HIROSHIMA MATH. J.
36 (2006), 365–386

On the construction and investigation of hierarchic models for elastic rods

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(Received Oct. 1, 2003)
(Revised Dec. 12, 2005)

ABSTRACT. In the present paper static and dynamical one-dimensional models for elastic rods are constructed. The existence and uniqueness of solutions to the corresponding boundary and initial boundary value problems are proved, the rate of approximation of the solutions to the original three-dimensional problems by vector-functions restored from the solutions of one-dimensional problems is estimated.

1. Introduction

Hierarchic modelling is widely used while constructing the lower-dimensional models in the theory of elasticity and mathematical physics ([1–3]). In the paper [4] I. Vekua proposed a new method of constructing the hierarchic two-dimensional models of elastic prismatic shells. In the static case the lower-dimensional model obtained in [4] first was investigated in the papers [5, 6]. More precisely, in [5] the estimate of accuracy was obtained in C^k spaces and existence and uniqueness of solution to the reduced two-dimensional boundary value problem in Sobolev spaces were studied in [6]. Further, static and dynamical two-dimensional hierarchical models for prismatic shells constructed by I. Vekua's reduction method were investigated using variational approach and modelling error estimates in Sobolev spaces were obtained in the paper [7]. Various lower-dimensional hierarchical models in mathematical physics were constructed and investigated in [8–18].

Generalizing an idea of I. Vekua, one-dimensional models for linearly elastic rods were obtained in [19, 20]. In the paper [19], expanding fields of displaycements, strains and stresses of the three-dimensional elastic body into double Fourier-Legendre series, one-dimensional mathematical models of bars were constructed. Note that in [19] main relations were obtained in the spaces of classical regular functions. Different approach were used in [20], where a hierarchy of static one-dimensional models was obtained in Sobolev spaces

2000 *Mathematics Subject Classification.* 42C10, 65M60, 74K10.
Key words and phrases. boundary and initial boundary value problems for elastic rod, Fourier-Legendre series, modelling error estimation.

directly from the variational formulation of the three-dimensional problem and the results of investigation of the constructed hierarchy were announced.

In the present paper we extend the methodology developed in [7] for elastic rods with variable cross-sections. We construct the one-dimensional models of the static and dynamical problems for elastic rod, prove existence and uniqueness of solutions to the corresponding boundary and initial boundary value problems. Moreover, we establish convergence in suitable spaces of the sequences of approximate solutions to the exact solutions of the original three-dimensional problems.

In order to simplify notations throughout the paper we assume that the indices i, j, p, q take their values in the set $\{1,2,3\}$, while the indices α, β vary in the set $\{1,2\}$ and the repeated index convention is used in conjunction with these rules. The partial derivative with respect to the p-th argument $\partial/\partial x_p$ we denote by ∂_p. For any Lipschitz domain $D \subset \mathbf{R}^s$, $L^2(D)$ denotes the space of real-valued square-integrable functions in D in the Lebesgue sense, $H^m(D) = W^{m,2}(D)$ denotes the Sobolev space of order m, $H_0^m(D)$ is the closure of the set of infinitely differentiable functions $C_0^\infty(D)$ with compact support in D in the space $H^m(D)$, and the spaces of vector-functions we denote by $\mathbf{H}^m(D) = [H^m(D)]^3$, $\mathbf{H}_0^m(D) = [H_0^m(D)]^3$, $\mathbf{L}^2(D) = [L^2(D)]^3$, $s, m \in \mathbf{N}$.

Let us consider an elastic rod with initial configuration $\bar{\Omega} \subset \mathbf{R}^3$,

$$\Omega = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3; h_\alpha^-(x_3) < x_\alpha < h_\alpha^+(x_3), x_3 \in I = (d_1, d_2)\},$$

$$h_1^+(x_3) > h_1^-(x_3), \quad h_2^+(x_3) > h_2^-(x_3), \qquad \forall x_3 \in [d_1, d_2],\ h_1^\pm, h_2^\pm \in C^1([d_1, d_2]),$$

where $d_1 < d_2$, Ω is a Lipschitz domain ([21]) and $\bar{\Omega}$ denotes the closure of the set $\Omega \subset \mathbf{R}^3$. The upper surface of the rod $\{x \in \bar{\Omega}; x_3 = d_2\}$ we denote by Γ_2 and the rest part of the boundary $\partial\Omega \backslash \Gamma_2$ is denoted by $\tilde{\Gamma}$.

We suppose that the material constituting the rod is linearly elastic, homogeneous and isotropic with Lamé constants λ, μ. The rod is clamped along the upper surface Γ_2. The density of the applied body forces acting on the rod we denote by $\boldsymbol{f} = (f_i) : \Omega \times (0, T) \to \mathbf{R}^3$ and applied surface force density is denoted by $\boldsymbol{g} = (g_i) : \tilde{\Gamma} \times [0, T] \to \mathbf{R}^3$. The linear three-dimensional model of the rod has the following form:

$$(1.1) \qquad \frac{\partial^2 u_i}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \{\lambda e_{pp}(\boldsymbol{u})\delta_{ij} + 2\mu e_{ij}(\boldsymbol{u})\} = f_i(x, t), \qquad (x, t) \in \Omega_T,$$

$$(1.2) \qquad \boldsymbol{u}(x, 0) = \boldsymbol{\varphi}(x), \quad \frac{\partial \boldsymbol{u}}{\partial t}(x, 0) = \boldsymbol{\psi}(x), \qquad x \in \Omega,$$

$$\boldsymbol{u} = \mathbf{0}, \qquad \text{on } \Gamma_2 \times [0, T],$$

$$(1.3) \qquad \sum_{j=1}^{3}(\lambda e_{pp}(\boldsymbol{u})\delta_{ij} + 2\mu e_{ij}(\boldsymbol{u}))\nu_j = g_i, \qquad \text{on } \tilde{\Gamma} \times [0, T],$$

where $\Omega_T = \Omega \times (0, T)$, $\boldsymbol{u} = (u_i) : \bar{\Omega}_T \to \mathbf{R}^3$ is the unknown displacement vector function, $\boldsymbol{\varphi}, \boldsymbol{\psi} : \Omega \to \mathbf{R}^3$ are the initial displacement and velocity vector fields of the rod, $\boldsymbol{\nu} = (\nu_j)$ denotes the outward unit normal to the boundary $\tilde{\Gamma}$, δ_{ij} is the Kronecker delta and $\boldsymbol{e}(\boldsymbol{u}) = \{e_{ij}(\boldsymbol{u})\}$ is the deformation tensor

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2, 3.$$

In Section 2 we consider the static case of problem (1.1)–(1.3), construct one-dimensional model of the rod and investigate convergence of the sequence of vector functions restored from the solutions of the corresponding boundary value problems to the solution of the original three-dimensional problem. Section 3 is devoted to study of dynamical problem (1.1)–(1.3), where we construct and investigate a hierarchy of dynamical one-dimensional models for the elastic rod.

2. Static boundary value problem

As we referred in the introduction in this section, we study the static case of problem (1.1)–(1.3), which admits the following variational formulation: find a vector function $\boldsymbol{u} \in V(\Omega) = \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega); \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_2\}$, such that

$$(2.1) \qquad B^{\Omega}(\boldsymbol{u}, \boldsymbol{v}) = L^{\Omega}(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in V(\Omega),$$

where

$$B^{\Omega}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega}(\lambda e_{pp}(\boldsymbol{u})e_{qq}(\boldsymbol{v}) + 2\mu e_{ij}(\boldsymbol{u})e_{ij}(\boldsymbol{v}))dx,$$

$$L^{\Omega}(\boldsymbol{v}) = \int_{\Omega} f_i v_i \, dx + \int_{\tilde{\Gamma}} g_i v_i \, d\tilde{\Gamma}.$$

The variational method of investigation of static problem (2.1) in the theory of linear elasticity is based on Korn's inequality first proved in [22]. Later on, many interesting papers were devoted to proof of Korn's type inequalities in various domains ([23–26]). Note that Korn's inequlaity directly follows from lemma of J.-L. Lions, which was proved for Lipschitz domains in [27, 28]. According to Korn's inequality, there exists a positive constant $c = const > 0$ such that

$$\sum_{i=1}^{3}\int_{\Omega} v_i v_i \, dx + \sum_{i,j=1}^{3}\int_{\Omega} e_{ij}(\boldsymbol{v})e_{ij}(\boldsymbol{v})dx \geq c\|\boldsymbol{v}\|^2_{\mathbf{H}^1(\Omega)}, \qquad \forall \boldsymbol{v} \in \mathbf{H}^1(\Omega).$$

Applying this inequality, it can be proved that for Lamé constants λ, μ satisfying conditions $\mu > 0$, $2\mu + 3\lambda > 0$, the bilinear form $B^{\Omega}(.,.)$ is coercive in $V(\Omega)$, i.e. $B^{\Omega}(\boldsymbol{v}, \boldsymbol{v}) \geq c_{\Omega}\|\boldsymbol{v}\|^2_{\mathbf{H}^1(\Omega)}$, $c_{\Omega} = const > 0$, for all $\boldsymbol{v} \in V(\Omega)$. Consequently, from Lax-Milgram theorem ([29]) it follows that three-dimensional problem (2.1) has a unique solution if $\mu > 0$, $2\mu + 3\lambda > 0$, $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{g} \in \mathbf{L}^2(\tilde{\Gamma})$, which is also a unique solution of the following minimization problem: find $\boldsymbol{u} \in V(\Omega)$ such that

$$J^{\Omega}(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in V(\Omega)} J^{\Omega}(\boldsymbol{v}), \quad J^{\Omega}(\boldsymbol{v}) = \frac{1}{2} B^{\Omega}(\boldsymbol{v}, \boldsymbol{v}) - L^{\Omega}(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in V(\Omega).$$

