6]) which is not generally valid on T. However, this converse to the Riesz representation is true on T (Theorem 2.4). Thirdly, for polyharmonic

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functions of order $m > 1$, we do not have results like the Dirichlet solution or the Harnack property. Consequently, the properties of polysuperharmonic functions and polypotentials on a tree T cannot be deduced, by treating T as another example of a Brelot harmonic space. Finally, in the theorems proved here, we do not always place the restriction that there should be positive potentials on T.

Section 2 studies the potential theory on a Cartier tree T with a transition probability structure P. One part of the paper [4] by Cartier deals with this, by starting with the definition of the Green function $G(x, y)$ as the kernel associated with the set of all the paths in T and developing the theory of superharmonic functions and potentials on T in the spirit of probability theory. In contrast, the development in this section follows closely the methods of the classical and the axiomatic potential theory, which is useful in the classification theory (Section 3) of determining whether there exist on T, positive potentials, positive non-constant harmonic functions, bounded non-constant harmonic functions etc..

Section 4 studies the potential theory associated with another structure P' on T, that is subordinate to the initial probability structure P. An example of the P'-potential theory on T is the potential theory on T associated with the Schrödinger equation $\Delta u(x) = Q(x)u(x)$ for some $Q \geq 0$ on T.

Section 5 studies the potenial theory on T associated with the operator Δ^m, m integer ≥ 2. After defining polypotentials on T, we obtain a necessary and sufficient condition for the existence of positive polypotentials on T; we discuss the balayage and the domination principle for polypotentials; and finally, in Section 6 a general representation of positive m-superharmonic functions on T is given, on the lines of the Riesz-Martin representation for positive superharmonic functions.

2. Preliminaries

Let T be a tree in the sense of Cartier's [4]: T is an infinite graph, connected, locally finite and without circuits. If $[x, y]$ is an edge on T, x and y are called neighbours, denoted by $x \sim y$. A vertex x in T is called terminal if and only if it has a single neighbour in T. We say that a transition probability structure P is given on T if for any two vertices x and y, there is associated a number $p(x, y) \geq 0$ such that for any x in T, $p(x) = \sum_{x \sim y_i} p(x, y_i) = 1$; $p(x, y) > 0$ if $x \sim y$; $p(x, y) = 0$ if x and y are not neighbours; and $p(x, y)$ need not be equal to $p(y, x)$.

With respect to a transition probability structure P, given any function $f(x)$ on T, define $\Delta f(x) = \sum_{x \sim y} p(x, y) f(y) - f(x)$ for any non-terminal vertex

x in T. Given a subset S of T, a vertex x is called an interior point of S (denoted by $x \in S^\circ$), if every neighbour of x in T belongs to S.

We stipulate that a terminal vertex x is not in S° for any subset S. u is said to be P-harmonic (resp. P-superharmonic) at a vertex x_0, if x_0 is not terminal and if u is defined at x_0 and on all its neighbours and verifies the condition $\Delta u(x_0) = 0$ (resp. $\Delta u(x_0) \le 0$).

A function $h(x)$ defined on a subset S of T is called P-harmonic (resp. P-superharmonic) on S, if h is real-valued on S and $\Delta h(x) = 0$ (resp. $h > -\infty$ on S and $\Delta h(x) \le 0$) at every interior point x of S. (As usual, $u \equiv \infty$ is not considered as P-superharmonic, and v is called P-subharmonic if $-v$ is P-superharmonic.)

If x and y are two vertices, then the length of the geodesic (see [4, p. 212]) joining x and y is called the distance between x and y, denoted by $d(x, y)$. We shall fix a non-terminal vertex e in T and denote $|x| = d(e, x)$, the distance of x measured from e. That is, if $\{e, x_1, x_2, \dots, x_n = x\}$ is the geodesic path connecting e and x, then $|x| = d(e, x) = n$.

In this section, we shall fix a transition probability structure P. Consequently, there will be no confusion if we drop the prefix "P-" from P-superharmonic, P-harmonic etc..

Lemma 2.1 (See the proof of [1, Proposition 4.2]). *Suppose $u(x)$ is defined on $n \le |x| \le n + m$, with an integer $m \ge 1$ and harmonic on $n < |x| < n + m$ (which is an empty set if $m = 1$). Then $u(x)$ extends as a harmonic function for $|x| > n$.*

Proof. Let $|x_1| = n + m$. Consider the neighbours of x_1. There is one x_0 with $|x_0| = n + m - 1$ and the others are finite in number. If x is a vertex in the latter group, then $|x| = n + m + 1$. Denote the set of these latter neighbours by N, which is empty if x_1 is terminal. Note that $u(x)$ is defined at x_0 and x_1. Let $u(x_0) = \alpha_0$ and $u(x_1) = \alpha_1$. Choose the constant α_2 such that if $u(x) = \alpha_2$ at each neighbour x in N, then $u(x)$ is P-harmonic at x_1; that is, $\alpha_1 = \alpha_0 p(x_1, x_0) + \alpha_2 \sum_{x \in N} p(x_1, x)$.

We repeat this procedure for each x_1 with $|x_1| = n + m$. We then get an extension of $u(x)$ as a harmonic function on a set that includes $|x| = n + m$. Proceeding step-by-step, we extend $u(x)$ as a harmonic function on $|x| > n$.

Remark 2.1. 1) *In the above type of construction of a harmonic extension, note the following: Let x and z be neighbours of y with $|x| < |y| < |z|$. If $0 \le u(x) < u(y)$, then we have $u(y) < u(z)$. For, with the notations in the above lemma,*

$$
\begin{aligned}
\alpha_2 - \alpha_1 &= \frac{\alpha_1 - \alpha_0 p(x_1, x_0)}{\sum_N p(x_1, x)} - \alpha_1 \\
&= \frac{\alpha_1 \left(1 - \sum_N p(x_1, x)\right) - \alpha_0 p(x_1, x_0)}{\sum_N p(x_1, x)} \\
&> \frac{\alpha_1 \left(1 - \sum_N p(x_1, x)\right) - \alpha_1 p(x_1, x_0)}{\sum_N p(x_1, x)} \qquad (\text{if } 0 \le \alpha_0 < \alpha_1) \\
&= 0.
\end{aligned}
$$

2) As a special case of the above Lemma, we state: Let s be a superharmonic function defined on $\overline{B_n} = \{x : |x| \le n\}$ with finite harmonic support on $A \subset B_n$; that is, $\Delta s(x) \le 0$ if $|x| < n$ and $\Delta s(x) = 0$ at each vertex x in $B_n \backslash A$. Then there exists a superharmonic function u on T such that $u(x) = s(x)$ for all x, $|x| \le n$ and $u(x)$ has the same harmonic support A. In particular, if $h(x)$ is harmonic on $\overline{B_n}$, $n \ge 1$, then we can find a harmonic function $H(x)$ on T such that $H(x) = h(x)$ if $|x| \le n$.

Definition 2.1. 1) *A simple set ω in T is a set consisting of points x such that x is an interior point of ω or has an interior point of ω as a neighbour. (A terminal point in T is not considered as an interior point.)*

2) *ω is said to be a connected simple set, if ω is simple and ω° is connected.*

Example 2.1. i) *With e as a fixed non-terminal vertex in T, if $|x| = d(x, e)$, then $|x| \le n$ is a connected simple set and $|x| \ge n$ is a simple set.*

ii) *The whole tree T is a connected simple set.*

iii) *If $x_0 \sim e$, then define the section determined by e and x_0 as*

$$[e, x_0] = \{x: \text{the geodesic path joining } e \text{ and } x \text{ passes through } x_0\};$$

e and x_0 are also included in $[e, x_0]$. Note that if e is not terminal, then T is divided into a finite number (>1) of disjoint sections with e as the joining vertex; also each section $[e, x_0]$ is a connected simple set.

Networks and trees

An infinite network (see M. Yamasaki [8] and [9]) consists of a countable set X of nodes and a countable set Y of directed arcs, each arc joining a pair of nodes. If two nodes x_1 and x_2 are joined by an arc, we shall say that x_1 and x_2 are neighbours and denote this situation by writing $x_1 \sim x_2$. With the terminology as in the case of a Cartier tree, $\{X, Y\}$ is locally finite and connected. Corresponding to the transition probability structure given on a tree, one is given a strictly positive real function r on Y. An infinite network N is determined by the set X of nodes, the set Y of directed arcs and the strictly positive function r.

Given a pair of distinct nodes x_1 and x_2 in X, using the function r, in [8, p. 34] a real number $t(x_1, x_2) = t(x_2, x_1) \geq 0$ is associated such that $t(x_1, x_2) = 0$ if and only if x_1 and x_2 are not neighbours. Set $t(x) = \sum_{x_i \sim x} t(x, x_i)$. Consequently, $t(x) > 0$ for any node x. Then, given a real-valued function $u(x)$ on X, the Laplacian of u is defined as $\Delta_N u(x) = -t(x)u(x) + \sum_{x_i \sim x} t(x, x_i)u(x_i)$; u is said to be harmonic (resp. superharmonic) on a set A if $\Delta_N u(x) = 0$ (resp. $\Delta_N u(x) \leq 0$) for all $x \in A$.

For any pair of vertices x and y in T, define $p(x, y) = \frac{t(x, y)}{t(x)}$. Then $p(x, y) \geq 0$ and $p(x, y) = 0$ if and only if x and y are not neighbours; $\sum_{x \sim y} p(x, y) = 1$ for every fixed $x \in X$; and $p(x, y)$ may not be equal to $p(y, x)$. For this probability structure, let us define the Laplacian by $\Delta_T u(x) = \sum_{x_i \sim x} p(x, x_i)u(x_i) - u(x)$.

If an infinite network N is thus considered with this probability structure, then the harmonic structures defined on N by Δ_N and Δ_T are the same. For,

$$\begin{aligned} \Delta_N u(x) &= -t(x)u(x) + \sum_{x_i \sim x} t(x, x_i)u(x_i) \\ &= t(x)\left[-u(x) + \sum_{x_i \sim x} p(x, x_i)u(x_i)\right] \\ &= t(x)\Delta_T u(x). \end{aligned}$$

Since $t(x) > 0$ for any x, for the definition of harmonic (resp. superharmonic) functions on N, one can use Δ_N or Δ_T.

Thus, Yamasaki's study of harmonic functions on an infinite network N ([8] and [9]) is useful while investigating the properties of harmonic and superharmonic functions on a Cartier tree T.

Lemma 2.2 (See [8, Lemma 2.3]). *Let $\{u_n\}$ be a sequence of superharmonic functions (resp. harmonic functions) defined on a connected simple set ω in a tree T. Suppose $u(x) = \lim u_n(x)$ exists on ω. If $u(x)$ is finite at some point in ω°, then $u(x)$ is finite on ω and superharmonic (resp. harmonic) on ω. Moreover,*

$$(-\Delta)u(x) = \lim(-\Delta)u_n(x) \qquad \text{on } \omega^\circ.$$

PROOF. First note that if $s(x)$ is superharmonic on ω, then $s(x)$ should be finite at each point. For suppose $s(x_0) = \infty$ for some $x_0 \in \omega$. If $x_0 \notin \omega^\circ$, then there is some $x_1 \in \omega^\circ$ such that $x_0 \sim x_1$. Then, because $s(x)$ is superharmonic at x_1 and $s(x_0) = \infty$, $s(x_1)$ should be ∞. Thus, if $s = \infty$ at some point x_0 in ω, then we can take without loss of generality $x_0 \in \omega^\circ$. Then, since ω° is connected, $s \equiv \infty$ on ω°. This is a contradiction.

Now to prove the lemma, since u_n is superharmonic on ω, for $x \in \omega^\circ$, $u_n(x) \geq \sum_{x \sim x_i} p(x, x_i) u_n(x_i)$. Take the limit as $n \to \infty$. Then $u(x) \geq \sum_{x \sim x_i} p(x, x_i) u(x_i)$. Since $u(x)$ satisfies the sub-mean-value property on ω° and is finite at some point, $u(x)$ is superharmonic on ω.

Finally, since for any $x \in \omega^\circ$, $(-\Delta) u_n(x) = u_n(x) - \sum_{x \sim y_i} p(x, y_i) u_n(y_i)$, by allowing $n \to \infty$ we obtain

$$(-\Delta) u(x) = u(x) - \sum_{x \sim y_i} p(x, y_i) u(y_i) = \lim_{n \to \infty} (-\Delta) u_n(x).$$

CONSEQUENCE: Let $\{u_n\}$ be a sequence of positive superharmonic functions on a connected simple set ω. Suppose $\sum_{n=1}^{\infty} u_n(y) < \infty$ for some $y \in \omega^\circ$. Then

$$u(x) = \sum_{n=1}^{\infty} u_n(x)$$

is finite for each $x \in \omega$ and $u(x)$ is superharmonic on ω.

THEOREM 2.1 (See [9, Theorem 2.3]). *Let ω be a connected simple set in a tree T. Let $a, b \in \omega^\circ$. Then there exist positive constants α and β, depending on a and b only, such that for any superharmonic function $s \geq 0$ on ω, $\alpha s(b) \leq s(a) \leq \beta s(b)$.*

PROOF. Let $s \geq 0$ be a superharmonic function on ω. Suppose $s(x) = 0$ for some $x \in \omega^\circ$. Then $s \equiv 0$ on ω. So, let us assume $s > 0$ on ω°. Since ω° is connected by assumption, there exists a path in ω° that connects a and b. Hence (see [4, Proposition 1.1]), the geodesic path connecting a and b lies in ω.

Let $\{a = x_0, x_1, \ldots, x_n = b\}$ be the geodesic path connecting a and b. Since s is superharmonic, $s(x_i) \geq \sum_{x_i \sim x} p(x_i, x) s(x)$. Since x_{i+1} is a neighbour of x_i $(0 \leq i \leq n-1)$,

$$\sum_{x_i \sim x} p(x_i, x) s(x) \geq p(x_i, x_{i+1}) s(x_{i+1}).$$

Hence $s(x_i) \geq p(x_i, x_{i+1}) s(x_{i+1})$. Writing such inequalities for all i, $0 \leq i \leq n-1$ and multiplying them we obtain $p(x_0, x_1) p(x_1, x_2) \ldots p(x_{n-1}, b) s(b) \leq s(a)$.

