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Polyharmonicity and algebraic support of measures

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Abstract. Our main result states that two signed measures μ and ν with bounded support contained in the zero set of a polynomial $P(x)$ are equal if they coincide on the subspace of all polynomials of polyharmonic degree N_P where the natural number N_P is explicitly computed by the properties of the polynomial $P(x)$. The method of proof depends on a definition of a multivariate Markov transform which is another major objective of the present paper. The classical notion of orthogonal polynomial of second kind is generalized to the multivariate setting: it is a polyharmonic function which has similar features to those in the one-dimensional case.

1. Introduction

Recall that a complex-valued function f defined on a domain G in the euclidean space $\mathbf{R}^n$ is *polyharmonic of order* N if f is $2N$-times continuously differentiable and

$$\Delta^N f(x) = 0 \qquad \text{for all } x \in G$$

where Δ^N is the N-th iterate of the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. For $N = 1$ this class of functions are just the harmonic functions, while for $N = 2$ the term biharmonic function is used which is important in elasticity theory. Fundamental work about polyharmonic functions is due to E. Almansi [2], M. Nicolesco (see e.g. [25]) and N. Aronszajn [3], and still this is an area of active research; see e.g. [7], [8], [9], [12], [17], [18], [23], [27], [28]. Polyharmonic functions are also important in applied mathematics, e.g. in approximation theory, radial basis functions and wavelet analysis; see e.g. [5], [19], [20], [21], [24].

In this paper we address the following question: Let μ and ν be signed measures with compact support. Suppose that there exists a polynomial $P(x)$ such that the supports of μ and ν are contained in the zero set of P. Under which conditions do μ and ν coincide? As motivating example consider the polynomial $P(x) = |x|^2 - 1$ where $|x| := r(x) := \sqrt{x_1^2 + \cdots + x_n^2}$ is the euclidean

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norm in $\mathbf{R}^n$. It is well known that two measures μ and ν with support in the unit sphere $\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$ coincide if they are equal on the set of all *harmonic* polynomials. We shall show that two measures μ and ν with support in the set $K_P(R)$ (defined below in (2)), are equal if the moments $\mu(f)$ and $\nu(f)$ are equal for polyharmonic polynomials f of a certain degree N_P which depends on the polynomial P. In order to formulate this precisely, let us introduce the *polyharmonic degree* $d(f)$ defined by

$$d(f) := \min\{N \in \mathbf{N}_0 : \Delta^{N+1}(f) = 0\}. \tag{1}$$

In the appendix we shall compare properties of the polyharmonic degree and the total degree. Note that f has **polyharmonic degree** $\leq N$ if and only if f is of **polyharmonic order** $N+1$.

Let us denote by $\mathscr{P}$ the set of all polynomials. One of the main results of this paper reads as follows:

THEOREM 1. *Let*

$$K_P(R) := \{x \in \mathbf{R}^n : P(x) = 0 \text{ and } |x| \leq R\} \tag{2}$$

for $R > 0$ *and for a polynomial* $P(x)$*, and define*

$$N_P = \sup\{d(P \cdot h) : h \text{ is a harmonic polynomial}\}. \tag{3}$$

Let μ *and* ν *be signed measures with support contained in the set* $K_P(R)$ *for some* $R > 0$*. If* $\int h \, d\mu = \int h \, d\nu$ *for all polynomials* h *in the subspace*

$$U_{N_P} = \{Q \in \mathscr{P} : \Delta^{N_P} Q = 0\}$$

then μ *and* ν *are identical.*

It is not difficult to see that N_P is lower or equal to the total degree of the polynomial $P(x)$, see Corollary 20. In the appendix we shall give a procedure to determine the number N_P explicitly.

An application of the Hahn-Banach theorem shows us the following consequence of Theorem 1: the space U_{N_P} is dense in the space $C(K_P(R), \mathbf{C})$ of all continuous complex-valued functions on the compact space $K_P(R)$ endowed with the supremum norm, see Corollary 18. Let us emphasize that Theorem 1 is only a sufficient criterion, and does not always give the expected result: As illustrating examples consider the case of a sphere and an ellipsoid. In the first case, the defining polynomial $P(x) = |x|^2 - 1$ has the property that $N_P = 1$, so U_{N_P} is equal to the space of all harmonic polynomials. In the case of an ellipsoid, N_P is equal to 2, although it would be sufficient to know that the measures μ and ν are identical for harmonic polynomials. However, density results for solutions to $\Delta^p h = 0$ in $C(K)$ for compact sets K for $p > 1$ are much more complicated and obtained with the techniques of Potential theory in the 1970s; see [13], [14] and the references therein. The following example shows that our approach delivers a nontrivial criterion for density which is not covered by the other approaches so far: take $P(x) = \langle a, x\rangle(|x|^2 - 1)$ where $\langle a, x\rangle = a_1x_1 + \cdots + a_nx_n$. Then $N_P = 2$, and we need now the space of all biharmonic polynomials to ensure that two measures σ and ν are equal. Indeed, harmonic polynomials are not sufficient: take σ as the usual measure $d\theta$ on the unit sphere $\mathbf{S}^{n-1}$ and ν as the point evaluation in $x = 0$. Then σ and ν coincide on the space of all harmonic polynomials and both measures have support in $P^{-1}(0)$. Clearly σ and ν are different measures.

The proof of Theorem 1 will be a by-product of our investigation of the so-called *multivariate Markov transform* which we will introduce below and which we consider as a suitable generalization of the univariate *Markov transform*, an important tool in the classical moment problem and its applications to Spectral theory. Recall that the *Markov transform*[1] of a finite measure σ with support in the interval $[-R, R]$ is defined on the upper half–plane by the formula

$$\hat{\sigma}(\zeta) := \int \frac{1}{\zeta - x}\, d\sigma(x) \qquad \text{for } \operatorname{Im}\zeta > 0, \tag{4}$$

see e.g. [1, Chapter 2], [26, Chapter 2.6]. Let us recall a central result called Markov's theorem: the N-th Padé approximant $\pi_N(\zeta) = Q_N(\zeta)/P_N(\zeta)$ of the asymptotic expansion of $\hat{\sigma}(\zeta)$ at infinity converges compactly in the upper half plane to $\hat{\sigma}(\zeta)$; here the polynomial P_N is the N-th orthogonal polynomial with respect to the measure σ and Q_N is the *orthogonal polynomial of the second kind* with respect to the measure σ given through the formula

$$Q_N(\zeta) = \int \frac{P_N(\zeta) - P_N(x)}{\zeta - x}\, d\sigma(x). \tag{5}$$

Further, to each $\pi_N(\zeta)$ there corresponds a (non-negative) measure σ_N with support in the zeros of the nominator P_N, thus leading to a proof of the famous Gauß quadrature formula.

