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GA-optimal partially balanced fractional $2^{m_1+m_2}$ factorial designs of resolution $R(\{00, 10, 01\} \mid \Omega)$ with $2 \le m_1, m_2 \le 4$

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ABSTRACT. Under the assumption that the three-factor and higher-order interactions are negligible, we consider a partially balanced fractional $2^{m_1+m_2}$ factorial design derived from a simple partially balanced array such that the general mean, all the $m_1 + m_2$ main effects, and some linear combinations of $\binom{m_1}{2}$ two-factor interactions, of the $\binom{m_2}{2}$ ones and of the $m_1 m_2$ ones are estimable, where $2 \le m_k$ for $k = 1, 2$. This paper presents optimal designs with respect to the generalized A-optimality criterion when the number of assemblies is less than the number of non-negligible factorial effects, where $2 \le m_1$, $m_2 \le 4$.

1. Introduction

As a special case of an asymmetrical balanced array as defined by Nishii [10], a partially balanced array (PBA) of 2 symbols and $m_1 + m_2$ constraints was presented by Kuwada [3]. A PBA of strength $m_1 + m_2$ is said to be *simple*, and it is written by SPBA$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ for brevity, where λ_{i_1,i_2} are the indices of an SPBA. A fractional factorial design derived from an array of 2 symbols and $m_1 + m_2$ constraints is called a *partially balanced fractional* $2^{m_1+m_2}$ *factorial* ($2^{m_1+m_2}$*-PBFF*) *design* if the variance-covariance matrix of the estimators of the factorial effects to be of interest is invariant under any permutation within m_k factors for each k $(k = 1, 2)$. Under certain conditions, a PBA of 2 symbols and $m_1 + m_2$ constraints turns out to be a $2^{m_1+m_2}$-PBFF design (see [3]). If the general mean $(= \boldsymbol{\theta}_{00}$, say), all the m_1 main effects $(= \boldsymbol{\theta}_{10}$, say), all the m_2 ones $(= \boldsymbol{\theta}_{01}$, say) and all the $\binom{m_1}{a_1}\binom{m_2}{a_2}$ two-factor interactions $(= \boldsymbol{\theta}_{a_1 a_2}$, say) are estimable for $a_1 a_2 \in \Omega^*$, and the factorial effects of the $\binom{m_1}{b_1}\binom{m_2}{b_2}$ ones $(= \boldsymbol{\theta}_{b_1 b_2}$, say) are confounded with each other for $b_1 b_2 \in \bar{\Omega}^*$, then a design is said to be of *resolutions* $R(\{00, 10, 01\} \cup \Omega^* \mid \Omega)$, where the three-factor and higher-order interactions are assumed to be negligible, $\Omega^* = \{20, 02\}, \{20, 11\}$ (or $\{02, 11\}$), $\{20\}$ (or $\{02\}$), $\{11\}$, ϕ, $\bar{\Omega}^* =$

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$\{20, 02, 11\} - \Omega^*$, and $\Omega = \{00, 10, 01, 20, 02, 11\}$. Optimal $2^{m_1+m_2}$-PBFF designs of resolutions $\mathrm{R}(\{00, 10, 01\} \cup \Omega^* \,|\, \Omega)$ with respect to the generalized A-optimality (GA-optimality) criterion (see [6]) have been obtained by Kuwada et al. [6, 7] and Lu et al. [8], where $2 \le m_1, m_2 \le 4$.

In this paper, we consider a $2^{m_1+m_2}$-PBFF design such that $\boldsymbol{\theta}_{00}$, $\boldsymbol{\theta}_{10}$ and $\boldsymbol{\theta}_{01}$ are estimable and the effects of $\boldsymbol{\theta}_{20}$, $\boldsymbol{\theta}_{02}$ and $\boldsymbol{\theta}_{11}$ are confounded with each other, where the three-factor and higher-order interactions are assumed to be negligible. Such a design is said to be of *resolution* $R(\{00, 10, 01\} \,|\, \Omega)$. A resolution $\mathrm{R}(\{00, 10, 01\} \,|\, \Omega)$ design considered here is a part of the ordinary resolution IV designs (e.g. [9]). Furthermore we present GA-optimal $2^{m_1+m_2}$-PBFF designs of resolution $\mathrm{R}(\{00, 10, 01\} \,|\, \Omega)$ when the number of assemblies (or treatment combinations) is less than the number of non-negligible factorial effects, where $2 \le m_1, m_2 \le 4$.

2. Preliminaries

We consider a fractional $2^{m_1+m_2}$ factorial design T with $m_1 + m_2$ factors each at 2 levels and N assemblies, where the three-factor and higher-order interactions are assumed to be negligible, and $2 \le m_1, m_2$. Let $\boldsymbol{\Theta} = (\boldsymbol{\theta}'_{00}; \boldsymbol{\theta}'_{10}; \boldsymbol{\theta}'_{01}; \boldsymbol{\theta}'_{20}; \boldsymbol{\theta}'_{02}; \boldsymbol{\theta}'_{11})'$, which is the $v(m_1, m_2) \times 1$ vector of the non-negligible factorial effects, where $v(m_1, m_2) = 1 + (m_1 + m_2) + \binom{m_1+m_2}{2}$ and A' denotes the transpose of a matrix A. Under the linear model: $\boldsymbol{y}(T) = E_T\boldsymbol{\Theta} + \boldsymbol{e}_T$, where $\boldsymbol{y}(T)$, E_T and $\boldsymbol{e}_T$ are an $N \times 1$ observation vector, the $N \times v(m_1, m_2)$ design matrix whose elements are either 1 or -1, and an $N \times 1$ error vector with mean $\boldsymbol{0}_N$ and variance-covariance matrix $\sigma^2 I_N$, respectively, the normal equations for estimating $\boldsymbol{\Theta}$ are given by

$$M_T\hat{\boldsymbol{\Theta}} = E'_T\boldsymbol{y}(T), \tag{2.1}$$

where M_T $(= E'_T E_T)$ is the information matrix of order $v(m_1, m_2)$.

Let T $(= (T^{(1)}; T^{(2)})$, say) be a $2^{m_1+m_2}$-PBFF design derived from an $\mathrm{SPBA}(m_1 + m_2; \{\lambda_{i_1, i_2}\})$, where $T^{(k)}$ are of size $N \times m_k$ $(k = 1, 2)$. Then M_T is given by

$$\begin{aligned} M_T &= \sum_{a_1a_2}\sum_{b_1b_2}\sum_{\alpha_1\alpha_2} \gamma_{|a_1-b_1|+2\alpha_1,\, |a_2-b_2|+2\alpha_2} D_{\alpha_1\alpha_2}^{(a_1a_2,\, b_1b_2)} \\ &= \sum_{a_1a_2}\sum_{b_1b_2}\sum_{\beta_1\beta_2} \kappa_{\beta_1\beta_2}^{a_1a_2,\, b_1b_2} D_{\beta_1\beta_2}^{\#(a_1a_2,\, b_1b_2)}, \end{aligned}$$

where the relationships between γ_{i_1, i_2} and λ_{j_1, j_2}, and $\kappa_{\beta_1\beta_2}^{a_1a_2, b_1b_2}$ and γ_{i_1, i_2} are, respectively, given by

$$\gamma_{i_1,i_2} = \sum_{j_1,j_2}\prod_{k=1}^{2}\left\{\sum_{p_k}(-1)^{p_k}\binom{i_k}{p_k}\binom{m_k-i_k}{j_k-i_k+p_k}\right\}\lambda_{j_1,j_2},$$
$$\kappa^{a_1a_2,b_1b_2}_{\beta_1\beta_2} = \sum_{\alpha_1\alpha_2} z^{(a_1a_2,b_1b_2)}_{\beta_1\beta_2\alpha_1\alpha_2}\gamma_{|a_1-b_1|+2\alpha_1,|a_2-b_2|+2\alpha_2} \tag{2.2}$$

(see [3]). Here $D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2}$ for $a_1a_2, b_1b_2 \in S_{\alpha_1\alpha_2}$ ($\alpha_1\alpha_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11) are the $v(m_1,m_2) \times v(m_1,m_2)$ ordered association matrices of the extended triangular multidimensional partially balanced (ETMDPB) association scheme, $D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ for $a_1a_2, b_1b_2 \in S_{\beta_1\beta_2}$ ($\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11) are the matrices of order $v(m_1,m_2)$ given by some linear combinations of $D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2}$, and $z^{(a_1a_2,b_1b_2)}_{\beta_1\beta_2\alpha_1\alpha_2} = z^{(a_1,b_1)}_{\beta_1\alpha_1}z^{(a_2,b_2)}_{\beta_2\alpha_2}$, where $S_{00} = \{00, 10, 01, 20, 02, 11\}$, $S_{10} = \{10, 20$ (if $m_1 \geq 3$), $11\}$, $S_{01} = \{01, 02$ (if $m_2 \geq 3$), $11\}$, $S_{20} = \{20\}$ (if $m_1 \geq 4$), $S_{02} = \{02\}$ (if $m_2 \geq 4$) and $S_{11} = \{11\}$, and

$$z^{(a,b)}_{\beta\alpha}\ (= z^{(b,a)}_{\beta\alpha}) = \sum_{p=0}^{\alpha}(-1)^{\alpha-p}\binom{a-\beta}{p}\binom{a-p}{a-\alpha}\binom{m-a-\beta+p}{p}$$
$$\times\sqrt{\binom{m-a-\beta}{b-a}\binom{b-\beta}{b-a}}\Big/\binom{b-a+p}{p} \qquad \text{for } a \leq b$$

(see [3, 12]). Thus from the properties of the ETMDPB association algebra $[D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2} \mid a_1a_2, b_1b_2 \in S_{\beta_1\beta_2}$ ($\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11)] ($= \mathscr{A}$, say), the information matrix M_T is isomorphic to $\|\kappa^{a_1a_2,b_1b_2}_{\beta_1\beta_2}\|$ ($= K_{\beta_1\beta_2}$, say) of order 6 for $\beta_1\beta_2 = 00$, of order 3 (if $m_1 \geq 3$) (or 2 (if $m_1 = 2$)) for $\beta_1\beta_2 = 10$, of order 3 (if $m_2 \geq 3$) (or 2 (if $m_2 = 2$)) for $\beta_1\beta_2 = 01$, of order 1 (if $m_1 \geq 4$) for $\beta_1\beta_2 = 20$, of order 1 (if $m_2 \geq 4$) for $\beta_1\beta_2 = 02$ and of order 1 for $\beta_1\beta_2 = 11$ with multiplicities ϕ_{00}, ϕ_{10}, ϕ_{01}, ϕ_{20}, ϕ_{02} and ϕ_{11}, respectively (see [3]). Here $\phi_{\beta_1\beta_2} = \phi_{\beta_1}\phi_{\beta_2}$ and $\phi_\beta = \binom{m}{\beta} - \binom{m}{\beta-1}$. Note that $K_{\beta_1\beta_2}$ are called the *irreducible representations* of M_T with respect to the ideals $[D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2} \mid a_1a_2, b_1b_2 \in S_{\beta_1\beta_2}]$ ($= \mathscr{A}_{\beta_1\beta_2}$, say) for $\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11 of the algebra $\mathscr{A}$ (see [3]). Let $A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ be the $n_{a_1a_2} \times n_{b_1b_2}$ submatrices of $D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ corresponding to the a_1a_2-th row block and the b_1b_2-th column block, where $n_{a_1a_2} = \binom{m_1}{a_1}\binom{m_2}{a_2}$. Then the properties of $A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ and $D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ are cited in the following:

$$A^{\#(a_1a_2,c_1c_2)}_{\beta_1\beta_2}A^{\#(c_1c_2,b_1b_2)}_{\gamma_1\gamma_2} = \delta_{\beta_1\gamma_1}\delta_{\beta_2\gamma_2}A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2},$$
$$\sum_{\beta_1\beta_2}A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2} = I_{n_{a_1a_2}}, \qquad \operatorname{rank}\{A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}\} = \phi_{\beta_1\beta_2}, \tag{2.3}$$

$$D^{\#(a_1a_2,c_1c_2)}_{\beta_1\beta_2} D^{\#(d_1d_2,b_1b_2)}_{\gamma_1\gamma_2} = \delta_{c_1d_1}\delta_{c_2d_2}\delta_{\beta_1\gamma_1}\delta_{\beta_2\gamma_2} D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2},$$

$$\sum_{a_1a_2}\sum_{\beta_1\beta_2} D^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2} = I_{v(m_1,m_2)}, \qquad \operatorname{rank}\{D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}\} = \phi_{\beta_1\beta_2}$$

(see [3]), where δ_{pq} is the Kronecker delta. From (2.2), we have

$$K_{\beta_1\beta_2} = (D_{\beta_1\beta_2}F_{\beta_1\beta_2}\Lambda_{\beta_1\beta_2})(D_{\beta_1\beta_2}F_{\beta_1\beta_2}\Lambda_{\beta_1\beta_2})' \tag{2.4}$$