Similarly we can prove that $p(b, x_{n-1}) \dots p(x_1, x_0)s(a) \le s(b)$. Since the geodesic path $\{x_0, \dots, x_n\}$ joining a and b is fixed, we conclude that there are two constants $\alpha > 0$ and $\beta > 0$ depending only on a and b such that $\alpha s(b) \le s(a) \le \beta s(b)$.

Theorem 2.2. *Let s and t be real-valued functions on a set ω in a tree T. Let s be superharmonic and t be subharmonic on ω such that $s \ge t$. Then there exists a harmonic function h on ω such that $s \ge h \ge t$.*

Proof. Take the family $\Im$ of all subharmonic functions u on ω majorized by s on ω°. Let $h(x) = \sup_{u \in \Im} u(x)$. Let $y \in \omega^\circ$. Then there exists a sequence $u_n(y)$ increasing to $h(y)$.

For $x \in \omega$, let $q(x) = \sup u_n(x)$. Then $-\infty < q(x) \le h(x) \le s(x) < \infty$ and $q(x) = \lim v_n(x)$ where $v_n = \sup_{1 \le i \le n} u_i$ is subharmonic. Hence $q(x)$ is a subharmonic function on ω (Lemma 2.2), $q(x) \le h(x)$, and $q(y) = h(y)$. Consequently,

$$h(y) = q(y) \le \sum_{y \sim y_i} p(y, y_i) q(y_i) \le \sum_{y \sim y_i} p(y, y_i) h(y_i).$$

Hence, $h(x)$ is subharmonic at $x = y$. Since y is arbitrary in ω°, $h(x)$ is subharmonic on ω. We claim that $h(x)$ is harmonic on ω.

Take any $y \in \omega^\circ$. By hypothesis, $h(y) \le \sum_{y \sim y_i} p(y, y_i) h(y_i) = v(y)$, say. Consider,

$$v_1(x) = \begin{cases} h(x) & \text{on } \omega \backslash \{y\} \\ v(y) & \text{at } x = y. \end{cases}$$

Then $v_1(x)$ is subharmonic on ω and harmonic at $x = y$. To see this we have only to check the inequalities at $x = y$, and at the neighbouring points of y in ω°. At $x = y$, $v_1(y) = v(y) = \sum_{y \sim y_i} p(y, y_i) h(y_i) = \sum_{y \sim y_i} p(y, y_i) v_1(y_i)$ so that $v_1(x)$ is harmonic at $x = y$. At $x = z \sim y$, $z \in \omega^\circ$,

$$\begin{aligned} v_1(z) = h(z) &\le \sum_{z \sim x_i} p(z, x_i) h(x_i) \qquad (\text{since } h \text{ is subharmonic on } \omega) \\ &= \sum_{z \sim x_i, x_i \ne y} p(z, x_i) h(x_i) + p(z, y) h(y) \\ &\le \sum_{z \sim x_i, x_i \ne y} p(z, x_i) h(x_i) + p(z, y) v(y) \qquad (\text{since } v(y) \ge h(y)) \\ &= \sum_{z \sim x_i} p(z, x_i) v_1(x_i). \end{aligned}$$

Hence v_1 is subharmonic at z.

Also, $v_1(x) \le s(x)$ on ω. To see this, we have only to check at $x = y$; for, on $\omega \backslash \{y\}$, $v_1(x) = h(x) \le s(x)$. Now at $x = y$,

$$\begin{aligned} v_1(y) = v(y) &= \sum_{y \sim y_i} p(y, y_i) h(y_i) \\ &\le \sum_{y \sim y_i} p(y, y_i) s(y_i) \qquad (\text{since } h \le s) \\ &\le s(y) \qquad (\text{since } s \text{ is superharmonic}). \end{aligned}$$

Thus, v_1 is subharmonic on ω and $v_1 \le s$. Hence $v_1 \in \Im$ and consequently $v_1 \le h$ on ω. But by the construction of v_1, $v_1 \ge h$ on ω. Hence $v_1 \equiv h$ on ω. This means that $h(x)$ is harmonic at $x = y$. Since y is arbitrary in ω°, we conclude that $h(x)$ is harmonic on ω and $h \le s$.

REMARK 2.2. *The harmonic function h constructed as above such that $s \ge h \ge t$ on ω is referred to as the greatest harmonic minorant (g.h.m.) of s on ω.*

DEFINITION 2.2. *A real-valued superharmonic function $s \ge 0$ defined on a set ω in a tree T is said to be a potential on ω, if the greatest harmonic minorant of s on ω is 0.*

RIESZ DECOMPOSITION: Suppose $s \ge 0$ is a real-valued superharmonic function on a set ω in a tree T. Then s can be written as a unique sum $s = p + h$ of a potential p on ω and a nonnegative harmonic function h on ω, by choosing h as the greatest harmonic minorant of s on ω.

The Dirichlet problem

Let ω be a finite connected simple set. Let f be a real-valued function on $\partial\omega = \omega \backslash \omega^\circ$. Choose constants α and β such that $\alpha \le f \le \beta$ on $\partial\omega$. Define

$$t(x) = \begin{cases} f(x) & \text{if } x \in \partial\omega \\ \alpha & \text{if } x \in \omega^\circ, \end{cases}$$

and

$$s(x) = \begin{cases} f(x) & \text{if } x \in \partial\omega \\ \beta & \text{if } x \in \omega^\circ. \end{cases}$$

Then on ω, $s(x)$ is superharmonic, $t(x)$ is subharmonic, and $s(x) \ge t(x)$.

Let $h(x)$ be the g.h.m. of $s(x)$ on ω. Since $s(x) = t(x) = f(x)$ on $\partial\omega$, $h(x) = f(x)$ on $\partial\omega$. Thus, h is the Dirichlet solution on ω with boundary value f; remark that h is uniquely determined.

To prove the uniqueness of h, it is enough to show that $H \equiv 0$ if H is harmonic on ω and vanishes on $\partial\omega$: Suppose $H > 0$ at some point. Since ω is finite, H attains its maximum M at some point in ω°. Since ω° is connected, $H \equiv M$ on ω°. Let $y \in \partial\omega$. Since ω is simple, y has a neighbour $x \in \omega^\circ$. If $M > 0$ then

$$M = H(x) = \sum_{x_i \sim x, x_i \neq y} p(x, x_i)H(x_i) + p(x, y)H(y) < M,$$

since $H(x_i) \leq M$ and $H(y) = 0$; this is a contradiction. Hence $H \leq 0$ on ω. Similarly, we show that $H \geq 0$ on ω. Hence $H \equiv 0$.

(Remark that the above method to prove the uniqueness part can be used to obtain the following property for superharmonic functions: Let ω be a connected simple set in T. Suppose s is a superharmonic function on ω, attaining its minimum value at a vertex in ω°. Then s is a constant on ω.)

In this context, we prove the following minimum principle also for superharmonic functions:

Minimum principle: Let s be a superharmonic function defined on a finite connected set ω. Then $\inf_{\partial\omega} s = \inf_\omega s$.

For, suppose $\inf_\omega s = \beta$. Assume that for some $x_0 \in \omega^\circ$, $s(x_0) = \beta$. Take a vertex $y \in \partial\omega$. Since ω is connected, there is a path $\{x_0, x_1, \ldots, x_n = y\}$ connecting x_0 and y. Let i be the smallest index such that $x_i \in \partial\omega$. Since $s(x_0) \geq \sum_{x_0 \sim z} p(x_0, z)s(z)$ and $s(x_0) = \beta$ is the minimum value, we should have $s(z) = \beta$ for every $z \sim x_0$. Since $x_1 \sim x_0$, we have $s(x_1) = \beta$. The same argument repeated, leads to the conclusion $s(x_i) = \beta$. Consequently, $\inf_{\partial\omega} s \leq \beta$ which implies that $\inf_{\partial\omega} s = \inf_\omega s$.

We remark that the above method of finding the Dirichlet solution is based on potential theoretic techniques on a tree T. For an alternate method, using the hitting distribution of the stochastic process generated by the transition probability structure of T, see Berenstein et al. [2, p. 461]. We remark also that the above method proves the existence (not necessarily the uniqueness) of the Dirichlet solution in the following general situation: Let ω be an arbitrary (finite or not) set in a tree T. Let f be a bounded function on $\partial\omega$. Then there exists a bounded function h on ω such that $h = f$ on $\partial\omega$ and h is harmonic on ω°.

We shall use the term P-tree T if there is a potential > 0 on T and the term S-tree if there is no positive potential on T.

Theorem 2.3. *Let e be a non-terminal vertex in a P-tree T. Then there exists a unique potential $G_e(x)$ on T with point harmonic support at e (that is $G_e(x)$ is harmonic outside e) and $(-\Delta)G_e(x) = \delta_e(x)$, the Dirac measure at e.*

PROOF. Let $p(x)$ be a positive potential on T such that $p(e) = 1$. Let $\Im$ be the family of all superharmonic functions $u(x) > 0$ on T such that $u(e) \geq 1$. Let $s(x) = \inf_{u \in \Im} u(x)$. Then an argument as in the proof of Theorem 2.2 shows that $s(x)$ is positive, superharmonic on T, harmonic on $T \setminus \{e\}$ and $s(e) = 1$; moreover since $p \in \Im$, s is a potential on T. Since $(-\Delta)s(e) > 0$, if we define $G_e(x) = \frac{s(x)}{(-\Delta)s(e)}$ on T, then $G_e(x)$ has all the properties stated in the theorem.

For the uniqueness, suppose $Q(x)$ is another such potential on T. Then $(-\Delta)[G_e(x) - Q(x)] = 0$ for all x, so that for a harmonic function $v(x)$ on T, $G_e(x) = Q(x) + v(x)$. We conclude $v \equiv 0$ on T, by using the uniqueness of decomposition of a positive superharmonic function as the sum of a potential and a non-negative harmonic function.

Pseudo-potentials

Let T be an S-tree. Fix a non-terminal vertex e and a function $H \geq 0$ on T such that H is harmonic on $T \setminus \{e\}$, $\Delta H(e) = 1$ and $H(e) = 0$. Since H is subharmonic on the S-tree, if H is bounded, then it should be a constant. This is not the case here. Hence H is unbounded on T. Then (using [1, Theorem 4.3]) for any non-terminal $y \in T$, there exists a unique superharmonic function $q_y(x)$ on T such that $(-\Delta)q_y(x) = \delta_y(x)$ for all x in T, $q_y(y) = 0$ and $q_y(x) - \alpha_y H(x)$ is bounded on T for a uniquely determined $\alpha_y < 0$.

To see the uniqueness of the function $q_y(x)$ and the constant α_y, suppose $s(x)$ is another such superharmonic function on T with the properties: $(-\Delta)s(x) = \delta_y(x)$, $s(y) = 0$ and $s(x) - \beta H(x)$ is bounded on T. Then $h(x) = q_y(x) - s(x)$ is harmonic on T and $|h(x) - (\alpha_y - \beta)H(x)|$ is bounded on T. Since H is positive and unbounded, this would imply that h is bounded at least on one side; and consequently h is a constant since T is an S-tree. Since $h(0) = 0$, $h \equiv 0$. Then $|(\alpha_y - \beta)H(x)|$ is bounded on T, but $H(x)$ is unbounded. Hence $\alpha_y = \beta$. We shall call $q_y(x)$ the (unique) *pseudo-potential* on T with point harmonic support $\{y\}$. Suppose A is a set of non-terminal vertices such that $q(x) = \sum_{x_i \in A} \alpha_i q_{x_i}(x)$, for $\alpha_i \geq 0$, is a superharmonic function on T. Then we refer to $q(x)$ as a pseudo-potential with harmonic support A.

THEOREM 2.4. *Let $f(x) \geq 0$ be a real-valued funtion on T. Then there exists a superharmonic function $s(x)$ on T such that $(-\Delta)s(x) = f(x)$ on T°. (T° is the set of all non-terminal vertices of T.)*

PROOF. In the proof of this theorem we shall write

$$Q_y(x) = \begin{cases} G_y(x) & \text{if } T \text{ is a } P\text{-tree} \\ q_y(x) & \text{if } T \text{ is an } S\text{-tree.} \end{cases}$$

Correspondingly, the term Q-potential refers to a (Green) potential if T is a P-tree and to a pseudo-potential if T is an S-tree.

1) Suppose $f(x) = 0$ outside a finite set A of non-terminal vertices. Let $s(x) = \sum_{a \in A} f(a) Q_a(x)$. Then $s(x)$ is a Q-potential on T such that $(-\Delta)s(a) = f(a)$ for each $a \in A$, and $(-\Delta)s(x) = 0$ if $x \notin A$. Thus, $(-\Delta)s(x) = f(x)$ for all $x \in T^{\circ}$.

2) Suppose $f(x) \geq 0$ is an arbitrary function on T. Fix a non-terminal vertex e and measure distances from e. (See the Preliminaries for the term "distances measured from e"). Recall that for a real-valued function $g(x)$ on T, $\Delta g(x)$ is defined only for the non-terminal vertices x of T.

Let

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| = n+2 \\ 0 & \text{if } |x| \neq n+2. \end{cases}$$

Then by 1) above, there exists a Q-potential $s_n(x)$ on T such that $(-\Delta)s_n(x) = f_n(x)$. Since $s_n(x)$ is harmonic on $|x| < n+2$, by Remark 2.1, there exists a harmonic function $v_n(x)$ on T such that $s_n(x) = v_n(x)$ on $|x| \leq n$.

Let

$$q(x) = \sum_{n=1}^{\infty} [s_n(x) - v_n(x)].$$

Now given any finite set $K \subset \{x : |x| < m\}$, $t_n = s_n - v_n$ is a superharmonic function on T and $t_n = 0$ on K if n is large. That is, in $\sum_{n=1}^{\infty} [s_n(x) - v_n(x)]$ all the terms except a finite number of them are 0 when $x \in K$. Consequently by Lemma 2.2, $q(x)$ is a superharmonic function on $\{x : |x| < m\} \supset K$. Hence $q(x)$ is a superharmonic function on T such that

$$(-\Delta)q(x) = \sum_{n=1}^{\infty} (-\Delta)s_n(x) = \begin{cases} f(x) & \text{if } |x| \geq 3 \\ 0 & \text{if } |x| < 3. \end{cases}$$

Let $v(x) = f(e)Q_e(x) + \sum_{|y|=1} f(y)Q_y(x) + \sum_{|y|=2} f(y)Q_y(x)$. Then $v(x)$ is a Q-potential on T such that

$$(-\Delta)v(x) = \begin{cases} f(x) & \text{if } |x| \leq 2 \\ 0 & \text{if } |x| > 2. \end{cases}$$

Hence, if $s(x) = v(x) + q(x)$, $s(x)$ is a superharmonic function on T such that $(-\Delta)s(x) = f(x)$ for all $x \in T^{\circ}$.