Our definition of a multivariate Markov transform depends on the work of *N. Aronszajn* [3] on polyharmonic functions, and of *L. K. Hua* [15] about harmonic analysis on Lie groups; the definition is related to the Poisson formula for the ball $B_R := \{x \in \mathbf{R}^n : |x| < R\}$ which we recall now: Let $R > 0$

[1] In some recent works in Approximation theory, Potential theory, and Probability theory this function is called the *Markov function* of a measure, see e.g. [29] or [11]. On the other hand apparently Widder [32] was the first who has given the name *Stieltjes transform* to this function. If μ has infinite support the transform is also called Stieltjes transform. This tradition has been followed by Akhiezer [1] and other Russian mathematicians.

and h be a function harmonic in the ball B_R and continuous on the closure $\overline{B_R}$; then for any $x \in \mathbf{R}^n$ with $|x| < R$

$$h(x) = \frac{1}{\omega_n} \int_{\mathbf{S}^{n-1}} \frac{(R^2 - |x|^2) R^{n-2}}{r(R\theta - x)^n} h(R\theta) d\theta, \tag{6}$$

where ω_n denotes the area of $\mathbf{S}^{n-1}$, $\theta \in \mathbf{S}^{n-1}$, and $r(x)$ is the euclidean norm of x. Note that for fixed x with $|x| < R$ the function $\rho \mapsto r(\rho\theta - x)$ defined for $\rho \in \mathbf{R}$ with $|\rho| > R$ has an analytic continuation for $\zeta \in \mathbf{C}$ with $|\zeta| > R$, so we can write $r(\zeta\theta - x)$ for $\zeta \in \mathbf{C}$ with $|\zeta| > R$. The following *Cauchy type integral formula*, proved in [3, p. 125], is important for our approach: for any polynomial $u(x)$ and for any $|x| < R$ the following identity holds

$$u(x) = \frac{1}{2\pi i \omega_n} \int_{\Gamma_R} \int_{\mathbf{S}^{n-1}} \frac{\zeta^{n-1}}{r(\zeta\theta - x)^n} u(\zeta\theta) d\theta d\zeta \tag{7}$$

where the contour $\Gamma_R(t) = R \cdot e^{it}$ for $t \in [0, 2\pi]$. A similar result is also valid for holomorphic functions u defined on the so-called harmonicity hull of B_R; we refer the reader to [3, p. 125] for details.

Assume now that μ is a signed measure with support in the closed ball $\{x \in \mathbf{R}^n : |x| \leq R\}$. The *multivariate Markov transform* $\hat{\mu}$ of μ is a function defined for all $\theta \in \mathbf{S}^{n-1}$ and all $\zeta \in \mathbf{C}$ with $|\zeta| > R$ by the formula

$$\hat{\mu}(\zeta, \theta) = \frac{1}{\omega_n} \int_{\mathbf{R}^n} \frac{\zeta^{n-1}}{r(\zeta\theta - x)^n} d\mu(x). \tag{8}$$

Since $\zeta \mapsto r(\zeta\theta - x)$ has no zeros for $|\zeta| > R$ the function $\zeta \mapsto \hat{\mu}(\zeta, \theta)$ is defined for all $|\zeta| > R$. In the following Section we shall show that the multivariate Markov transform $\hat{\mu}$ determines the measure μ uniquely, cf. Theorem 3.

Our second main innovation is the introduction of the notion of the *function* $Q_P(\zeta, \theta)$ *of the second kind* with respect to a given polynomial $P(x)$ which is the multivariate analogue of (5), defined by

$$Q_P(\zeta, \theta) = \int_{\mathbf{R}^n} \frac{P(\zeta\theta) - P(x)}{r(\zeta\theta - x)^n} \zeta^{n-1} d\mu(x) \tag{9}$$

for all $|\zeta| > R$, $\theta \in \mathbf{S}^{n-1}$. Let us emphasize that Q_P is in general *not* a polynomial. However, we shall show the surprising and interesting result that the function $r\theta \mapsto r^{-(n-1)} Q_P(r\theta)$ is a *polyharmonic* function of order $\leq \deg P(x)$ where deg denotes the total degree of a polynomial.

One further *main result* of the paper, Theorem 13, is concerned with measures μ having their supports in algebraic sets: Let us assume that the measure μ has support in $K_P(R)$. Then the Markov transform $\hat{\mu}$ has the representation

$$\hat{\mu}(\zeta,\theta)=\frac{Q_P(\zeta,\theta)}{P(\zeta\theta)}\qquad \text{for } |\zeta|>R, \tag{10}$$

where Q_P is the function of second kind with respect to $P(x)$. The reverse statement holds as well, i.e. if the measure μ with $\operatorname{supp}(\mu)\subset\overline{B_R}$ satisfies (10) for some polynomial P where Q_P is defined by (9), then $\operatorname{supp}(\mu)\subset K_P(R)$. By means of these characterizations we can deduce our main result Theorem 1.

Finally let us recall some terminology from measure theory: a signed measure on $\mathbf{R}^d$ is a set function on the Borel σ-algebra on $\mathbf{R}^d$ which takes real values and is σ-additive. By the *Jordan decomposition* [6, p. 125], a signed measure μ is the difference of two non-negative finite measures, say $\mu=\mu^+-\mu^-$ with the property that there exists a Borel set A such that $\mu^+(A)=0$ and $\mu^-(\mathbf{R}^n\backslash A)=0$. The *variation* of μ is defined as $|\mu|:=\mu^++\mu^-$. The *support of a non-negative measure* μ on $\mathbf{R}^d$ is defined as the complement of the largest open set U such that $\mu(U)=0$. The *support of a signed measure* σ is defined as the support of the total variation $|\sigma|=\sigma_++\sigma_-$ (see [6, p. 226]). Recall that in general, the supports of σ_+ and σ_- are not disjoint (cf. exercise 2 in [6, p. 231]). Note that if a signed measure μ has compact support then all polynomials are integrable with respect to μ^+, μ^-, and $|\mu|$.

2. The multivariate Markov transform

Recall that the univariate Markov transform has, for $|\zeta|>R$, the asymptotic expansion

$$\hat{\sigma}(\zeta)=\sum_{k=0}^{\infty}\frac{1}{\zeta^{k+1}}\int t^k\,d\sigma(t). \tag{11}$$

Let Γ_R denote the contour in $\mathbf{C}$ defined by $\Gamma_R(t)=R\cdot e^{it}$ for $t\in[0,2\pi]$. By means of standard facts from complex analysis the following identity may be proved:

$$\frac{1}{2\pi i}\int_{\Gamma_{R_1}}p(\zeta)\hat{\sigma}(\zeta)d\zeta=\int p(x)d\sigma(x) \tag{12}$$

for all polynomials p and any $R_1>R$.