In order to reduce three-dimensional problem (2.1) to one-dimensional problem, let us consider equation (2.1) on the subspace of $V(\Omega)$, which consists of polynomials of degree N_1, N_2 with respect to the variables x_1 and x_2, i.e.

$$\boldsymbol{v}_{N_1N_2} = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} a_1 a_2 \left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right) \overset{k_1k_2}{\boldsymbol{v}} P_{k_1}(a_1x_1 - b_1) P_{k_2}(a_2x_2 - b_2),$$

where $\overset{k_1k_2}{\boldsymbol{v}} = (\overset{k_1k_2}{v_i}) \in \mathbf{H}^1(I)$, $k_1 = \overline{0, N_1}$, $k_2 = \overline{0, N_2}$, $a_\alpha = \dfrac{2}{h_\alpha^+ - h_\alpha^-}$, $b_\alpha = \dfrac{h_\alpha^+ + h_\alpha^-}{h_\alpha^+ - h_\alpha^-}$, $\alpha = 1, 2$, and P_k is the Legendre polynomial of order $k \in \mathbf{N} \cup \{0\}$ ([30]). Hence we obtain the following problem

$$B^{\Omega}(\boldsymbol{w}_{N_1N_2}, \boldsymbol{v}_{N_1N_2}) = L^{\Omega}(\boldsymbol{v}_{N_1N_2}), \qquad \forall \boldsymbol{v}_{N_1N_2} \in V_{N_1N_2}(\Omega), \tag{2.2}$$

$$V_{N_1N_2}(\Omega) = \Bigg\{ \boldsymbol{v}_{N_1N_2} = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} a_1 a_2 \left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right) \overset{k_1k_2}{\boldsymbol{v}} P_{k_1}(\omega_1) P_{k_2}(\omega_2);$$
$$\overset{k_1k_2}{\boldsymbol{v}} \in \mathbf{H}^1(I),\ \overset{k_1k_2}{\boldsymbol{v}} = \mathbf{0} \text{ for } x_3 = d_2, k_1 = \overline{0, N_1}, k_2 = \overline{0, N_2} \Bigg\},$$

$\omega_\alpha = a_\alpha x_\alpha - b_\alpha$, $\alpha = 1, 2$. In problem (2.2) the unknown is the vector function $\boldsymbol{w}_{N_1N_2} \in V_{N_1N_2}(\Omega)$,

$$\boldsymbol{w}_{N_1N_2} = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} a_1 a_2 \left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right) \overset{k_1k_2}{\boldsymbol{w}} P_{k_1}(\omega_1) P_{k_2}(\omega_2).$$

Therefore we have to find the vector function

$$\vec{w}_{N_1N_2} = (\overset{00}{\boldsymbol{w}}, \dots, \overset{N_1N_2}{\boldsymbol{w}}) \in \vec{V}_{N_1N_2}(I) = \{\vec{v}_{N_1N_2} = (\overset{00}{\boldsymbol{v}}, \dots, \overset{N_1N_2}{\boldsymbol{v}});\ \overset{k_1k_2}{\boldsymbol{v}} \in \mathbf{H}^1(I),\ \overset{k_1k_2}{\boldsymbol{v}} = \mathbf{0}$$
$$\text{for } x_3 = d_2, k_1 = \overline{0, N_1}, k_2 = \overline{0, N_2}\},$$

which satisfies the following equation

$$B^{\Omega}_{N_1N_2}(\vec{w}_{N_1N_2}, \vec{v}_{N_1N_2}) = L^{\Omega}_{N_1N_2}(\vec{v}_{N_1N_2}), \qquad \forall \vec{v}_{N_1N_2} \in \vec{V}_{N_1N_2}(I), \tag{2.3}$$

where $B^{\Omega}_{N_1N_2}$, $L^{\Omega}_{N_1N_2}$ are the forms $B^{\Omega}(\boldsymbol{w}_{N_1N_2}, \boldsymbol{v}_{N_1N_2})$ and $L^{\Omega}(\boldsymbol{v}_{N_1N_2})$, which are written in terms of the components $\overset{k_1k_2}{\boldsymbol{w}}$ and $\overset{k_1k_2}{\boldsymbol{v}}$ of $\vec{w}_{N_1N_2}$ and $\vec{v}_{N_1N_2}$.

Thus, three-dimensional problem (2.1) have been reduced to one-dimensional problem, for which the following theorem is true.

Theorem 2.1. *Assume that Lamé constants satisfy conditions* $\mu > 0$, $2\mu + 3\lambda > 0$ *and* $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{g} \in \mathbf{L}^2(\tilde{\Gamma})$, *then reduced one-dimensional problem* (2.3) *has a unique solution, which is also a unique solution of the following minimization problem*

$$\vec{w}_{N_1N_2} \in \vec{V}_{N_1N_2}(I), \qquad J_{N_1N_2}(\vec{w}_{N_1N_2}) = \inf_{\vec{v}_{N_1N_2} \in \vec{V}_{N_1N_2}(I)} J_{N_1N_2}(\vec{v}_{N_1N_2}),$$

$$J_{N_1N_2}(\vec{v}_{N_1N_2}) = \frac{1}{2} B^{\Omega}_{N_1N_2}(\vec{v}_{N_1N_2}, \vec{v}_{N_1N_2}) - L^{\Omega}_{N_1N_2}(\vec{v}_{N_1N_2}), \qquad \forall \vec{v}_{N_1N_2} \in \vec{V}_{N_1N_2}(I).$$

Proof. In order to prove the theorem first let us show that $V_{N_1N_2}(\Omega)$ is a closed subset of $V(\Omega)$. Let $\{\boldsymbol{v}^{(l)}_{N_1N_2}\}_{l=1}^{\infty}$ be a Cauchy sequence in the space $V_{N_1N_2}(\Omega)$, i.e.

$$\|\boldsymbol{v}^{(l)}_{N_1N_2} - \boldsymbol{v}^{(m)}_{N_1N_2}\|_{\mathbf{H}^1(\Omega)} \to 0, \qquad \text{as } l, m \to \infty. \tag{2.4}$$

Consequently, $\{\boldsymbol{v}^{(l)}_{N_1N_2}\}_{l=1}^{\infty}$ is a Cauchy sequence in the space $\mathbf{L}^2(\Omega)$ and the orthogonality of the Legendre polynomials imply that $\{\overset{k_1k_2}{\boldsymbol{v}}{}^{(l)}\}_{l=1}^{\infty}$, $0 \le k_1 \le N_1$, $0 \le k_2 \le N_2$, are Cauchy sequences in the space $\mathbf{L}^2(I)$. Moreover,

$$\begin{aligned}
&\frac{1}{2}\left\| \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2} a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)(\overset{k_1k_2}{\boldsymbol{v}})' P_{k_1}(\omega_1)P_{k_2}(\omega_2)\right\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le \left\| \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2} \partial_3\left(a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)P_{k_1}(\omega_1)P_{k_2}(\omega_2)\right)\overset{k_1k_2}{\boldsymbol{v}}\right\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + \left\|\frac{\partial \boldsymbol{v}_{N_1N_2}}{\partial x_3}\right\|^2_{\mathbf{L}^2(\Omega)}, \qquad \forall \boldsymbol{v}_{N_1N_2} \in V_{N_1N_2}(\Omega),
\end{aligned}$$

where the prime denotes differentiation with respect to the argument. Applying the last inequality, we obtain that $\{(\overset{k_1k_2}{\boldsymbol{v}}{}^{(l)})'\}_{l=1}^{\infty}$ are Cauchy sequences in the space $\mathbf{L}^2(I)$ $(0 \le k_1 \le N_1, 0 \le k_2 \le N_2)$.

Therefore $\{\overset{k_1k_2(l)}{\boldsymbol{v}}\}_{l=1}^{\infty}$ are Cauchy sequences in the space $\mathbf{H}^1(I)$ and

$$\overset{k_1k_2(l)}{\boldsymbol{v}} \to \overset{k_1k_2}{\boldsymbol{z}} \qquad \text{in } \mathbf{H}^1(I), \text{ as } l \to \infty,$$

$$\overset{k_1k_2(l)}{\boldsymbol{v}} = \mathbf{0}, \qquad \text{for } x_3 = d_2,\ 0 \le k_1 \le N_1,\ 0 \le k_2 \le N_2,$$

from which it immediately follows that

$$\boldsymbol{v}_{N_1N_2}^{(l)} \to \boldsymbol{z}_{N_1N_2} \qquad \text{in } V(\Omega), \text{ as } l \to \infty,$$

where $\boldsymbol{z}_{N_1N_2} \in V_{N_1N_2}(\Omega)$ is defined by

$$\boldsymbol{z}_{N_1N_2} = \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2} a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\overset{k_1k_2}{\boldsymbol{z}} P_{k_1}(\omega_1)P_{k_2}(\omega_2).$$

So, the space $V_{N_1N_2}(\Omega)$ is closed and taking into account that $V(\Omega)$ is complete, we obtain that $V_{N_1N_2}(\Omega)$ and $\vec{V}_{N_1N_2}(I)$ are Hilbert spaces.

Since B^{Ω} is coercive in $V(\Omega)$, we have that it is coercive in the subspace $V_{N_1N_2}(\Omega) \subset V(\Omega)$. Hence the bilinear form $B^{\Omega}_{N_1N_2}(.,.)$ is coercive in $\vec{V}_{N_1N_2}(I)$ and applying Lax-Milgram theorem we obtain that problem (2.3) has a unique solution, which is a unique solution of energy functional $J_{N_1N_2}$ minimization problem. □

So, we have investigated the well-posedness of the obtained one-dimensional problems. Now, let us prove the following approximation theorem.