COROLLARY 2.1. *Let F be any subset of T. Let $f(x) \geq 0$ be defined on F. Then there exists a superharmonic function s on T such that $(-\Delta)s(x) = f(x)$ for each $x \in F^\circ$.*

PROOF. Extend f as a positive function on T, by defining $f(y) = 0$ if $y \notin F$. Then, apply the above Theorem 2.4.

COROLLARY 2.2. *Let v be a potential on a P-tree T. Then, for $x \in T$,*

$$v(x) = \sum_{y \in T^\circ} (-\Delta)v(y)G_y(x).$$

PROOF. Fix a non-terminal vertex e in T, and measure distances from e. Construct two superharmonic functions s_1 and s_2 on T such that $(-\Delta)s_1(x) = f_1(x)$ and $(-\Delta)s_2(x) = f_2(x)$ where

$$f_1(x) = \begin{cases} (-\Delta)v(x) & \text{if } |x| \leq n \\ 0 & \text{if } |x| > n, \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & \text{if } |x| \leq n \\ (-\Delta)v(x) & \text{if } |x| > n. \end{cases}$$

Then $v(x) = s_1(x) + s_2(x) + h(x)$ on T where $h(x)$ is harmonic on T. Since $v \geq 0$, $s_2 \geq -s_1 - h$; that is, s_2 has a subharmonic minorant on T. Hence by Theorem 2.2, $s_2 = q_2 + h_2$ where q_2 is a potential and h_2 is a (not necessarily positive) harmonic function on T.

For similar reasons, $s_1 = q_1 + h_1$ where q_1 is a potential and h_1 is harmonic. Then, by using the uniqueness of decomposition property, from $v = q_1 + q_2 + (h_1 + h_2 + h)$, we conclude that $v = q_1 + q_2$ on T. Hence $q_1 \leq v$. Since $q_1(x) = \sum_{|y| \leq n} (-\Delta)v(y)G_y(x)$, by allowing $n \to \infty$ we obtain $Q(x) = \sum_{y \in T^\circ} (-\Delta)v(y)G_y(x) \leq v(x)$. Since $Q(x)$ is a non-negative superharmonic function, majorized by the potential $v(x)$, $Q(x)$ is a potential. Further, $(-\Delta)Q(x) = (-\Delta)v(x)$ so that $Q(x) = v(x) + u(x)$ where $u(x)$ is a harmonic function on T. Again the uniqueness of decomposition implies that $u \equiv 0$. Thus, $v(x) = \sum_{y \in T^\circ} (-\Delta)v(y)G_y(x)$ for every $x \in T$.

REMARK 2.3. *The above representation of a positive potential on T is taken as the definition of a potential in Cartier [4, Sections 2.2 and 2.3]. There, by starting with the notion of the kernel associated to a collection of paths, the Green function $G(x, y)$ is defined. Then, for any function $f \geq 0$ on T, $Gf(x) =$*

$\sum_y G(x, y) f(y)$ *is either always* ∞ *or always finite on* T *satisfying* $\Delta G f = -f$; *in the latter case,* $Gf(x)$ *is termed as the potential of* f.

LEMMA 2.3. *Let* $s(x)$ *be a positive superharmonic function and* $q(x)$ *be a potential on a P-tree* T *such that* $(-\Delta)s \geq (-\Delta)q$ *on* T°. *Then* $s \geq q$ *on* T.

PROOF. By hypothesis, $s = q + s_1$ where s_1 is superharmonic on T. Since $s \geq 0$, $q \geq -s_1$ on T. Since q is a potential and $-s_1$ is subharmonic, $-s_1 \leq 0$. Hence $s \geq q$ on T.

THEOREM 2.5 (Domination Principle). *Let* $s \geq 0$ *be a superharmonic function and* $q \geq 0$ *be a potential on* T. *Let* E *be the harmonic support of* q. *If* $s \geq q$ *on* E, *then we have* $s \geq q$ *on* T.

PROOF. Let $u = \inf(s, q)$. Then u is a potential on T satisfying $u \leq q$ on T and $u = q$ on E. Hence, if $x \in E \cap T^\circ$, then

$$\begin{aligned}(-\Delta)u(x) &= u(x) - \sum_{x \sim y_i} p(x, y_i) u(y_i) \\ &\geq q(x) - \sum_{x \sim y_i} p(x, y_i) q(y_i) \qquad (\text{since } u(x) = q(x) \text{ and } u \leq q \text{ on } T) \\ &= (-\Delta)q(x).\end{aligned}$$

Now, if $x \in E^c \cap T^\circ$, then $(-\Delta)q(x) = 0$, but $(-\Delta)u(x) \geq 0$ always. Thus, for all $x \in T^\circ$, $(-\Delta)u(x) \geq (-\Delta)q(x)$. This implies (Lemma 2.3) that $u(x) \geq q(x)$ on T. Hence $\inf(s, q) = u = q$ on T, so that $q \leq s$ on T.

REMARK 2.4. *In [7, Theorem 5.3], this Domination Principle is proved on* T, *under some restrictions on the transition probability structure* P, *namely: There exists a constant* δ *such that* $0 < \delta < \frac{1}{2}$ *and for all* $s, t \in T$ *with* $s \sim t$, *we should have* $\delta \leq p(s, t) \leq \frac{1}{2} - \delta$.

NOTATION: With the standard notations of balayage, let us write

$$\widehat{R_1^e}(x) = \inf\{s(x) : s \geq 0 \text{ is superharmonic on } T \text{ and } s(e) \geq 1\}.$$

THEOREM 2.6. *Let* T *be a P-tree. For a non-terminal vertex* e, $G_e(x) = G_e(e)\widehat{R_1^e}(x)$ *on* T. *In particular,* $G_e(x) \leq G_e(e)$ *for all* x *in* T.

PROOF. Let $s(x) \geq 0$ be a superharmonic function on T with $s(e) \geq 1$. Let $u(x) = \frac{G_e(x)}{G_e(e)}$. Let $v(x) = \inf(s(x), u(x))$. Then $v(x)$ is a superharmonic function on T such that $v(e) = 1$.

Now,

$$(-\Delta)v(e) = v(e) - \sum_{e\sim x_i} p(e,x_i)v(x_i)$$

$$\geq u(e) - \sum_{e\sim x_i} p(e,x_i)u(x_i) \qquad (\text{since } u \geq v \text{ and } u(e) = v(e) = 1)$$

$$= (-\Delta)u(e).$$

Since $(-\Delta)v(x) \geq 0$ for all x in T° and $(-\Delta)u(x) = 0$ if $x \neq e$, we have $(-\Delta)v(x) \geq (-\Delta)u(x)$ for all x in T°. Hence by Lemma 2.3, $v(x) \geq u(x)$ on T, so that $u(x) = v(x) = \inf(s(x), u(x))$; that is $u(x) \leq s(x)$ on T. Taking the infimum over all such funtions s, we conclude that $u(x) \leq \widehat{R_1^e}(x)$ on T. In particular, $u(x) \leq 1$. Thus $u(x)$ is a positive superharmonic function on T and $u(e) = 1$ so that $u(x) \geq \widehat{R_1^e}(x)$ on T. Hence we have $u(x) = \widehat{R_1^e}(x)$ for all x in T.

PROPOSITION 2.1. *Let u be a subharmonic function defined outside a finite set in a P-tree (resp. S-tree) T. Then there exist a subharmonic function v and two potentials (resp. pseudo-potentils) p_1 and p_2 with finite harmonic support on T such that $u = v + p_1 - p_2$ near infinity and $p_1 - p_2$ is bounded on T if it is a P-tree. Moreover, in case u is harmonic near infinity, v is harmonic on T; and in this case the harmonic function v is uniquely determined if T is a P-tree, but v is uniquely defined only up to an additive constant if T is an S-tree.*

PROOF. For $y \in T^\circ$ let $Q_y(x)$ denote the potential (resp pseudo-potential) with harmonic support at y if T is a P-tree (resp. an S-tree). We have $(-\Delta)Q_y(y) = 1$ in all cases. Fix a non-terminal vertex e and let $|x| = d(e,x)$ be the distance of x from e. For large n, let $B_n u$ denote the Dirichlet solution on $|x| < n$ with boundary values $u(x)$ on $|x| = n$.

Define

$$s(x) = \begin{cases} u(x) & \text{if } |x| > n \\ B_n u & \text{if } |x| \leq n. \end{cases}$$

Then $\Delta s(x) \geq 0$ if $|x| > n$ and $\Delta s(x) = 0$ if $|x| < n$. Let $v(x) = s(x) + \sum_{|y|=n} \Delta s(y) Q_y(x)$. Clearly $\Delta v(x) \geq 0$ if $|x| \neq n$ and if $|x| = n$, $x = y$, $\Delta v(y) = 0$. Thus $\Delta v(x) \geq 0$ for every $x \in T^\circ$. Hence v is subharmonic on T and when $|x| > n$, $u(x) = v(x) + p_1(x) - p_2(x)$ where $p_1(x) = \sum_{|y|=n} [\Delta s(y)]^- Q_y(x)$ and $p_2(x) = \sum_{|y|=n} [\Delta s(y)]^+ Q_y(x)$ so that p_1 and p_2 are potentials (resp. pseudo-potentials) with finite harmonic support if T is a P-tree (resp. an S-tree). By Theorem 2.6, $p_1 - p_2$ is bounded if T is a P-tree.

Finally, suppose u is harmonic near infinity. Then $\Delta v(x) = \Delta s(x) = 0$ if $|x| \neq n$ also, so that $\Delta v = 0$ on T°. Hence v is harmonic on T. Now, suppose $u = v' + p_1' - p_2'$ is another such representation outside a finite set. Then,

(i) if T is a P-tree, then the subharmonic function $|v - v'|$ on T is majorized by the potential $p_1 + p_2 + p_1' + p_2'$ outside a finite set A. Choose a potential $L > 0$ on T. Since A is finite, we can find a constant $\lambda > 0$ such that $|v - v'| \leq \lambda L$ on A. This implies that $|v - v'| \leq \lambda L + (p_1 + p_2 + p_1' + p_2')$ on T. Since the subharmonic function $|v - v'|$ is majorized by a potential on T, $|v - v'| \leq 0$ and hence $v - v' \equiv 0$; and

(ii) if T is an S-tree, since the p's are pseudo-potentials with finite harmonic support, for some α, $v - v' - \alpha H$ is bounded near infinity. Since $v - v'$ is harmonic on T, $\alpha = 0$. This implies that $v - v'$ is a constant c.

Corollary 2.3 (Laurent decomposition). *Let e be a fixed non-terminal vertex and $|x| = d(e, x)$. Suppose $u(x)$ is defined on $n \leq |x| \leq n + m$, with an integer $m \geq 1$ and harmonic on $n < |x| < n + m$. Then there exists a harmonic function $t(x)$ on $|x| \leq n + m$ and a harmonic function $s(x)$ on $|x| \geq n$ such that $u(x) = s(x) - t(x)$ on $n \leq |x| \leq n + m$. Moreover, $s(x)$ can be chosen as follows:*

(i) *if T is a P-tree, then there exists a potential $p(x)$ on T such that $|s(x)| \leq p(x)$ outside a finite set. Hence the decomposition is unique.*

(ii) *if T is an S-tree, then there exists a unique α such that $s(x) - \alpha H(x)$ is bounded outside a finite set. Hence the decomposition is unique up to an additive constant.*

Proof. First use Lemma 2.1 to extend u as a harmonic function on all of $|x| > n$. Then by the above Proposition 2.1, there exists a harmonic function v on T such that $u = v + p_1 - p_2$ (when $|x| > n$), where p_1 and p_2 are potentials on T with finite harmonic support if T is a P-tree and p_1 and p_2 are pseudo-potentials with finite harmonic support if T is an S-tree; the harmonic supports of p_1 and p_2 are in $|x| = n$.

Define $s(x) = u(x) - v(x)$ on $|x| \geq n$ and $t(x) = -v(x)$ on $|x| \leq n + m$. Then the properties stated in the corollary can be verified.

3. Harmonic measure of the point at infinity of a section

The sections in a tree which we are going to define now, are useful to classify the trees with or without positive potentials. They correspond to the "ends" of Cartier's (see "bouts" in [4, p. 212]) and hence to the boundary points in the compactification given in [4, p. 219].

Definition 3.1. *Let e be a vertex in T. Let e_1 be a neighbour of e, such that there exists an infinite geodesic chain $R = \{e, e_1, x_1, x_2, \dots\}$ consisting of distinct elements. Let $\sigma = \sigma_R[e, e_1]$ be the union of R and all the neighbours of* *the vertices in* $R\backslash\{e\}$*. Then, we say that* $\sigma_R[e,e_1]$ *is a section determined by* e *and* e_1 *containing* R*.*

LEMMA 3.1. *Let* $\sigma_R[e,e_1]$ *be a section determined by* e *and* e_1 *containing* $R=\{e,e_1,x_1,x_2,\dots\}$*. Then there exists a function* h *on* $\sigma_R[e,e_1]$ *such that* $h(e)=0$*,* $h(e_1)=1$ *and* $h(x)=c_n$ *if* $d(e_1,x)=n$ *where* c_n *is a sequence of positive numbers such that* $1<c_n<c_{n+1}$ *for all* $n\geq 1$*. Moreover,* $h(x)$ *is harmonic on* $\sigma_R[e,e_1]$*, that is harmonic at every interior point of* $\sigma_R[e,e_1]$*.*

PROOF. Let $x\in\sigma_R[e,e_1]$, $x\neq e$ be such that $d(e_1,x)=1$. Take $h(x)=c_1$ so that h is harmonic at e_1. That is, if t is the probability of transition from e_1 to e, then we have $0+(1-t)c_1=1$ so that $c_1=\frac{1}{1-t}>1$.