In this Section we want to show that similar results hold for the multivariate Markov transform $\hat{\mu}$; in particular the following is the analogue of formula (12) in the multivariate case:

Proposition 2. *Let μ be a signed measure over $\mathbf{R}^n$ with support in $\overline{B_R}$ and let $R_1>R$. Then for every polynomial $P(x)$*

$$M_\mu(P):=\frac{1}{2\pi i}\int_{\Gamma_{R_1}}\int_{\mathbf{S}^{n-1}}P(\zeta\theta)\hat{\mu}(\zeta,\theta)d\zeta d\theta=\int_{\mathbf{R}^n}P(x)d\mu(x). \tag{13}$$

Proof. Replace $\hat{\mu}(\zeta,\theta)$ in (13) by (8) and interchange integration. Then

$$M_\mu(P) = \int_{\mathbf{R}^n} \frac{1}{2\pi i\omega_n} \int_{\Gamma_{R_1}} \int_{\mathbf{S}^{n-1}} P(\zeta\theta) \frac{\zeta^{n-1}}{r(\zeta\theta - x)^n} \, d\zeta d\theta d\mu(x). \tag{14}$$

According to (7) we obtain $M_\mu(P) = \int P(x)d\mu(x)$. ∎

Theorem 3. *Let μ, v be finite signed measures over $\mathbf{R}^n$ with compact support. If the multivariate Markov transforms of μ and v coincide for large ζ, i.e., if there exists $R > 0$ such that $\hat{\mu}(\zeta,\theta) = \hat{v}(\zeta,\theta)$ for all $|\zeta| > R$ and for all $\theta \in \mathbf{S}^{n-1}$, then μ and v are identical.*

Proof. Since the multivariate Markov transforms coincide for large $|\zeta|$ it is clear that the functionals M_μ and M_v in (13) are identical by taking the radius R_1 of the path Γ_{R_1} large enough. Then Proposition 2 shows that $\int P(x)d\mu(x) = \int P(x)dv(x)$ for all polynomials $P(x)$. Further we apply a standard argument: since μ and v have compact supports we may apply the Stone–Weierstrass theorem according to which the polynomials are dense in the space $C(\mathrm{supp}(\mu) \cup \mathrm{supp}(v))$ which implies that $\mu = v$. ∎

Next we want to determine the asymptotic expansion of the multivariate Markov transform and we need some notations from harmonic analysis; for a detailed account we refer to [4] or [30]. Recall that a function $Y : \mathbf{S}^{n-1} \to \mathbf{C}$ is called a *spherical harmonic* of degree $k \in \mathbf{N}_0$ if there exists a *homogeneous harmonic* polynomial $P(x)$ of degree k (in general, with complex coefficients[2]) such that $P(\theta) = Y(\theta)$ for all $\theta \in \mathbf{S}^{n-1}$. Throughout the paper we assume that $Y_{k,m}(x)$, $m = 1, \ldots, a_k$, is a basis of the set of all harmonic homogeneous polynomials of degree k which are orthonormal with respect to scalar product

$$\langle f, g \rangle_{\mathbf{S}^{n-1}} := \int_{\mathbf{S}^{n-1}} f(\theta)\overline{g(\theta)} d\theta.$$

For a continuous function $f : \mathbf{S}^{n-1} \to \mathbf{C}$ we define the *Laplace-Fourier series* by

$$f(\theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} f_{k,m} Y_{k,m}(\theta)$$

and $f_{k,m} = \int_{\mathbf{S}^{n-1}} f(\theta)\overline{Y_{k,m}(\theta)} d\theta$ are the *Laplace-Fourier coefficients* of f.

Using the *Gauss decomposition* of a polynomial (see Theorem 5.5 in [4]) it is easy to see that the system

$$|x|^{2t} Y_{k,m}(x), \qquad t, k \in \mathbf{N}_0, \, m = 1, \ldots, a_k$$

is a basis of the set of all polynomials. The numbers

[2] One may restrict the attention to real valued spherical harmonics and this does not change the results essentially.

$$c_{t,k,m} := \int_{\mathbf{R}^n} |x|^{2t}\overline{Y_{k,m}(x)}d\mu(x), \qquad t,k \in \mathbf{N}_0,\ m = 1,\dots,a_k \tag{15}$$

are sometimes called the *distributed moments*, see [16]. For a treatment and formulation of the *multivariate moment problem* we refer to [10], see also [31].

Theorem 4. *Let μ be a signed measure over $\mathbf{R}^n$ with support in the closed ball $\overline{B_R}$. Then for all $|\zeta| > R$ and for all $\theta \in \mathbf{S}^{n-1}$ the following relation holds*

$$\hat{\mu}(\zeta,\theta) = \sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=1}^{a_k}\frac{Y_{k,m}(\theta)}{\zeta^{2t+k+1}}\int_{\mathbf{R}^n}|x|^{2t}\overline{Y_{k,m}(x)}d\mu(x). \tag{16}$$

Proof. A zonal harmonic of degree k with pole $\theta \in \mathbf{S}^{n-1}$ is the unique spherical harmonic $Z_\theta^{(k)}$ of degree k such that for all spherical harmonics Y of degree k the relation $Y(\theta) = \int_{\mathbf{S}^{n-1}} Z_\theta^{(k)}(\eta)\overline{Y(\eta)}d\eta$ holds. Let $p_n(\theta,x) = \frac{1}{\omega_n}\frac{1-|x|^2}{|x-\theta|^n}$ be the Poisson kernel for $0 \le |x| < 1 = |\theta|$. Theorem 2.10 in [30, p. 145] gives $p_n(\theta,x) = \sum_{k=0}^{\infty}|x|^k Z_\theta^{(k)}(x')$ for all $\theta, x' \in \mathbf{S}^{n-1}$, where $x = |x|\cdot x'$, $|x| < 1$. Lemma 2.8 in [30] shows that $Z_\theta^{(k)}(x') = \sum_{m=1}^{a_k}\overline{Y_{k,m}(x')}Y_{k,m}(\theta)$ where $x',\theta \in \mathbf{S}^{n-1}$, so

$$p_n(\theta,x) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k}|x|^k\,\overline{Y_{k,m}(x')}\,Y_{k,m}(\theta). \tag{17}$$

for $|x| < 1$. Let R be as in the theorem, and replace now x in (17) by x/ρ, $\rho \in \mathbf{R}$ such that $|x| < R < \rho$; one obtains that

$$\frac{1}{\omega_n}\frac{\rho^{n-2}(\rho^2 - |x|^2)}{r(\rho\theta - x)^n} = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k}\frac{1}{\rho^k}\,\overline{Y_{k,m}(x)}\,Y_{k,m}(\theta). \tag{18}$$

The real variable ρ can now be replaced by a complex variable ζ with $|\zeta| > R$. We multiply by $\zeta(\zeta^2 - |x|^2)^{-1}$, and integrate over the closed ball $\overline{B_R}$ with respect to μ. This gives

$$\hat{\mu}(\zeta,\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k}Y_{k,m}(\theta)\zeta^{-k+1}\int_{\mathbf{R}^n}\frac{\overline{Y_{k,m}(x)}}{\zeta^2 - |x|^2}\,d\mu(x), \tag{19}$$

and we have determined the Laplace-Fourier series of $\theta \mapsto \hat{\mu}(\zeta,\theta)$. Since $|\zeta| > R \ge |x|$ we can expand $1/\left(1 - \frac{|x|^2}{\zeta^2}\right)$ in a geometric series and we obtain

$$\hat{\mu}(\zeta,\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k}\frac{Y_{k,m}(\theta)}{\zeta^{k+1}}\int_{\mathbf{R}^n}\overline{Y_{k,m}(x)}\left(\sum_{t=0}^{\infty}\frac{|x|^{2t}}{\zeta^{2t}}\right)d\mu(x). \tag{20}$$

After interchanging summation and integration the claim is obvious. ■

3. The function of the second kind

In the following we want to a give a multivariate analogue of the polynomial of second kind. It turns out that in the multivariate case the corresponding definition does not lead to a polynomial but to a polyharmonic function $Q_P(\zeta,\theta)$ which is defined only for all $|\zeta| > R$, $\theta \in \mathbf{S}^{n-1}$.