THEOREM 2.2. *If conditions of Theorem* 2.1 *hold, then the vector function* $\boldsymbol{w}_{N_1N_2} = \sum\limits_{k_1=0}^{N_1}\sum\limits_{k_2=0}^{N_2} a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\overset{k_1k_2}{\boldsymbol{w}} P_{k_1}(\omega_1)P_{k_2}(\omega_2)$ *corresponding to the solution* $\vec{w}_{N_1N_2} = (\overset{00}{\boldsymbol{w}},\ldots,\overset{N_1N_2}{\boldsymbol{w}})$ *of reduced problem* (2.3) *tends to the solution* $\boldsymbol{u}$ *of three-dimensional problem* (2.1) $\boldsymbol{w}_{N_1N_2} \to \boldsymbol{u}$ *in the space* $\mathbf{H}^1(\Omega)$, *as* $\min\{N_1,N_2\} \to \infty$. *Moreover, if* $\boldsymbol{u} \in \mathbf{H}^{s,s,1}(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega); \partial_\alpha^k \boldsymbol{v} \in \mathbf{H}^1(\Omega), 0 \le k \le s-1, \alpha = 1,2\}$, $s \ge 2$, *then the following estimate is valid*

$$\|\boldsymbol{u} - \boldsymbol{w}_{N_1N_2}\|^2_{\mathbf{H}^1(\Omega)} \le \left(\frac{1}{N_1^{2s-3}} + \frac{1}{N_2^{2s-3}}\right)\delta_1(h_1^{\pm}, h_2^{\pm}, N_1, N_2),$$

where $\delta_1(h_1^{\pm}, h_2^{\pm}, N_1, N_2) \to 0$, *as* $\min\{N_1,N_2\} \to \infty$. *In addition, if* $\|\boldsymbol{u}\|^2_{\mathbf{H}^{s,s,1}(\Omega)} = \sum\limits_{k=0}^{s-1}\sum\limits_{\alpha=1}^{2} \|\partial_\alpha^k \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} \le c$, *where* c *is independent of* $h_1 = \max\limits_{x_3 \in \bar{I}}(h_1^+(x_3) - h_1^-(x_3))$, $h_2 = \max\limits_{x_3 \in \bar{I}}(h_2^+(x_3) - h_2^-(x_3))$, *then the following estimate holds*

$$\|\boldsymbol{u} - \boldsymbol{w}_{N_1N_2}\|^2_{E(\Omega)} \le \left(\frac{h_1^{2(s-1)}}{N_1^{2s-3}} + \frac{h_2^{2(s-1)}}{N_2^{2s-3}}\right)\delta_2(N_1, N_2),$$

where $\delta_2(N_1, N_2) \to 0$, *as* $\min\{N_1, N_2\} \to \infty$, $\|\boldsymbol{v}\|^2_{E(\Omega)} = B^\Omega(\boldsymbol{v}, \boldsymbol{v})$, $\boldsymbol{v} \in V(\Omega)$.

Proof. From Theorem 2.1 we have that $\vec{w}_{N_1N_2}$ is a solution of the minimization problem of energy functional $J_{N_1N_2}$, i.e.,

$$J_{N_1N_2}(\vec{w}_{N_1N_2}) \le J_{N_1N_2}(\vec{v}_{N_1N_2}), \qquad \forall \vec{v}_{N_1N_2} \in \vec{V}_{N_1N_2}(I). \tag{2.5}$$

Since

$$\begin{aligned} B^\Omega_{N_1N_2}(\vec{v}_{N_1N_2}, \vec{v}_{N_1N_2}) &= B^\Omega(\boldsymbol{v}_{N_1N_2}, \boldsymbol{v}_{N_1N_2}), \\ L^\Omega_{N_1N_2}(\vec{v}_{N_1N_2}) &= L^\Omega(\boldsymbol{v}_{N_1N_2}), \end{aligned} \qquad \forall \vec{v}_{N_1N_2} \in \vec{V}_{N_1N_2}(I),$$

where $\boldsymbol{v}_{N_1N_2} = \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2} a_1a_2\left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right)\overset{k_1k_2}{\boldsymbol{v}} P_{k_1}(\omega_1)P_{k_2}(\omega_2)$, then applying (2.5), we have

$$B^\Omega(\boldsymbol{u} - \boldsymbol{w}_{N_1N_2}, \boldsymbol{u} - \boldsymbol{w}_{N_1N_2}) \le B^\Omega(\boldsymbol{u}, \boldsymbol{u}) - 2L^\Omega(\boldsymbol{v}_{N_1N_2}) + B^\Omega(\boldsymbol{v}_{N_1N_2}, \boldsymbol{v}_{N_1N_2}).$$

From the last inequality we obtain, that for all $\boldsymbol{v}_{N_1N_2} \in V_{N_1N_2}(\Omega)$,

$$B^\Omega(\boldsymbol{u} - \boldsymbol{w}_{N_1N_2}, \boldsymbol{u} - \boldsymbol{w}_{N_1N_2}) \le B^\Omega(\boldsymbol{u} - \boldsymbol{v}_{N_1N_2}, \boldsymbol{u} - \boldsymbol{v}_{N_1N_2}). \tag{2.6}$$

By the trace theorems for Sobolev spaces ([21]), for any $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{v} = \mathbf{0}$ on Γ_2, there exists continuation $\tilde{\boldsymbol{v}} \in \mathbf{H}^1_0(\Omega_1)$ of the vector function $\boldsymbol{v}$, where $\Omega_1 \supset \Omega$, $\partial\Omega_1 \supset \Gamma_2$. From the density of $C_0^\infty(\Omega_1)$ in $\mathbf{H}^1_0(\Omega_1)$, we obtain that the set of infinitely differentiable functions in Ω, which are equal to zero on Γ_2, is dense in $V(\Omega)$. The relations, which we obtain below to prove the estimates of the theorem, imply that $\bigcup_{N_1, N_2 \ge 0} V_{N_1N_2}(\Omega)$ is dense in $V(\Omega)$ and thus $\boldsymbol{w}_{N_1N_2} \to \boldsymbol{u}$ in the space $\mathbf{H}^1(\Omega)$, as $\min\{N_1, N_2\} \to \infty$.

Now let us estimate the rate of approximation of $\boldsymbol{u}$ by $\boldsymbol{w}_{N_1N_2}$, if $\partial_\alpha^k \boldsymbol{u} \in \mathbf{H}^1(\Omega)$, $0 \le k \le s-1$, $s \ge 2$. Denote by

$$\boldsymbol{\varepsilon}_{N_1N_2} = \boldsymbol{u} - \boldsymbol{u}_{N_1N_2} = \boldsymbol{u} - \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2} a_1a_2\left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right)\overset{k_1k_2}{\boldsymbol{u}} P_{k_1}(\omega_1)P_{k_2}(\omega_2),$$

where $\overset{k_1k_2}{\boldsymbol{u}} = \int_{h_2^-}^{h_2^+}\int_{h_1^-}^{h_1^+} \boldsymbol{u}P_{k_1}(\omega_1)P_{k_2}(\omega_2)dx_1dx_2$, $0 \le k_1 \le N_1$, $0 \le k_2 \le N_2$.

Applying the following recurrence relations for the Legendre polynomials ([30])

$$P_r(t) = \frac{1}{2r+1}(P'_{r+1}(t) - P'_{r-1}(t)), \qquad r \geq 1,$$
$$tP'_r(t) = P'_{r+1}(t) - (r+1)P_r(t), \qquad r \geq 0, \tag{2.7}$$

we infer, that for almost all $x_3 \in I$,

$$\overset{k_1k_2}{\boldsymbol{u}} = \frac{\tilde{h}_1}{2k_1+1}(\overset{k_1-1,k_2}{\partial_1 \boldsymbol{u}} - \overset{k_1+1,k_2}{\partial_1 \boldsymbol{u}}) = \frac{\tilde{h}_2}{2k_2+1}(\overset{k_1,k_2-1}{\partial_2 \boldsymbol{u}} - \overset{k_1,k_2+1}{\partial_2 \boldsymbol{u}}). \tag{2.8}$$

$$\partial_3\left(\frac{1}{\tilde{h}_1\tilde{h}_2}\overset{k_1k_2}{\boldsymbol{u}}\right) = \frac{1}{\tilde{h}_1\tilde{h}_2}\left(\overset{k_1k_2}{\partial_3 \boldsymbol{u}} + (\bar{h}_1)'\overset{k_1k_2}{\partial_1 \boldsymbol{u}} + (\tilde{h}_1)'\left(\frac{k_1}{\tilde{h}_1}\overset{k_1k_2}{\boldsymbol{u}} + \overset{k_1+1,k_2}{\partial_1 \boldsymbol{u}}\right)\right.$$
$$\left. + (\bar{h}_2)'\overset{k_1k_2}{\partial_2 \boldsymbol{u}} + (\tilde{h}_2)'\left(\frac{k_2}{\tilde{h}_2}\overset{k_1k_2}{\boldsymbol{u}} + \overset{k_1,k_2+1}{\partial_2 \boldsymbol{u}}\right)\right), \tag{2.9}$$

where $\tilde{h}_\alpha = \frac{1}{2}(h^+_\alpha - h^-_\alpha)$, $\bar{h}_\alpha = \frac{1}{2}(h^+_\alpha + h^-_\alpha)$, $\alpha = 1, 2$. Applying the formulas (2.7)–(2.9), we obtain

$$\frac{\partial \boldsymbol{u}_{N_1N_2}}{\partial x_1} = \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2}\frac{1}{\tilde{h}_1\tilde{h}_2}\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\overset{k_1k_2}{\partial_1 \boldsymbol{u}}P_{k_1}(\omega_1)P_{k_2}(\omega_2)$$
$$- \sum_{k_1=N_1}^{N_1+1}\sum_{k_2=0}^{N_2}\frac{1}{2\tilde{h}_1\tilde{h}_2}\left(k_2+\frac{1}{2}\right)\overset{k_1k_2}{\partial_1 \boldsymbol{u}}P'_{k_1-1}(\omega_1)P_{k_2}(\omega_2),$$