(see [6]), where

$$D_{00} = \operatorname{diag}[1; -1/\sqrt{m_1}; -1/\sqrt{m_2}; 1/\sqrt{2m_1(m_1-1)}; 1/\sqrt{2m_2(m_2-1)}; 1/\sqrt{m_1m_2}],$$

$$D_{10} = \begin{cases} \operatorname{diag}[2; -2/\sqrt{m_2}] & \text{if } m_1 = 2, \\ \operatorname{diag}[2; -2/\sqrt{m_1-2}; -2/\sqrt{m_2}] & \text{if } m_1 \geq 3, \end{cases}$$

$$D_{01} = \begin{cases} \operatorname{diag}[2; -2/\sqrt{m_1}] & \text{if } m_2 = 2, \\ \operatorname{diag}[2; -2/\sqrt{m_2-2}; -2/\sqrt{m_1}] & \text{if } m_2 \geq 3, \end{cases}$$

$$D_{20} = \begin{cases} \text{vanishes} & \text{if } m_1 = 2,3, \\ 2^2 & \text{if } m_1 \geq 4, \end{cases} \qquad D_{02} = \begin{cases} \text{vanishes} & \text{if } m_2 = 2,3, \\ 2^2 & \text{if } m_2 \geq 4, \end{cases} \qquad D_{11} = 2^2,$$

the column vector of F_{00} corresponding to $\lambda_{a,x}$ $(0 \leq a \leq m_1; 0 \leq x \leq m_2)$ is $\sqrt{\lambda_{a,x}}\begin{pmatrix} 1 \\ m_1 - 2a \\ m_2 - 2x \\ (m_1-2a)^2 - m_1 \\ (m_2-2x)^2 - m_2 \\ (m_1-2a)(m_2-2x) \end{pmatrix}$, the column one of F_{10} corresponding to $\lambda_{b,y}$ $(1 \leq b \leq m_1 - 1; 0 \leq y \leq m_2)$ is $\sqrt{\lambda_{b,y}}\begin{pmatrix} 1 \\ m_1 - 2b \\ m_2 - 2y \end{pmatrix}$ (if $m_1 \geq 3$) (or $\sqrt{\lambda_{1,y}}\begin{pmatrix} 1 \\ m_2 - 2y \end{pmatrix}$ (if $m_1 = 2$)), the column one of F_{01} corresponding to $\lambda_{c,z}$ $(0 \leq c \leq m_1; 1 \leq z \leq m_2 - 1)$ is $\sqrt{\lambda_{c,z}}\begin{pmatrix} 1 \\ m_2 - 2z \\ m_1 - 2c \end{pmatrix}$ (if $m_2 \geq 3$) (or $\sqrt{\lambda_{c,1}}\begin{pmatrix} 1 \\ m_1 - 2c \end{pmatrix}$ (if $m_2 = 2$)), and the columns of F_{20} corresponding to $\lambda_{d,u}$ $(2 \leq d \leq m_1 - 2; 0 \leq u \leq m_2)$, of F_{02} corresponding to $\lambda_{e,v}$ $(0 \leq e \leq m_1; 2 \leq v \leq m_2 - 2)$ and of F_{11} corresponding to $\lambda_{f,w}$ $(1 \leq f \leq m_1 - 1; 1 \leq w \leq m_2 - 1)$ are $\sqrt{\lambda_{d,u}}$, $\sqrt{\lambda_{e,v}}$ and $\sqrt{\lambda_{f,w}}$, respectively, and the diagonal elements of $\Lambda_{\beta_1\beta_2}$ $(\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), $11)$ corresponding to $\lambda_{g,s}$ $(\beta_1 \leq g \leq m_1 - \beta_1; \beta_2 \leq s \leq m_2 - \beta_2)$ are given by $\sqrt{\binom{m_1-2\beta_1}{g-\beta_1}\binom{m_2-2\beta_2}{s-\beta_2}}$ and the off-diagonal elements of them are all zero. Note that F_{00} is of size $6 \times \{(m_1+1)(m_2+1)\}$, F_{10} is of size $3 \times \{(m_1-1)(m_2+1)\}$ (if $m_1 \geq 3$) (or $2 \times (m_2+1)$ (if $m_1 = 2$)), F_{01} is of size $3 \times \{(m_1+1)(m_2-1)\}$ (if $m_2 \geq 3$) (or $2 \times (m_1+1)$ (if $m_2 = 2$)), F_{20} (if $m_1 \geq 4$) is of size $1 \times (m_1-3)$, F_{02} (if $m_2 \geq 4$) is of size $1 \times (m_2-3)$ and F_{11} is of size $1 \times \{(m_1-1)(m_2-1)\}$, and $\Lambda_{\beta_1\beta_2}$ are of order $(m_1+1-2\beta_1)(m_2+1-2\beta_2)$.

It follows from the definitions of $D_{\beta_1\beta_2}$, $F_{\beta_1\beta_2}$ and $\Lambda_{\beta_1\beta_2}$ that $\text{rank}\{K_{\beta_1\beta_2}\} = \text{r-rank}\{F_{\beta_1\beta_2}\}$, where r-rank$\{A\}$ denotes the row rank of a matrix A.

Let $T = (T^{(1)}; T^{(2)})$ be an SPBA$(m_1+m_2; \{\lambda_{i_1,i_2}\})$, and further let $\tilde{T} = (\overline{T}^{(1)}; T^{(2)}), \breve{T} = (T^{(1)}; \overline{T}^{(2)})$ and $\overline{T} = (\overline{T}^{(1)}; \overline{T}^{(2)})$, where $\overline{T}^{(k)}$ denotes the complement of $T^{(k)}$. Then $\tilde{T}$, $\breve{T}$ and $\overline{T}$ are called the *former complementary array* (*FCA*) of T, the *latter complementary array* (*LCA*) of T and the *completely complementary array* (*CCA*) of T, respectively. Note that when T is an SPBA$(m_1+m_2; \{\lambda_{i_1,i_2}\})$, $\tilde{T}$, $\breve{T}$ and $\overline{T}$ are the SPBA$(m_1+m_2; \{\lambda_{m_1-i_1,i_2}\})$, the SPBA$(m_1+m_2; \{\lambda_{i_1,m_2-i_2}\})$ and the SPBA$(m_1+m_2; \{\lambda_{m_1-i_1,m_2-i_2}\})$, respectively. Let $M_{\tilde{T}}$, $M_{\breve{T}}$ and $M_{\overline{T}}$ be the information matrices associated with $\tilde{T}$, $\breve{T}$ and $\overline{T}$, respectively, where T is an SPBA$(m_1+m_2; \{\lambda_{i_1,i_2}\})$, and further let $\tilde{K}_{\beta_1\beta_2}$, $\breve{K}_{\beta_1\beta_2}$ and $\overline{K}_{\beta_1\beta_2}$ be the irreducible representations of $M_{\tilde{T}}$, $M_{\breve{T}}$ and $M_{\overline{T}}$ with respect to the ideals $\mathscr{A}_{\beta_1\beta_2}$ of the ETMDPB association algebra $\mathscr{A}$, respectively. Then we have the following (see [6]):

Lemma 2.1. *Let T be an* SPBA$(m_1+m_2; \{\lambda_{i_1,i_2}\})$, *where* $2 \leq m_k$ $(k = 1,2)$. *Then* $\tilde{K}_{\beta_1\beta_2} = \tilde{\Delta}_{\beta_1\beta_2} K_{\beta_1\beta_2} \tilde{\Delta}_{\beta_1\beta_2}$, $\breve{K}_{\beta_1\beta_2} = \breve{\Delta}_{\beta_1\beta_2} K_{\beta_1\beta_2} \breve{\Delta}_{\beta_1\beta_2}$ *and* $\overline{K}_{\beta_1\beta_2} = \overline{\Delta}_{\beta_1\beta_2} K_{\beta_1\beta_2} \overline{\Delta}_{\beta_1\beta_2}$ *for* $\beta_1\beta_2 = 00, 10, 01, 20$ (*if* $m_1 \geq 4$), 02 (*if* $m_2 \geq 4$), 11, *where* $\tilde{\Delta}_{00} = \text{diag}[+;-;+;+;+;-]$, $\breve{\Delta}_{00} = \text{diag}[+;+;-;+;+;-]$, $\overline{\Delta}_{00} = \text{diag}[+;-;-;+;+;+]$, $\tilde{\Delta}_{10} = \text{diag}[+;-;+]$ (*if* $m_1 \geq 3$) (*or* $\text{diag}[+;+]$ (*if* $m_1 = 2$)), $\breve{\Delta}_{10} = \text{diag}[+;+;-]$ (*if* $m_1 \geq 3$) (*or* $\text{diag}[+;-]$ (*if* $m_1 = 2$)), $\overline{\Delta}_{10} = \text{diag}[+;-;-]$ (*if* $m_1 \geq 3$) (*or* $\text{diag}[+;-]$ (*if* $m_1 = 2$)), $\tilde{\Delta}_{01} = \text{diag}[+;+;-]$ (*if* $m_2 \geq 3$)) (*or* $\text{diag}[+;-]$ (*if* $m_2 = 2$)), $\breve{\Delta}_{01} = \text{diag}[+;-;+]$ (*if* $m_2 \geq 3$) (*or* $\text{diag}[+;+]$ (*if* $m_2 = 2$)), $\overline{\Delta}_{01} = \text{diag}[+;-;-]$ (*if* $m_2 \geq 3$) (*or* $\text{diag}[+;-]$ (*if* $m_2 = 2$)), $\tilde{\Delta}_{20} = \breve{\Delta}_{20} = \overline{\Delta}_{20} = (+)$ (*if* $m_1 \geq 4$) (*or vanishes* (*if* $m_1 = 2,3$)), $\tilde{\Delta}_{02} = \breve{\Delta}_{02} = \overline{\Delta}_{02} = (+)$ (*if* $m_2 \geq 4$)) (*or vanishes* (*if* $m_2 = 2,3$)) *and* $\tilde{\Delta}_{11} = \breve{\Delta}_{11} = \overline{\Delta}_{11} = (+)$. *Here* $+$ *and* $-$ *denote* 1 *and* -1, *respectively*.

3. Parametric functions

We now consider a $2^{m_1+m_2}$-PBFF design of resolution $\text{R}(\{00,10,01\} \mid \Omega)$ derived from an SPBA$(m_1+m_2; \{\lambda_{i_1,i_2}\})$. A necessary and sufficient condition for parametric functions $C\boldsymbol{\Theta}$ of $\boldsymbol{\Theta}$ to be estimable for some matrix C of order $v(m_1,m_2)$ is that there exists a matrix X of order $v(m_1,m_2)$ such that $XM_T = C$ (e.g. [11]). Since M_T belongs to the ETMDPB association algebra $\mathscr{A}$, we impose some restrictions on C such that it belongs to $\mathscr{A}$, and hence X also belongs to it (e.g. [6]). Thus we define C and X as follows:

$$
\begin{aligned}
C = {} & D_{00}^{\#(00,00)} + \{D_{00}^{\#(10,10)} + D_{10}^{\#(10,10)}\} + \{D_{00}^{\#(01,01)} + D_{01}^{\#(01,01)}\} \\
& + \{g_{00}^{20,20} D_{00}^{\#(20,20)} + g_{10}^{20,20} D_{10}^{\#(20,20)} \ \ (\text{if } \ m_1 \geq 3) \\
& \quad + g_{20}^{20,20} D_{20}^{\#(20,20)} \ \ (\text{if } \ m_1 \geq 4)\} \\
& + \{g_{00}^{20,02} D_{00}^{\#(20,02)} + g_{00}^{02,20} D_{00}^{\#(02,20)}\} \\
& + \{g_{00}^{20,11} D_{00}^{\#(20,11)} + g_{10}^{20,11} D_{10}^{\#(20,11)} \ \ (\text{if } \ m_1 \geq 3) \\
& \quad + g_{00}^{11,20} D_{00}^{\#(11,20)} + g_{10}^{11,20} D_{10}^{\#(11,20)} \ \ (\text{if } \ m_1 \geq 3)\} \\
& + \{g_{00}^{02,02} D_{00}^{\#(02,02)} + g_{01}^{02,02} D_{01}^{\#(02,02)} \ \ (\text{if } \ m_2 \geq 3) \\
& \quad + g_{02}^{02,02} D_{02}^{\#(02,02)} \ \ (\text{if } \ m_2 \geq 4)\} \\
& + \{g_{00}^{02,11} D_{00}^{\#(02,11)} + g_{01}^{02,11} D_{01}^{\#(02,11)} \ \ (\text{if } \ m_2 \geq 3) \\
& \quad + g_{00}^{11,02} D_{00}^{\#(11,02)} + g_{01}^{11,02} D_{01}^{\#(11,02)} \ \ (\text{if } \ m_2 \geq 3)\} \\
& + \{g_{00}^{11,11} D_{00}^{\#(11,11)} + g_{10}^{11,11} D_{10}^{\#(11,11)} + g_{01}^{11,11} D_{01}^{\#(11,11)} \\
& \quad + g_{11}^{11,11} D_{11}^{\#(11,11)}\}, \qquad (3.1) \\
X = {} & \sum_{a_1a_2} \sum_{b_1b_2} \sum_{\beta_1\beta_2} \chi_{\beta_1\beta_2}^{a_1a_2,b_1b_2} D_{\beta_1\beta_2}^{\#(a_1a_2,b_1b_2)},
\end{aligned}
$$