Suppose $c_1,\dots,c_n$ are chosen such that $h(x)=c_k$ if $d(e_1,x)=k$ for all k, $1\leq k\leq n$ and $h(x)$ is harmonic at all interior points x of $\sigma_R[e,e_1]$ for which $d(e_1,x)\leq n-1$. Let us fix c_{n+1} so that $h(x)=c_{n+1}$ if $d(e_1,x)=n+1$ and $h(x)$ is harmonic at all interior points x of $\sigma_R[e,e_1]$ for which $d(e_1,x)=n$. For this, we should have the following: Let $d(e_1,x)=n+1$. Let t_n be the probability of transition from x_n to x_{n-1} (take $x_0=e_1$ and $c_0=1$). Then $t_nc_{n-1}+(1-t_n)c_{n+1}=c_n$ so that $c_{n+1}=\frac{c_n-t_nc_{n-1}}{1-t_n}>c_n$ since $c_{n-1}<c_n$. This completes the proof of the lemma.

DEFINITION 3.2. *The section* $\sigma_R[e,e_1]$ *determined by the geodesic chain* R *is called a P-section if the harmonic function* $h(x)$ *in Lemma* 3.1 *is bounded; otherwise it is called an S-section.*

EXAMPLE 3.1. i) *Let* T *be a homogeneous tree of degree* $q+1$ $(q\geq 2)$*; that is, every vertex in* T *has exactly* $q+1$ *neighbours with the transition probability from one vertex to another being* $\frac{1}{q+1}$ *(Cartier [*4*, p.* 262*]). In this case, any section is a P-section.*

For, let e_1 *be a neighbour of* e*. Let* $\sigma_R[e,e_1]$ *be a section determined by an infinite geodesic chain* $\{e,e_1,x_1,x_2,\dots\}$*. Then the construction of the harmonic function as in Lemma* 3.1 *shows that* $h(e)=0$*,* $h(e_1)=1$*, and* $h(x_n)=\sum_{k=0}^{n}q^{-k}$*. Since* $q\geq 2$*,* h *is bounded and hence* $\sigma_R[e,e_1]$ *is a P-section.*

ii) *Let* T *be a star tree with centre* e *(Cartier [*4*, p.* 251*]). Let* e_1 *be a neighbour of* e*. Then the section* $\sigma_R[e,e_1]$ *is an infinite geodesic ray which we shall denote by* $R=\{e,e_1,x_1,x_2,\dots.\}$*. Suppose the transition probability from* x_n *to* x_{n+1} *is* p_n *and from* x_{n+1} *to* x_n *is* q_{n+1} *for* $n\geq 0$ *(taking* $x_0=e_1$*). Let* q_0 *be the transition probability from* e_1 *to* e*. Hence* $p_n+q_n=1$ *for* $n\geq 0$*.*

Then the function h *for* $\sigma_R[e,e_1]$ *constructed as in Lemma* 3.1 *is given by:* $h(e)=0$*, and* $h(e_1)=1$ *and if* $n\geq 1$*,* $h(x_n)=1+\sum_{k=1}^{n}\frac{q_0q_1\dots q_{k-1}}{p_0p_1\dots p_{k-1}}$*. Hence* $\sigma_R[e,e_1]$ *is a P-section if and only if* $\sum_{k=1}^{\infty}\frac{q_0q_1\dots q_{k-1}}{p_0p_1\dots p_{k-1}}$ *is convergent.*

iii) *Let T be a star tree with centre e and N branches, $N \geq 2$, $1 \leq t \leq N$, denoted by $C_t = [s_{0,t}, s_{1,t}, \ldots]$, $s_{0,t} = e$ for all t; and $p_{n,t}$ is the transition probability from $s_{n,t}$ to $s_{n+1,t}$ for $n \geq 0$. Like in (2), on each branch C_t, construct a harmonic function $H_t \geq 0$ such that $H_t(e) = 0$ and $H_t(s_{1,t}) = 1$. Note that H_t is unique and if u is defined on C_t and harmonic at each $s_{n,t}$ for $n \geq 1$, then $u(s_{n,t}) = [u(s_{1,t}) - u(e)]H_t(s_{n,t}) + u(e)$.*

Consequently, any harmonic function h on T is uniquely determined by its N values at $\{s_{1,t}\}$. For, since h is harmonic at e, $h(e) = \sum p_{0,t}h(s_{1,t})$. Then at any vertex $s_{n,t}$ we should have $h(s_{n,t}) = [h(s_{1,t}) - h(e)]H_t(s_{n,t}) + h(e)$.

Now assume that on such a star tree T, there exists a harmonic function $v > 0$ such that $v(e) > v(s_{1,t})$. Then C_t is a P-section. For, from the above representation of a harmonic function on T, we should have $H_t(s_{n,t}) < v(e)[v(e) - v(s_{1,t})]^{-1}$. Since H_t is bounded, C_t is a P-section.

Let e be a vertex in T. Let e_1 be a neighbour of e such that $\sigma_R[e, e_1]$ is a section determined by $R = \{e, e_1, x_1, \ldots\}$. Let H_n be the Dirichlet solution in $\omega_n = \{d(e, x) \leq n\} \cap \sigma_R[e, e_1]$ with boundary values 0 at e and 1 at all points x in $\sigma_R[e, e_1]$ such that $d(e, x) = n$, so that $\{H_n\}$ is a decreasing sequence. Let $H_R(x) = \inf_n H_n(x)$ for each $x \in \sigma_R[e, e_1]$. Clearly $0 \leq H_R(x) \leq 1$.

Definition 3.3. *The harmonic measure of the point at infinity of the section $\sigma_R[e, e_1]$ is said to be 0 if and only if $H_R \equiv 0$.*

Proposition 3.1. *A section $\sigma_R[e, e_1]$ determined by the geodesic chain R, is an S-section if and only if the harmonic measure of the point at infinity of this section is 0.*

Proof. Let $\sigma_R[e, e_1]$ be an S-section. Then by the construction given in Lemma 3.1, there exists a harmonic function h increasing to infinity. Let $x_0 \in \sigma_R[e, e_1]$ be arbitrary. For any N, we can find n such that $h(x) \geq N$ if $d(e, x) \geq n$. Hence (with the above notations) $H_n(x) \leq \frac{h(x)}{N}$ on $\omega_n = \{d(e, x) \leq n\} \cap \sigma_R[e, e_1]$, so that $H_R(x) = \inf_n H_n(x) \leq \frac{h(x)}{N}$ on ω_n. Choose n large so that $x_0 \in \omega_n$. Hence $H_R(x_0) \leq \frac{h(x_0)}{N}$. Since N is arbitrary, $H_R(x_0) = 0$. Hence $H_R \equiv 0$, that is, the harmonic measure of the point at infinity of the section $\sigma_R[e, e_1]$ is 0.

Conversely, suppose $H_R \equiv 0$. Then $\sigma_R[e, e_1]$ should be an S-section. For, otherwise, it is a P-section in which case the function h constructed in Lemma 3.1 should be bounded by a constant M. Let $u(x) = \frac{h(x)}{M}$. Then $u(x) \leq 1$ and $u(e) = 0$. Hence if H_n is harmonic on $\omega_n = \{d(e, x) \leq n\} \cap \sigma_R[e, e_1]$ with boundary value 1 on $d(e, x) = n$ and 0 at e, then $H_n(x) \geq u(x)$ on ω_n. Consequently, $H_R(x) = \inf_n H_n(x) \geq u(x)$. This contradicts the assumption $H_R \equiv 0$. Hence $\sigma_R[e, e_1]$ is not a P-section.

In Bajunaid et al. [1, Theorem 5.2], it is proved that if a particular function $H^*(\omega)$ defined on the boundary of a tree T has some special property, then T is a P-tree. In a similar vein, we have the following sufficient condition for T to be a P-tree.

THEOREM 3.1. *Let T be a tree. Suppose e is a vertex such that for a neighbour e_1 of e, $\sigma_R[e, e_1]$ defines a P-section. Then T is a P-tree.*

PROOF. Since $\sigma_R[e, e_1]$ is a P-section, there exists a bounded harmonic function $h \geq 0$ on $\sigma_R[e, e_1]$, $h(e) = 0$, and $h(e_1) = 1$. Hence h extended by 0 on $T \backslash \sigma_R[e, e_1]$ is a bounded non-constant subharmonic function on T. This implies that there exists a positive potential on T.

COROLLARY 3.1. *If T is an S-tree, then every section $\sigma_R[e, e_1]$ defined by every vertex e in T is an S-section.*

COROLLARY 3.2. *Let h be a harmonic function on a star S-tree T. Then on each branch, h is constant or tends to $+\infty$ or $-\infty$.*

PROOF. Since T is an S-tree, each branch is an S-section. Hence $H_t \to \infty$ (see Example 3.1 (3)) for each t. Now h can be represented as

$$h(s_{n,t}) = [h(s_{1,t}) - h(e)]H_t(s_{n,t}) + h(e).$$

Consequently on the branch C_t, (i) h is a constant if $h(s_{1,t}) = h(e)$, (ii) $h \to \infty$ if $h(s_{1,t}) > h(e)$ and (iii) $h \to -\infty$ if $h(s_{1,t}) < h(e)$.

We shall now introduce a method of dividing a tree T into a finite number of subsets, starting with a non-terminal vertex. Let e be a non-terminal vertex and $x_0 \sim e$. Write $[e, x_0] = \{x \in T$: the geodesic joining e and x passes through $x_0\}$; we assume that e and x_0 are also in $[e, x_0]$. Note that $T = \bigcup_{x_i \sim e} [e, x_i]$ and some of these subsets $[e, x_i]$ can contain only a finite number of vertices. (We have a situation here where the sets $[e, x_i]$ correspond to the connected components of the complement of a nonempty compact set in a harmonic space. This finite division of T can be used to obtain some sufficient conditions for the existence of non-constant positive harmonic functions and bounded non-constant harmonic functions on trees with positive potentials.)

Given a subset $[e, x_0]$, let $H_0'[e, x_0]$ denote the family of functions $h \geq 0$ on $[e, x_0]$ such that $h(e) = 0$, $h(x_0) = 1$ and $h(x)$ is harmonic in the interior of $[e, x_0]$. Note that a construction as in Lemma 2.1 shows that $H_0'[e, x_0] \neq \phi$.

DEFINITION 3.4. *Let $[e, x_0]$ be a set containing an infinite number of vertices. Then, $[e, x_0]$ is called a P-set if and only if there exists a bounded function h in $H_0'[e, x_0]$; otherwise $[e, x_0]$ is called an S-set.*

Theorem 3.2. *A tree T is a P-tree if and only if some non-terminal vertex e determines a P-set $[e, x_0]$.*

Proof. Suppose $[e, x_0]$ is a P-set for a non-terminal vertex e. That is, there exists a bounded function $h \geq 0$ on $[e, x_0]$ such that $h(e) = 0$, $h(x_0) = 1$ and $h(x)$ is harmonic in the interior of $[e, x_0]$. Extend h as a function H on T, by taking $H(x) = 0$ if $x \notin [e, x_0]$. Then $H(x)$ is a bounded (non-harmonic) subharmonic function on T, which (by Theorem 2.2) implies that there is a positive potential on T. For, if $0 \leq H(x) \leq M$ on T, then there exists a harmonic function $u(x)$ on T such that $-M \leq u(x) \leq -H(x)$. By the construction in Theorem 2.2, $u(x)$ is the greatest harmonic minorant of $-H(x)$ on T. Hence, $p(x) = -H(x) - u(x)$ is a potential on T.

Suppose now that there is a potential $p > 0$ on T. Choose a non-terminal vertex e and let $u(x) = \widehat{R_1^e}(x) = \inf\{s(x) : s$ positive superharmonic on T, $s(e) \geq 1\}$. Then $u(x)$ is a potential, $u(x) \leq 1$ on T, harmonic on $T \backslash \{e\}$ and $u(e) = 1$. Since u cannot be a constant, there exists a set $[e, x_0]$, $e \sim x_0$, with infinite vertices such that for some y in the interior of $[e, x_0]$, $u(y) < 1$. (To see that $[e, x_0]$ contains an infinite number of vertices: Suppose $u = 1$ in the interior of every one of the sets $[e, x_i]$ with infinite vertices. Then $u \equiv 1$ on each $[e, x_i]$ with infinite vertices; it means that $u = 1$ outside a finite set in T. Since u is superharmonic on T, by the minimum principle, $u \geq 1$ on T; hence $u \equiv 1$ on T. This is a contradiction.)

Note $u(x_0) \neq 1$. For if $u(x_0) = 1$, since $u(x) \leq 1$, $u(e) = 1$ and u is harmonic outside e, then we should have $u \equiv 1$ on the connected set $[e, x_0]$; a contradiction, since $u(y) < 1$ for some y in the interior of $[e, x_0]$. Define $h(x) = \frac{1-u(x)}{1-u(x_0)}$ for $x \in [e, x_0]$. Then $h \geq 0$, $h(e) = 0$, $h(x_0) = 1$ and $h(x)$ is bounded harmonic in the interior of $[e, x_0]$. Hence $[e, x_0]$ is a P-set.

Let now $\{x_i\}$ be the finite set of neighbours of a non-terminal vertex e in a tree T with positive potentials. On each $[e, x_i]$, construct a harmonic function u_i such that $u_i(e) = 0$ and $u_i(x_i) = 1$. Then the collection $\{u_i\}$ of functions defines a function $u(x) \geq 0$ on T such that $u(e) = 0$, $u(x_i) = 1$ for each neighbour x_i of e, $u(x)$ is harmonic on $T \backslash \{e\}$ and $u(x) \geq 0$ is subharmonic on T. Then, using Proposition 2.1, we can find a harmonic function H on T such that $0 \leq u(x) \leq H(x)$ on T. Hence we can now construct the least harmonic majorant $h > 0$ of $u(x)$ on T. Conversely, supppose $h > 0$ is a harmonic function on T. Then $u = h - \widehat{R_h^e}$ is a positive subharmonic function on T and $u(e) = 0$. Recall that for any function $t(x) \geq 0$ on T and any subset A of T, we write $\widehat{R_t^A}(x) = \inf\{s(x) : s \geq 0$ is superharmonic on$T$ and $s \geq t$ on $A\}$.