DEFINITION 5. *Let $P(x)$ be a polynomial and μ be a non-negative measure with support in $\overline{B_R}$. Then the* function $Q_P(\zeta,\theta)$ of the second kind *is defined by*

$$Q_P(\zeta,\theta) = \frac{1}{\omega_n}\int_{\mathbf{R}^n} \frac{P(\zeta\theta) - P(x)}{r(\zeta\theta - x)^n}\zeta^{n-1}\, d\mu(x) \tag{21}$$

for all $|\zeta| > R$, $\theta \in \mathbf{S}^{n-1}$. Similarly we define the function $R_P(\zeta,\theta)$ by

$$R_P(\zeta,\theta) = \frac{1}{\omega_n}\int_{\mathbf{R}^n} \frac{P(x)}{r(\zeta\theta - x)^n}\zeta^{n-1}\, d\mu(x) \tag{22}$$

for all $|\zeta| > R$, $\theta \in \mathbf{S}^{n-1}$.

The last definitions immediately give the identity

$$P(\zeta\theta)\hat{\mu}(\zeta,\theta) = Q_P(\zeta,\theta) + R_P(\zeta,\theta). \tag{23}$$

THEOREM 6. *Let $P(x)$ be a polynomial, μ be a signed measure with support in $\overline{B_R}$ and $Q_P(\zeta,\theta)$ the function of the second kind. Then for any $R_1 > R$ and for each polynomial $h(x)$*

$$\frac{1}{2\pi i}\int_{\Gamma_{R_1}}\int_{\mathbf{S}^{n-1}} h(\zeta\theta)Q_P(\zeta,\theta)d\zeta d\theta = 0. \tag{24}$$

PROOF. Let us denote the integral in (24) by $I(h)$. By (23) we obtain that $I(h) = I_1(h) - I_2(h)$ where

$$I_1(h) = \frac{1}{2\pi i}\int_{\Gamma_{R_1}}\int_{\mathbf{S}^{n-1}} h(\zeta\theta)P(\zeta\theta)\hat{\mu}(\zeta,\theta)d\zeta d\theta, \tag{25}$$

$$I_2(h) = \frac{1}{2\pi i\omega_n}\int_{\Gamma_{R_1}}\int_{\mathbf{S}^{n-1}} h(\zeta\theta)\int_{\mathbf{R}^n} \frac{P(x)}{r(\zeta\theta - x)^n}\zeta^{n-1}\, d\mu(x)d\zeta d\theta. \tag{26}$$

Proposition 2 yields $I_1(h) = \int_{\mathbf{R}^n} h(x)P(x)d\mu(x)$. Change the integration order in (26) and use formula (7). Then we obtain $I_2(h) = I_1(h)$, therefore $I(h) = 0$ which was our claim. ■

A similar argument to that in the proof of formula (16) proves the following:

Theorem 7. *The function $R_P(\zeta,\theta)$ has the asymptotic expansion*

$$\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=1}^{a_k}\frac{Y_{k,m}(\theta)}{\zeta^{2t+k+1}}\int_{\mathbf{R}^n}P(x)|x|^{2t}\overline{Y_{k,m}(x)}d\mu(x). \tag{27}$$

Note that the map $\zeta \mapsto R_P(\zeta,\theta)$ for $|\zeta| > R$ and $\theta \in \mathbf{S}^{n-1}$ is holomorphic in the complex variable ζ. So we can consider the Laurent series of the function $\zeta \mapsto R_P(\zeta,\theta)$ and we write for $|\zeta| > R$ and fixed $\theta \in \mathbf{S}^{n-1}$

$$R_P(\zeta,\theta) = \sum_{s=0}^{\infty} r_s[P](\theta)\frac{1}{\zeta^{s+1}}. \tag{28}$$

From (27), by putting $s = 2t + k$, it follows that

$$r_s[P](\theta) = \sum_{t=0}^{[s/2]}\sum_{m=1}^{a_{s-2t}} Y_{s-2t,m}(\theta)\int_{\mathbf{R}^n}P(x)|x|^{2t}\overline{Y_{s-2t,m}(x)}d\mu(x). \tag{29}$$

Hence the coefficient function $r_s[P]$ is a sum of spherical harmonics with degree $\leq s$.

We can now formulate a characterization of orthogonality in asymptotic analysis:

Theorem 8. *Let μ be a signed measure with compact support and $P(x)$ be a polynomial. Then P is orthogonal to all polynomials of degree $< M$ with respect to μ if and only if*

$$r_0[P] = \cdots = r_{M-1}[P] = 0$$

where $r_s[P]$ are the functions defined in (28)–(29).

Proof. From (29) we see that $r_0[P] = \cdots = r_{M-1}[P] = 0$ if and only for all $s = 0,\dots,M-1$

$$\int_{\mathbf{R}^n}P(x)|x|^{2t}\overline{Y_{s-2t,m}(x)}d\mu(x) = 0.$$

But the polynomials $|x|^{2t}Y_{s-2t,m}(x)$ with $s = 0,\dots,M-1$, $t = 0,\dots,[s/2]$, $m = 1,\dots,a_{s-2t}$, span up the space of polynomials of degree $\leq M-1$. ■

The next theorem, interesting in its own right, is not needed later, and therefore the proof will be omitted.

Theorem 9. *Let μ be a signed measure with compact support and let $P(x)$ be a polynomial of degree $2N$. If P is orthogonal to all polynomials of degree $\leq 2N$ and polyharmonic degree $< N$ then $r_0[P] = \cdots = r_{2N-1}[P] = 0$ and $r_{2N}[P](\theta)$ is constant.*

4. Polyharmonicity of the function of second kind

In this Section we want to show that the function $Q_P(\zeta,\theta)$ of the second kind, multiplied by $\zeta^{-(n-1)}$, is a polyharmonic function.

Recall that we have defined $N_P = \sup\{d(P\cdot h) : h \text{ harmonic polynomial}\}$ for a polynomial $P(x)$. In the Appendix we will show that $N_P \leq \deg P(x)$ and an explicit determination of N_P will be given there as well.