where $g_{\gamma_1\gamma_2}^{a_1a_2,b_1b_2}$ are some constants. Then C and X are isomorphic to $\Gamma_{\beta_1\beta_2}$ and $X_{\beta_1\beta_2}$ ($\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11), respectively, where

$$
\Gamma_{00} = \operatorname{diag}\left[I_3; \begin{pmatrix} g_{00}^{20,20} & g_{00}^{20,02} & g_{00}^{20,11} \\ g_{00}^{02,20} & g_{00}^{02,02} & g_{00}^{02,11} \\ g_{00}^{11,20} & g_{00}^{11,02} & g_{00}^{11,11} \end{pmatrix} \right],
$$

$$
\Gamma_{10} = \begin{cases} \operatorname{diag}[1; g_{10}^{11,11}] & \text{if } m_1 = 2, \\ \operatorname{diag}\left[1; \begin{pmatrix} g_{10}^{20,20} & g_{10}^{20,11} \\ g_{10}^{11,20} & g_{10}^{11,11} \end{pmatrix}\right] & \text{if } m_1 \geq 3, \end{cases}
$$

$$
\Gamma_{01} = \begin{cases} \operatorname{diag}[1; g_{01}^{11,11}] & \text{if } m_2 = 2, \\ \operatorname{diag}\left[1; \begin{pmatrix} g_{01}^{02,02} & g_{01}^{02,11} \\ g_{01}^{11,02} & g_{01}^{11,11} \end{pmatrix}\right] & \text{if } m_2 \geq 3, \end{cases}
$$

$$\Gamma_{20} = \begin{cases} \text{vanishes} & \text{if } m_1 = 2,3, \\ g_{20}^{20,20} & \text{if } m_1 \ge 4, \end{cases} \qquad \Gamma_{02} = \begin{cases} \text{vanishes} & \text{if } m_2 = 2,3, \\ g_{02}^{02,02} & \text{if } m_2 \ge 4, \end{cases}$$

$$\Gamma_{11} = g_{11}^{11,11}, \qquad X_{\beta_1\beta_2} = \|\chi_{\beta_1\beta_2}^{a_1a_2,b_1b_2}\|.$$

Thus $XM_T = C$ is isomorphic to $X_{\beta_1\beta_2}K_{\beta_1\beta_2} = \Gamma_{\beta_1\beta_2}$. Note that if $C\boldsymbol{\Theta}$ is estimable, where C is given by (3.1), then a design is of resolution $\mathrm{R}(\{00,10,01\}\,|\,\Omega)$.

If $N \ge \nu(m_1,m_2)$, then there exists a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\Omega|\Omega)$ (e.g. [3]). A resolution $\mathrm{R}(\Omega|\Omega)$ design is of resolution V. Thus we focus on obtaining a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\{00,10,01\}\,|\,\Omega)$ derived from an $\mathrm{SPBA}(m_1+m_2;\{\lambda_{i_1,i_2}\})$ with $N < \nu(m_1,m_2)$ and $2 \le m_k$ $(k=1,2)$. Since $N < \nu(m_1,m_2)$, the information matrix M_T is singular, and hence at least one of $K_{\beta_1\beta_2}$ is singular. Therefore at least one of $F_{\beta_1\beta_2}$ is not of full row rank. The system of equations $(m_1-2a)^2 - m_1 = 0$, $(m_2-2x)^2 - m_2 = 0$ and $(m_1-2a)(m_2-2x) = 0$ with parameters a and x has no solution. Thus there do not exist the indices $\lambda_{a,x}$ of an SPBA such that r-rank$\{F_{00}\} = 3$ and the last three rows of F_{00} are zero. Applying Lemma A.1 to the matrix equations $X_{\beta_1\beta_2}K_{\beta_1\beta_2} = \Gamma_{\beta_1\beta_2}$ with parameter matrices $X_{\beta_1\beta_2}$, we have the following:

Lemma 3.1. *Let T be a $2^{m_1+m_2}$-PBFF design of resolution $R(\{00,10,01\}\,|\,\Omega)$ derived from an $\mathrm{SPBA}(m_1+m_2;\{\lambda_{i_1,i_2}\})$ with $N < \nu(m_1,m_2)$ and $2 \le m_k$ for $k=1,2$. Then the first three rows of F_{00} and each of the first row of F_{10} and of F_{01} must be linearly independent, and furthermore*

(A) (i) *if* r-rank$\{F_{00}\} = 4$, *then* (a) *the last two rows of F_{00} are zero,*
- (b) *the fourth and the last rows of F_{00} are zero,*
- (c) *the fourth and the fifth rows of F_{00} are zero,*
- (d) *the last row of F_{00} is zero and the fifth equals u_{00} $(\neq 0)$ times the fourth,*
- (e) *the fifth row of F_{00} is zero and the last equals v_{00} $(\neq 0)$ times the fourth,*
- (f) *the fourth row of F_{00} is zero and the last equals w_{00} $(\neq 0)$ times the fifth, or*
- (g) *the fifth and the last rows of F_{00} equal u_{00} $(\neq 0)$ times the fourth and v_{00} $(\neq 0)$ times the fourth, respectively, and*

(ii) *if* r-rank$\{F_{00}\} = 5$, *then* (a) *the last row of F_{00} is zero,*
- (b) *the fifth row of F_{00} is zero,*
- (c) *the fourth row of F_{00} is zero,*
- (d) *the last row of F_{00} equals w_{00} $(\neq 0)$ times the fifth,*
- (e) *the last row of F_{00} equals v_{00} $(\neq 0)$ times the fourth,*
- (f) *the fifth row of F_{00} equals u_{00} $(\neq 0)$ times the fourth, or*

(g) *the last row of* F_{00} *equals the sum of* v_{00} $(\neq 0)$ *times the fourth and* w_{00} $(\neq 0)$ *times the fifth,*

(B) (i) *if* r-rank$\{F_{10}\} = 1$, *then* (1) *when* $m_1 = 2$, *the last row of* F_{10} *is zero, and*

(2) *when* $m_1 \geq 3$, *the last two rows of* F_{10} *are zero, and*

(ii) *if* $m_1 \geq 3$ *and* r-rank$\{F_{10}\} = 2$, *then* (a) *the last row of* F_{10} *is zero,*

(b) *the second row of* F_{10} *is zero, or*

(c) *the last row of* F_{10} *equals* v_{10} $(\neq 0)$ *times the second, and*

(C) (i) *if* r-rank$\{F_{01}\} = 1$, *then* (1) *when* $m_2 = 2$, *the last row of* F_{01} *is zero, and*

(2) *when* $m_2 \geq 3$, *the last two rows of* F_{01} *are zero, and*

(ii) *if* $m_2 \geq 3$ *and* r-rank$\{F_{01}\} = 2$, *then* (a) *the last row of* F_{01} *is zero,*

(b) *the second row of* F_{01} *is zero, or*

(c) *the last row of* F_{01} *equals* w_{01} $(\neq 0)$ *times the second.*

If $F_{\beta_1\beta_2}$ is of full row rank for some $\beta_1\beta_2$ (and hence $K_{\beta_1\beta_2}$ is of full rank), then in the matrix equation $X_{\beta_1\beta_2}K_{\beta_1\beta_2} = \Gamma_{\beta_1\beta_2}$, there always exists $X_{\beta_1\beta_2}$ such that $X_{\beta_1\beta_2} = K_{\beta_1\beta_2}^{-1}$, and hence $\Gamma_{\beta_1\beta_2}$ is the identity matrix. Thus if $F_{\gamma_1\gamma_2}$ is of full row rank, then without loss of generality, we can put $g_{\gamma_1\gamma_2}^{a_1a_2,b_1b_2} = 1$ $(\gamma_1\gamma_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11) if $a_1a_2 = b_1b_2$ and $g_{\gamma_1\gamma_2}^{a_1a_2,b_1b_2} = 0$ $(\gamma_1\gamma_2 = 00, 10$ (if $m_1 \geq 3$), 01 (if $m_2 \geq 3$)) if $a_1a_2 \neq b_1b_2$. Therefore (2.3), (2.4), and Lemmas 3.1 and A.1 yield the following:

THEOREM 3.2. *Let* T *be a* $2^{m_1+m_2}$-PBFF *design of resolution* $R(\{00, 10, 01\} \mid \Omega)$ *derived from an* SPBA$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ *with* $N < v(m_1, m_2)$ *and* $2 \leq m_k$ $(k = 1, 2)$. *Then we have the following:*

(I) *If the matrix* $F_{\beta_1\beta_2}$ $(\beta_1\beta_2 = 00, 10, 01, 20$ (*if* $m_1 \geq 4$), 02 (*if* $m_2 \geq 4$), 11) *is of full row rank, then* $A_{\beta_1\beta_2}^{\#(a_1a_2,a_1a_2)}\boldsymbol{\theta}_{a_1a_2}$ $(a_1a_2 \in S_{\beta_1\beta_2})$ *are estimable,*

(II) *if the first three rows of* F_{00} *are linearly independent, then* $A_{00}^{\#(a_1a_2,a_1a_2)}\boldsymbol{\theta}_{a_1a_2}$ $(a_1a_2 = 00, 10, 01)$ *are estimable, and in addition*

(A) (i) *if* r-rank$\{F_{00}\} = 4$, *and furthermore*

(a) *if the last two rows of* F_{00} *are zero, then*

$$g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}\boldsymbol{\theta}_{20} = g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}(A_{00}^{\#(20,20)}\boldsymbol{\theta}_{20})$$
$$(b_1b_2 = 20, 02, 11)$$

are estimable,

(b) *if the fourth and the last rows of* F_{00} *are zero, then*

$$g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}\boldsymbol{\theta}_{02} = g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}(A_{00}^{\#(02,02)}\boldsymbol{\theta}_{02})$$

are estimable,

(c) *if the fourth and the fifth rows of* F_{00} *are zero, then*

$$g_{00}^{b_1b_2,11} A_{00}^{\#(b_1b_2,11)} \boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,11} A_{00}^{\#(b_1b_2,11)} (A_{00}^{\#(11,11)} \boldsymbol{\theta}_{11})$$

are estimable,

(d) *if the last row of* F_{00} *is zero and the fifth equals* u_{00} $(\neq 0)$ *times the fourth, then*

$$g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} \boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,02} A_{00}^{\#(b_1b_2,02)} \boldsymbol{\theta}_{02} = g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} (A_{00}^{\#(20,20)} \boldsymbol{\theta}_{20} + u_{00}^* A_{00}^{\#(20,02)} \boldsymbol{\theta}_{02})$$

are estimable, where $u_{00}^* = \sqrt{m_1(m_1-1)/\{m_2(m_2-1)\}} u_{00}$,

(e) *if the fifth row of* F_{00} *is zero and the last equals* v_{00} $(\neq 0)$ *times the fourth, then*

$$g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} \boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,11} A_{00}^{\#(b_1b_2,11)} \boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} (A_{00}^{\#(20,20)} \boldsymbol{\theta}_{20} + v_{00}^* A_{00}^{\#(20,11)} \boldsymbol{\theta}_{11})$$

are estimable, where $v_{00}^* = \sqrt{2(m_1-1)/m_2} v_{00}$,

(f) *if the fourth row of* F_{00} *is zero and the last equals* w_{00} $(\neq 0)$ *times the fifth, then*

$$g_{00}^{b_1b_2,02} A_{00}^{\#(b_1b_2,02)} \boldsymbol{\theta}_{02} + g_{00}^{b_1b_2,11} A_{00}^{\#(b_1b_2,11)} \boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,02} A_{00}^{\#(b_1b_2,02)} (A_{00}^{\#(02,02)} \boldsymbol{\theta}_{02} + w_{00}^* A_{00}^{\#(02,11)} \boldsymbol{\theta}_{11})$$

are estimable, where $w_{00}^* = \sqrt{2(m_2-1)/m_1} w_{00}$, *and*

(g) *if the fifth and the last rows of* F_{00} *equal* u_{00} $(\neq 0)$ *times the fourth and* v_{00} $(\neq 0)$ *times the fourth, respectively, then*

$$g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} \boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,02} A_{00}^{\#(b_1b_2,02)} \boldsymbol{\theta}_{02} + g_{00}^{b_1b_2,11} \times A_{00}^{\#(b_1b_2,11)} \boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} (A_{00}^{\#(20,20)} \boldsymbol{\theta}_{20} + u_{00}^* A_{00}^{\#(20,02)} \boldsymbol{\theta}_{02} + v_{00}^* A_{00}^{\#(20,11)} \boldsymbol{\theta}_{11})$$

are estimable, and

(ii) *if* r-rank$\{F_{00}\} = 5$, *and furthermore*

(a) *if the last row of* F_{00} *is zero, then*

$$g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} \boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,02} A_{00}^{\#(b_1b_2,02)} \boldsymbol{\theta}_{02} = g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} (A_{00}^{\#(20,20)} \boldsymbol{\theta}_{20}) + g_{00}^{b_1b_2,02} A_{00}^{\#(b_1b_2,02)} \times (A_{00}^{\#(02,02)} \boldsymbol{\theta}_{02})$$