We prove now that this relation between $u \in H_0^+(T \backslash \{e\})$ (that is, $u \geq 0$ is subharmonic on T, $u(e) = 0$ and u is harmonic on $T \backslash \{e\}$) and $h \in H^+(T)$ (that is, h is non-negative harmonic on T) is isomorphic.

THEOREM 3.3. *In a tree T with positive potentials, let e be a non-terminal vertex. Then the map $H^+(T) \to H_0^+(T\backslash\{e\})$ is one-one and onto.*

PROOF. 1) One-one: Suppose $h - \widehat{R_h^e} = H - \widehat{R_H^e}$ for two positive harmonic functions h and H on T. Then $h + \widehat{R_H^e} = H + \widehat{R_h^e}$. Since $\widehat{R_h^e}$ and $\widehat{R_H^e}$ are potentials, by the uniqueness of decomposition we have $h = H$.

2) Onto: Let now $u \in H_0^+(T\backslash\{e\})$. We shall show that $u = h - \widehat{R_h^e}$ for the least harmonic majorant h of u (and hence if u is bounded, then h is bounded).

Let $v = h - u \geq 0$. Then v is superharmonic on T such that $v(e) = h(e)$ so that $v(x) \geq \widehat{R_h^e}(x)$ on T; that is, $h(x) \geq u(x) + \widehat{R_h^e}(x)$.

Now, define

$$q(x) = \begin{cases} h(e) & \text{if } x = e \\ u(x) + \widehat{R_h^e}(x) & \text{if } x \neq e. \end{cases}$$

Then $q(x)$ is harmonic on $T\backslash\{e\}$. At $x = e$,

$$\begin{aligned} q(e) = h(e) &= \sum_{e \sim x_i} p(e, x_i) h(x_i) \quad (\text{since } h \text{ is harmonic}) \\ &\geq \sum_{e \sim x_i} p(e, x_i)[u(x_i) + \widehat{R_h^e}(x_i)] \quad (\text{since } h \geq u + \widehat{R_h^e} \text{ (proved above)}) \\ &= \sum_{e \sim x_i} p(e, x_i) q(x_i) \quad (\text{since } x_i \neq e). \end{aligned}$$

Hence $q(x)$ is superharmonic at $x = e$. Thus, $q(x) > 0$ is superharmonic on T and $q(e) = h(e)$. Clearly $q(x) \geq u(x)$ on T since $u(e) = 0$. Hence $h(x)$ being the least harmonic majorant of $u(x)$, we have $q(x) \geq h(x)$. Hence $u(x) + \widehat{R_h^e}(x) \geq h(x)$ on T. Consequently, $u(x) + \widehat{R_h^e}(x) = h(x)$ on T. The theorem is proved.

COROLLARY 3.3. *Let e be a non-terminal vertex in a tree T. Suppose e determines at least two infinite sets $[e, x_1]$ and $[e, x_2]$, of which one is a P-set and the other is an S-set. Then there exists a non-constant positive harmonic function on T.*

PROOF. Since there is a P-set in T, T is a P-tree (by Theorem 3.2). By the definitions of a P-set and an S-set, there exist a bounded harmonic function $h_1 \geq 0$ on $H_0'[e, x_1]$ and an unbounded harmonic function $h_2 \geq 0$ on $H_0'[e, x_2]$. Extend h_1 as a function h_1^* on T by giving the value 0 on $T\backslash[e, x_1]$; similarly h_2 is extended as h_2^* on T. Now, h_1^* and h_2^* are non-proportional and are in $H_0^+(T\backslash\{e\})$. Then by using the above Theorem 3.3, we can find two non-proportional positive harmonic functions H_1 and H_2 on T. This proves the existence of a non-constant positive harmonic function on T.

Corollary 3.4. *Let e be a non-terminal vertex in a tree T. Suppose e determines at least two P-sets* $[e, x_1]$ *and* $[e, x_2]$. *Then there exists a non-constant bounded harmonic function on T.*

Proof. With the same notations as in the proof of the above Corollary 3.3, we obtain two non-proportional harmonic functions H_1 (corresponding to h_1^*) and H_2 (corresponding to h_2^*) on T. Since h_1^* and h_2^* are bounded, by the construction (see the "onto" proof of Theorem 3.3) H_1 and H_2 are bounded on T. Thus, $H_1 > 0$ and $H_2 > 0$ are bounded, non-proportional harmonic functions on T. Hence at least one of them is non-constant.

4. Subordinate structures

In this section, we introduce the notion of a structure P' subordinate to the transition probability structure P given on a Cartier tree T. The potential theory corresponding to P' can be used to study the properties of the solutions of $\Delta u(x) = Q(x)u(x)$ on T, where $Q(x) \geq 0$ is a finite-valued function on T.

Let T be a tree in the sense of Cartier's with a probability structure P giving the nearest neighbour transition probablity $p(x, y)$ for x, y in T. If the neighbouring points x and y are indicated by $x \sim y$, then recall that $p(x, y) > 0$ if $x \sim y$, $p(x, y) = 0$ if x and y are not neighbours, $0 \leq p(x, y) \leq 1$ and $\sum_{x \sim y_i} p(x, y_i) = 1$ for all x in T. We shall refer to a tree T with such a probability structure P as a Cartier tree (T, P).

In such a tree we shall introduce another structure P' such that:

(1) For any pair x, y in T, there is an associated number $p'(x, y)$ such that $0 \leq p'(x, y) \leq 1$;

(2) $p'(x, y) \leq p(x, y)$;

(3) $p'(x, y) \neq 0$ if $x \sim y$; and

(4) $p'(x) = \sum_{x \sim y_i} p'(x, y_i) < 1$ for at least one non-terminal vertex x in T.

This structure P' giving the transition numbers $p'(x, y)$ will be refered to as the structure P' on T *subordinate* to P.

In a tree (T, P) with a subordinate structure P', given a function $f(x)$ on T, and a non-terminal vertex x, define $\Delta' f(x) = \sum_{x \sim y_i} p'(x, y_i) f(y_i) - f(x)$.

Then a lower-finite (resp. finite) function $u \not\equiv \infty$, defined on a neighbourhood of a non-terminal vertex $x_0 \in T$ is called a *P'-superharmonic* (resp. *P'-harmonic*) function at x_0 if and only if $\Delta' u(x_0) \leq 0$ (resp. $\Delta' u(x_0) = 0$). A lower-finite function $v \not\equiv \infty$ on a set ω is said to be P'-superharmonic on ω, if $\Delta' v(x_0) \leq 0$ for each $x_0 \in \omega^\circ$. With these definitions, the constant 1 is P'-superharmonic but not P'-harmonic on T. Hence, there always exists a

P'-potential $u > 0$ on T; that is, the greatest P'-harmonic minorant of the P'-superharmonic function u is 0.

Consequently, if a Cartier tree (T, P) has a structure P' subordinate to P, then the potential theory associated with the P'-structure resembles that of a Cartier tree with potentials. For example, we can prove the following results as in Section 2:

1. Let e be a non-terminal vertex in T. Then there exists a unique P'-potential $G'_e(x)$ on T such that $(-\Delta')G'_e(x) = \delta_e(x)$ for x in T.
2. $G'_e(x) \leq G'_e(e)$ for all x in T, if e is a non-terminal vertex.
3. If v is a P'-potential on T, then for x in T, $v(x) = \sum_{y \in T^\circ} (-\Delta')v(y)G'_y(x)$.
4. If $s(x) \geq 0$ is a P'-superharmonic function and $q(x)$ is a P'-potential on T such that $(-\Delta')s \geq (-\Delta')q$ on E, the P'-harmonic support of q, then $s \geq q$ on T.
5. If $s(x)$ is P'-superharmonic and $t(x)$ is P'-subharmonic on a set ω such that $s \geq t$, then there exists the greatest P'-harmonic minorant (g.P'-h.m.) h of s, such that $s \geq h \geq t$.

Let (T, P) be a Cartier tree with a probability structure P. Let P' be a structure subordinate to P. Then we have two different sets of superharmonic functions on T: one is with respect to the P-structure on T and the other is with respect to the subordinate structure P'. We shall use the prefix P (like the term P-superharmonic functions) with respect to the potential theory associated with the structure P. Similarly the prefix P' (like the term P'-potential) is used with respect to the potential theory associated with the subordinate structure P'. We shall prove that there always exists a positive non-constant P'-harmonic function on T and give some sufficient conditions for the existence of bounded non-constant P'-harmonic functions on T.

PROPOSITION 4.1. *Let ω be a set in a tree (T, P), and P' be a structure subordinate to P. Then every P-potential on ω is a P'-potential.*

PROOF. Let $u > 0$ be a P-potential on ω. Then, for every $y \in \omega^\circ$,

$$u(y) \geq \sum_{y \sim y_i} p(y, y_i)u(y_i) \geq \sum_{y \sim y_i} p'(y, y_i)u(y_i).$$

Hence $u(x)$ is P'-superharmonic at y. Since y is arbitrary in ω°, $u(x)$ is P'-superharmonic and $u(x) > 0$ on ω. Let $h \geq 0$ be the g.P'-h.m. of u on ω. Then for $y \in \omega^\circ$,

$$h(y) = \sum_{y \sim y_i} p'(y, y_i)h(y_i) \leq \sum_{y \sim y_i} p(y, y_i)h(y_i).$$

Hence $h(x)$ is P-subharmonic at y. Since $0 \le h(x) \le u(x)$, $h(x)$ is P-subharmonic and $u(x)$ is P-potential, we find $h \equiv 0$. Hence u is a P'-potential on ω.

THEOREM 4.1. *Let (T, P) be a Cartier tree on which positive P-potentials exist. Let P' be a structure on T subordinate to P. Then for any non-terminal vertex e, $G'_e(x) \le G_e(x)$ for x in T.*

PROOF. Since any positive P-superharmonic function on T is P'-superharmonic, $G_e(x)$ is a P'-superharmonic function on T. Now, for $x \in T^\circ$,

$$\begin{aligned}(-\Delta')G_e(x) &= G_e(x) - \sum_{x\sim y_i} p'(x, y_i)G_e(y_i)\\ &\ge G_e(x) - \sum_{x\sim y_i} p(x, y_i)G_e(y_i)\\ &= (-\Delta)G_e(x) = \delta_e(x) = (-\Delta')G'_e(x).\end{aligned}$$

Hence, as in Lemma 2.3, $G'_e \le G_e$ on T.

THEOREM 4.2. *There always exists a non-constant positive P'-harmonic function on T.*

PROOF. By the definition of the subordinate structure P', there exists at least one non-terminal vertex e on T such that $p'(e) = \sum_{e\sim x_i} p'(e, x_i) < 1$. Choose a function h such that $h(x_i) = 1$ for all $x_i \sim e$ and $h(e) = p'(e) < 1$. Note that h is P'-harmonic at e. Then by the method used in the proof of Lemma 2.1, we construct a P'-harmonic extension function $h > 0$ on T.

REMARK 4.1. *Since $h(e) < h(x_i)$ if $x_i \sim e$, we should have $h(x) < h(y)$ if $x \sim y$ and $|x| < |y|$ where the distances are measured from e. However, h may or may not be bounded on T. We shall now give some sufficient conditions for the existence of bounded non-constant positive P'-harmonic functions on T.*

LEMMA 4.1. *For any $x \in T$, $\sum_{y\in T^\circ} (1 - p'(y))G'_y(x) \le 1$, where $p'(y) = \sum_{y\sim y_i} p'(y, y_i)$.*

PROOF. $s \equiv 1$ is a P'-superharmonic function on T and

$$(-\Delta')s(x) = s(x) - \sum_{x\sim y_i} p'(x, y_i)s(y_i) = 1 - p'(x).$$

Then by using the expression for the potential part of s as in Corollary 2.2, we have,

$$1 = s(x) = \sum_{y \in T^\circ.} (-\Delta')s(y)G'_y(x) + h(x)$$

$$= \sum_{y \in T^\circ.} (1 - p'(y))G'_y(x) + h(x),$$

where h is a non-negative P'-harmonic function. Hence $\sum_{y \in T^\circ.} (1 - p'(y))G'_y(x) \le 1$ for every x in T.

THEOREM 4.3. *In a tree* (T, P) *with a subordinate structure* P', *the following are equivalent:*

1) $\sum_{y \in T^\circ.} (1 - p'(y))G'_y(x) = 1$ *for some* x *in* T°.
2) $\sum_{y \in T^\circ.} (1 - p'(y))G'_y(x) = 1$ *for all* x *in* T.
3) *The constant function* 1 *is a* P'*-potential on* T.
4) *The only bounded* P'*-harmonic function on* T *is* 0.

PROOF. Using the above Lemma 4.1, we write $1 = q(x) + h(x)$ where

$$q(x) = \sum_{y \in T^\circ.} (1 - p'(y))G'_y(x)$$

is a P'-potential on T and $h(x)$ is a non-negative P'-harmonic function.

1) $\Rightarrow$ 2): If $h(x) = 0$ for some x in T°, $h \equiv 0$.

2) $\Rightarrow$ 3): Since $h \equiv 0$, $1 = q(x)$ is a P'-potential on T.

3) $\Rightarrow$ 4): Let $u(x)$ be a bounded P'-harmonic function on T. Let $|u(x)| \le M$. Since M is a P'-potential and $|u(x)|$ is a P'-subharmonic function on T, $|u(x)| = 0$ for $x \in T$.

4) $\Rightarrow$ 1): Write $1 = q(x) + h(x)$. Since $0 \le h(x) \le 1$, the P'-harmonic function $h \equiv 0$ by the assumption 4). Hence $1 = \sum_{y \in T^\circ.} (1 - p'(y))G'_y(x)$.

COROLLARY 4.1. *Suppose there is no positive* P*-potential on* T. *Then the only bounded* P'*-harmonic function on* T *is* 0.

PROOF. Suppose h is a bounded P'-harmonic function on T, say $|h| \le M$. Then for $x \in T^\circ$

$$|h(x)| = \left| \sum_{x \sim x_i} p'(x, x_i)h(x_i) \right|$$

$$\le \sum_{x \sim x_i} p'(x, x_i)|h(x_i)|$$

$$\le \sum_{x \sim x_i} p(x, x_i)|h(x_i)|.$$

Hence $|h(x)|$ is a bounded P-subharmonic function on T. Since by the assumption there is no positive P-potential, $|h(x)| = \alpha$, a constant. Suppose $\alpha \neq 0$. Since α is P'-superharmonic (but not P'-harmonic) while $|h|$ is P'-subharmonic, we have a contradiction. Hence $\alpha = 0$, that is $h \equiv 0$.