Proposition 10. *Let $Y_{k,m}$, $m=1,\dots,a_k$, be an orthonormal basis of the space of all homogeneous harmonic polynomials. Then*

$$N_P = \sup_{k\in\mathbf{N}_0,\, m=1,\dots,a_k} d(P(x)\,Y_{k,m}(x)). \tag{30}$$

Proof. Let us denote the right hand side by M_P. Then the inequality $M_P \leq N_P$ is trivial. For the converse let $h(x)$ be a harmonic polynomial and write $h(x) = \sum_{k=0}^{N}\sum_{m=1}^{a_k}\lambda_{k,m}Y_{k,m}(x)$. Then

$$d(P\cdot h) \leq \sup_{k\in\mathbf{N}_0,\, m=1,\dots,a_k} d(P(x)\,Y_{k,m}(x)) \leq M_P. \quad \blacksquare$$

Note that $N_P = \sup_{k\in\mathbf{N}_0,m=1,\dots,a_k} d(P(x)\overline{Y_{k,m}(x)})$ since $\overline{Y_{k,m}}$, $m=1,\dots,a_k$ is an orthonormal basis as well. Now we determine the asymptotic expansion of the function of the second kind:

Theorem 11. *Let $P(x)$ be a polynomial and μ be a signed measure with support in $\overline{B_R}$. Then $\theta \mapsto Q_P(\zeta,\theta)$, the function of the second kind, possesses a Laplace-Fourier series of the form*

$$Q_P(\zeta,\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k}\frac{1}{\zeta^{k-1}}p_{k,m}(\zeta^2)\,Y_{k,m}(\theta) \tag{31}$$

where $p_{k,m}(t)$ are univariate polynomials of degree strictly smaller than $N_{k,m} := d(P(x)\,Y_{k,m}(x))$. The function $Q_P(\zeta,\theta)$ of the second kind depends on those distributed moments

$$\int_{\mathbf{R}^n} h(x)|x|^{2t}d\mu(x) \tag{32}$$

where $t \leq \sup_{k\in\mathbf{N}_0}\deg p_{k,m}$ and $h(x)$ is a harmonic polynomial.

Proof. For each fixed ζ with $|\zeta| > R$ the function $\theta \mapsto Q_P(\zeta,\theta)$ possesses a Laplace-Fourier expansion, say

$$Q_P(\zeta,\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k} e_{km}(\zeta)\,Y_{k,m}(\theta).$$

Recall that $Q_P(\zeta,\theta) = P(\zeta\theta)\hat{\mu}(\zeta,\theta) - R_P(\zeta,\theta)$, see (23). Formula (27) easily yields the Laplace-Fourier expansion of $\theta \mapsto R_P(\zeta,\theta)$: in (27) one has only to compute the sum over the variable t obtaining

$$R_P(\zeta,\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k} Y_{k,m}(\theta)\frac{1}{\zeta^{k-1}}\int_{\mathbf{R}^n}\frac{P(x)\overline{Y_{k,m}(x)}}{\zeta^2 - |x|^2}\,d\mu(x). \tag{33}$$

The Laplace-Fourier coefficients of $\theta \mapsto P(\zeta\theta)\hat{\mu}(\zeta,\theta)$ are given through

$$f_{k,m}(\zeta) := \int_{\mathbf{S}^{n-1}} P(\zeta\theta)\hat{\mu}(\zeta,\theta)\,\overline{Y_{k,m}(\theta)}d\theta. \tag{34}$$

Let us write $P(x)\overline{Y_{k,m}(x)}$ in the Gauß decomposition, see Theorem 5.5 in [4], in the form

$$P(x)\overline{Y_{k,m}(x)} = \sum_{j=0}^{N_{k,m}} h_{j,k,m}(x)|x|^{2j}, \tag{35}$$

where $h_{j,k,m}$ are harmonic polynomials and $N_{k,m}$ is the polyharmonic degree of $P(x)Y_{k,m}(x)$. Then (34) and (35) yield

$$\begin{aligned}
f_{k,m}(\zeta) &= \frac{1}{\zeta^k}\int_{\mathbf{S}^{n-1}} P(\zeta\theta)\zeta^k\,\overline{Y_{k,m}(\theta)}\hat{\mu}(\zeta,\theta)d\theta \\
&= \frac{1}{\zeta^k}\sum_{j=0}^{N_{k,m}}\zeta^{2j}\int_{\mathbf{S}^{n-1}} h_{j,k,m}(\zeta\theta)\hat{\mu}(\zeta,\theta)d\theta \\
&= \frac{1}{\zeta^k}\sum_{j=0}^{N_{k,m}}\zeta^{2j}\int_{\mathbf{R}^n}\int_{\mathbf{S}^{n-1}} h_{j,k,m}(\zeta\theta)\frac{1}{\omega_n}\frac{\zeta^{n-1}}{r(\zeta\theta - x)^n}\,d\theta d\mu(x).
\end{aligned}$$

Since $h_{j,k,m}$ is a harmonic polynomial the Poisson formula shows that for real $\zeta > R$ holds

$$h_{j,k,m}(x) = \frac{1}{\omega_n}\int_{\mathbf{S}^{n-1}} h_{j,k,m}(\zeta\theta)\frac{\zeta^{n-2}(\zeta^2 - |x|^2)}{r(\zeta\theta - x)^n}\,d\theta.$$

Since the integrand is holomorphic in ζ this holds for all complex values ζ with $|\zeta| > R$ as well. Thus

$$f_{k,m}(\zeta) = \frac{1}{\zeta^k}\sum_{j=0}^{N_{k,m}}\zeta^{2j}\int_{\mathbf{R}^n}\frac{\zeta}{\zeta^2 - |x|^2}h_{j,k,m}(x)d\mu(x) \tag{36}$$

are the Laplace Fourier coefficients of $\theta \mapsto P(\zeta\theta)\hat{\mu}(\zeta,\theta)$.

Replace now $P(x)\overline{Y_{k,m}(x)}$ in (33) by the right hand side of (35) and take the difference of the Laplace-Fourier coefficients we computed so far. Then the Laplace-Fourier coefficients of $Q_P(\zeta,\theta)$ are given by

$$e_{k,m}(\zeta)=\frac{1}{\zeta^{k-1}}\sum_{j=0}^{N_{k,m}}\int_{\mathbf{R}^n}\frac{1}{\zeta^2-|x|^2}h_{j,k,m}(x)(\zeta^{2j}-|x|^{2j})d\mu(x).$$

Note that for $j=0$ the summand is just zero. For $j\geq 1$ we have

$$\frac{\zeta^{2j}-|x|^{2j}}{\zeta^2-|x|^2}=|x|^{2(j-1)}+|x|^{2(j-1)}\zeta^2+\cdots+\zeta^{2(j-1)}.$$

We conclude that $\zeta\mapsto\zeta^{k-1}e_{k,m}(\zeta)=:P_{k,m}(\zeta^2)$ is a polynomial in ζ^2 of degree at most $N_{k,m}-1$. It follows that $e_{k,m}(\zeta)$ can be computed if we know all moments of the form (32) where $t\leq\deg p_{k,m}$ and $h(x)$ is a harmonic polynomial. The proof is complete. ■

From this we have the following interesting consequence:

Corollary 12. *Let $P(x)$ be a polynomial, μ be a signed measure with support in $\overline{B_R}$ and $Q_P(\zeta,\theta)$ be the corresponding function of the second kind. Then the function $r\theta\mapsto r^{-(n-1)}Q_P(r,\theta)$ defined for $r>R$ and $\theta\in\mathbf{S}^{n-1}$, is a polyharmonic function of polyharmonic degree $<N_P$ where N_P is defined in* (3).