are estimable,

(b) *if the fifth row of* F_{00} *is zero, then*

$$g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} \boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,11} A_{00}^{\#(b_1b_2,11)} \boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,20} A_{00}^{\#(b_1b_2,20)} (A_{00}^{\#(20,20)} \boldsymbol{\theta}_{20}) + g_{00}^{b_1b_2,11} A_{00}^{\#(b_1b_2,11)} \times (A_{00}^{\#(11,11)} \boldsymbol{\theta}_{11})$$

are estimable,

(c) *if the fourth row of* F_{00} *is zero, then*

$$g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}\boldsymbol{\theta}_{02} + g_{00}^{b_1b_2,11}A_{00}^{\#(b_1b_2,11)}\boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}(A_{00}^{\#(02,02)}\boldsymbol{\theta}_{02}) + g_{00}^{b_1b_2,11}A_{00}^{\#(b_1b_2,11)} \times (A_{00}^{\#(11,11)}\boldsymbol{\theta}_{11})$$

are estimable,

(d) *if the last row of* F_{00} *equals* w_{00} $(\neq 0)$ *times the fifth, then*

$$g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}\boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}\boldsymbol{\theta}_{02} + g_{00}^{b_1b_2,11} \times A_{00}^{\#(b_1b_2,11)}\boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}(A_{00}^{\#(20,20)}\boldsymbol{\theta}_{20}) + g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}(A_{00}^{\#(02,02)}\boldsymbol{\theta}_{02} + w_{00}^{*}A_{00}^{\#(02,11)}\boldsymbol{\theta}_{11})$$

are estimable,

(e) *if the last row of* F_{00} *equals* v_{00} $(\neq 0)$ *times the fourth, then*

$$g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}\boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}\boldsymbol{\theta}_{02} + g_{00}^{b_1b_2,11} \times A_{00}^{\#(b_1b_2,11)}\boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}(A_{00}^{\#(20,20)}\boldsymbol{\theta}_{20} + v_{00}^{*}A_{00}^{\#(20,11)}\boldsymbol{\theta}_{11}) + g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}(A_{00}^{\#(02,02)}\boldsymbol{\theta}_{02})$$

are estimable,

(f) *if the fifth row of* F_{00} *equals* u_{00} $(\neq 0)$ *times the fourth, then*

$$g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}\boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}\boldsymbol{\theta}_{02} + g_{00}^{b_1b_2,11} \times A_{00}^{\#(b_1b_2,11)}\boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}(A_{00}^{\#(20,20)}\boldsymbol{\theta}_{20} + u_{00}^{*}A_{00}^{\#(20,02)}\boldsymbol{\theta}_{02}) + g_{00}^{b_1b_2,11}A_{00}^{\#(b_1b_2,11)}(A_{00}^{\#(11,11)}\boldsymbol{\theta}_{11})$$

are estimable, and

(g) *if the last row of* F_{00} *equals the sum of* v_{00} $(\neq 0)$ *times the fourth and* w_{00} $(\neq 0)$ *times the fifth, then*

$$g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}\boldsymbol{\theta}_{20} + g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}\boldsymbol{\theta}_{02} + g_{00}^{b_1b_2,11} \times A_{00}^{\#(b_1b_2,11)}\boldsymbol{\theta}_{11} = g_{00}^{b_1b_2,20}A_{00}^{\#(b_1b_2,20)}(A_{00}^{\#(20,20)}\boldsymbol{\theta}_{20} + v_{00}^{*}A_{00}^{\#(20,11)}\boldsymbol{\theta}_{11}) + g_{00}^{b_1b_2,02}A_{00}^{\#(b_1b_2,02)}(A_{00}^{\#(02,02)}\boldsymbol{\theta}_{02} + w_{00}^{*}A_{00}^{\#(02,11)}\boldsymbol{\theta}_{11})$$

are estimable,

(B) (i) *if* r-rank$\{F_{10}\} = 1$, *and furthermore if* $m_1 = 2$ *and the last row of* F_{10} *is zero, or if* $m_1 \geq 3$ *and the last two rows of* F_{10} *are zero, then* $A_{10}^{\#(10,10)}\boldsymbol{\theta}_{10}$ *is estimable, and*

(ii) *if* $m_1 \geq 3$, r-rank$\{F_{10}\} = 2$ *and the first row of* F_{10} *is linearly independent, then* $A_{10}^{\#(10,10)}\boldsymbol{\theta}_{10}$ *is estimable, and furthermore*

(a) *if the last row of* F_{10} *is zero, then*

$$g_{10}^{c_1c_2,20}A_{10}^{\#(c_1c_2,20)}\boldsymbol{\theta}_{20} = g_{10}^{c_1c_2,20}A_{10}^{\#(c_1c_2,20)}(A_{10}^{\#(20,20)}\boldsymbol{\theta}_{20})$$
$$(c_1c_2 = 20, 11)$$

are estimable,

(b) *if the second row of* F_{10} *is zero, then*

$$g_{10}^{c_1c_2,11}A_{10}^{\#(c_1c_2,11)}\boldsymbol{\theta}_{11} = g_{10}^{c_1c_2,11}A_{10}^{\#(c_1c_2,11)}(A_{10}^{\#(11,11)}\boldsymbol{\theta}_{11})$$

are estimable, and

(c) *if the last row of* F_{10} *equals* v_{10} $(\neq 0)$ *times the second, then*

$$g_{10}^{c_1c_2,20}A_{10}^{\#(c_1c_2,20)}\boldsymbol{\theta}_{20} + g_{10}^{c_1c_2,11}A_{10}^{\#(c_1c_2,11)}\boldsymbol{\theta}_{11}$$
$$= g_{10}^{c_1c_2,20}A_{10}^{\#(c_1c_2,20)}(A_{10}^{\#(20,20)}\boldsymbol{\theta}_{20} + v_{10}^*A_{10}^{\#(20,11)}\boldsymbol{\theta}_{11})$$

are estimable, where $v_{10}^* = \sqrt{(m_1-2)/m_2}v_{10}$, *and*

(C) (i) *if* r-rank$\{F_{01}\} = 1$, *and furthermore if* $m_2 = 2$ *and the last row of* F_{01} *is zero, or if* $m_2 \geq 3$ *and the last two rows of* F_{01} *are zero, then* $A_{01}^{\#(01,01)}\boldsymbol{\theta}_{01}$ *is estimable, and*

(ii) *if* $m_2 \geq 3$, r-rank$\{F_{01}\} = 2$ *and the first row of* F_{01} *is linearly independent, then* $A_{01}^{\#(01,01)}\boldsymbol{\theta}_{01}$ *is estimable, and furthermore*

(a) *if the last row of* F_{01} *is zero, then*

$$g_{01}^{d_1d_2,02}A_{01}^{\#(d_1d_2,02)}\boldsymbol{\theta}_{02} = g_{01}^{d_1d_2,02}A_{01}^{\#(d_1d_2,02)}(A_{01}^{\#(02,02)}\boldsymbol{\theta}_{02})$$
$$(d_1d_2 = 02, 11)$$

are estimable,

(b) *if the second row of* F_{01} *is zero, then*

$$g_{01}^{d_1d_2,11}A_{01}^{\#(d_1d_2,11)}\boldsymbol{\theta}_{11} = g_{01}^{d_1d_2,11}A_{01}^{\#(d_1d_2,11)}(A_{01}^{\#(11,11)}\boldsymbol{\theta}_{11})$$

are estimable, and

(c) *if the last row of* F_{01} *equals* w_{01} $(\neq 0)$ *times the second, then*

$$g_{01}^{d_1d_2,02}A_{01}^{\#(d_1d_2,02)}\boldsymbol{\theta}_{02} + g_{01}^{d_1d_2,11}A_{01}^{\#(d_1d_2,11)}\boldsymbol{\theta}_{11}$$
$$= g_{01}^{d_1d_2,02}A_{01}^{\#(d_1d_2,02)}(A_{01}^{\#(02,02)}\boldsymbol{\theta}_{02} + w_{01}^*A_{01}^{\#(02,11)}\boldsymbol{\theta}_{11})$$

are estimable, where $w_{01}^* = \sqrt{(m_2-2)/m_1}w_{01}$.

REMARK 3.1. It follows from Lemma A.1 that in Theorem 3.2(II), since $g_{\gamma_1\gamma_2}^{a_1a_2,b_1b_2}$ $(\gamma_1\gamma_2 = 00, 10, 01)$ are arbitrary, without loss of generality, we can put

(A) $g_{00}^{a_1a_2,a_1a_2}=1$ for (i)(a) and (ii)(a),(b),(d) $(a_1a_2=20)$, (i)(b) and (ii)(a),(c), (e) $(a_1a_2=02)$, (i)(c) and (ii)(b),(c),(f) $(a_1a_2=11)$, $g_{00}^{a_1a_2,b_1b_2}=0$ for (ii)(a),(d),(e),(g) $((a_1a_2,b_1b_2)=(20,02),(02,20))$, (b),(f) $((a_1a_2,b_1b_2)=(20,11),(11,20))$, (c) $((a_1a_2,b_1b_2)=(02,11),(11,02))$, $g_{00}^{a_1a_2,b_1b_2}\neq 0$ for (i)(a),(d),(e),(g) $(a_1a_2=02,11;b_1b_2=20)$, (b),(f) $(a_1a_2=20,11;b_1b_2=02)$, (c) $(a_1a_2=20,02;b_1b_2=11)$, and $(g_{00}^{a_1a_2,b_1b_2},g_{00}^{a_1a_2,c_1c_2})\neq(0,0)$ for (ii)(c) $(a_1a_2=20;b_1b_2=02;c_1c_2=11)$, (b),(f) $(a_1a_2=02;b_1b_2=20;c_1c_2=11)$, (a),(d),(e),(g) $(a_1a_2=11;b_1b_2=20;c_1c_2=02)$,

(B) $g_{10}^{a_1a_2,a_1a_2}=1$ for (ii)(a) $(a_1a_2=20)$, (b) $(a_1a_2=11)$, and $g_{10}^{a_1a_2,b_1b_2}\neq 0$ for (ii)(a),(c) $((a_1a_2,b_1b_2)=(11,20))$, (b) $((a_1a_2,b_1b_2)=(20,11))$, and

(C) $g_{01}^{a_1a_2,a_1a_2}=1$ for (ii)(a) $(a_1a_2=02)$, (b) $(a_1a_2=11)$, and $g_{01}^{a_1a_2,b_1b_2}\neq 0$ for (ii)(a),(c) $((a_1a_2,b_1b_2)=(11,02))$, (b) $((a_1a_2,b_1b_2)=(02,11))$.

Furthermore we define $g_{00}^{a_1a_2,a_1a_2}$ $(=g_{00}^{a_1a_2,a_1a_2}(\alpha)$, say) for (A)(i)(d),(e),(g) and (ii)(e),(f),(g) $(a_1a_2=20)$, (i)(f) and (ii)(d),(g) $(a_1a_2=02)$, $g_{10}^{20,20}$ $(=g_{10}^{20,20}(\alpha)$, say) for (B)(ii)(c), and $g_{01}^{02,02}$ $(=g_{01}^{02,02}(\alpha)$, say) for (C)(ii)(c) as follows: $g_{\gamma_1\gamma_2}^{a_1a_2,a_1a_2}(0)=1$ $(\gamma_1\gamma_2=00,10,01)$ for all the cases, $g_{00}^{20,20}(\alpha)=1/(1+|u_{00}^*|)$ if $\alpha=1$ and $1/\sqrt{1+(u_{00}^*)^2}$ if $\alpha=2$ for (A)(i)(d) and (ii)(f), $g_{00}^{20,20}(\alpha)=1/(1+|v_{00}^*|)$ if $\alpha=1$ and $1/\sqrt{1+(v_{00}^*)^2}$ if $\alpha=2$ for (A)(i)(e) and (ii)(e),(g), $g_{00}^{02,02}(\alpha)=1/(1+|w_{00}^*|)$ if $\alpha=1$ and $1/\sqrt{1+(w_{00}^*)^2}$ if $\alpha=2$ for (A)(i)(f) and (ii)(d),(g), $g_{00}^{20,20}(\alpha)=1/(1+|u_{00}^*|+|v_{00}^*|)$ if $\alpha=1$ and $1/\sqrt{1+(u_{00}^*)^2+(v_{00}^*)^2}$ if $\alpha=2$ for (A)(i)(g), $g_{10}^{20,20}(\alpha)=1/(1+|v_{10}^*|)$ if $\alpha=1$ and $1/\sqrt{1+(v_{10}^*)^2}$ if $\alpha=2$ for (B)(ii)(c), and $g_{01}^{02,02}(\alpha)=1/(1+|w_{01}^*|)$ if $\alpha=1$ and $1/\sqrt{1+(w_{01}^*)^2}$ if $\alpha=2$ for (C)(ii)(c).