Corollary 4.2. *Suppose there are positive P-potentials on T and if $G_y(x)$ denotes the P-potential with $(-\Delta)G_y(x) = \delta_y(x)$, then assume $\sup_{x \in T^\circ} G_x(x) = M < \infty$. Suppose P' is a structure on T subordinate to P with $\sum_{x \in T^\circ.} (1 - p'(x)) < \infty$. Then there is a bounded positive P'-harmonic function on T.*

Proof. Suppose every bounded non-negative P'-harmonic function on T is 0. Then by the above Theorem 4.3,

$$\begin{aligned} 1 &= \sum_{y \in T^\circ.} (1 - p'(y)) G'_y(x) \\ &\leq \sum_{y \in T^\circ.} (1 - p'(y)) G_y(x) \qquad \text{(by Theorem 4.1)} \\ &\leq \sum_{y \in T^\circ.} (1 - p'(y)) G_y(y) \qquad (G_y(x) \leq G_y(y) \text{ as in Theorem 2.6}) \\ &\leq M \sum_{y \in T^\circ.} (1 - p'(y)) \\ &< \infty. \end{aligned}$$

Hence $u(x) = \sum_{y \in T^\circ.} (1 - p'(y)) G_y(x)$ should be a P-potential. Since $u(x) \geq 1$ and 1 is P-harmonic, this is a contradiction.

Using the P'-sets and the S'-sets, we shall now give a sufficient condition for the existence of bounded P'-harmonic functions on T.

Recall the definition of $[e, x_0]$: Given a non-terminal vertex e and $x_0 \sim e$, $[e, x_0] = \{x$: the geodesic joining e and x passes through $x_0\}$; e and x_0 are also in $[e, x_0]$. Suppose $h(x)$ is a function such that $h(e) = 0$ and $h(x_0) = 1$. Then (as in Lemma 2.1) $h(x)$ can be extended as a P'-harmonic function on the whole set $[e, x_0]$. This P'-harmonic function on $[e, x_0]$ may or may not be bounded. We say that an infinite set $[e, x_0]$ is a P'-set if $h(x)$ is bounded and it is an S'-set if $h(x)$ is unbounded.

Lemma 4.2. *Let $h(x)$ be P'-harmonic outside a finite set $\subset \{x : |x| \leq n - 2\}$. Then there exist a P'-harmonic function H on T and two P'-potentials p_1 and p_2 on T with harmonic support on $|x| = n$ such that $h(x) = H(x) + p_1(x) - p_2(x)$ when $|x| > n$ and $|p_1 - p_2|$ is bounded on T.*

THEOREM 4.4. *There exists a bounded positive P'-harmonic function on T if and only if there exists a P'-set with respect to a non-terminal vertex e.*

PROOF. Suppose $[e, x_0]$ is a P'-set. Then there exists a bounded P'-harmonic function $u \geq 0$ on $[e, x_0]$ such that $u(e) = 0$ and $u(x_0) = 1$. Define

$$v = \begin{cases} u & \text{on } [e, x_0] \\ 0 & \text{outside } [e, x_0]. \end{cases}$$

Then $v \in H_0^+(T \backslash \{e\})$ and v is bounded on T. By the above Lemma 4.2, there exists a P'-harmonic function h on T such that $|v - h| \leq M$, for a constant M. Hence the least P'-harmonic majorant H of v verifies the inequalities $0 \leq v \leq H \leq h + M$ on T. Hence $0 \leq H - v \leq (h + M) - v \leq 2M$. Since v is bounded on T, H is bounded on T. Since $H \geq v \geq 0$ and $H(x_0) \geq v(x_0) = 1$, by the minimum principle $H > 0$ on T.

Conversely, suppose there exists a bounded P'-harmonic function $h > 0$ on T. For a non-terminal vertex e, let

$$u(x) = \widehat{R_h^e}'(x) = \inf\{s(x) : s \text{ is positive } P'\text{-superharmonic on } T,\ s(e) \geq h(e)\}.$$

Then $u(x)$ is a P'-potential on T such that $u(x) \leq h(x)$ on T, $u(x)$ is P'-harmonic on $T \backslash \{e\}$ and $u(e) = h(e)$. Since $u \not\equiv h$, there exists a set $[e, x_0]$, $e \sim x_0$, with infinite vertices such that for some y in the interior of $[e, x_0]$, $u(y) < h(y)$. To prove this statement:

(i) Write $T = \bigcup_{e \sim x_i} [e, x_i]$. Suppose $u(x) = h(x)$ at every interior point of $[e, x_i]$. Then, since $h(x) - u(x) \geq 0$ is P'-harmonic on $[e, x_i]$, we should have $h \equiv u$ on $[e, x_i]$ and hence on T. This is a contradiction. Hence for some interior point y in some $[e, x_0]$, $u(y) < h(y)$.

(ii) We show now that it can be assumed that $[e, x_0]$ contains an infinite number of vertices. For suppose $u = h$ on every one of the sets $[e, x_i]$ with infnite vertices. It means that $u(x) = h(x)$ outside a finite set in T. Since $v(x) = u(x) - h(x)$ is P'-superharmonic on T, and equals 0 outside a finite set, by the minimun principle, $v \geq 0$ on T. This leads to the conclusion $u \equiv h$ on T, a contradiction.

Note $u(x_0) < h(x_0)$. For otherwise by the minimum principle for the P'-harmonic function, $u - h \geq 0$ on $[e, x_0]$ and $u = h$ in the interior of $[e, x_0]$; this is a contradiction, since $u(y) < h(y)$. Define $H(x) = \frac{h(x) - u(x)}{h(x_0) - u(x_0)}$ for $x \in [e, x_0]$. Then $H \geq 0$, $H(e) = 0$, $H(x_0) = 1$ and $H(x)$ is bounded harmonic in the interior of $[e, x_0]$. Hence $[e, x_0]$ is a P'-set.

COROLLARY 4.3. *For a subordinate structure P' on (T, P), suppose* 1 *is a P'-potential on T. Then with respect to any non-terminal vertex e, every infinite set $[e, x_0]$ is an S'-set.*

Proof. If $[e, x_0]$ is a P'-set, then there exists a bounded P'-harmonic function $H > 0$ on T by the above theorem. However 1 cannot be a P'-potential on T (Theorem 4.3), which is a contradiction.

An application to $\Delta u(x) = Q(x)u(x)$ (See Yamasaki [8].)

Let $Q(x) \geq 0$ be a function defined on T with a Cartier structure P. Assume $Q \not\equiv 0$ on T°. We say that a function $u(x)$ defined on T is *Q-harmonic* (resp. *Q-superharmonic*) if $(1 + Q(x))u(x) = \sum_{x \sim x_i} p(x, x_i)u(x_i)$ (resp. $(1 + Q(x))u(x) \geq \sum_{x \sim x_i} p(x, x_i)u(x_i)$) for every $x \in T^\circ$.

Now define a subordinate structure P' on T as follows: For any pair x, y in T, define $p'(x, y) = \frac{p(x,y)}{1+Q(x)}$. Then $p'(x) = \sum_{x \sim x_i} p'(x, x_i) = \frac{1}{1+Q(x)}$. Hence $0 < p'(x) \leq 1$ and $p'(x) < 1$ for some x in T° since $Q \not\equiv 0$ on T°. With respect to this subordinate structure P', $v(x)$ is P'-harmonic if and only if $v(x) = \sum_{x \sim x_i} p'(x, x_i)v(x_i)$ for every $x \in T^\circ$; that is, if and only if $(1 + Q(x))v(x) = \sum_{x \sim x_i} p(x, x_i)v(x_i)$ which means that $v(x)$ is Q-harmonic. Thus, v is P'-harmonic (resp. P'-superharmonic) if and only if v is Q-harmonic (resp. Q-superharmonic). Consequently, the potential theory associated with the Q-harmonic functions on T becomes a particular case of the potential theory associated with a subordinate structure P' on T.

5. Polypotentials on trees

In this section we study the properties of functions u defined on a connected simple set ω in a Cartier tree T, satisfying the condition $\Delta^m u \geq 0$ at the interior points of ω. For the discussion below, we do not place the restriction that there are positive potentials on T. Remark that for polyharmonic functions u ($\Delta^m u = 0$) defined on a homogeneous tree (and hence a tree with positive potentials), Cohen et al. [5] give an integral representation (inspired by the Almansi representation in the classical case), establish one-one corespondence with polymartingales and study the boundary behaviour of u.

Let ω be a connected simple set in T (see Definition 2.1). Let $f \geq 0$ be a real-valued function on ω. Then, by Corollary 2.1, there exists a superharmonic function s on ω such that $(-\Delta)s(x) = f(x)$ for each $x \in \omega^\circ$. Hence, if g is an arbitrary real-valued function on ω, we can find two superharmonic functions s_1 and s_2 on ω such that $(-\Delta)s_1(x) = g^+(x)$ and $(-\Delta)s_2(x) = g^-(x)$ at each $x \in \omega^\circ$. Thus, for g on ω, there exists a δ-subharmonic function $s = s_1 - s_2$ on ω such that $(-\Delta)s(x) = g(x)$ at each $x \in \omega^\circ$.

By the same procedure, we can find a δ-subharmonic function u on ω such that $(-\Delta)u(x) = s(x)$ on ω°, so that $(-\Delta)^2 u(x) = g(x)$ on ω°. Continuing thus, given a real-valued function on a connected simple set ω, and an interger $m \geq 1$, we can find a δ-subharmonic function v on ω such that $(-\Delta)^m v(x) = g(x)$ on ω°.

Definition 5.1. *Let* $(s_i)_{1 \leq i \leq m}$ *be a set of real-valued functions on a connected simple set* ω *such that* $(-\Delta)s_i = s_{i-1}$ *on* ω° *for* $2 \leq i \leq m$. *Then* $s = (s_i)_{1 \leq i \leq m}$ *is said to be an m-superharmonic (resp. m-harmonic) function on* ω *if and only if* s_1 *is superharmonic (resp. harmonic) on* ω. *We say that the m-superharmonic (resp. m-harmonic) function* $s = (s_i)_{1 \leq i \leq m}$ *is generated by* s_1.

Theorem 5.1. *Let* $h = (h_i)_{1 \leq i \leq m}$ *be an m-harmonic function defined on* $|x| \leq n$ *(distances measured from a fixed non-terminal vertex e) in a Cartier tree* T. *Then there exists an m-harmonic function* $H = (H_i)_{1 \leq i \leq m}$ *on* T *such that* $H_i(x) = h_i(x)$ *for* $|x| \leq n$ *and* $1 \leq i \leq m$.

Proof. Note that $h_1(x)$ is harmonic on $|x| \leq n$. Hence by Remark 2.1, we can find a harmonic function H_1 on T such that $H_1(x) = h_1(x)$ if $|x| \leq n$. Let $(-\Delta)u = H_1$.

Write $t(x) = u(x) - h_2(x)$ if $|x| \leq n$. Then, if $|x| < n$, $(-\Delta)t(x) = (-\Delta)u(x) - (-\Delta)h_2(x) = H_1(x) - h_1(x) = 0$. Hence $t(x)$ is harmonic on $|x| \leq n$. Again by Remark 2.1 (2), we can find a harmonic function v on T such that $v(x) = t(x)$ for $|x| \leq n$. Write $H_2(x) = u(x) - v(x)$ for x in T. Then, for $|x| \leq n$, $H_2(x) = h_2(x)$; and in T°, $(-\Delta)H_2(x) = (-\Delta)u(x) = H_1$.

Proceeding thus, we construct an m-harmonic function $H = (H_i)_{1 \leq i \leq m}$ on T such that $H_i(x) = h_i(x)$ if $|x| \leq n$, $1 \leq i \leq m$.

Theorem 5.2 (Laurent decomposition for m-harmonic functions). *Let e be a fixed non-terminal vertex in* T *and* $d(e,x) = |x|$. *Suppose* $u = (u_i)_{1 \leq i \leq m}$ *is m-harmonic on* $n \leq |x| \leq N$, *where* N *is an integer. Then there exists an m-harmonic function* $s = (s_i)_{1 \leq i \leq m}$ *on* $|x| \geq n$ *and an m-harmonic function* $t = (t_i)_{1 \leq i \leq m}$ *on* $|x| \leq N$ *such that* $u(x) = s(x) - t(x)$ *on* $n \leq |x| \leq N$.

Proof. Since $u_1(x)$ is harmonic on $n < |x| < N$, by Corollary 2.3, there exist a harmonic function $s_1(x)$ on $|x| \geq n$ and a harmonic function $t_1(x)$ on $|x| \leq N$ such that $u_1(x) = s_1(x) - t_1(x)$ on $n \leq |x| \leq N$. Choose (see Corollary 2.1) the functions f_1 and g_1 on T such that $(-\Delta)f_1(x) = s_1(x)$ for $|x| > n$ and $(-\Delta)g_1(x) = t_1(x)$ for $|x| < N$. Then, $(-\Delta)u_2(x) = u_1(x) = (-\Delta)f_1(x) - (-\Delta)g_1(x)$ on $n < |x| < N$, so that $u_2(x) = f_1(x) - g_1(x) + H(x)$ where $H(x)$ is harmonic on $n \leq |x| \leq N$.

Again by Corollary 2.3, there exist $f_2(x)$ harmonic on $|x| \geq n$ and $g_2(x)$ harmonic on $|x| \leq N$ such that $H(x) = f_2(x) - g_2(x)$ on $n \leq |x| \leq N$. Write $s_2(x) = f_1(x) + f_2(x)$ and $t_2(x) = g_1(x) + g_2(x)$. Then $(-\Delta)s_2(x) = s_1(x)$ on $|x| > n$, $(-\Delta)t_2(x) = t_1(x)$ on $|x| < N$, and $u_2(x) = s_2(x) - t_2(x)$ on $n \leq |x| \leq N$.

Proceeding thus, we construct $s = (s_i)_{1 \le i \le m}$ and $t = (t_i)_{1 \le i \le m}$ as stated in the theorem so that $u(x) = s(x) - t(x)$ on $n \le |x| \le N$. This proves the theorem.