$$\frac{\partial \boldsymbol{u}_{N_1N_2}}{\partial x_2} = \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2-1}\frac{1}{\tilde{h}_1\tilde{h}_2}\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\overset{k_1k_2}{\partial_2 \boldsymbol{u}}P_{k_1}(\omega_1)P_{k_2}(\omega_2)$$
$$- \sum_{k_1=0}^{N_1}\sum_{k_2=N_2}^{N_2+1}\frac{1}{2\tilde{h}_1\tilde{h}_2}\left(k_1+\frac{1}{2}\right)\overset{k_1k_2}{\partial_2 \boldsymbol{u}}P'_{k_2-1}(\omega_2)P_{k_1}(\omega_1),$$

$$\frac{\partial \boldsymbol{u}_{N_1N_2}}{\partial x_3} = \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2}\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\partial_3\left(\frac{1}{\tilde{h}_1\tilde{h}_2}\overset{k_1k_2}{\boldsymbol{u}}\right)P_{k_1}(\omega_1)P_{k_2}(\omega_2)$$
$$- \sum_{k_1=1}^{N_1}\sum_{k_2=0}^{N_2}\frac{1}{\tilde{h}_1\tilde{h}_2}\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\frac{\overset{k_1k_2}{\boldsymbol{u}}}{\tilde{h}_1}((\bar{h}_1)'P'_{k_1}(\omega_1)$$
$$+ (\tilde{h}_1)'(k_1P_{k_1}(\omega_1) + P'_{k_1-1}(\omega_1)))P_{k_2}(\omega_2)$$
$$- \sum_{k_1=0}^{N_1}\sum_{k_2=1}^{N_2}\frac{1}{\tilde{h}_1\tilde{h}_2}\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\frac{\overset{k_1k_2}{\boldsymbol{u}}}{\tilde{h}_2}((\bar{h}_2)'P'_{k_2}(\omega_2)$$
$$+ (\tilde{h}_2)'(k_2P_{k_2}(\omega_2) + P'_{k_2-1}(\omega_2)))P_{k_1}(\omega_1)$$

$$
\begin{aligned}
&= \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2} \frac{1}{\tilde{h}_1\tilde{h}_2}\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\overset{k_1k_2}{\partial_3\boldsymbol{u}}P_{k_1}(\omega_1)P_{k_2}(\omega_2) \\
&\quad + \sum_{k_1=0}^{N_1} \frac{(\bar{h}_2)'}{\tilde{h}_1\tilde{h}_2}\left(k_1+\frac{1}{2}\right)\left(N_2+\frac{1}{2}\right)\overset{k_1N_2}{\partial_2\boldsymbol{u}}P_{k_1}(\omega_1)P_{N_2}(\omega_2) \\
&\quad + \sum_{k_2=0}^{N_2} \frac{(\bar{h}_1)'}{\tilde{h}_1\tilde{h}_2}\left(k_2+\frac{1}{2}\right)\left(N_1+\frac{1}{2}\right)\overset{N_1k_2}{\partial_1\boldsymbol{u}}P_{N_1}(\omega_1)P_{k_2}(\omega_2) \\
&\quad + \sum_{k_1=N_1}^{N_1+1}\sum_{k_2=0}^{N_2} \frac{1}{2\tilde{h}_1\tilde{h}_2}\left(k_2+\frac{1}{2}\right)\overset{k_1k_2}{\partial_1\boldsymbol{u}}((\bar{h}_1)'P'_{k_1-1}(\omega_1)+(\tilde{h}_1)'P'_{k_1}(\omega_1))P_{k_2}(\omega_2) \\
&\quad + \sum_{k_2=N_2}^{N_2+1}\sum_{k_1=0}^{N_1} \frac{1}{2\tilde{h}_1\tilde{h}_2}\left(k_1+\frac{1}{2}\right)\overset{k_1k_2}{\partial_2\boldsymbol{u}}((\bar{h}_2)'P'_{k_2-1}(\omega_2)+(\tilde{h}_2)'P'_{k_2}(\omega_2))P_{k_1}(\omega_1).
\end{aligned}
$$

Hence, taking into account the expressions for derivatives of the Legendre polynomials

$$
P'_r(t) = \sum_{k=0}^{r-1}\left(k+\frac{1}{2}\right)(1-(-1)^{r+k})P_k(t), \qquad r \geq 1,
$$

and $\int_{-1}^{1}|P'_r(t)|^2dt = \sum_{k=0}^{r-1}\left(k+\frac{1}{2}\right)(1-(-1)^{k+r})^2 = r(r+1),\ r \in \mathbf{N}$, we have

$$
\begin{aligned}
\|\boldsymbol{\varepsilon}_{N_1N_2}\|^2_{\mathbf{L}^2(\Omega)} &= \sum_{(k_1,k_2)\in K_{N_1+1,N_2+1}} \int_I a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\|\overset{k_1k_2}{\boldsymbol{u}}\|^2_{\mathbf{R}^3}dx_3, \\
\left\|\frac{\partial\boldsymbol{\varepsilon}_{N_1N_2}}{\partial x_1}\right\|^2_{\mathbf{L}^2(\Omega)} &= \sum_{(k_1,k_2)\in K_{N_1,N_2+1}} \int_I a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\|\overset{k_1k_2}{\partial_1\boldsymbol{u}}\|^2_{\mathbf{R}^3}dx_3 \\
&\quad + \sum_{k_1=N_1}^{N_1+1}\sum_{k_2=0}^{N_2}\int_I a_1a_2\left(k_2+\frac{1}{2}\right)\frac{k_1}{4}(k_1-1)\|\overset{k_1k_2}{\partial_1\boldsymbol{u}}\|^2_{\mathbf{R}^3}dx_3, \\
\left\|\frac{\partial\boldsymbol{\varepsilon}_{N_1N_2}}{\partial x_2}\right\|^2_{\mathbf{L}^2(\Omega)} &= \sum_{(k_1,k_2)\in K_{N_1+1,N_2}} \int_I a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\|\overset{k_1k_2}{\partial_2\boldsymbol{u}}\|^2_{\mathbf{R}^3}dx_3 \\
&\quad + \sum_{k_2=N_2}^{N_2+1}\sum_{k_1=0}^{N_1}\int_I a_1a_2\left(k_1+\frac{1}{2}\right)\frac{k_2}{4}(k_2-1)\|\overset{k_1k_2}{\partial_2\boldsymbol{u}}\|^2_{\mathbf{R}^3}dx_3,
\end{aligned}
$$

$$\left\|\frac{\partial \boldsymbol{\varepsilon}_{N_1N_2}}{\partial x_3}\right\|^2_{\mathbf{L}^2(\Omega)} \le 5\Bigg(\sum_{(k_1,k_2)\in K_{N_1+1,N_2+1}} \int_I a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)\|\overset{k_1k_2}{\partial_3\boldsymbol{u}}\|^2_{\mathbf{R}^3}dx_3$$

$$+\sum_{k_1=N_1}^{N_1+1}\sum_{k_2=0}^{N_2}\int_I a_1a_2\left(k_2+\frac{1}{2}\right)\frac{N_1+1}{4}((2k_1-N_1)(\tilde{h}_1')^2$$

$$+(3N_1-2k_1+2)(\bar{h}_1')^2)\|\overset{k_1k_2}{\partial_1\boldsymbol{u}}\|^2_{\mathbf{R}^3}dx_3$$

$$+\sum_{k_2=N_2}^{N_2+1}\sum_{k_1=0}^{N_1}\int_I a_1a_2\left(k_1+\frac{1}{2}\right)\frac{N_2+1}{4}((2k_2-N_2)(\tilde{h}_2')^2$$

$$+(3N_2-2k_2+2)(\bar{h}_2')^2)\|\overset{k_1k_2}{\partial_2\boldsymbol{u}}\|^2_{\mathbf{R}^3}dx_3\Bigg),$$

where $K_{N_1,N_2}=\{(k_1,k_2)\in\mathbf{N}\times\mathbf{N};k_1\ge N_1 \text{ or } k_2\ge N_2\}$, $\|.\|_{\mathbf{R}^3}$ denotes the norm in Euclidean space $\mathbf{R}^3$.

Therefore applying (2.8), we infer that

$$\|\boldsymbol{\varepsilon}_{N_1N_2}\|^2_{\mathbf{L}^2(\Omega)} \le \left(\frac{1}{N_1^{2s}}+\frac{1}{N_2^{2s}}\right)\delta(h_1^{\pm},h_2^{\pm},N_1,N_2),$$

$$\left\|\frac{\partial \boldsymbol{\varepsilon}_{N_1N_2}}{\partial x_i}\right\|^2_{\mathbf{L}^2(\Omega)} \le \left(\frac{1}{N_1^{2s-3}}+\frac{1}{N_2^{2s-3}}\right)\delta(h_1^{\pm},h_2^{\pm},N_1,N_2),$$

where $i=1,2,3,$ $\delta(h_1^{\pm},h_2^{\pm},N_1,N_2)\to 0,$ as $\min\{N_1,N_2\}\to\infty$.

From (2.6) and coerciveness of the bilinear form $B^{\Omega}(.,.)$ we obtain

$$\|\boldsymbol{u}-\boldsymbol{w}_{N_1N_2}\|^2_{\mathbf{H}^1(\Omega)} \le \left(\frac{1}{N_1^{2s-3}}+\frac{1}{N_2^{2s-3}}\right)\delta_1(h_1^{\pm},h_2^{\pm},N_1,N_2),$$

where $\delta_1(h_1^{\pm},h_2^{\pm},N_1,N_2)\to 0,$ as $\min\{N_1,N_2\}\to\infty$.