Using Lemma 3.1, we can obtain the following:

THEOREM 3.3. *Let T be an* SPBA$(m_1+m_2;\{\lambda_{i_1,i_2}\})$ *with* $N<\nu(m_1,m_2)$, *where* $2\leq m_1\leq m_2\leq 4$. *Then T is a* $2^{m_1+m_2}$*-PBFF design of resolution* $R(\{00,10,01\}\,|\,\Omega)$ *if and only if one of the following holds:*

(I) *When* $m_1=m_2=2$ $(\nu(2,2)=11)$,

(i) $\lambda_{0,1},\lambda_{1,0},\lambda_{1,2},\lambda_{2,1}\geq 1$ *and* $\lambda_{0,2}=\lambda_{1,1}=\lambda_{2,0}=0$, *and furthermore*

(1) $\lambda_{0,1}+\lambda_{1,0}+\lambda_{1,2}+\lambda_{2,1}\leq 5$ *and* $\lambda_{0,0}=\lambda_{2,2}=0$, *or*

(2) $\lambda_{0,0}+\lambda_{2,2}\geq 1$ *and* $2(\lambda_{0,1}+\lambda_{1,0}+\lambda_{1,2}+\lambda_{2,1})+\lambda_{0,0}+\lambda_{2,2}\leq 10$, *or its FCA, or*

(ii) $\lambda_{0,0},\lambda_{0,2},\lambda_{2,0},\lambda_{2,2}\geq 1$, $\lambda_{1,1}=1$, $\lambda_{0,0}+\lambda_{0,2}+\lambda_{2,0}+\lambda_{2,2}\leq 6$ *and* $\lambda_{0,1}=\lambda_{1,0}=\lambda_{1,2}=\lambda_{2,1}=0$,

(II) *when* $m_1=2$ *and* $m_2=3$ $(\nu(2,3)=16)$, $\lambda_{0,0}+\lambda_{2,3}\geq 1$, $\lambda_{0,1},\lambda_{1,0},\lambda_{1,3},\lambda_{2,2}\geq 1$, $3(\lambda_{0,1}+\lambda_{2,2})+2(\lambda_{1,0}+\lambda_{1,3})+\lambda_{0,0}+\lambda_{2,3}\leq 15$ *and* $\lambda_{0,2}=\lambda_{0,3}=\lambda_{1,1}=\lambda_{1,2}=\lambda_{2,0}=\lambda_{2,1}=0$, *or its FCA*,

(III) *when* $m_1 = 2$ *and* $m_2 = 4$ $(v(2,4) = 22)$, $\lambda_{0,0} + \lambda_{2,4} \geq 1$, $\lambda_{0,1}, \lambda_{1,0}, \lambda_{1,4}, \lambda_{2,3} \geq 1$, $4(\lambda_{0,1} + \lambda_{2,3}) + 2(\lambda_{1,0} + \lambda_{1,4}) + \lambda_{0,0} + \lambda_{2,4} \leq 21$ *and* $\lambda_{0,x} = \lambda_{1,y} = \lambda_{2,z} = 0$ $(2 \leq x \leq 4; 1 \leq y \leq 3; 0 \leq z \leq 2)$, *or its FCA,*

(IV) *when* $m_1 = m_2 = 3$ $(v(3,3) = 22)$, $\lambda_{1,3} = \lambda_{2,0} = 0$, *and in addition*

- (i) $\lambda_{0,1}, \lambda_{1,0}, \lambda_{2,3}, \lambda_{3,2} \geq 1$ *and* $\lambda_{0,2} = \lambda_{a,x} = \lambda_{3,1} = 0$ $(1 \leq a, x \leq 2)$, *and furthermore*
  - (1) *at least two out of* $\{\lambda_{0,0}, \lambda_{0,3}, \lambda_{3,0}, \lambda_{3,3}\}$ *except for* $\{\lambda_{0,0}, \lambda_{3,3}\}$ *and* $\{\lambda_{0,3}, \lambda_{3,0}\}$ *are nonzero and* $3(\lambda_{0,1} + \lambda_{1,0} + \lambda_{2,3} + \lambda_{3,2}) + \lambda_{0,0} + \lambda_{0,3} + \lambda_{3,0} + \lambda_{3,3} \leq 21$, *or its FCA, or*
  - (2) $\lambda_{0,0} + \lambda_{3,3} \geq 1$, $3(\lambda_{0,1} + \lambda_{1,0} + \lambda_{2,3} + \lambda_{3,2}) + \lambda_{0,0} + \lambda_{3,3} \leq 21$ *and* $\lambda_{0,3} = \lambda_{3,0} = 0$, *or its FCA,*
- (ii) $\lambda_{0,0} = \lambda_{0,3} = \lambda_{1,1} = \lambda_{2,2} = \lambda_{3,0} = 1$ *and* $\lambda_{0,1} = \lambda_{0,2} = \lambda_{1,0} = \lambda_{1,2} = \lambda_{2,1} = \lambda_{2,3} = \lambda_{3,x} = 0$ $(1 \leq x \leq 3)$, *or its FCA, LCA and CCA, or*
- (iii) *at least two out of* $\{\lambda_{0,0}, \lambda_{0,3}, \lambda_{3,0}, \lambda_{3,3}\}$ *except for* $\{\lambda_{0,0}, \lambda_{3,3}\}$ *and* $\{\lambda_{0,3}, \lambda_{3,0}\}$ *are nonzero,* $\lambda_{0,2}, \lambda_{1,0}, \lambda_{2,3}, \lambda_{3,1} \geq 1$, $3(\lambda_{0,2} + \lambda_{1,0} + \lambda_{2,3} + \lambda_{3,1}) + \lambda_{0,0} + \lambda_{0,3} + \lambda_{3,0} + \lambda_{3,3} \leq 21$ *and* $\lambda_{0,1} = \lambda_{a,x} = \lambda_{3,2} = 0$ $(1 \leq a, x \leq 2)$, *or its FCA,*

(V) *when* $m_1 = 3$ *and* $m_2 = 4$ $(v(3,4) = 29)$, $\lambda_{1,1} = \lambda_{1,2} = \lambda_{2,2} = \lambda_{2,3} = 0$, *and in addition*

- (i) $\lambda_{0,1}, \lambda_{1,0}, \lambda_{2,4}, \lambda_{3,3} \geq 1$ *and* $\lambda_{0,2} = \lambda_{0,3} = \lambda_{1,4} = \lambda_{2,0} = \lambda_{3,1} = \lambda_{3,2} = 0$, *and furthermore*
  - (1) *at least two out of* $\{\lambda_{0,0}, \lambda_{0,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *except for* $\{\lambda_{0,0}, \lambda_{3,4}\}$ *and* $\{\lambda_{0,4}, \lambda_{3,0}\}$ *are nonzero* $4(\lambda_{0,1} + \lambda_{3,3}) + 3(\lambda_{1,0} + \lambda_{2,4}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{3,0} + \lambda_{3,4} \leq 28$ *and* $\lambda_{1,3} = \lambda_{2,1} = 0$, *or its FCA,*
  - (2) $\lambda_{0,0} + \lambda_{3,4} \geq 1$, $4(\lambda_{0,1} + \lambda_{3,3}) + 3(\lambda_{1,0} + \lambda_{2,4}) + \lambda_{0,0} + \lambda_{3,4} \leq 28$ *and* $\lambda_{0,4} = \lambda_{1,3} = \lambda_{2,1} = \lambda_{3,0} = 0$, *or its FCA, or*
  - (3) $\lambda_{1,3} = 1$, $\lambda_{0,0} + \lambda_{3,4} \leq 2$ *and* $\lambda_{0,4} = \lambda_{2,1} = \lambda_{3,0} = 0$, *or its FCA, LCA and CCA,*
- (ii) $\lambda_{0,1} = \lambda_{1,3} = \lambda_{1,4} = \lambda_{2,0} = \lambda_{3,3} = 1$ *and* $\lambda_{0,x} = \lambda_{1,0} = \lambda_{2,1} = \lambda_{2,4} = \lambda_{3,y} = 0$ $(x = 0, 2, 3, 4; y = 0, 1, 2, 4)$, *or its FCA, LCA and CCA, or*
- (iii) $\lambda_{1,0}, \lambda_{2,4} \geq 1$ *and* $\lambda_{1,3} = \lambda_{1,4} = \lambda_{2,0} = \lambda_{2,1} = 0$, *and furthermore*
  - (1) *at least two out of* $\{\lambda_{0,0}, \lambda_{0,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *except for* $\{\lambda_{0,0}, \lambda_{3,4}\}$ *and* $\{\lambda_{0,4}, \lambda_{3,0}\}$ *are nonzero and* $\lambda_{0,1} = \lambda_{3,3} = 0$, *and moreover*
    - (a) $\lambda_{0,2}, \lambda_{3,2} \geq 1$, $6(\lambda_{0,2} + \lambda_{3,2}) + 3(\lambda_{1,0} + \lambda_{2,4}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{3,0} + \lambda_{3,4} \leq 28$ *and* $\lambda_{0,3} = \lambda_{3,1} = 0$, *or its FCA, or*
    - (b) $\lambda_{0,3}, \lambda_{3,1} \geq 1$, $4(\lambda_{0,3} + \lambda_{3,1}) + 3(\lambda_{1,0} + \lambda_{2,4}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{3,0} + \lambda_{3,4} \leq 28$ *and* $\lambda_{0,2} = \lambda_{3,2} = 0$, *or its FCA, or*
  - (2) $\lambda_{0,0} + \lambda_{0,4} + \lambda_{3,0} + \lambda_{3,4} \geq 1$, *at least three out of* $\{\lambda_{0,1}, \lambda_{0,3}, \lambda_{3,1}, \lambda_{3,3}\}$ *are nonzero,* $4(\lambda_{0,1} + \lambda_{0,3} + \lambda_{3,1} + \lambda_{3,3}) + 3(\lambda_{1,0} + \lambda_{2,4}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{3,0} + \lambda_{3,4} \leq 28$ *and* $\lambda_{0,2} = \lambda_{3,2} = 0$, *or its FCA,*

(VI) *when* $m_1 = m_2 = 4$ $(v(4,4) = 37)$, $\lambda_{1,2} = \lambda_{1,3} = \lambda_{2,x} = \lambda_{3,1} = \lambda_{3,2} = 0$ $(1 \le x \le 3)$, *and in addition*

- (i) $\lambda_{0,0} + \lambda_{4,4} \ge 1$, $\lambda_{0,4}, \lambda_{4,0} \ge 1$, $\lambda_{1,1} = \lambda_{3,3} = 1$, $\lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 4$ *and* $\lambda_{a,x} = \lambda_{b,0} = \lambda_{b,4} = 0$ $(a = 0,4; 1 \le b, x \le 3)$, *or its FCA,*
- (ii) $\lambda_{0,0} + \lambda_{4,4} \ge 1$, $\lambda_{0,1}, \lambda_{1,0}, \lambda_{3,4}, \lambda_{4,3} \ge 1$, $4(\lambda_{0,1} + \lambda_{1,0} + \lambda_{3,4} + \lambda_{4,3}) + \lambda_{0,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{0,y} = \lambda_{1,1} = \lambda_{1,4} = \lambda_{2,0} = \lambda_{2,4} = \lambda_{3,0} = \lambda_{3,3} = \lambda_{4,z} = 0$ $(2 \le y \le 4; 0 \le z \le 2)$, *or its FCA,*
- (iii) *at least two out of* $\{\lambda_{0,0}, \lambda_{0,4}, \lambda_{4,0}, \lambda_{4,4}\}$ *except for* $\{\lambda_{0,0}, \lambda_{4,4}\}$ *and* $\{\lambda_{0,4}, \lambda_{4,0}\}$ *are nonzero and* $\lambda_{1,1} = \lambda_{1,4} = \lambda_{3,0} = \lambda_{3,3} = 0$, *and furthermore*
  - (1) $\lambda_{2,0}, \lambda_{2,4} \ge 1$ *and* $\lambda_{0,3} = \lambda_{1,0} = \lambda_{3,4} = \lambda_{4,1} = 0$, *and moreover*
    - (a) $\lambda_{0,2}, \lambda_{4,2} \ge 1$, $6(\lambda_{0,2} + \lambda_{2,0} + \lambda_{2,4} + \lambda_{4,2}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{0,1} = \lambda_{4,3} = 0$, *or*
    - (b) $\lambda_{0,1}, \lambda_{4,3} \ge 1$, $6(\lambda_{2,0} + \lambda_{2,4}) + 4(\lambda_{0,1} + \lambda_{4,3}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{0,2} = \lambda_{4,2} = 0$, *or its FCA, or*
  - (2) $\lambda_{1,0}, \lambda_{3,4} \ge 1$ *and* $\lambda_{2,0} = \lambda_{2,4} = 0$, *and moreover*
    - (a) $\lambda_{0,2}, \lambda_{4,2} \ge 1$, $6(\lambda_{0,2} + \lambda_{4,2}) + 4(\lambda_{1,0} + \lambda_{3,4}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{0,1} = \lambda_{0,3} = \lambda_{4,1} = \lambda_{4,3} = 0$, *or its FCA,*
    - (b) $\lambda_{0,1}, \lambda_{4,3} \ge 1$, $4(\lambda_{0,1} + \lambda_{1,0} + \lambda_{3,4} + \lambda_{4,3}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{0,2} = \lambda_{0,3} = \lambda_{4,1} = \lambda_{4,2} = 0$, *or its FCA, or*
    - (c) $\lambda_{0,3}, \lambda_{4,1} \ge 1$, $4(\lambda_{0,3} + \lambda_{1,0} + \lambda_{3,4} + \lambda_{4,1}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{0,1} = \lambda_{0,2} = \lambda_{4,2} = \lambda_{4,3} = 0$, *or its FCA, or*
- (iv) $\lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \ge 1$ *and* $\lambda_{1,1} = \lambda_{3,3} = 0$, *and furthermore*
  - (1) *at least three out of* $\{\lambda_{0,1}, \lambda_{0,3}, \lambda_{4,1}, \lambda_{4,3}\}$ *are nonzero and* $\lambda_{0,2} = \lambda_{4,2} = 0$, *and moreover*
    - (a) $\lambda_{2,0}, \lambda_{2,4} \ge 1$, $6(\lambda_{2,0} + \lambda_{2,4}) + 4(\lambda_{0,1} + \lambda_{0,3} + \lambda_{4,1} + \lambda_{4,3}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{1,0} = \lambda_{1,4} = \lambda_{3,0} = \lambda_{3,4} = 0$,
    - (b) $\lambda_{1,0}, \lambda_{3,4} \ge 1$, $4(\lambda_{0,1} + \lambda_{0,3} + \lambda_{1,0} + \lambda_{3,4} + \lambda_{4,1} + \lambda_{4,3}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{1,4} = \lambda_{2,0} = \lambda_{2,4} = \lambda_{3,0} = 0$, *or its FCA, or*
    - (c) *at least three out of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *are nonzero,* $4(\lambda_{0,1} + \lambda_{0,3} + \lambda_{1,0} + \lambda_{1,4} + \lambda_{3,0} + \lambda_{3,4} + \lambda_{4,1} + \lambda_{4,3}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 36$ *and* $\lambda_{2,0} = \lambda_{2,4} = 0$, *or*
  - (2) *at least three out of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *are nonzero and* $\lambda_{0,3} = \lambda_{2,0} = \lambda_{2,4} = \lambda_{4,1} = 0$, *and moreover*