Proof. By the last theorem the function $\theta\mapsto r^{-(n-1)}Q_P(r,\theta)$ has the following Laplace-Fourier expansion

$$f(r\theta):=r^{-(n-1)}Q_P(r,\theta)=\sum_{k=0}^{\infty}\sum_{m=1}^{a_k}\frac{1}{r^{n+k-2}}p_{k,m}(r^2)\,Y_{k,m}(\theta).$$

Let us define the differential operator

$$L_{(k)}:=\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}-\frac{k(k+n-2)}{r^2}. \tag{37}$$

It is known that a function $g(r\theta)$ is a solution of $\Delta^p g(x)=0$ if and only if the coefficient functions $g_{k,m}(r)$ of its Laplace-Fourier expansion are solutions of the equation $[L_{(k)}]^p g_{k,m}(r)=0$; an elaboration of these classical results can be found in [19]. Further the polynomials r^j with $j=-k-n+2,-k-n+4,\dots,-k-n+2p$ are solutions of this equation. It follows that

$$f_{k,m}(r)=\frac{1}{r^{n+k-2}}p_{k,m}(r^2)$$

are solutions of the equation $[L_{(k)}]^p g_{k,m}(r)=0$ when $p\geq N_k$. The proof is complete. ■

5. Measures with algebraic support

A measure μ over $\mathbf{R}^n$ has *algebraic support* if the support of the measure is contained in an algebraic set, i.e. if the support of μ is contained in $P^{-1}(0)$ for some polynomial $P(x)$. Further we say that μ has *finite support* if the support has only finitely many elements. The following gives a characterization of algebraic support of a measure in terms of the Markov function:

Theorem 13. *Let μ be a measure with support in $\overline{B_R}$ and let $P(x)$ be a polynomial. Then μ has support in $P^{-1}(0)$ if and only if*

$$P(\zeta\theta)\hat{\mu}(\zeta,\theta) = Q_P(\zeta,\theta) \qquad \textit{for all } \theta \in \mathbf{S}^{n-1},\ |\zeta| > R, \tag{38}$$

where $Q_P(\zeta,\theta)$ is the function of the second kind.

Proof. If μ has support in $P^{-1}(0)$ it follows that the function $R_P(\zeta,\theta)$ is equal to zero and (38) is evident by (23). For the converse assume that $P(\zeta\theta)\hat{\mu}(\zeta,\theta) = Q_P(\zeta,\theta)$. Define the polynomial P^* by $P^*(x) := \overline{P(x)}$ for $x \in \mathbf{R}^n$. By Proposition 2 and Theorem 6

$$\begin{aligned}\int hP^*P\,d\mu &= \frac{1}{2\pi i}\int_{\Gamma_{R_1}}\int_{\mathbf{S}^{n-1}} h(\zeta\theta)P^*(\zeta\theta)P(\zeta\theta)\hat{\mu}(\zeta,\theta)d\zeta d\theta \\ &= \frac{1}{2\pi i}\int_{\Gamma_{R_1}}\int_{\mathbf{S}^{n-1}} h(\zeta\theta)P^*(\zeta\theta)Q_P(\zeta,\theta)d\zeta d\theta = 0 \end{aligned}\tag{39}$$

for any polynomial $h(x)$. Since the polynomials are dense it follows that $P^*P\,d\mu$ is the zero measure. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition. It follows that $P^*P\,d\mu^+$ and $P^*P\,d\mu^-$ are zero measures, and it is easy to see that this implies that the support of μ^+ and μ^- is contained in $P^{-1}(0)$. Thus μ has support in $P^{-1}(0)$. ■

Let U be an open non-empty subset of the complex plane $\mathbf{C}$ and f be a function defined on $U \times \mathbf{S}^{n-1}$. We say that f is *pointwise rational* if there exists a polynomial $P(x)$ in n variables such that for each fixed $\theta \in \mathbf{S}^{n-1}$ the function $\zeta \mapsto P(\zeta\theta)f(\zeta,\theta)$ is a polynomial in the variable ζ.

Proposition 14. *Let μ be a signed measure with bounded support and suppose that the Markov function $\hat{\mu}(\zeta,\theta)$ is pointwise rational. Then μ has algebraic support.*

Proof. Let $P(x)$ be a polynomial such that the map $\zeta \mapsto P(\zeta\theta)\hat{\mu}(\zeta,\theta)$ is a polynomial in the variable ζ. Then the integral over Γ_{R_1} in (39) is already zero and as in the last proof we obtain that μ has support in $P^{-1}(0)$. ■

The converse of the last proposition is not true as the following result with σ equal to the Lebesgue measure on the unit interval shows:

Proposition 15. *Let σ be a measure over $\mathbf{R}$ with compact support, δ_0 the Dirac measure over $\mathbf{R}$ at the point 0 and let $\mu = \sigma \otimes \delta_0$. Then for $|\zeta| > R$ the multivariate Markov transform is given by*

$$\widehat{\sigma \otimes \delta_0}(\zeta, e^{it}) = \frac{1}{\omega_2} \sum_{l=0}^{\infty} \int x^l \, d\sigma(x) \frac{\sin(l+1)t}{\sin t} \frac{1}{\zeta^{l+1}}. \tag{40}$$

The measure μ has algebraic support. Its multivariate Markov transform $\widehat{\sigma \otimes \delta_0}$ is pointwise rational if and only if the measure σ has finite support.

Proof. Let $\theta = e^{it}$ with $t \in \mathbf{R}$. It is straightforward to verify that for $|\zeta| > R$ holds

$$\begin{aligned}\widehat{\sigma \otimes \delta_0}(\zeta, \theta) &= \frac{1}{\omega_2} \int_{\mathbf{R}^2} \frac{\zeta}{r(\zeta\theta - (x, y))^2} \, d(\sigma \otimes \delta_0) \\ &= \frac{1}{\omega_2} \int \frac{\zeta}{\zeta^2 - 2\zeta x \cos t + x^2} \, d\sigma.\end{aligned}$$

Note that

$$\frac{2i\zeta \sin t}{\zeta^2 - 2\zeta x \cos t + x^2} = \frac{1}{\zeta\bar{\theta} - x} - \frac{1}{\zeta\theta - x}.$$

Define for the measure σ the one-dimensional Markov transform by $\tilde{\sigma}(\zeta) = \int \frac{1}{\zeta - x} \, d\sigma(x)$. Then $2i\omega_2 \sin t \cdot \widehat{\sigma \otimes \delta_0}(\zeta, \theta) = \tilde{\sigma}(\zeta\bar{\theta}) - \tilde{\sigma}(\zeta\theta)$ and the asymptotic expansion of $\tilde{\sigma}$ leads to (40).