In addition, if $\sum\limits_{k=0}^{s-1}\sum\limits_{\alpha=1}^{2}\|\partial_\alpha^k\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)}\le c$, where c is independent of h_1, h_2, then from (2.8) we have

$$\|\boldsymbol{\varepsilon}_{N_1N_2}\|^2_{\mathbf{L}^2(\Omega)} \le \left(\frac{h_1^{2s}}{N_1^{2s}}+\frac{h_2^{2s}}{N_2^{2s}}\right)\bar{\delta}(N_1,N_2),$$

$$\left\|\frac{\partial \boldsymbol{\varepsilon}_{N_1N_2}}{\partial x_i}\right\|^2_{\mathbf{L}^2(\Omega)} \le \left(\frac{h_1^{2(s-1)}}{N_1^{2s-3}}+\frac{h_2^{2(s-1)}}{N_2^{2s-3}}\right)\bar{\delta}(N_1,N_2),$$

where $i=\overline{1,3}$, $\bar{\delta}(N_1,N_2)\to 0,$ as $\min\{N_1,N_2\}\to\infty$. From the latter inequalities, taking into account (2.6), we obtain the second estimate of the theorem

$$\|\boldsymbol{u} - \boldsymbol{w}_{N_1N_2}\|^2_{E(\Omega)} \leq \left(\frac{h_1^{2(s-1)}}{N_1^{2s-3}} + \frac{h_2^{2(s-1)}}{N_2^{2s-3}}\right)\delta_2(N_1, N_2),$$

where $\delta_2(N_1, N_2) \to 0$, as $\min\{N_1, N_2\} \to \infty$, $\|\boldsymbol{v}\|_{E(\Omega)} = \sqrt{B^\Omega(\boldsymbol{v}, \boldsymbol{v})}$. □

3. Dynamical initial boundary value problem

In the present section we construct a hierarchy of dynamical one-dimensional models of elastic rod and investigate the corresponding initial boundary value problems. In addition, we prove, that the sequence of vector functions restored from the solutions of the reduced problems converges to the solution of the original three-dimensional problem.

Let us consider initial boundary value problem (1.1)–(1.3), the weak formulation of which is of the following form: Find the unknown vector function $\boldsymbol{u} \in C^0([0, T]; V(\Omega))$, $\boldsymbol{u}' \in C^0([0, T]; \mathbf{L}^2(\Omega))$, which satisfies the equation

$$\frac{d}{dt}(\boldsymbol{u}', \boldsymbol{v})_{\mathbf{L}^2(\Omega)} + B^\Omega(\boldsymbol{u}, \boldsymbol{v}) = L^\Omega(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in V(\Omega), \tag{3.1}$$

in the sense of distributions in $(0, T)$ together with the following initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{\varphi}, \qquad \boldsymbol{u}'(0) = \boldsymbol{\psi}, \tag{3.2}$$

where $\boldsymbol{\varphi} \in V(\Omega)$, $\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$ and $C^0([0, T]; H)$ is a space of continuous vector functions from $[0, T]$ to a Banach space H. Note that each $\zeta \in C^0([0, T]; H)$ can be identified with distribution in $(0, T)$ with values in H and its generalized derivative we denote by ζ'.

The formulated three-dimensional dynamical problem (3.1), (3.2) has a unique solution $\boldsymbol{u}$ if $2\mu + 3\lambda > 0$, $\mu > 0$, $\boldsymbol{f} \in \mathbf{L}^2(\Omega \times (0, T))$, $\boldsymbol{g}$, $\dfrac{\partial \boldsymbol{g}}{\partial t} \in \mathbf{L}^2(\tilde{\Gamma} \times (0, T))$, which satisfies the following energy equality: for all $t \in [0, T]$,

$$(\boldsymbol{u}'(t), \boldsymbol{u}'(t))_{\mathbf{L}^2(\Omega)} + B^\Omega(\boldsymbol{u}(t), \boldsymbol{u}(t)) = (\boldsymbol{\psi}, \boldsymbol{\psi})_{\mathbf{L}^2(\Omega)} + B^\Omega(\boldsymbol{\varphi}, \boldsymbol{\varphi}) + \tilde{L}^\Omega(\boldsymbol{u})(t),$$

where

$$\tilde{L}^\Omega(\boldsymbol{u})(t) = 2\int_0^t (\boldsymbol{f}(\tau), \boldsymbol{u}'(\tau))_{\mathbf{L}^2(\Omega)} d\tau + 2(\boldsymbol{g}(t), \boldsymbol{u}(t))_{\mathbf{L}^2(\tilde{\Gamma})}$$
$$- 2(\boldsymbol{g}(0), \boldsymbol{u}(0))_{\mathbf{L}^2(\tilde{\Gamma})} - 2\int_0^t \left(\frac{\partial \boldsymbol{g}}{\partial t}(\tau), \boldsymbol{u}(\tau)\right)_{\mathbf{L}^2(\tilde{\Gamma})} d\tau, \qquad \forall t \in [0, T].$$

As in the case of static problem, to reduce three-dimensional problem (3.1), (3.2) to a hierarchy of one-dimensional problems, let us consider equation (3.1) on the subspace $V_{N_1N_2}(\Omega)$ ($V_{N_1N_2}(\Omega)$ is defined in Section 2) and take $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ from the subspaces $V_{N_1N_2}(\Omega)$ and $H_{N_1N_2}(\Omega)$, respectively, where

$$H_{N_1N_2}(\Omega) = \left\{ \boldsymbol{v}_{N_1N_2} = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} a_1 a_2 \left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right) \overset{k_1k_2}{\boldsymbol{v}} P_{k_1}(\omega_1) P_{k_2}(\omega_2); \right.$$

$$\left. \overset{k_1k_2}{\boldsymbol{v}} \in \mathbf{L}^2(I), \omega_\alpha = a_\alpha x_\alpha - b_\alpha, \alpha = 1,2, k_1 = \overline{0, N_1}, k_2 = \overline{0, N_2} \right\}.$$

Thus, we obtain the following problem: Find $\boldsymbol{w}_{N_1N_2} \in C^0([0,T]; V_{N_1N_2}(\Omega))$, $\boldsymbol{w}'_{N_1N_2} \in C^0([0,T]; H_{N_1N_2}(\Omega))$, which satisfies the equation

$$\frac{d}{dt}(\boldsymbol{w}'_{N_1N_2}, \boldsymbol{v}_{N_1N_2})_{\mathbf{L}^2(\Omega)} + B^\Omega(\boldsymbol{w}_{N_1N_2}, \boldsymbol{v}_{N_1N_2}) = L^\Omega(\boldsymbol{v}_{N_1N_2}), \tag{3.3}$$

for all $\boldsymbol{v}_{N_1N_2} \in V_{N_1N_2}(\Omega)$, in the sense of distributions in $(0,T)$, together with the following initial conditions

$$\boldsymbol{w}_{N_1N_2}(0) = \boldsymbol{\varphi}_{N_1N_2}, \qquad \boldsymbol{w}'_{N_1N_2}(0) = \boldsymbol{\psi}_{N_1N_2}, \tag{3.4}$$

where $\boldsymbol{\varphi}_{N_1N_2} \in V_{N_1N_2}(\Omega)$, $\boldsymbol{\psi}_{N_1N_2} \in H_{N_1N_2}(\Omega)$.

Note, that problem (3.3), (3.4) is equivalent to the following one: Find a vector function $\vec{w}_{N_1N_2} = (\overset{00}{\boldsymbol{w}}, \dots, \overset{N_1N_2}{\boldsymbol{w}}) \in C^0([0,T]; \vec{V}_{N_1N_2}(I))$, $\vec{w}'_{N_1N_2} \in C^0([0,T]; [\mathbf{L}^2(I)]^{(N_1+1)(N_2+1)})$, which satisfies the equation

$$\frac{d}{dt}(\mathbf{M}\vec{w}'_{N_1N_2}, \vec{v}_{N_1N_2})_{[\mathbf{L}^2(I)]^{(N_1+1)(N_2+1)}} + B^\Omega_{N_1N_2}(\vec{w}_{N_1N_2}, \vec{v}_{N_1N_2}) = L^\Omega_{N_1N_2}(\vec{v}_{N_1N_2}), \qquad \forall \vec{v}_{N_1N_2} \in \vec{V}_{N_1N_2}(I), \tag{3.5}$$

in the sense of distributions in $(0,T)$, together with the initial conditions

$$\vec{w}_{N_1N_2}(0) = \vec{\varphi}_{N_1N_2}, \qquad \vec{w}'_{N_1N_2}(0) = \vec{\psi}_{N_1N_2}, \tag{3.6}$$

where

$$\vec{\varphi}_{N_1N_2} = (\overset{00}{\boldsymbol{\varphi}}, \dots, \overset{N_1N_2}{\boldsymbol{\varphi}}) \in \vec{V}_{N_1N_2}(I), \qquad \vec{\psi}_{N_1N_2} = (\overset{00}{\boldsymbol{\psi}}, \dots, \overset{N_1N_2}{\boldsymbol{\psi}}) \in [\mathbf{L}^2(I)]^{(N_1+1)(N_2+1)},$$

$$\boldsymbol{\varphi}_{N_1N_2} = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} a_1 a_2 \left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right) \overset{k_1k_2}{\boldsymbol{\varphi}} P_{k_1}(\omega_1) P_{k_2}(\omega_2),$$

$$\boldsymbol{\psi}_{N_1N_2} = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} a_1 a_2 \left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right) \overset{k_1k_2}{\boldsymbol{\psi}} P_{k_1}(\omega_1) P_{k_2}(\omega_2),$$

$$\mathbf{M}\vec{w}_{N_1N_2} = (M_{00}\overset{00}{\boldsymbol{w}}, \dots, M_{N_1N_2}\overset{N_1N_2}{\boldsymbol{w}}), \qquad M_{k_1k_2} = a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right),$$

$k_1 = \overline{0, N_1}$, $k_2 = \overline{0, N_2}$ and $B^{\Omega}_{N_1N_2}$, $L^{\Omega}_{N_1N_2}$ are defined in Section 2.