(a) $\lambda_{0,2}, \lambda_{4,2} \geq 1$, $6(\lambda_{0,2} + \lambda_{4,2}) + 4(\lambda_{1,0} + \lambda_{1,4} + \lambda_{3,0} + \lambda_{3,4}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \leq 36$ *and* $\lambda_{0,1} = \lambda_{4,3} = 0$, *or*

(b) $\lambda_{0,1}, \lambda_{4,3} \geq 1$, $4(\lambda_{0,1} + \lambda_{1,0} + \lambda_{1,4} + \lambda_{3,0} + \lambda_{3,4} + \lambda_{4,3}) + \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \leq 36$ *and* $\lambda_{0,2} = \lambda_{4,2} = 0$, *or its FCA.*

Remark 3.2. In Theorem 3.3,

(A) (a) the matrix F_{00} is of full row rank for (IV)(i)(1),(iii), (V)(i)(1),(iii) and (VI)(iii),(iv),

(b) r-rank$\{F_{00}\} = 5$, and furthermore

(1) the fifth row of F_{00} is equal to u_{00} $(= 1)$ times the fourth for (I)(ii), (IV)(ii) and (VI)(i),

(2) the last row of F_{00} is equal to the sum of v_{00} $(\neq 0)$ times the fourth and w_{00} $(\neq 0)$ times the fifth for (I)(i)(2), (II), (III), (IV)(i)(2), (V)(i)(2),(3),(ii) and (VI)(ii), where $v_{00} = w_{00} = 1$ for (I)(i)(2), $v_{00} = 3/2$ and $w_{00} = 1/2$ for (II), $v_{00} = 2$ and $w_{00} = 1/3$ for (III), $v_{00} = w_{00} = 3/4$ for (IV)(i)(2), $v_{00} = 1$ and $w_{00} = 1/2$ for (V)(i)(2),(3), $v_{00} = 1$ and $w_{00} = -1/6$ for (V)(ii), and $v_{00} = w_{00} = 2/3$ for (VI)(ii),

(c) r-rank$\{F_{00}\} = 4$, the last row of F_{00} is zero, and the fifth row of F_{00} is equal to u_{00} $(= -1)$ times the fourth for (I)(i)(1),

(B) (a) the matrix F_{10} is of full row rank for (I)(i), (II), (III), (V)(i)(3),(ii) and (VI)(iv)(1)(c),(2),

(b) r-rank$\{F_{10}\} = 2$, and furthermore

(1) the second row of F_{10} is zero for (VI)(iii)(1),(iv)(1)(a),

(2) the last row of F_{10} is equal to v_{10} $(\neq 0)$ times the second for (IV)(i),(ii),(iii), (V)(i)(1),(2),(iii) and (VI)(i),(ii),(iii)(2),(iv)(1)(b), where $v_{10} = 3$ for (IV)(i),(iii), $v_{10} = 1$ for (IV)(ii) and (VI)(i), $v_{10} = 4$ for (V)(i)(1),(2),(iii), and $v_{10} = 2$ for (VI)(ii),(iii)(2), (iv)(1)(b),

(c) r-rank$\{F_{10}\} = 1$ and the last row of F_{10} is zero for (I)(ii),

(C) (a) the matrix F_{01} is of full row rank for (I)(i), (V)(i)(3),(ii),(iii)(2) and (VI)(iv)(1),

(b) r-rank$\{F_{01}\} = 2$, and furthermore

(1) the second row of F_{01} is zero for (V)(iii)(1)(a) and (VI)(iii)(1)(a), (2)(a),(iv)(2)(a),

(2) the last row of F_{01} is equal to w_{01} $(\neq 0)$ times the second for (II), (III), (IV)(i),(ii),(iii), (V)(i)(1),(2),(iii)(1)(b) and (VI)(i),(ii), (iii)(1)(b),(2)(b),(c),(iv)(2)(b), where $w_{01} = 2$ for (II) and (VI)(ii), (iii)(1)(b),(2)(b),(iv)(2)(b), $w_{01} = 1$ for (III), (IV)(ii) and (VI)(i), $w_{01} = 3$ for (IV)(i), $w_{01} = -3$ for (IV)(iii), $w_{01} = 3/2$ for (V)(i)(1), (2), $w_{01} = -3/2$ for (V)(iii)(1)(b), and $w_{01} = -2$ for (VI)(iii)(2)(c),

(c) r-rank$\{F_{01}\} = 1$ and the last row of F_{01} is zero for (I)(ii),

(D) (a) the matrix F_{20} is of full row rank for (VI)(iii)(1),(iv)(1)(a),

(b) r-rank$\{F_{20}\} = 0$ for (VI)(i),(ii),(iii)(2),(iv)(1)(b),(c),(2),

(c) the matrix F_{20} vanishes for (I), (II), (III), (IV) and (V),

(E) (a) the matrix F_{02} is of full row rank for (V)(iii)(1)(a) and (VI)(iii)(1)(a), (2)(a),(iv)(2)(a),

(b) r-rank$\{F_{02}\} = 0$ for (III), (V)(i),(ii),(iii)(1)(b),(2) and (VI)(i),(ii), (iii)(1)(b),(2)(b),(c),(iv)(1),(2)(b),

(c) the matrix F_{02} vanishes for (I), (II) and (IV),

(F) (a) the matrix F_{11} is of full row rank for (I)(ii), (IV)(ii), (V)(i)(3),(ii) and (VI)(i),

(b) r-rank$\{F_{11}\} = 0$ except for (I)(ii), (IV)(ii), (V)(i)(3),(ii) and (VI)(i).

4. GA-optimal designs

In this section, we present GA-optimal $2^{m_1+m_2}$-PBFF designs of resolution $R(\{00, 10, 01\} \mid \Omega)$ derived from SPBAs$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ with $N < v(m_1, m_2)$, where $2 \le m_1 \le m_2 \le 4$. If $C\boldsymbol{\Theta}$ is estimable (and hence there exists a matrix X such that $XM_T = C$), then its unbiased estimator is given by $C\hat{\boldsymbol{\Theta}}$, where $\hat{\boldsymbol{\Theta}}$ is a solution of the Eqs. (2.1), and $\mathrm{Var}[C\hat{\boldsymbol{\Theta}}] = \sigma^2 XM_TX'$. By utilizing the algebraic structure of the ETMDPB association scheme, XM_TX' is isomorphic to $X_{\beta_1\beta_2}K_{\beta_1\beta_2}X'_{\beta_1\beta_2}$ ($\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \ge 4$), 02 (if $m_2 \ge 4$), 11).

Let $K_{00}(a_1a_2)$ ($a_1a_2 = 20, 02, 11$) and $K_{00}(b_1b_2, c_1c_2)$ ($(b_1b_2, c_1c_2) = (20, 02), (20, 11), (02, 11)$) be the 4×4 submatrix of K_{00} corresponding to the first three and the $\langle a_1a_2\rangle$-th rows and columns, and the 5×5 one corresponding to the first three, the $\langle b_1b_2\rangle$-th and the $\langle c_1c_2\rangle$-th rows and columns, respectively, where the $\langle x_1x_2\rangle$-th denotes "the fourth", "the fifth" and "the last" according as $x_1x_2 = 20, 02$ and 11, respectively, and further let $K_{10}(d_1d_2)$ ($d_1d_2 = 20, 11$) and $K_{01}(e_1e_2)$ ($e_1e_2 = 02, 11$) be the 2×2 submatrices of K_{10} and of K_{01} corresponding to the first and the $[y_1y_2]$-th rows and columns, respectively, where the $[y_1y_2]$-th denotes "the second" and "the last" according as $y_1y_2 = 20$ (if $m_1 \ge 3$) (or 02 (if $m_2 \ge 3$)) and 11, respectively. Now we define the matrix $\sigma^2 V_T$ as the variance-covariance matrix of the linearly independent estimators in $C\hat{\boldsymbol{\Theta}}$. Then from Theorem 3.2, Lemma A.2 and the properties of the ETMDPB association algebra $\mathscr{A}$, we get the following:

Lemma 4.1. *Let T be a $2^{m_1+m_2}$-PBFF design of resolution $R(\{00, 10, 01\} \mid \Omega)$ derived from an SPBA$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ with N assemblies, where $N < v(m_1, m_2)$ and $2 \le m_k$ for $k = 1, 2$. Then the matrix V_T $(= V_T(\alpha)$, say) is isomorphic to $V_{\beta_1\beta_2}(\alpha)$ ($\beta_1\beta_2 = 00, 10, 01, 20$ (if $m_1 \ge 4$), 02 (if $m_2 \ge 4$), 11) for $0 \le \alpha \le 2$, where*

$$V_{\beta_1\beta_2}(\alpha) = K_{\beta_1\beta_2}^{-1} \qquad \textit{if } F_{\beta_1\beta_2} \textit{ is of full row rank},$$

$$V_{00}(\alpha) = \begin{cases} K_{00}(a_1a_2)^{-1} \\ \quad \textit{if } \text{r-rank}\{F_{00}\} = 4, \textit{ and furthermore if exactly two rows of } F_{00} \textit{ out} \\ \quad \textit{of last three except for the } \langle a_1a_2\rangle\textit{-th are zero } (a_1a_2 = 20, 02, 11), \\ \begin{pmatrix} I_3 & 0 \\ 0 & g_{00}^{20,20}(\alpha) \end{pmatrix} K_{00}(20)^{-1} \begin{pmatrix} I_3 & 0 \\ 0 & g_{00}^{20,20}(\alpha) \end{pmatrix} \\ \quad \textit{if } \text{r-rank}\{F_{00}\} = 4, \textit{ and furthermore } (1) \textit{ if the last row of } F_{00} \textit{ is} \\ \quad \textit{zero and the fifth equals } u_{00} \ (\neq 0) \textit{ times the fourth}, \\ \quad (2) \quad \textit{if the fifth row of } F_{00} \textit{ is zero and the last equals } v_{00} \ (\neq 0) \\ \qquad\quad \textit{times the fourth, or} \\ \quad (3) \quad \textit{if the fifth and the last rows of } F_{00} \textit{ equal } u_{00} \ (\neq 0) \textit{ times the} \\ \qquad\quad \textit{fourth and } v_{00} \ (\neq 0) \textit{ times the fourth, respectively}, \\ \begin{pmatrix} I_3 & 0 \\ 0 & g_{00}^{02,02}(\alpha) \end{pmatrix} K_{00}(02)^{-1} \begin{pmatrix} I_3 & 0 \\ 0 & g_{00}^{02,02}(\alpha) \end{pmatrix} \\ \quad \textit{if } \text{r-rank}\{F_{00}\} = 4, \textit{ and furthermore if the fourth row of } F_{00} \textit{ is zero} \\ \quad \textit{and the last equals } w_{00} \ (\neq 0) \textit{ times the fifth}, \\ K_{00}(a_1a_2, b_1b_2)^{-1} \\ \quad \textit{if } \text{r-rank}\{F_{00}\} = 5, \textit{ and furthermore if exactly one row of } F_{00} \textit{ out} \\ \quad \textit{of the last three except for the } \langle a_1a_2\rangle\textit{-th and } \langle b_1b_2\rangle\textit{-th is zero} \\ \quad ((a_1a_2, b_1b_2) = (20, 02), (20, 11), (02, 11)), \\ \begin{pmatrix} I_4 & 0 \\ 0 & g_{00}^{02,02}(\alpha) \end{pmatrix} K_{00}(20, 02)^{-1} \begin{pmatrix} I_4 & 0 \\ 0 & g_{00}^{02,02}(\alpha) \end{pmatrix} \\ \quad \textit{if } \text{r-rank}\{F_{00}\} = 5, \textit{ and furthermore if the last row of } F_{00} \textit{ equals} \\ \quad w_{00} \ (\neq 0) \textit{ times the fifth}, \\ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & g_{00}^{20,20}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} K_{00}(20, 02)^{-1} \begin{pmatrix} I_3 & 0 & 0 \\ 0 & g_{00}^{20,20}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \quad \textit{if } \text{r-rank}\{F_{00}\} = 5, \textit{ and furthermore if the last row of } F_{00} \textit{ equals} \\ \quad v_{00} \ (\neq 0) \textit{ times the fourth}, \\ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & g_{00}^{20,20}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} K_{00}(20, 11)^{-1} \begin{pmatrix} I_3 & 0 & 0 \\ 0 & g_{00}^{20,20}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{cases}$$