Assume now that $\widehat{\sigma \otimes \delta_0}(\zeta, \theta)$ is pointwise rational. Then for $t = \pi/2$ the function

$$\widehat{\sigma \otimes \delta_0}(\zeta, \pi/2) = \frac{1}{\omega_2} \sum_{k=0}^{\infty} \int x^{2k} \, d\sigma(x) \frac{(-1)^k}{\zeta^{2k+1}} = \frac{1}{\omega_2} \zeta \int \frac{1}{\zeta^2 + x^2} \, d\sigma(x)$$

must be a rational functional in ζ. As it is known from univariate Padé approximation this implies that σ must have finite support, [26, chapter 2, section 3, Theorem 3.1]. Conversely, if a measure μ over $\mathbf{R}^n$ has finite support, and the dimension n is even then it is easy to see that $\zeta\hat{\mu}(\zeta, \theta)$ is a quotient of two polynomials, in particular it is pointwise rational. ∎

We finish this section with the following example:

Example 16. *Let μ be the Lebesgue measure on the unit circle $\mathbf{S}^1$. Since the measure is rotation-invariant it follows that $\hat{\mu}(\zeta, \theta) = \frac{\zeta}{\zeta^2 - 1}$. Hence the multivariate Markov transform $\zeta\hat{\mu}(\zeta, \theta)$ is pointwise rational but μ does not have finite support.*

6. Proof of Theorem 1

PROOF. In Theorem 11 we have seen that $Q_{\mu,P}$ and $Q_{\nu,P}$ only depend on the moments $c_{t,k,m}$ defined in (15) where $t < N_P$. It follows that $Q_{\mu,P} = Q_{\nu,P}$. By Theorem 13 $P(\zeta\theta)\hat{\mu}(\zeta,\theta) = Q_{\mu,P}(\zeta,\theta)$ and $P(\zeta\theta)\hat{\nu}(\zeta,\theta) = Q_{\nu,P}(\zeta,\theta)$ for all large ζ and for all $\theta \in \mathbf{S}^{n-1}$, therefore $P(\zeta\theta)\hat{\mu}(\zeta,\theta) = P(\zeta\theta)\hat{\nu}(\zeta,\theta)$. We want to conclude that $\hat{\mu}(\zeta,\theta) = \hat{\nu}(\zeta,\theta)$; in that case Theorem 3 yields $\mu = \nu$. If $P(\zeta\theta)$ has no zeros for large ζ it is clear that $\hat{\mu}(\zeta,\theta) = \hat{\nu}(\zeta,\theta)$. In the general case, it suffices to show that $A := \{(\zeta,\theta) \in \mathbf{C} \times \mathbf{S}^{n-1} : P(\zeta\theta) = 0\}$ is nowhere dense since then a continuity argument leads to $\hat{\mu}(\zeta,\theta) = \hat{\nu}(\zeta,\theta)$. This fact will be proven in the next Proposition. ∎

PROPOSITION 17. *The set* $A := \{(\zeta,\theta) \in \mathbf{C} \times \mathbf{S}^{n-1} : P(\zeta\theta) = 0\}$ *is closed and has no interior point, i.e. A is nowhere dense in* $\mathbf{C} \times \mathbf{S}^{n-1}$.

PROOF. Clearly A is closed. Suppose that there $\theta_0 \in \mathbf{S}^{n-1}$ and ζ_0 such that $P(\zeta\theta) = 0$ for all ζ in a neighborhood U of ζ_0 and for all θ in a neighborhood V of θ_0. For fixed $\theta \in V$ it follows that $\zeta \to P(\zeta\theta)$ must be the zero polynomial since for all $\zeta \in U$ (hence uncountably many ζ) we have $P(\zeta\theta) = 0$. It follows that $P(\zeta\theta) = 0$ for all $\zeta \in \mathbf{C}$ and for all $\theta \in V$. Hence $P(x) = 0$ for all x in an open set W of $\mathbf{R}^n$ and, by the properties of real analytic functions, we conclude that $P \equiv 0$. ∎

COROLLARY 18. *Let* $P(x)$ *be a polynomial and* N_P *be given by* (30). *Then the space*

$$U_{N_P} := \{Q \in \mathscr{P} : \Delta^{N_P} Q = 0\}$$

is dense in the space $C(K_P(R), \mathbf{C})$ *of all continuous complex-valued functions on* $K_P(R)$ *endowed with the supremum norm.*

PROOF. Since U_{N_P} is closed under complex conjugation we may reduce the problem to the case of real-valued continuous functions. Suppose that U_{N_P} is not dense in $C(K_P(R), \mathbf{R})$. By the Hahn-Banach theorem there exists a continuous non-trivial real-valued functional L which vanishes on U_{N_P}. By Riesz's Theorem there exists a signed measures σ representing the functional L with support in K_P. By Theorem 1 (applied to σ and the zero measure) we conclude that $\sigma = 0$, a contradiction. ∎

7. Appendix: The polyharmonic degree

We want to list some of the properties of the polyharmonic degree map. Note that the inequality $d(P+Q) \leq \max\{d(P), d(Q)\}$ is trivial. In [3] the important equality

$$d(Q \cdot |x|^2) = d(Q) + d(|x|^2) = d(Q) + 1. \tag{41}$$

is proved for any polyharmonic function Q defined on a domain containing zero. The following inequality is implicitly contained in [3, Theorem 1.2, p. 31]. For completeness we give the short proof.

PROPOSITION 19. *Let f, g be harmonic. Then $d(fg) \leq \min\{\deg f, \deg g\}$ and $d(ff^*) = \deg f$.*

PROOF. Let ∇f be the gradient of f. Then $\Delta(fg) = (\Delta f)g + 2\langle \nabla f, \nabla g\rangle + f\Delta g$. If h and g are harmonic it is easy to show by induction that

$$\Delta^p(fg) = 2^p \sum_{i_1,\dots,i_p=1}^{n} \left(\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} f\right)\left(\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} g\right). \tag{42}$$

Suppose that $s := \deg f \leq \deg g$. Then $\frac{\partial^\beta}{\partial x^\beta} f = 0$ for all $\beta \in \mathbf{N}_0^n$ with $|\beta| = s+1$. It follows from (42) that $\Delta^{s+1}(fg) = 0$. Hence $d(fg) \leq s$ and the first statement is proved. Clearly this implies also that $d(ff^*) \leq \deg f$. Suppose that $\Delta^{p+1}(ff^*) = 0$ for some $p \in \mathbf{N}$. Then $\sum_{i_1,\dots,i_{p+1}=1}^{n} \left|\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{p+1}}} f\right|^2 = 0$. It follows that $\frac{\partial^\beta}{\partial x^\beta} f = 0$ for all $\beta \in \mathbf{N}_0^n$ with $|\beta| = p+1$. Hence $\deg f \leq p$, and we have proved that $\deg f \leq d(ff^*)$. ■