NOTATION: Let $s = (s_i)_{1 \le i \le m}$ and $t = (t_i)_{1 \le i \le m}$ be two sets of real-valued functions defined on a simple set ω. We say that $s \ge t$ if and only if $s_i \ge t_i$ for every i; $s \ge 0$ if $s_i \ge 0$ for all i.

THEOREM 5.3. *Let s be an m-superharmonic function on a set ω in T, and let t be an m-subharmonic function on ω such that $t \le s$ on ω. Then there exists an m-harmonic function h on ω such that $t \le h \le s$ on ω.*

PROOF. Let $s = (s_i)_{1 \le i \le m}$ and $t = (t_i)_{1 \le i \le m}$. Let $\Im$ be the family of subharmonic functions u on ω such that $t_1 \le u \le s_1$. Let $h_1 = \sup_{\Im} u$. Then h_1 is harmonic on ω and it is the greatest harmonic minorant of s_1 on ω.

Let

$$f = \begin{cases} h_1 & \text{on } \omega \\ 0 & \text{on } T \backslash \omega. \end{cases}$$

Then, there exists a δ-subharmonic function g on T such that $(-\Delta)g = f$. Let $H_2 = g|_{\omega}$. Then $(-\Delta)H_2 = h_1$ on ω°.

Similarly choose f_2 and g_2 on ω such that on ω°, $(-\Delta)f_2 = s_1 - h_1$ and $(-\Delta)g_2 = t_1 - h_1$. Then f_2 is superharmonic on ω and g_2 is subharmonic on ω such that

$$(-\Delta)s_2 = s_1 = (-\Delta)f_2 + (-\Delta)H_2 \qquad \text{on } \omega^\circ$$

and

$$(-\Delta)t_2 = t_1 = (-\Delta)g_2 + (-\Delta)H_2 \qquad \text{on } \omega^\circ.$$

Consequently, $s_2 = f_2 + H_2 +$ (a harmonic function) on ω; write $s_2 = f_2' + H_2$ where f_2' is superharmonic on ω. Similarly write $t_2 = g_2' + H_2$ where g_2' is subharmonic on ω. Since $s_2 \ge t_2$ by hypothesis, $f_2' \ge g_2'$. Let u be the g.h.m. of f_2' so that $f_2' \ge u \ge g_2'$. Let $h_2 = H_2 + u$. Then $(-\Delta)h_2 = h_1$ on ω° and $s_2 \ge h_2 \ge t_2$. Suppose h_2' is a function such that $(-\Delta)h_2' = h_1$ on ω° and $s_2 \ge h_2' \ge t_2$ on ω. Then $h_2' = h_2 +$ (a harmonic function v on ω) $= (H_2 + u) + v$. Since $s_2 \ge h_2' \ge t_2$ we should have $f_2' \ge u + v \ge g_2'$. Since u is the g.h.m. of f_2', $v \le 0$ so that $h_2' \le h_2$.

Proceeding thus, we construct $h = (h_i)_{1 \le i \le m}$ which is an m-harmonic function such that $t \le h \le s$ on ω. This function has the additional property that if h' is any m-harmonic function on ω such that $t \le h' \le s$, then $h' \le h$.

DEFINITION 5.2. *Let s be m-superharmonic and t be m-subharmonic defined on a set ω in T, such that $s \geq t$. Then the m-harmonic function h constructed in the above Theorem* 5.3 *such that $s \geq h \geq t$ is called the greatest m-harmonic minorant of s on ω.*

DEFINITION 5.3. *An m-superharmonic function $s \geq 0$ on a set ω in T is said to be a polypotential of order m (an m-potential, for short) if its greatest m-harmonic minorant on ω is* 0. *We say that T is an m-potential tree if there exists a positive m-potential on T.*

THEOREM 5.4. *Let $s \geq 0$ be an m-superharmonic function on a set ω in T. Then $s = (s_i)_{1 \leq i \leq m}$ is an m-potential if and only if s_i is a potential for each i.*

PROOF. Let s be an m-potential. Suppose s_j is not a potential for some j, $1 \leq j \leq m$. Let h_j be the g.h.m. of s_j on ω. Then as in Theorem 5.3, we can construct an m-harmonic minorant $h = (h_i)_{1 \leq i \leq m}$ with $h_i = 0$ if $i < j$. Since $h \not\equiv 0$, s is not an m-potential. This is a contradiction.

Conversely, if each s_i is a potential in the m-superharmonic function $s = (s_i)_{1 \leq i \leq m}$ on ω, then s is an m-potential. For, let $h = (h_i)_{1 \leq i \leq m}$ be the greatest m-harmonic minorant of s. Since $0 \leq h_1 \leq s_1$ and s_1 is a potential, $h_1 \equiv 0$. Since $(-\Delta)h_2 = h_1$ on ω°, h_2 should be harmonic on ω. Since $0 \leq h_2 \leq s_2$ and s_2 is a potential, $h_2 \equiv 0$. Proceeding thus, we show that $h \equiv 0$. Hence s is an m-potential.

COROLLARY 5.1. *Let $s = (s_i)_{1 \leq i \leq m}$ be an m-superharmonic function on a set ω in T. Suppose q is a potential on ω such that for each i, $|s_i| \leq q$, outside a finite set in ω. Then s is an m-potential.*

PROOF. Since s_1 has a subharmonic minorant $(-q)$ outside a finite set in ω, $s_1 = p_1 + h_1$ where p_1 is a potential and h_1 is harmonic. Since $|h_1| \leq p_1 + q$ outside a finite set, $h_1 \equiv 0$. That is, s_1 is a potential on ω.

Since $(-\Delta)s_2 = s_1 \geq 0$, s_2 is a superharmonic function on ω; and since $|s_2| \leq q$ outside a finite set in ω, s_2 is a potential. Proceeding thus, we find that in $s = (s_i)$, each s_i is a potential. Hence s is an m-potential.

THEOREM 5.5. *Let s be an m-superharmonic function and t an m-subharmonic function defined on a set, such that $s \geq t$. Then s is the unique sum of an m-potential p and an m-harmonic function h, being the greatest m-harmonic minorant of s.*

PROOF. This is a consequence of Theorem 5.3 and Definition 5.3. For, if h is the greatest m-harmonic minorant of s, then let $p = s - h$. Then $p = (p_i) \geq 0$ is an m-superharmonic function whose greatest harmonic minorant is 0, and hence an m-potential on T.

Theorem 5.6. *Let $Q = (Q_i)_{1\le i\le m}$ be an m-potential on a set ω. Let p_1 be a potential on ω such that $p_1 \le Q_1$. Then there exists a unique m-potential $p = (p_i)_{1\le i\le m}$ generated by p_1 such that $p_i \le Q_i$ for $1 \le i \le m$.*

Proof. As in Theorem 2.4, choose a superharmonic function s on ω such that $(-\Delta)s = p_1$ on ω°. By hypothesis, $(-\Delta)Q_2 = Q_1 \ge p_1$ on ω°. Choose a superharmonic function t on ω, such that $(-\Delta)t = Q_1 - p_1$ on ω°. Then $Q_2 = s + t +$ (a harmonic function h_1). Since $Q_2 \ge 0$, s has a subharmonic minorant on ω, so that $s =$ (a potential p_2) + (a harmonic function h_2) on ω; similarly, $t =$ (a potential p_2') + (a harmonic function h_2') on ω. Thus $Q_2 = p_2 + p_2' + (h_1 + h_2 + h_2')$. Equating the potential parts, we have $Q_2 = p_2 + p_2'$. Hence $p_2 \le Q_2$; note that $(-\Delta)p_2 = (-\Delta)s = p_1$ on ω°.

Proceeding similarly, we find the potential p_3 on ω such that $p_3 \le Q_3$ and $(-\Delta)p_3 = p_2$ on ω°. Continuing thus, we construct the m-potential $p = (p_i)_{1\le i\le m}$ on ω such that $p_i \le Q_i$.

As for the uniqueness, suppose $(q_m, q_{m-1}, \dots, q_2, p_1)$ is another m-potential generated by p_1. Since $(-\Delta)p_2 = p_1 = (-\Delta)q_2$, $p_2 = q_2 +$ (a harmonic function h) on ω; and since p_2 and q_2 are potentials, $h \equiv 0$. Proceeding similarly, we find that the m-potential generated by p_1 is unique.

A special case: Let p_1 be a potential with finite harmonic support in a set ω on which a positive m-potential $Q = (Q_i)_{1\le i\le m}$ exists. Then p_1 generates an m-potential $p = (p_i)_{1\le i\le m}$ on ω. For, since p_1 has finite harmonic support, we can choose $\lambda > 0$ so that $p_1 \le \lambda Q_1$ on ω (Theorem 2.5, Domination Principle); and $\lambda Q = (\lambda Q_i)$ is an m-potential on ω.

Theorem 5.7 (Balayage). *Let $p = (p_i)_{1\le i\le m}$ be an m-potential on a set ω. Let E be a subset of ω. Then there exists an m-potential $\hat{R}_p^E = (q_i)_{1\le i\le m}$ on ω such that*

1. *$\hat{R}_p^E \le p$ on ω,*
2. *$\hat{R}_p^E = p +$ (an $(m-1)$-harmonic function) on E°,*
3. *$\hat{R}_p^E$ is m-harmonic on $(\omega\backslash E)^\circ$.*

Proof. Take $q_1 = \hat{R}_{p_1}^E$ on ω (which is the infimum of all positive superharmonic functions s on ω such that $s \ge p_1$ on E). Then q_1 generates an m-potential $q = (q_i)_{1\le i\le m}$ as indicated in the above Theorem 5.6 such that $q \le p$ on ω. Since q_1 is harmonic on $(\omega\backslash E)^\circ$, q is m-harmonic on $(\omega\backslash E)^\circ$. Further $q_1 = p_1$ on E°, so that $(-\Delta)q_2 = q_1 = p_1 = (-\Delta)p_2$ and hence $q_2 = p_2 +$ a harmonic function h_2 on E°.

Let $(-\Delta)H_3 = h_2$ on E°, so that $(-\Delta)q_3 = (-\Delta)p_3 + (-\Delta)H_3$ on E° and $q_3 = p_3 + H_3 +$ (a harmonic function u) on E°. Write $h_3 = H_3 + u$ on E°. Then $q_3 = p_3 + h_3$ on E° and $(-\Delta)h_3 = h_2$. Thus proceeding, we obtain an

$(m-1)$-harmonic function $(h_m, h_{m-1}\dots, h_2)$ on E° which can be identified with the m-harmonic function $h = (h_m, h_{m-1}\dots, h_2, 0)$ so that $q = p + h$ on E°.

REMARK 5.1. *In the above theorem, if E is a finite set and p is any positive m-superharmonic function on ω, then also $\hat{R}_p^E$ is an m-potential with the stated properties.*

Suppose $s = (s_i)_{1\le i\le m}$ is an m-superharmonic function on a set ω in T. We say that E is the m-harmonic support of s if the superharmonic function s_1 has E as its harmonic support.

THEOREM 5.8 (Domination Principle). *Let $s = (s_i)_{1\le i\le m}$ be a positive m-superharmonic function and $p = (p_i)_{1\le i\le m}$ be an m-potential on a set ω in T with its m-harmonic support E. Suppose $s_1 \ge p_1$ on E. Then $s \ge p$ on ω.*

PROOF. As in Theorem 2.5, if $s_1 \ge p_1$ on E, then $s_1 \ge p_1$ on ω. Let $(-\Delta)u = s_1 - p_1 = (-\Delta)s_2 - (-\Delta)p_2$. Then u is superharmonic on ω and $s_2 = p_2 + u + (\text{a harmonic function}) \ge 0$. Hence u has a subharmonic minorant on ω, so that u is the sum of a potential q and a harmonic function on ω. Hence $s_2 = p_2 + q + (\text{a harmonic function})$ on ω. Use now the uniqueness of decomposition of s_2 as the sum of a potential and a harmonic function to conclude that $s_2 \ge p_2$ on ω. Proceeding thus, we conclude that $s_i \ge p_i$ on ω for every i.

THEOREM 5.9. *Let T be a tree with positive potentials. Then T is an m-potential tree if and only if given an m-harmonic function $h = (h_i)_{1\le i\le m}$ near infinity, there exists a (unique) m-harmonic function $H = (H_i)_{1\le i\le m}$ on T such that for some potential $q > 0$ on T, $|h_i - H_i| \le q$ near infinity for each i.*

PROOF. Let T be an m-potential tree. Since h_1 is harmonic near infinity, by Proposition 2.1, there exists a harmonic function H_1 on T and a potential p_1 with finite harmonic support on T such that $|h_1 - H_1| < p_1$ near infinity. Since p_1 has finite harmonic support, it generates an m-potential $p = (p_i)_{1\le i\le m}$ on T. Let $(-\Delta)H_2' = H_1$ on T°. $(-\Delta)h_2 = h_1$ outside a finite set by the assumption. Then, $|h_1 - H_1| < p_1$ near infinity means that $-(-\Delta)p_2 \le (-\Delta)h_2 - (-\Delta)H_2' \le (-\Delta)p_2$ outside a finite set A on T. Write $s = h_2 - H_2' + p_2$ and $t = h_2 - H_2' - p_2$. Then on $T\backslash A$, s is superharmonic and t is subharmonic such that $t \le s$. Hence there exists a harmonic function h_0 on $T\backslash A$ such that $t \le h_0 \le s$ (Theorem 2.2). Consequently, we can as before find a harmonic function u on T and a potential v with finite harmonic support such that $|h_0 - u| \le v$ outside a finite set.

Set $H_2 = H_2' + u$ so that $(-\Delta)H_2 = (-\Delta)H_2' = H_1$ and note that

$$|h_2 - H_2| = |(h_2 - H_2' - h_0) - (u - h_0)| \le p_2 + v \text{ near infinity.}$$

Now, since v is a potential with finite harmonic support, there exists a potential v_1 such that $(-\Delta)v_1 = v$ on T°. Hence, if $q_3 = p_3 + v_1$, then $(-\Delta)q_3 = p_2 + v$. Set $q_2 = p_2 + v$. Thus far, we have proved that there exist H_2 such that $(-\Delta)H_2 = H_1$ and the potential q_3 such that $(-\Delta)q_3 = q_2$ and $|h_2 - H_2| \leq q_2$ outside a finite set in T. Then by induction we prove that for $m \geq i \geq 3$, there exist H_i such that $(-\Delta)H_i = H_{i-1}$ and a potential q_i such that $|h_i - H_i| \leq q_i$ outside a finite in T. Write $q = p_1 + q_2 + \cdots + q_m$. Then $H = (H_i)_{1 \leq i \leq m}$ is an m-harmonic function on T such that for every $1 \leq i \leq m$, $|h_i - H_i| \leq q$ outside a finite set.