$$V_{00}(\alpha) = \begin{cases} \quad \textit{if } \text{r-rank}\{F_{00}\} = 5, \textit{ and furthermore if the fifth row of } F_{00} \textit{ equals } u_{00}\ (\neq 0) \textit{ times the fourth}, \\ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & g_{00}^{20,20}(\alpha) & 0 \\ 0 & 0 & g_{00}^{02,02}(\alpha) \end{pmatrix} K_{00}(20,02)^{-1} \begin{pmatrix} I_3 & 0 & 0 \\ 0 & g_{00}^{20,20}(\alpha) & 0 \\ 0 & 0 & g_{00}^{02,02}(\alpha) \end{pmatrix} \\ \quad \textit{if } \text{r-rank}\{F_{00}\} = 5, \textit{ and furthermore if the last row of } F_{00} \textit{ equals the sum of } v_{00}\ (\neq 0) \textit{ times the fourth and } w_{00}\ (\neq 0) \textit{ times the fifth}, \end{cases}$$

$$V_{10}(\alpha) = \begin{cases} 1/\kappa_{10}^{10,10} \\ \quad \textit{if } \text{r-rank}\{F_{10}\} = 1, \textit{ and furthermore } (1) \textit{ if } m_1 = 2 \textit{ and the last row of } F_{10} \textit{ is zero, or} \\ \quad (2) \quad \textit{if } m_1 \geq 3 \textit{ and the last two rows of } F_{10} \textit{ are zero}, \\ K_{10}(c_1c_2)^{-1} \\ \quad \textit{if } m_1 \geq 3 \textit{ and } \text{r-rank}\{F_{10}\} = 2, \textit{ and furthermore if exactly one row of } F_{10} \textit{ out of the last two except for the } [c_1c_2]\textit{-th is zero } (c_1c_2 = 20, 11), \\ \begin{pmatrix} 1 & 0 \\ 0 & g_{10}^{20,20}(\alpha) \end{pmatrix} K_{10}(20)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & g_{10}^{20,20}(\alpha) \end{pmatrix} \\ \quad \textit{if } m_1 \geq 3 \textit{ and } \text{r-rank}\{F_{10}\} = 2, \textit{ and furthermore if the last row of } F_{10} \textit{ equals } v_{10}\ (\neq 0) \textit{ times the second}, \end{cases}$$

$$V_{01}(\alpha) = \begin{cases} 1/\kappa_{01}^{01,01} \\ \quad \textit{if } \text{r-rank}\{F_{01}\} = 1, \textit{ and furthermore } (1) \textit{ if } m_2 = 2 \textit{ and the last row of } F_{01} \textit{ is zero, or} \\ \quad (2) \quad \textit{if } m_2 \geq 3 \textit{ and the last two rows of } F_{01} \textit{ are zero}, \\ K_{01}(d_1d_2)^{-1} \\ \quad \textit{if } m_2 \geq 3 \textit{ and } \text{r-rank}\{F_{01}\} = 2, \textit{ and furthermore if exactly one row of } F_{01} \textit{ out of the last two except for the } [d_1d_2]\textit{-th is zero } (d_1d_2 = 02, 11), \\ \begin{pmatrix} 1 & 0 \\ 0 & g_{01}^{02,02}(\alpha) \end{pmatrix} K_{01}(02)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & g_{01}^{02,02}(\alpha) \end{pmatrix} \\ \quad \textit{if } m_2 \geq 3 \textit{ and } \text{r-rank}\{F_{01}\} = 2, \textit{ and furthermore if the last row of } F_{01} \textit{ equals } w_{01}\ (\neq 0) \textit{ times the second}, \end{cases}$$

$$V_{20}(\alpha) = \begin{cases} \textit{vanishes} & \textit{if } m_1 = 2, 3, \\ 0 & \textit{if } m_1 \geq 4 \textit{ and } \text{r-rank}\{F_{20}\} = 0, \end{cases}$$

$$V_{02}(\alpha) = \begin{cases} \textit{vanishes} & \textit{if } m_2 = 2, 3, \\ 0 & \textit{if } m_2 \geq 4 \textit{ and } \text{r-rank}\{F_{02}\} = 0, \end{cases}$$

Table 4.1. GA_α-optimal 2^{2+2}-PBFF designs.

N	λ'	$\text{tr}\{V_T(0)\}$	$\text{tr}\{V_T(1)\}$	$\text{tr}\{V_T(2)\}$	Theorem
8	010 101 010	0.87500	0.78125	0.81250	(I)(i)(1)
	101 010 101	0.87500	0.78125	0.81250	(I)(ii)
9	201 010 101	0.82031	0.73242	0.76172	(I)(ii)
10	010 201 010	0.76563	0.68359	0.71094	(I)(i)(1)
	020 101 010	0.76563	0.68359	0.71094	(I)(i)(1)
	201 010 102	0.76563	0.68359	0.71094	(I)(ii)
	201 010 201	0.76563	0.68359	0.71094	(I)(ii)
	202 010 101	0.76563	0.68359	0.71094	(I)(ii)

Table 4.2. GA_α-optimal 2^{2+3}-PBFF designs.

N	λ'	$\text{tr}\{V_T(0)\}$	$\text{tr}\{V_T(1)\}$	$\text{tr}\{V_T(2)\}$	Theorem
11	1100 1001 0010	1.73611	1.04633	1.25000	(II)
12	1100 1001 0011	1.32738	0.88855	1.00655	(II)
13	2100 1001 0011	1.24911	0.85834	0.95993	(II)
14	2100 1001 0012	1.19298			(II)
	1100 2001 0011		0.80558	0.92037	(II)
15	2100 2001 0011	1.14653		0.86986	(II)
	1200 1001 0011		0.77088		(II)

Table 4.3. GA_α-optimal 2^{2+4}-PBFF designs.

N	λ'	$\text{tr}\{V_T(0)\}$	$\text{tr}\{V_T(1)\}$	$\text{tr}\{V_T(2)\}$	Theorem
13	11000 10001 00010	1.93750	1.36833	1.54948	(III)
14	11000 10001 00011	1.47064	1.11239	1.22585	(III)
15	21000 10001 00011	1.36690	1.05551	1.15394	(III)
16	21000 10001 00012	1.29911	1.01755	1.10640	(III)
17	21000 20001 00011	1.26540	0.97053	1.06570	(III)
18	21000 20001 00012	1.20515	0.93702	1.02324	(III)
19	22000 10001 00011	1.16330	0.89233	0.97610	(III)
20	22000 10001 00012	1.10617		0.93830	(III)
	21000 20002 00012		0.85611		(III)
21	22000 20001 00011	1.06005	0.80589	0.88632	(III)

$V_{11}(\alpha) = 0 \qquad if\ \text{r-rank}\{F_{11}\} = 0,$

and $g_{\gamma_1\gamma_2}^{a_1a_2,a_1a_2}(\alpha)$ $(\gamma_1\gamma_2 = 00, 10, 01)$ *for* $0 \leq \alpha \leq 2$ *are given in Remark* 3.1.

By Theorem 3.2(II) and Lemma 4.1, we obtain the following:

THEOREM 4.2. *Let* T *be a* $2^{m_1+m_2}$-PBFF *design of resolution* $R(\{00, 10, 01\} \mid \Omega)$ *derived from an* $\text{SPBA}(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ *with* N *assemblies, where* $N < v(m_1, m_2)$ *and* $2 \leq m_k$ $(k = 1, 2)$. *Then we get*

$$\begin{aligned}\text{tr}\{V_T(\alpha)\} = {} & \phi_{00}\,\text{tr}\{V_{00}(\alpha)\} + \phi_{10}\,\text{tr}\{V_{10}(\alpha)\} + \phi_{01}\,\text{tr}\{V_{01}(\alpha)\} \\ & + \phi_{20}\,\text{tr}\{V_{20}(\alpha)\}\ (if\ \ m_1 \geq 4) + \phi_{02}\,\text{tr}\{V_{02}(\alpha)\}\ (if\ \ m_2 \geq 4) \\ & + \phi_{11}\,\text{tr}\{V_{11}(\alpha)\} \qquad for\ \ 0 \leq \alpha \leq 2.\end{aligned}$$

As a generalization of the A-optimality criterion, Kuwada et al. [5] introduced the GA-optimality criterion for the selection of a design. For resolution $R(\{00, 10, 01\} \mid \Omega)$ designs, we define the GA_α-optimality criteria

Table 4.4. GA_α-optimal 2^{3+3}-PBFF designs.

N	λ'	$\text{tr}\{V_T(0)\}$	$\text{tr}\{V_T(1)\}$	$\text{tr}\{V_T(2)\}$	Theorem
13	1100 1000 0001 0010	2.14844	1.35345	1.55580	(IV)(i)(2)
14	1100 1000 0001 0011	1.71023	1.09901	1.22808	(IV)(i)(2)
15	2100 1000 0001 0011	1.61285	1.04247	1.15526	(IV)(i)(2)
16	1011 1000 0001 1101	1.50694			(IV)(iii)
	2100 1000 0001 0012		1.00472	1.10714	(IV)(i)(2)
17	1012 1000 0001 1101	1.47467			(IV)(iii)
	2011 1000 0001 1101	1.47467			(IV)(iii)
	1100 2000 0001 0011		0.98323		(IV)(i)(2)
	1200 1000 0001 0011		0.98323		(IV)(i)(2)
	3100 1000 0001 0012			1.08435	(IV)(i)(2)
18	2100 2000 0001 0011	1.42822	0.92004	1.02235	(IV)(i)(2)
	2200 1000 0001 0011	1.42822	0.92004	1.02235	(IV)(i)(2)
19	1011 2000 0001 1101	1.34207			(IV)(iii)
	1021 1000 0001 1101	1.34207			(IV)(iii)
	2100 2000 0001 0012		0.88885	0.98168	(IV)(i)(2)
	2200 1000 0001 0012		0.88885	0.98168	(IV)(i)(2)
20	1022 1000 0001 1101	1.30316			(IV)(iii)
	2011 2000 0001 1101	1.30316			(IV)(iii)
	1200 2000 0001 0011		0.86662		(IV)(i)(2)
	3100 2000 0001 0012			0.95531	(IV)(i)(2)
	3200 1000 0001 0012			0.95531	(IV)(i)(2)
21	1001 0100 0010 1000	0.85185	0.66537	0.75231	(IV)(ii)

$(0 \leq \alpha \leq 2)$ as follows: If $\mathrm{tr}\{V_T(\alpha)\} \leq \mathrm{tr}\{V_{T^*}(\alpha)\}$ for any T^*, where both T and T^* are $2^{m_1+m_2}$-PBFF designs of resolution $\mathrm{R}(\{00,10,01\} \mid \Omega)$ derived from an $\mathrm{SPBA}(m_1+m_2;\{\lambda_{i_1,i_2}\})$ with N assemblies and an $\mathrm{SPBA}(m_1+m_2;\{\lambda^*_{i_1,i_2}\})$ with the same number of assemblies, respectively, and $N < v(m_1,m_2)$ and $2 \leq m_k$ $(k=1,2)$, then T is said to be *GA_α-optimal* for $0 \leq \alpha \leq 2$. It follows from Theorem 3.2(II) and Lemma 4.1 that the GA_1- and GA_2-optimality criteria reflect the confounding (or aliasing) relationship among $A^{\#(a_1a_2,20)}_{\gamma_1\gamma_2}\boldsymbol{\theta}_{20}$, $A^{\#(a_1a_2,02)}_{\gamma_1\gamma_2}\boldsymbol{\theta}_{02}$ and $A^{\#(a_1a_2,11)}_{\gamma_1\gamma_2}\boldsymbol{\theta}_{11}$ $(a_1a_2=20,02,11)$ for $\gamma_1\gamma_2=00,10,01$. In this sense, the authors believe that the GA_γ-optimality criteria for $\gamma=1,2$ are suitable for comparison of designs.