So, we have obtained a hierarchy of dynamical one-dimensional models of the rod. In order to investigate initial boundary value problem (3.5), (3.6) let us consider more general variational problem and formulate theorem on the existence and uniqueness of its solution, from which we obtain the corresponding result for reduced problem (3.5), (3.6).

Let V and H be separable real Hilbert spaces, V is dense in H and is continuously imbedded in it. The dual space of V we denote by V' and H is identified with its dual with respect to the scalar product in H, then $V \hookrightarrow H \hookrightarrow V'$ with continuous and dense imbeddings. The duality relation between the spaces V' and V we denote by $\langle.,.\rangle$.

Assume that A, B, L are linear continuous operators, such that

$$B = B_1 + B_2, \qquad B_1 \in \mathfrak{L}(V; V'),\ B_2 \in \mathfrak{L}(V; H) \cap \mathfrak{L}(H; V'),\ A, L \in \mathfrak{L}(H; H),$$

B_1 is self-adjoint and $B_1 + \beta_1 I$ is coercive for some real number β_1, A is self-adjoint and coercive, i.e.,

$$\left.\begin{aligned} &b_1(u,v) = b_1(v,u), \quad |b_1(u,v)| \le c_{b_1}\|u\|_V\|v\|_V, \\ &b_1(u,u) \ge \beta\|u\|_V^2 - \beta_1\|u\|_H^2, \quad \beta > 0, \end{aligned}\right. \qquad \forall u, v \in V,$$

$$(3.7) \quad |b_2(\tilde{u},\tilde{v})| \le \begin{cases} c_{b_2}\|\tilde{u}\|_V\|\tilde{v}\|_H, & \forall \tilde{u} \in V,\ \tilde{v} \in H, \\ c_{b_2}\|\tilde{u}\|_H\|\tilde{v}\|_V, & \forall \tilde{u} \in H,\ \tilde{v} \in V, \end{cases}$$

$$\left.\begin{aligned} &a(u_1,v_1) = a(v_1,u_1), \quad a(u_1,u_1) \ge \alpha\|u_1\|_H^2, \quad \alpha > 0, \\ &|a(u_1,v_1)| \le c_a\|u_1\|_H\|v_1\|_H, \quad |l(u_1,v_1)| \le c_l\|u_1\|_H\|v_1\|_H, \end{aligned}\right. \qquad \forall u_1, v_1 \in H,$$

where $b_1(u,v) = \langle B_1u, v\rangle$, $b_2(u,v) = \langle B_2u, v\rangle$, $l(u_1,v_1) = (Lu_1, v_1)_H$, $a(u_1,v_1) = (Au_1, v_1)_H$, $b(u,v) = b_1(u,v) + b_2(u,v)$, for all $u, v \in V$, $u_1, v_1 \in H$.

Let us consider the following variational problem: Find a vector function $z \in C^0([0,T]; V)$, $z' \in C^0([0,T]; H)$, which satisfies the equation

$$(3.8) \qquad \frac{d}{dt}a(z',v) + b(z,v) + l(z',v) = (F,v)_H + \langle\tilde{F}, v\rangle, \qquad \forall v \in V,$$

in the sense of distributions in $(0,T)$, together with the following initial conditions

$$(3.9) \qquad z(0) = z_0, \qquad z'(0) = z_1,$$

where $z_0 \in V$, $z_1 \in H$, $F \in L^2(0,T;H)$, $\tilde{F}, \tilde{F}' \in L^2(0,T;V')$.

For the formulated problem the following theorem is true.

THEOREM 3.1. *If conditions* (3.7) *are satisfied, then problem* (3.8), (3.9) *possesses a unique solution, which satisfies the energy equality*

$$a(z'(t),z'(t))+b_1(z(t),z(t))+2\int_0^t b_2(z(\tau),z'(\tau))d\tau+2\int_0^t l(z'(\tau),z'(\tau))d\tau$$
$$=a(z_1,z_1)+b_1(z_0,z_0)+2\int_0^t (F(\tau),z'(\tau))_H d\tau+2\langle\tilde{F}(t),z(t)\rangle$$
$$-2\langle\tilde{F}(0),z_0\rangle-2\int_0^t\langle\tilde{F}'(\tau),z(\tau)\rangle d\tau,\qquad \forall t\in[0,T].$$

The existence result of Theorem 3.1 can be proved in a standard way applying Faedo-Galerkin's method (Chap. 18, sect. 5 of [31]), while the energy equality can be obtained through the usual regularization and limiting procedure.

Applying Theorem 3.1 for one-dimensional problem (3.5), (3.6), we obtain the following theorem.

THEOREM 3.2. *Assume that Lamé constants satisfy conditions* $2\mu+3\lambda>0$, $\mu>0$ *and* $\boldsymbol{f}\in\mathbf{L}^2(\Omega\times(0,T))$, $\boldsymbol{g}$, $\partial\boldsymbol{g}/\partial t\in\mathbf{L}^2(\tilde{\Gamma}\times(0,T))$, $\vec{\varphi}_{N_1N_2}\in\vec{V}_{N_1N_2}(I)$, $\vec{\psi}_{N_1N_2}\in[\mathbf{L}^2(I)]^{(N_1+1)(N_2+1)}$, *then problem* (3.5), (3.6) *has a unique solution* $\vec{w}_{N_1N_2}(t)$ *and the following energy equality is valid*

$$(\boldsymbol{w}'_{N_1N_2}(t),\boldsymbol{w}'_{N_1N_2}(t))_{\mathbf{L}^2(\Omega)}+B^{\Omega}(\boldsymbol{w}_{N_1N_2}(t),\boldsymbol{w}_{N_1N_2}(t))$$
$$=(\boldsymbol{\psi}_{N_1N_2},\boldsymbol{\psi}_{N_1N_2})_{\mathbf{L}^2(\Omega)}+B^{\Omega}(\boldsymbol{\varphi}_{N_1N_2},\boldsymbol{\varphi}_{N_1N_2})$$
$$+\tilde{L}^{\Omega}(\boldsymbol{w}_{N_1N_2})(t),\qquad\forall t\in[0,T]. \tag{3.10}$$

PROOF. The formulated theorem is a consequence of Theorem 3.1. Indeed, it suffices to take $V=\vec{V}_{N_1N_2}(I)$, $H=[\mathbf{L}^2(I)]^{(N_1+1)(N_2+1)}$,

$$z(t)=\vec{w}_{N_1N_2}(t),\qquad v=\vec{v}_{N_1N_2},\qquad z_0=\vec{\varphi}_{N_1N_2},\qquad z_1=\vec{\psi}_{N_1N_2},$$
$$b_1(\vec{w}_{N_1N_2},\vec{v}_{N_1N_2})=B^{\Omega}_{N_1N_2}(\vec{w}_{N_1N_2},\vec{v}_{N_1N_2}),\qquad b_2\equiv 0,\qquad l\equiv 0,$$
$$a(\vec{w}_{N_1N_2},\vec{v}_{N_1N_2})=(\mathbf{M}\vec{w}_{N_1N_2},\vec{v}_{N_1N_2})_{[\mathbf{L}^2(I)]^{(N_1+1)(N_2+1)}},\qquad F=(\overset{k_1k_2}{\boldsymbol{F}}),$$
$$\overset{k_1k_2}{\boldsymbol{F}}=\int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+}\boldsymbol{f}a_1a_2\left(k_1+\frac{1}{2}\right)\left(k_2+\frac{1}{2}\right)P_{k_1}(\omega_1)P_{k_2}(\omega_2)dx_1dx_2,$$
$$\langle\tilde{F},\vec{v}_{N_1N_2}\rangle=(\boldsymbol{g},\boldsymbol{v}_{N_1N_2})_{\mathbf{L}^2(\tilde{\Gamma})},\qquad\forall\vec{v}_{N_1N_2}\in\vec{V}_{N_1N_2}(I).$$

Note that since the norm $\|.\|_{[\mathbf{H}^1(I)]^{(N_1+1)(N_2+1)}}$ in the space $\vec{V}_{N_1N_2}(I)$ is equivalent to the norm $\|.\|_*$, $\|\vec{v}_{N_1N_2}\|_*=\|\boldsymbol{v}_{N_1N_2}\|_{\mathbf{H}^1(\Omega)}$, where $\boldsymbol{v}_{N_1N_2}\in V_{N_1N_2}(\Omega)$

corresponds to $\vec{v}_{N_1N_2} \in \vec{V}_{N_1N_2}(I)$, then all conditions of Theorem 3.1 are fulfilled. Therefore problem (3.5), (3.6) has a unique solution, $\vec{w}_{N_1N_2}$ satisfies the energy equality

$$
\begin{aligned}
&(\mathbf{M}\vec{w}'_{N_1N_2}(t), \vec{w}'_{N_1N_2}(t))_{[\mathbf{L}^2(I)]^{(N_1+1)(N_2+1)}} + B^{\Omega}_{N_1N_2}(\vec{w}_{N_1N_2}(t), \vec{w}_{N_1N_2}(t)) \\
&\quad = (\mathbf{M}\vec{\psi}_{N_1N_2}, \vec{\psi}_{N_1N_2})_{[\mathbf{L}^2(I)]^{(N_1+1)(N_2+1)}} + B^{\Omega}_{N_1N_2}(\vec{\varphi}_{N_1N_2}, \vec{\varphi}_{N_1N_2}) \\
&\qquad + 2\int_0^t (\boldsymbol{f}, \boldsymbol{w}'_{N_1N_2}(\tau))_{\mathbf{L}^2(\Omega)} d\tau + 2(\boldsymbol{g}(t), \boldsymbol{w}_{N_1N_2}(t))_{\mathbf{L}^2(\tilde{\Gamma})} - 2(\boldsymbol{g}(0), \boldsymbol{\varphi}_{N_1N_2})_{\mathbf{L}^2(\tilde{\Gamma})} \\
&\qquad - 2\int_0^t \left(\frac{\partial \boldsymbol{g}}{\partial t}(\tau), \boldsymbol{w}_{N_1N_2}(\tau)\right)_{\mathbf{L}^2(\tilde{\Gamma})} d\tau, \qquad \forall t \in [0, T],
\end{aligned}
$$

which is equivalent to equality (3.10). □

Thus, we have reduced three-dimensional problem (3.1), (3.2) to one-dimensional problem (3.5), (3.6) and have proved the existence and uniqueness of its solution. Now we estimate the rate of approximation of the exact solution $\boldsymbol{u}$ of the three-dimensional problem by the vector functions $\boldsymbol{w}_{N_1N_2}(t)$ restored from the solutions $\vec{w}_{N_1N_2}(t)$ of the reduced problems. For simplicity of notes we denote by $\|.\|$ and $|.|$ norms in the spaces $V(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively, and the scalar product in $\mathbf{L}^2(\Omega)$ we denote by $(.,.)$.