Conversely, let p_1 be a potential with finite harmonic support on T. For example, if A is a finite subset of T, then take $p_1(x) = \sum_{y \in A} \alpha_y G_y(x)$ where $\alpha_y > 0$ are constants. Let $p = (p_i)_{1 \leq i \leq m}$ be an m-superharmonic function generated by p_1. Since p_1 is harmonic outside a finite set A, p is an m-harmonic function on $T \backslash A$. Hence by the assumption, there exist an m-harmonic function $H = (H_i)_{1 \leq i \leq m}$ on T and a potential q on T such that $|p_i - H_i| \leq q$ near infinity. Set $s_i = p_i - H_i$ on T. Then by Corollary 5.1, $s = (s_i)_{1 \leq i \leq m}$ is an m-potential on T; hence T is an m-potential tree.

Theorem 5.10. *Let T be a tree with positive potentials. For $y \in T^\circ$, let $G_y(x) = G(x, y)$ be the potential such that $(-\Delta)G_y(x) = \delta_y(x)$. Then T is an m-potential tree if and only if there exist two vertices u and v on T° such that*

$$\sum_{x_1, x_2, \ldots, x_{m-1} \in T^\circ} G(u, x_{m-1})G(x_{m-1}, x_{m-2}) \ldots G(x_2, x_1)G(x_1, v) < \infty.$$

Proof. Take $u = x_{m-1} = \cdots = x_3 = x_2$ in the above sum. Since $G(u, u) < \infty$, we see that $\sum_{x_1 \in T^\circ} G(x_2, x_1)G(x_1, v) < \infty$. Hence $q_2(x) = \sum_{x_1 \in T^\circ} G(x, x_1)G(x_1, v)$ is a potential on T, and $(-\Delta)q_2(x) = G(x, v) = G_v(x)$. Similarly, since $\sum_{x_1, x_2 \in T^\circ} G(x_3, x_2)G(x_2, x_1)G(x_1, v) < \infty$, we find that $\sum_{x_2 \in T^\circ} G(x_3, x_2)q_2(x_2)$ is finite. Hence $q_3(x) = \sum_{x_2 \in T^\circ} G(x, x_2)q_2(x_2)$ is a potential on T such that $(-\Delta)q_3(x) = q_2(x)$. Thus proceeding, we construct q_i as a potential such that

$$(-\Delta)q_i(x) = q_{i-1}, 2 \leq i \leq m,$$

where $q_1(x) = G_v(x)$. This means that $q = (q_i)_{1 \leq i \leq m}$ is an m-potential on T.

Conversely, let $(p_m, \ldots, p_1)$ be an m-potential on T. Since

$$p_m(x) = \sum_y G(x, y)(-\Delta p_m)(y) = \sum_y G(x, y)p_{m-1}(y)$$

is a potential by hypothesis, $\sum_{x_{m-1} \in T^\circ} G(u, x_{m-1}) p_{m-1}(x_{m-1}) < \infty$ for $u \in T$. This means

$$\sum_{x_{m-1} \in T^\circ} G(u, x_{m-1}) \left[\sum_{x_{m-2}} G(x_{m-1}, x_{m-2}) p_{m-2}(x_{m-2}) \right] < \infty.$$

Hence

$$\sum_{x_{m-1}, x_{m-2} \in T^\circ} G(u, x_{m-1}) G(x_{m-1}, x_{m-2}) p_{m-2}(x_{m-2}) < \infty.$$

Proceeding thus, we arrive at the conclusion

$$\sum_{x_{m-1}, \dots, x_1 \in T^\circ} G(u, x_{m-1}) G(x_{m-1}, x_{m-2}) \dots G(x_2, x_1) p_1(x_1) < \infty.$$

But for v fixed in T° and $x \in T$, we can find $\lambda > 0$ such that $G(x, v) \leq \lambda p_1(x)$. Consequently, $\sum_{x_{m-1}, \dots, x_1 \in T^\circ} G(u, x_{m-1}) G(x_{m-1}, x_{m-2}) \dots G(x_2, x_1) G(x_1, v) < \infty$.

REMARK 5.2. 1) *From the above proof it follows that if T is an m-potential tree, then for any pair of vertices u and v in T°,*

$$\sum_{x_{m-1}, \dots, x_1 \in T^\circ} G(u, x_{m-1}) G(x_{m-1}, x_{m-2}) \dots G(x_2, x_1) G(x_1, v) < \infty.$$

Hence, if we write for $y \in T^\circ$

$$Q_y(x) = \sum_{x_{m-1}, \dots, x_1 \in T^\circ} G(x, x_{m-1}) G(x_{m-1}, x_{m-2}) \dots G(x_2, x_1) G(x_1, y)$$

in an m-potential tree T, then $Q_y(x)$ is an m-potential such that $(-\Delta)^{m-1} Q_y(x) = G_y(x)$. We term $Q_y(x)$ as the m-harmonic Green function on T with pole $\{y\}$. Since $Q_y(x)$ is an m-potenial generated by $G_y(x)$, the m-harmonic Green function is uniquely determined (Theorem 5.6*).*

2) *If T is an m-potential tree, then for any $z \in T^\circ$, there exists a potential u on T such that $(-\Delta)^m u(x) = \delta_z(x)$ and $(-\Delta)^i u$ is a potential on T for $1 \leq i \leq m-1$.*

3) *Let $q = (q_i)_{1 \leq i \leq m-1}$ be an $(m-1)$-potential on T such that $(-\Delta) q_1 = 1$. Then we say that q is a quasi $(m-1)$-harmonic potential on T. It can be seen that such a potential q exists on T if and only if for one (and hence any) u in T°,*

$$\sum_{x_{m-1}, \dots, x_1 \in T^\circ} G(u, x_{m-1}) G(x_{m-1}, x_{m-2}) \dots G(x_2, x_1) < \infty.$$

Since $G(x_1, v) \leq G(v, v)$, *from the above Theorem* 5.10 *it follows that if there is a quasi* $(m-1)$*-harmonic potential on* T, *then* T *is an* m*-potential tree.*

Proposition 5.1. *A homogeneous tree is a bipotential tree.*

Proof. Let T be a homogeneous tree of degree $(q+1)$ with $q \geq 2$ (see Cartier [4, p. 262]); that is, each vertex has exactly $q+1$ neighbours. Let $d(s,t)$ be the length of the geodesic joining s to t. Then $d(s,t) = 1$ if and only if s and t are neighbours and $d(s, s_2) \equiv d(s, s_1) + d(s_1, s_2) \pmod 2$. Also T is a P-tree and the Green function is $G(s,t) = \frac{q}{q-1} \cdot \frac{1}{q^{d(s,t)}}$.

Let us fix u and v two neighbours in T. Then for $x \in T$,

$$G(u,x)G(x,v) = \frac{q^2}{(q-1)^2} \frac{1}{q^{d(u,x)+d(x,v)}}.$$

Since $d(u,x) + d(x,v) \equiv 1 \pmod 2$, if we write $A_n = \{x : d(u,x) + d(x,v) = n\}$, then n is odd and $T = \bigcup A_n$. Now

$$|d(u,x) - d(x,v)| \leq 1, d(u,x) = d(x,v) \pm 1,$$

so that if $x \in A_n$, then $d(u,x) = \frac{n-1}{2}$ or $d(x,v) = \frac{n-1}{2}$. This implies that card $A_n \leq 2(q+1)^{(n-1)/2}$. Hence

$$\sum_{x \in A_n} G(u,x)G(x,v) \leq 2(q+1)^{(n-1)/2} \frac{q^2}{(q-1)^2} \frac{1}{q^n} = \lambda \frac{(q+1)^{n/2}}{q^n}$$

where λ is the constant $2(q+1)^{-1/2} \frac{q^2}{(q-1)^2}$. Consequently,

$$\sum_{x \in T} G(u,x)G(x,v) \leq \lambda \sum_n \frac{(q+1)^{n/2}}{q^n}.$$

Note that this last series is convergent since $q \geq 2$. Consequently, Theorem 5.10 implies that T is a bipotential tree.

6. Riesz-Martin representation for positive m-superharmonic functions on T

In the classical potential theory in $\mathscr{R}^n$, $n \geq 2$, let $s \geq 0$ be a superharmonic function defined on a bounded domain Ω in $\mathscr{R}^n$. Then (see for example Brelot [3, pp. 150–152]), there exist two uniquely determined positive Radon measures μ on Ω and ν on the Martin boundary Δ with support in the minimal boundary Δ_1, such that $s(x) = \int_\Omega G(x,y) d\mu(y) + \int_{\Delta_1} K(x,y) d\nu(y)$; here $G(x,y)$ is the Green function of Ω and $K(x,y)$ is the Martin kernel. This representation is referred to as the Riesz-Martin representation for the positive superharmonic function s on Ω. For such a representation on a tree, see Cartier [4, pp. 235–

237]. Below, we give a similar representation for positive m-superharmonic functions defined on a tree T.

Let T be an m-potential tree. Let M be the class of measures $\mu \geq 0$ on T such that $p_1(x) = \sum_{y \in T^\circ} G_y(x)\mu(\{y\})$ is a potential that generates an m-potential $(p_i)_{m \geq i \geq 1}$ on T. Let Λ_i $(1 \leq i \leq m)$ be the class of measures $\nu_i \geq 0$ on the Martin boundary (with the usual normalization) such that the harmonic function $h_1 \geq 0$ on T associated with ν_i (see Cartier [4, Theorem 2.1, p. 232]) generates an i-harmonic function $(h_i, h_{i-1}, \dots, h_1) \geq 0$ where $h_2, \dots, h_i$ are all potentials. Such a function can be identified with the positive m-harmonic function $(h_i, h_{i-1}, \dots, h_1, 0, \dots, 0)$. Note that if such a function exists, then the potentials $h_2, \dots, h_i$ are all uniquely determined.

THEOREM 6.1 (Riesz-Martin representation for positive m-superharmonic functions). *Any positive m-superharmonic function* $s = (s_i)_{1 \leq i \leq m}$ *in an m-potential tree T can be uniquely identified with* $(m+1)$*-measures* $(\mu, \nu_m, \dots, \nu_1) \in M \times \Lambda_m \times \cdots \times \Lambda_1$.

PROOF. As in Theorem 5.5, s can be written uniquely as the sum of an m-potential $P = (P_m, \dots, P_1)$ and a positive m-harmonic function $H = (H_m, \dots, H_1)$. Since $(-\Delta)P_{i+1} = P_i$, $1 \leq i \leq m-1$, the potentials $P_2, \dots, P_m$ are uniquely determined once P_1 is known. And P_1 is determined if its associated measure μ, $(-\Delta)P_1 = \mu$, is known. Thus the measure μ on T determines uniquely the m-potential $P = (P_m, \dots, P_1)$.

Consider now $H = (H_m, \dots, H_1) \geq 0$. Since $(-\Delta)H_2 = H_1 \geq 0$, H_2 is a positive superharmonic function and hence is a potential q_2 up to an additive harmonic function h_2, so that $H_2 = q_2 + h_2$ and $(-\Delta)q_2 = H_1$. Since $H_2 \geq q_2$, we can find a potential q_3 on T such that $(-\Delta)q_3 = q_2$ and $q_3 \leq H_3$. Thus proceeding, we construct an m-harmonic function $(q_m, \dots, q_2, H_1) \geq 0$ with all the functions $q_2, \dots, q_m$ as potentials. Clearly this function is uniquely determined if H_1 is known. And H_1 is determined if its associated measure ν_m on the Martin boundary (with the usual normalization) is known. Thus the measure ν_m on the Martin boundary determines uniquely the m-harmonic function $(q_m, \dots, q_2, H_1)$; that is $\nu_m \in \Lambda_m$. Then

$$\begin{aligned} H &= (H_m, \dots, H_2, H_1) \\ &= (q_m, \dots, q_2, H_1) + (u_m, \dots u_3, h_2, 0) \end{aligned}$$

where $H_j = q_j + u_j$ for $3 \leq j \leq m$. Since $h_2 \geq 0$ is harmonic, following the construction in the above paragraph, we obtain $(q'_m, \dots, q'_3, h_2, 0)$ where $q'_m, \dots, q'_3$ are potentials, $(-\Delta)q'_{j+1} = q'_j$, $3 \leq j \leq m-1$, and $(-\Delta)q'_3 = h_2$ is harmonic. Again this function is uniquely determined if h_2 is known. And

h_2 is determined if its associated measure v_{m-1} on the Martin boundary is known. Thus the measure v_{m-1} on the Matrin boundary determines uniquely the function $(q'_m, \dots, q'_3, h_2, 0)$; that is $v_{m-1} \in \Lambda_{m-1}$. Moreover, if we write $(u_m, \dots, u_3, h_2, 0) = (q'_m, \dots, q'_3, h_2, 0) + (r_m, \dots, r_4, h_3, 0, 0)$, we find that each r_i is positive superharmonic such that $(-\Delta)r_{j+1} = r_j$ for $4 \le j \le m-1$, $(-\Delta)r_4 = h_3$ which is a positive harmonic function.

Proceeding thus, we find that we can write

$$\begin{aligned} H &= (H_m, \dots, H_2, H_1) \\ &= (q_m, \dots, q_2, H_1) + (q'_m, \dots, q'_3, h_2, 0) \\ &\quad + (q'_m, \dots, q'_4, h_3, 0, 0) + \cdots + (h_m, 0, \dots, 0) \\ &= \sum_{i=1}^{m} v_i \end{aligned}$$

where each v_i is uniquely determined by a measure $v_i \in \Lambda_i$ in the Martin boundary.

Finally, since $s = P + H$, we conclude that s is uniquely determined by the measures $\mu \in M$ and $v_i \in \Lambda_i$.

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