Now we can prove the following:

COROLLARY 20. *Let $P(x)$ be a polynomial with the Gauß decomposition*

$$P(x) = h_0(x) + |x|^2 h_1(x) + \cdots + |x|^{2N} h_N(x). \tag{43}$$

Then for N_p defined in (3) *the following inequality holds:*

$$N_P \leq \max_{r=0,\dots,N} \{r + \deg h_r\} \leq \deg P(x). \tag{44}$$

PROOF. Recall formula (30) for N_P and let Y_k be a harmonic homogeneous polynomial of degree k. An iteration argument in (41) implies that $d(|x|^{2r} h_r Y_k) = r + d(h_r Y_k)$. By Proposition 19 $d(h_r Y_k) \leq \deg h_r$. Hence $d(P \cdot Y_k) \leq \max_{r=0,\dots,N}\{r + \deg h_r\}$, and this proves the first inequality. Further we know that $\deg(|x|^{2r} h_r) = 2r + \deg h_r \leq \deg P$ for $r = 0, \dots, N$. Hence the second inequality is established. ■

In the following we want to give an explicit formula for N_P. We need the following result which is interesting in its own right:

THEOREM 21. *Let $Y_{k,m}(x)$ be an orthonormal basis of spherical harmonics with $k \in \mathbf{N}_0$ and $m = 1, \dots, a_k$. Then $d(Y_{k,m}(x) Y_{k,m_1}(x)) = k$ if and only if $m = m_1$.*

PROOF. We start with a general remark: Let Y_k and Y_l be harmonic homogeneous polynomials of degree k and l respectively. Clearly $Y_k(x)\,Y_l(x)$ is a homogeneous polynomial of degree $k+l$. By Proposition 19 it has polyharmonic degree at most $\min\{k,l\}$. By Gauß decomposition there exist harmonic homogeneous polynomials h_{k+l-2u}, either h_{k+l-2u} is zero or of exact degree $k+l-2u$ for $u=0,\dots,\min\{k,l\}$, such that

$$Y_k(x)\,Y_l(x) = \sum_{u=0}^{\min\{k,l\}} |x|^{2u} h_{k+l-2u}(x). \tag{45}$$

Now assume that $Y_k(x) = Y_{k,m}(x)$ and $Y_l(x) = Y_{k,m_1}(x)$. Let us consider the summand $|x|^{2k} h_0(x)$ for $u=k$. Then h_0 must have degree 0, hence it is a constant polynomial. Integrate equation (45) with respect to $d\theta$. Since h_{2k-2u} is either 0 or of exact degree $2k-2u>0$ for $u=0,\dots,k-1$ the integral over the sphere of $|x|^{2u} h_{k+l-2u}(x)$ will vanish. Then we obtain with the orthogonality relations for spherical harmonics

$$\delta_{m,m_1} = \int_{\mathbf{S}^{n-1}} h_0\, d\theta = h_0\omega_n.$$

Hence for $m \neq m_1$ we see that the polyharmonic degree is less than k, for $m = m_1$ it is exactly k. The proof is finished. ■

THEOREM 22. *Let $P(x)$ be a homogeneous polynomial of degree N, say of the form*

$$P(x) = \sum_{t,k\in\mathbf{N}_0,\,2t+k=N}\ \sum_{m=1}^{a_k} b_{t,k,m}|x|^{2t}\, Y_{k,m}(x).$$

Let $k_0 = k_0(P)$ be the largest natural number such that $b_{t_0,k_0,m_0} \neq 0$ for some m_0 and t_0 in the above sum. Then

$$N_P = \frac{1}{2}(N + k_0(P)).$$

PROOF. Let k_0 be as specified in the theorem. Let $k_1 \in \mathbf{N}_0$ and $m_1 \in \{1,\dots,a_{k_1}\}$, then

$$d(P(x)\,Y_{k_1,m_1}(x)) \leq \max\, d(|x|^{2t}\, Y_{k,m}\, Y_{k_1,m_1}(x)) \tag{46}$$

where the maximum ranges over all indices t, k, m with $b_{t,k,m} \neq 0$. Using (41) and the inequality $d(Y_{k,m}\,Y_{k_1,m_1}) \leq k$ in (46) we arrive at (note that $2t+k=N$)

$$d(P(x)\,Y_{k_1,m_1}(x)) \leq \max\{t+k\} = \frac{1}{2}\max\{N+k\} \leq \frac{1}{2}(N+k_0),$$

where the last inequality follows from the choice of k_0. Now (30) yields $N_P \leq \frac{1}{2}(N + k_0)$. For the other direction it suffices to show that $P(x)Y_{k_0,m_0}$ has polyharmonic degree $\geq \frac{1}{2}(N + k_0)$. Clearly it suffices to show that there exists a polynomial $R(x)$ of polyharmonic degree $< \frac{1}{2}(N + k_0)$ such that

$$P(x)Y_{k_0,m_0} = b_{t_0,k_0,m_0}|x|^{2t_0}Y_{k_0,m_0}Y_{k_0,m_0} + R(x) \tag{47}$$

since (41) and Theorem 21 imply that $b_{t_0,k_0,m_0}|x|^{2t_0}Y_{k_0,m_0}Y_{k_0,m_0}$ has polyharmonic degree

$$t_0 + d(Y_{k_0,m_0}Y_{k_0,m_0}) = t_0 + k_0 = \frac{1}{2}(N + k_0)$$

using the fact that $2t_0 + k_0 = N$. It remains to prove that $R(x)$ has polyharmonic degree less than $\frac{1}{2}(N + k_0)$. It suffices to show that for each nonzero summand $b_{t,k,m}|x|^{2t}Y_{k,m}Y_{k_0,m_0}$ in $R(x)$

$$d(b_{t,k,m}|x|^{2t}Y_{k,m}Y_{k_0,m_0}) = t + d(Y_{k,m}Y_{k_0,m_0}) < \frac{1}{2}(N + k_0). \tag{48}$$

If $k < k_0$ this is clear since $d(Y_{k,m}Y_{k_0,m_0}) \leq k$ and $t + k = \frac{1}{2}(N + k)$. If $k = k_0$ we know that $m \neq m_0$, and by Theorem 21 we have again strict inequality. By choice of k_0 we always have $k \leq k_0$, so the theorem is proved. ■

In the last theorem it is essential that the polynomial $P(x)$ is homogeneous. If $P(x)$ is arbitrary, we can write $P(x) = \sum_{j=0}^{N} P_j(x)$ where $P_j(x)$ are homogeneous polynomials. It is not very difficult to see that

$$d(P \cdot Y_{k,m}) = \max_{j=0,\ldots,N} d(P_j \cdot Y_{k,m}),$$

see e.g. the proof of Theorem 1.27 in [4]. Hence N_P is the maximum of N_{P_j} for $j = 0, \ldots, N$.

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