THEOREM 3.3. *If conditions of Theorem* 3.2 *are fulfilled and* $\boldsymbol{\varphi}_{N_1N_2}$, $\boldsymbol{\psi}_{N_1N_2}$ *corresponding to* $\vec{\varphi}_{N_1N_2}$, $\vec{\psi}_{N_1N_2}$ *tend to* $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ *in the spaces* $V(\Omega)$ *and* $\mathbf{L}^2(\Omega)$, *respectively, then the vector function* $\boldsymbol{w}_{N_1N_2}(t)$ *corresponding to the solution* $\vec{w}_{N_1N_2}(t) = (\overset{00}{\boldsymbol{w}}(t), \dots, \overset{N_1N_2}{\boldsymbol{w}}(t))$ *of reduced problem* (3.5), (3.6) *tends to the solution* $\boldsymbol{u}(t)$ *of three-dimensional problem* (3.1), (3.2) *in the space* $V(\Omega)$,

$$
\begin{aligned}
&\boldsymbol{w}_{N_1N_2}(t) \to \boldsymbol{u}(t) \quad \textit{strongly in } V(\Omega), \\
&\boldsymbol{w}'_{N_1N_2}(t) \to \boldsymbol{u}'(t) \quad \textit{strongly in } \mathbf{L}^2(\Omega),
\end{aligned}
\qquad \textit{as } \min\{N_1, N_2\} \to \infty,\ \forall t \in [0, T].
$$

Moreover, if components of $\vec{\varphi}_{N_1N_2}$, $\vec{\psi}_{N_1N_2}$ *are moments of* $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ *with respect to the Legendre polynomials, i.e.* $\vec{\varphi}_{N_1N_2} = (\overset{00}{\boldsymbol{\varphi}}, \dots, \overset{N_1N_2}{\boldsymbol{\varphi}})$, $\vec{\psi}_{N_1N_2} = (\overset{00}{\boldsymbol{\psi}}, \dots, \overset{N_1N_2}{\boldsymbol{\psi}})$,

$$
\overset{k_1k_2}{\boldsymbol{\varphi}} = \int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+} \boldsymbol{\varphi} P_{k_1}(\omega_1)P_{k_2}(\omega_2)dx_1dx_2, \quad \overset{k_1k_2}{\boldsymbol{\psi}} = \int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+} \boldsymbol{\psi} P_{k_1}(\omega_1)P_{k_2}(\omega_2)dx_1dx_2,
$$

$k_1 = \overline{0, N_1}$, $k_2 = \overline{0, N_2}$, *and* $\boldsymbol{u}$ *satisfies additional regularity properties with respect to the spatial variables* $\boldsymbol{u} \in L^2(0, T; \mathbf{H}^{s_0, s_0, 1}(\Omega))$, $\boldsymbol{u}' \in L^2(0, T; \mathbf{H}^{s_1, s_1, 1}(\Omega))$, $\boldsymbol{u}'' \in L^2(0, T; \mathbf{H}^{s_2, s_2, 1}(\Omega))$, $s_0 \ge s_1 \ge s_2 \ge 1$, $s_1 \ge 2$, *then the following estimate is valid:* $s = \min\{s_2, s_1 - 3/2\}$,

$$
|\boldsymbol{u}' - \boldsymbol{w}'_{N_1N_2}|^2 + \|\boldsymbol{u} - \boldsymbol{w}_{N_1N_2}\|^2 \le \left(\frac{1}{N_1^{2s}} + \frac{1}{N_2^{2s}}\right)\eta(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2),
$$

where $\eta(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2) \to 0$, *as* $\min\{N_1, N_2\} \to \infty$. *If additionally the following conditions are fulfilled* $\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}^{s_0,s_0,1}(\Omega))} \le \tilde{c}$, $\|\boldsymbol{u}'\|_{L^2(0,T;\mathbf{H}^{s_1,s_1,1}(\Omega))} \le \tilde{c}$, $\|\boldsymbol{u}''\|_{L^2(0,T;\mathbf{H}^{s_2,s_2,1}(\Omega))} \le \tilde{c}$, *where* $\tilde{c}$ *is independent of* $h_1 = \max_{x_3 \in \bar{I}}(h_1^+(x_3) - h_1^-(x_3))$ *and* $h_2 = \max_{x_3 \in \bar{I}}(h_2^+(x_3) - h_2^-(x_3))$, *then*

$$|\boldsymbol{u}' - \boldsymbol{w}'_{N_1N_2}|^2 + \|\boldsymbol{u} - \boldsymbol{w}_{N_1N_2}\|^2_{E(\Omega)} \le \left(\frac{h_1^{2\bar{s}}}{N_1^{2s}} + \frac{h_2^{2\bar{s}}}{N_2^{2s}}\right)\bar{\eta}(T, N_1, N_2),$$

where $\bar{\eta}(T, N_1, N_2) \to 0$, *as* $\min\{N_1, N_2\} \to \infty$, $\bar{s} = \min\{s_2, s_1 - 1\}$.

Proof. From Theorem 3.2 we have, that the vector function $\boldsymbol{w}_{N_1N_2}(t)$ corresponding to the solution $\vec{w}_{N_1N_2}(t)$ of reduced problem (3.5), (3.6) satisfies energy equality (3.10) and since $\boldsymbol{\varphi}_{N_1N_2} \to \boldsymbol{\varphi}$ in $V(\Omega)$, $\boldsymbol{\psi}_{N_1N_2} \to \boldsymbol{\psi}$ in $\mathbf{L}^2(\Omega)$, for all $t \in [0, T]$, we have

$$|\boldsymbol{w}'_{N_1N_2}(t)|^2 + \|\boldsymbol{w}_{N_1N_2}(t)\|^2 \le c_1\Bigg(|\boldsymbol{\psi}|^2 + \|\boldsymbol{\varphi}\|^2 + \int_0^t |\boldsymbol{f}(\tau)|^2 d\tau + \|\boldsymbol{g}(t)\|^2_{\mathbf{L}^2(\tilde{\Gamma})}$$
$$+ \|\boldsymbol{g}(0)\|^2_{\mathbf{L}^2(\tilde{\Gamma})} + \int_0^t \left\|\frac{\partial \boldsymbol{g}}{\partial t}(\tau)\right\|^2_{\mathbf{L}^2(\tilde{\Gamma})} d\tau + \int_0^t (|\boldsymbol{w}'_{N_1N_2}(\tau)|^2 + \|\boldsymbol{w}_{N_1N_2}(\tau)\|^2) d\tau\Bigg).$$

Applying Gronwall's lemma ([32]), from the last inequality, we obtain

$$|\boldsymbol{w}'_{N_1N_2}(t)|^2 + \|\boldsymbol{w}_{N_1N_2}(t)\|^2 < c_2, \qquad \forall N_1, N_2 \in \mathbf{N},\ t \in [0, T]. \tag{3.11}$$

It should be pointed out, that the method of constructing of the approximate solutions $\{\boldsymbol{w}_{N_1N_2}\}$ doesn't coincide with Faedo-Galerkin's method, because for each pair (N_1, N_2) the unknown vector functions $\overset{k_1k_2}{\boldsymbol{w}}$ $(0 \le k_1 \le N_1,$ $0 \le k_2 \le N_2)$ depend on two variables. However, in order to prove strong pointwise with respect to the variable t convergence of the sequence of approximate solutions $\{\boldsymbol{w}_{N_1N_2}\}$ it is possible to use the arguments which are applied to prove the same property when the approximate solutions are constructed by Faedo-Galerkin's method (Chap. 18, sect. 5 of [31]). Therefore we present only the scheme of the proof.