Using Theorems 3.3 and 4.2, we can obtain GA_α-optimal 2^{2+2}-, 2^{2+3}-, 2^{2+4}-, 2^{3+3}-, 2^{3+4}- and 2^{4+4}-PBFF designs of resolution $\mathrm{R}(\{00,10,01\} \mid \Omega)$ for $0 \leq \alpha \leq 2$ derived from $\mathrm{SPBAs}(m_1+m_2;\{\lambda_{i_1,i_2}\})$ with $N < v(m_1,m_2)$, which are given by Tables 4.1 through 4.6, respectively. In each table, all GA_α-optimal designs except for $m_1=2$, $m_2=3$ and $N=14,15$, $m_1=2$, $m_2=4$ and $N=20$, $m_1=m_2=3$ and $N=16,17,19,20$, $m_1=3$, $m_2=4$ and $N=18,19,20,24,27$, and $m_1=m_2=4$ and $N=20,21,22,35,36$ are also GA_γ-optimal for other two γ $(0 \leq \gamma \neq \alpha \leq 2)$. Here GA_1-optimal designs with $m_1=2$, $m_2=3$ and $N=14$, $m_1=m_2=3$ and $N=16,19$, $m_1=3$, $m_2=4$ and $N=18,24,27$, and

Table 4.5. GA_α-optimal 2^{3+4}-PBFF designs.

N	λ'	$\mathrm{tr}\{V_T(0)\}$	$\mathrm{tr}\{V_T(1)\}$	$\mathrm{tr}\{V_T(2)\}$	Theorem
15	11000 10000 00001 00010	2.52333	1.82926	2.01806	(V)(i)(2)
16	11000 10000 00001 00011	1.93229	1.39828	1.52257	(V)(i)(2)
17	21000 10000 00001 00011	1.78883	1.29367	1.40231	(V)(i)(2)
18	10011 10000 00001 11001	1.66301			(V)(iii)(1)(b)
	21000 10000 00001 00012		1.22883	1.32792	(V)(i)(2)
19	10012 10000 00001 11001	1.60252		1.29002	(V)(iii)(1)(b)
	31000 10000 00001 00012		1.19688		(V)(i)(2)
20	10101 10000 00001 00100	1.47608			(V)(iii)(1)(a)
	21000 20000 00001 00011		1.17029		(V)(i)(2)
	10012 10000 00001 21001			1.24974	(V)(iii)(1)(b)
21	10101 10000 00001 00101	1.29396	1.07174	1.09396	(V)(iii)(1)(a)
22	10101 10000 00001 10101	1.21277	0.99055	1.01277	(V)(iii)(1)(a)
23	20101 10000 00001 10101	1.17014	0.94791	0.97014	(V)(iii)(1)(a)
24	10101 10000 00002 00101	1.11404			(V)(iii)(1)(a)
	20101 10000 00001 10102		0.91905	0.94128	(V)(iii)(1)(a)
25	10101 20000 00001 10101	1.04905	0.88239	0.89905	(V)(iii)(1)(a)
26	20101 20000 00001 10101	0.99858	0.83191	0.84858	(V)(iii)(1)(a)
27	10101 20000 00002 00101	0.96078			(V)(iii)(1)(a)
	30101 20000 00001 10101		0.80938	0.82605	(V)(iii)(1)(a)
28	10101 20000 00002 10101	0.89116	0.78005	0.79116	(V)(iii)(1)(a)

Table 4.6. GA_α-optimal 2^{4+4}-PBFF designs.

N	$\boldsymbol{\lambda}'$	tr$\{V_T(0)\}$	tr$\{V_T(1)\}$	tr$\{V_T(2)\}$	Theorem
17	11000 10000 00000 00001 00010	3.03704	2.43921	2.62222	(VI)(ii)
18	11000 10000 00000 00001 00011	2.21970	1.76384	1.88636	(VI)(ii)
19	21000 10000 00000 00001 00011	2.00952	1.59017	1.69714	(VI)(ii)
20	10011 10000 00000 00001 11001	1.82217		1.57217	(VI)(iii)(2)(c)
	21000 10000 00000 00001 00012		1.48684		(VI)(ii)
21	10011 10000 00000 00001 11002	1.76435		1.51435	(VI)(iii)(2)(c)
	10012 10000 00000 00001 11001	1.76435		1.51435	(VI)(iii)(2)(c)
	31000 10000 00000 00001 00012		1.43442		(VI)(ii)
22	10011 10000 00000 00001 21002	1.70652		1.45652	(VI)(iii)(2)(c)
	10012 10000 00000 00001 11002	1.70652		1.45652	(VI)(iii)(2)(c)
	31000 10000 00000 00001 00013		1.39380		(VI)(ii)
23	10101 10000 00000 00001 00101	1.45909	1.30376	1.33409	(VI)(iii)(2)(a)
	11001 00000 10001 00000 00011	1.45909	1.30376	1.33409	(VI)(iii)(1)(b)
24	20101 10000 00000 00001 00101	1.37318	1.21785	1.24818	(VI)(iii)(2)(a)
	21000 00000 10001 00000 10011	1.37318	1.21785	1.24818	(VI)(iii)(1)(b)
25	20101 10000 00000 00001 10101	1.30938	1.15405	1.18438	(VI)(iii)(2)(a)
	21001 00000 10001 00000 10011	1.30938	1.15405	1.18438	(VI)(iii)(1)(b)
26	10100 00000 10001 00000 10100	1.03646	1.03646	1.03646	(VI)(iii)(1)(a)
	10101 00000 10001 00000 00100	1.03646	1.03646	1.03646	(VI)(iii)(1)(a)
27	10101 00000 10001 00000 10100	0.94792	0.94792	0.94792	(VI)(iii)(1)(a)
28	10101 00000 10001 00000 10101	0.85938	0.85938	0.85938	(VI)(iii)(1)(a)
29	20101 00000 10001 00000 10101	0.83894	0.83894	0.83894	(VI)(iii)(1)(a)
30	20101 00000 10001 00000 20101	0.81851	0.81851	0.81851	(VI)(iii)(1)(a)
	20102 00000 10001 00000 10101	0.81851	0.81851	0.81851	(VI)(iii)(1)(a)
31	20102 00000 10001 00000 20101	0.80457	0.80457	0.80457	(VI)(iii)(1)(a)
32	20102 00000 10001 00000 20102	0.79063	0.79063	0.79063	(VI)(iii)(1)(a)
33	30102 00000 10001 00000 20102	0.78391	0.78391	0.78391	(VI)(iii)(1)(a)
34	10101 00000 20001 00000 10101	0.76224	0.76224	0.76224	(VI)(iii)(1)(a)
	10201 00000 10001 00000 10101	0.76224	0.76224	0.76224	(VI)(iii)(1)(a)
35	20101 00000 20001 00000 10101	0.73933			(VI)(iii)(1)(a)
	20201 00000 10001 00000 10101	0.73933			(VI)(iii)(1)(a)
	10001 01000 00000 00010 10000		0.68362	0.73648	(VI)(i)
36	20101 00000 20001 00000 20101	0.71616			(VI)(iii)(1)(a)
	20202 00000 10001 00000 10101	0.71616			(VI)(iii)(1)(a)
	10001 01000 00000 00010 10001		0.65079	0.69349	(VI)(i)

$m_1 = m_2 = 4$ and $N = 35, 36$ are GA_2-optimal, and GA_0-optimal designs with $m_1 = 2$, $m_2 = 3$ and $N = 15$, $m_1 = 2$, $m_2 = 4$ and $N = 20$, $m_1 = 3$, $m_2 = 4$ and $N = 19$, and $m_1 = m_2 = 4$ and $N = 20, 21, 22$ are GA_2-optimal. On the other hand, GA_α-optimal designs with $m_1 = m_2 = 3$ and $N = 17, 20$, and $m_1 = 3$, $m_2 = 4$ and $N = 20$ are all different for each α. Note that in each table, $\boldsymbol{\lambda}' = (\lambda_{0,0}, \lambda_{0,1}, \ldots, \lambda_{0,m_2}, \lambda_{1,0}, \lambda_{1,1}, \ldots, \lambda_{1,m_2}, \ldots, \lambda_{m_1,0}, \lambda_{m_1,1}, \ldots, \lambda_{m_1,m_2})$ and the number (*)(**)(***)(****) of the last column corresponds to Theorem 3.3 (*)(**)(***)(****). Furthermore from Lemma 2.1, in each table, if a design derived from an SPBA is GA_α-optimal for $0 \le \alpha \le 2$, then the designs derived from its FCA, LCA and/or CCA are also GA_α-optimal. Interchanging i_1 and i_2 of the indices λ_{i_1,i_2} of Tables 4.2, 4.3 and 4.5, we can obtain GA_α-optimal designs of resolution $R(\{00, 10, 01\} \mid \Omega)$ for $2 \le m_2 < m_1 \le 4$.

Note that GA-optimal $2^{m_1+m_2}$-PBFF designs with $\det(K_{\gamma_1\gamma_2}) \ne 0$ ($\gamma_1\gamma_2 = 00, 10, 01, 20$ (if $m_1 \ge 4$), 02 (if $m_2 \ge 4$)) and $K_{11} = 0$ for $4 \le m_1 + m_2 \le 6$, and with $\det(K_{\gamma_1\gamma_2}) \ne 0$ ($\gamma_1\gamma_2 = 00, 10, 01$) and furthermore (A) $K_{20} \ne 0$ (if $m_1 \ge 4$) or vanishes (if $m_1 = 2, 3$) and $K_{02} = K_{11} = 0$ for $2 \le m_1 \le 4$ and $m_2 = 4$, and (B) $K_{20} = K_{02} = K_{11} = 0$ for $m_1 = m_2 = 4$ were obtained by Kuwada [2] and Kuwada and Matsuura [4], respectively, where $\det(A)$ denotes the determinant of a matrix A. Moreover GA_α-optimal $2^{m_1+m_2}$-PBFF designs of resolutions $R(\{00, 10, 01, 20, 02\} \mid \Omega)$ and $R(\{00, 10, 01, 20, 11\} \mid \Omega)$ with $N < \nu(m_1, m_2)$ and $2 \le m_k \le 4$, of resolution $R(\{00, 10, 01, 20\} \mid \Omega)$ with $N < \nu(m_1, m_2)$ and $2 \le m_k \le 4$, and of resolution $R(\{00, 10, 01, 11\} \mid \Omega)$ with $N < \nu(m_1, m_2)$ and $2 \le m_1 \le m_2 \le 4$ have been obtained by Kuwada et al. [6, 7] and Lu et al. [8], respectively.

5. Appendix

Let us consider a matrix equation $ZL = H$ with a parameter matrix Z of order n, where $L = \|L_{ij}\|$ is the positive semidefinite matrix of order n with $\operatorname{rank}\{L\} = \operatorname{rank}\left\{\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\right\} = n_1 + n_2$ (≥ 1) and $H = \|H_{ij}\|$ $(i, j = 1, 2, 3)$ is a matrix of order n with $H_{11} = I_{n_1}$, $H_{12} = H'_{21} = 0_{n_1 \times n_2}$ and $H_{13} = H'_{31} = 0_{n_1 \times n_3}$. Here both L_{ij} and H_{ij} are of size $n_i \times n_j$, $n_1 + n_2 + n_3 = n$, and $0_{p \times q}$ is the zero matrix of size $p \times q$. The equation $ZL = H$ has a solution if and only if $\operatorname{rank}\{L'\} = \operatorname{rank}\{(L'; H')\}$. Thus we have the following (see [1]):

Lemma A.1. *A matrix equation $ZL = H$ has a solution, where Z is a parameter matrix of order n, if and only if*

(I) *$n_3 = 0$, where H_{22} (if $n_2 \ge 1$) is arbitrary, or*

(II) *$n_3 \ge 1$, and in addition*

  (i) *when $n_2 = 0$, $L_{33} = 0_{n_3 \times n_3}$, and furthermore $H_{33} = 0_{n_3 \times n_3}$, or*

  (ii) *when $n_2 \ge 1$, there exists a matrix W of size $n_3 \times n_2$ such that $(L_{31}; L_{32}; L_{33}) = W(L_{21}; L_{22}; L_{23})$, and furthermore $H'_{23} = WH'_{22}$ and $H'_{33} = WH'_{32}$, where H_{22} and H_{32} are arbitrary.*

By use of a solution Z of the $ZL = H$, we obtain the following (see [6]):

Lemma A.2.

$$ZLZ' = \begin{cases} L_{11}^{-1} & \text{if } n_2 = n_3 = 0, \\ \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} L_{11}^{-1}(I_{n_1}; 0) & \text{if } n_2 = 0 \text{ and } n_3 \geq 1, \\ \begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22}' \end{pmatrix} & \text{if } n_2 \geq 1 \text{ and } n_3 = 0, \\ \begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \\ 0 & H_{32} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & H_{22}' & H_{32}' \end{pmatrix} & \text{if } n_2 \geq 1 \text{ and } n_3 \geq 1, \end{cases}$$

where H_{22} *and* H_{32} *are arbitrary.*

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