Since the sequence $\{\boldsymbol{w}_{N_1N_2}(t)\}$ satisfies (3.11), it is bounded in the space $L^\infty(0, T; V(\Omega)) \cap L^2(0, T; V(\Omega))$, while $\{\boldsymbol{w}'_{N_1N_2}(t)\}$ belongs to the bounded set of the space $L^2(0, T; \mathbf{L}^2(\Omega)) \cap L^\infty(0, T; \mathbf{L}^2(\Omega))$. Hence, taking into account the density of the union $\bigcup_{N_1, N_2 \ge 0} V_{N_1N_2}(\Omega)$ in $V(\Omega)$, we obtain that as $\min\{N_1, N_2\} \to \infty$,

$$\begin{aligned} &\boldsymbol{w}_{N_1N_2} \to \boldsymbol{u} \ \text{ weakly in } L^2(0, T; V(\Omega)), \text{ weakly-}* \text{ in } L^\infty(0, T; V(\Omega)),\\ &\boldsymbol{w}'_{N_1N_2} \to \boldsymbol{u}' \ \text{ weakly in } L^2(0, T; \mathbf{L}^2(\Omega)), \text{ weakly-}* \text{ in } L^\infty(0, T; \mathbf{L}^2(\Omega)). \end{aligned} \tag{3.12}$$

Applying energy equalities for $\boldsymbol{u}(t)$ and $\boldsymbol{w}_{N_1N_2}(t)$, we obtain the following equality for their difference $\boldsymbol{\delta}_{N_1N_2}(t) = \boldsymbol{u}(t) - \boldsymbol{w}_{N_1N_2}(t)$,

$$(3.13)\qquad \begin{aligned}(\boldsymbol{\delta}'_{N_1N_2}(t), \boldsymbol{\delta}'_{N_1N_2}(t)) &+ B^{\Omega}(\boldsymbol{\delta}_{N_1N_2}(t), \boldsymbol{\delta}_{N_1N_2}(t)) \\ &= (\boldsymbol{\delta}'_{N_1N_2}(0), \boldsymbol{\delta}'_{N_1N_2}(0)) + B^{\Omega}(\boldsymbol{\delta}_{N_1N_2}(0), \boldsymbol{\delta}_{N_1N_2}(0)) \\ &\quad + \tilde{L}^{\Omega}(\boldsymbol{\delta}_{N_1N_2})(t) + 2\tilde{J}_{N_1N_2}(t),\end{aligned}$$

where

$$\begin{aligned}\tilde{J}_{N_1N_2}(t) = (\boldsymbol{u}'(0), \boldsymbol{w}'_{N_1N_2}(0)) &+ B^{\Omega}(\boldsymbol{u}(0), \boldsymbol{w}_{N_1N_2}(0)) - B^{\Omega}(\boldsymbol{u}(t), \boldsymbol{w}_{N_1N_2}(t)) \\ &- (\boldsymbol{u}'(t), \boldsymbol{w}'_{N_1N_2}(t)) + \tilde{L}^{\Omega}(\boldsymbol{w}_{N_1N_2})(t).\end{aligned}$$

Since $\boldsymbol{u}$ and $\boldsymbol{w}_{N_1N_2}$ are solutions of problems (3.1), (3.2) and (3.3), (3.4), respectively, from (3.11) we obtain that for any fixed $t \in [0, T]$,

$$\left.\begin{aligned}\boldsymbol{w}_{N_1N_2}(t) &\to \boldsymbol{u}(t) \quad \text{weakly in } V(\Omega), \\ \boldsymbol{w}'_{N_1N_2}(t) &\to \boldsymbol{u}'(t) \quad \text{weakly in } \mathbf{L}^2(\Omega),\end{aligned}\right. \qquad \text{as } \min\{N_1, N_2\} \to \infty.$$

Applying the energy equality for $\boldsymbol{u}$ and passing to the limit in $J_{N_1N_2}(t)$ as N_1 and N_2 tend to infinity, we get

$$(3.14)\qquad \begin{aligned}\tilde{J}_{N_1N_2}(t) \to (\boldsymbol{u}'(0), \boldsymbol{u}'(0)) &+ B^{\Omega}(\boldsymbol{u}(0), \boldsymbol{u}(0)) + \tilde{L}^{\Omega}(\boldsymbol{u})(t) \\ &- (\boldsymbol{u}'(t), \boldsymbol{u}'(t)) - B^{\Omega}(\boldsymbol{u}(t), \boldsymbol{u}(t)) = 0.\end{aligned}$$

Thus, from (3.13) we deduce

$$(3.15)\qquad \begin{aligned}|\boldsymbol{\delta}'_{N_1N_2}(t)|^2 + \|\boldsymbol{\delta}_{N_1N_2}(t)\|^2 \le c_3(2\tilde{J}_{N_1N_2}(t) &+ (\boldsymbol{\delta}'_{N_1N_2}(0), \boldsymbol{\delta}'_{N_1N_2}(0)) \\ &+ B^{\Omega}(\boldsymbol{\delta}_{N_1N_2}(0), \boldsymbol{\delta}_{N_1N_2}(0)) + \tilde{L}^{\Omega}(\boldsymbol{\delta}_{N_1N_2})(t)).\end{aligned}$$

From the conditions of the theorem it follows that $\boldsymbol{\delta}_{N_1N_2}(0) \to \mathbf{0}$ strongly in $V(\Omega)$ and $\boldsymbol{\delta}'_{N_1N_2}(0) \to \mathbf{0}$ strongly in $\mathbf{L}^2(\Omega)$. Applying (3.12), (3.14), we obtain

$$(\boldsymbol{\delta}'_{N_1N_2}(0), \boldsymbol{\delta}'_{N_1N_2}(0)) + B^{\Omega}(\boldsymbol{\delta}_{N_1N_2}(0), \boldsymbol{\delta}_{N_1N_2}(0)) + 2\tilde{J}_{N_1N_2}(t) + \tilde{L}^{\Omega}(\boldsymbol{\delta}_{N_1N_2})(t) \to 0,$$

as $\min\{N_1, N_2\} \to \infty$, and from (3.15) we have

$$|\boldsymbol{\delta}'_{N_1N_2}(t)|^2 + \|\boldsymbol{\delta}_{N_1N_2}(t)\|^2 \to 0, \qquad \text{as } \min\{N_1, N_2\} \to \infty.$$

Therefore, for all $t \in [0, T]$,

$$\left.\begin{aligned}\boldsymbol{w}_{N_1N_2}(t) &\to \boldsymbol{u}(t) \quad \text{strongly in } V(\Omega), \\ \boldsymbol{w}'_{N_1N_2}(t) &\to \boldsymbol{u}'(t) \quad \text{strongly in } \mathbf{L}^2(\Omega),\end{aligned}\right. \qquad \text{as } \min\{N_1, N_2\} \to \infty.$$

Now we prove the estimates of the theorem. The solution $\boldsymbol{u}$ of the three-dimensional problem satisfies equation (3.1) for all $\boldsymbol{v} \in V(\Omega)$ and hence satisfies it for all $\boldsymbol{v}_{N_1N_2} \in V_{N_1N_2}(\Omega) \subset V(\Omega)$, i.e.

$$\frac{d}{dt}(\boldsymbol{u}', \boldsymbol{v}_{N_1N_2}) + B^{\Omega}(\boldsymbol{u}, \boldsymbol{v}_{N_1N_2}) = L^{\Omega}(\boldsymbol{v}_{N_1N_2}), \qquad \forall \boldsymbol{v}_{N_1N_2} \in V_{N_1N_2}(\Omega).$$

Since the vector function $\boldsymbol{w}_{N_1N_2}$ corresponds to the solution $\vec{w}_{N_1N_2}$ of problem (3.5), (3.6) and satisfies equation (3.3), we have

$$\frac{d}{dt}((\boldsymbol{u} - \boldsymbol{w}_{N_1N_2})', \boldsymbol{v}_{N_1N_2}) + B^{\Omega}(\boldsymbol{u} - \boldsymbol{w}_{N_1N_2}, \boldsymbol{v}_{N_1N_2}) = 0, \qquad \forall \boldsymbol{v}_{N_1N_2} \in V_{N_1N_2}(\Omega).$$

Suppose that $\boldsymbol{u} \in L^2(0,T;\mathbf{H}^{s_0,s_0,1}(\Omega))$, $\boldsymbol{u}' \in L^2(0,T;\mathbf{H}^{s_1,s_1,1}(\Omega))$, $\boldsymbol{u}'' \in L^2(0,T;\mathbf{H}^{s_2,s_2,1}(\Omega))$, $s_0 \geq s_1 \geq s_2 \geq 1$, $s_1 \geq 2$. From the regularity theorems we obtain $\boldsymbol{u} \in C^0([0,T];\mathbf{H}^{s_1,s_1,1}(\Omega))$, $\boldsymbol{u}' \in C^0([0,T];\mathbf{H}^{s_2,s_2,1}(\Omega))$. Let us consider the Fourier-Legendre expansion of the vector function $\boldsymbol{u}$ with respect to the variables x_1, x_2. We denote by $\boldsymbol{u}_{N_1N_2}$ the piece of series, consisting of the first $N_1 + N_2 + 2$ terms, while the remainder term is denoted by $\gamma_{N_1N_2}$, i.e. $\boldsymbol{u} = \boldsymbol{u}_{N_1N_2} + \gamma_{N_1N_2}$,

$$\boldsymbol{u}_{N_1N_2} = \sum_{k_1=0}^{N_1}\sum_{k_2=0}^{N_2} a_1 a_2 \left(k_1 + \frac{1}{2}\right)\left(k_2 + \frac{1}{2}\right) \overset{k_1k_2}{\boldsymbol{u}} P_{k_1}(\omega_1) P_{k_2}(\omega_2),$$

$$\overset{k_1k_2}{\boldsymbol{u}} = \int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+} \boldsymbol{u} P_{k_1}(\omega_1) P_{k_2}(\omega_2)\,dx_1 dx_2, \qquad \omega_1 = a_1 x_1 - b_1,\ \omega_2 = a_2 x_2 - b_2,$$

$k_1 = \overline{0, N_1}$, $k_2 = \overline{0, N_2}$. Let us take initial conditions $\vec{\varphi}_{N_1N_2}$, $\vec{\psi}_{N_1N_2}$ of the problem (3.5), (3.6) such that $\vec{\varphi}_{N_1N_2} = (\overset{00}{\boldsymbol{\varphi}}, \dots, \overset{N_1N_2}{\boldsymbol{\varphi}})$, $\vec{\psi}_{N_1N_2} = (\overset{00}{\boldsymbol{\psi}}, \dots, \overset{N_1N_2}{\boldsymbol{\psi}})$,

$$\overset{k_1k_2}{\boldsymbol{\varphi}} = \int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+} \boldsymbol{\varphi} P_{k_1}(\omega_1) P_{k_2}(\omega_2)\,dx_1 dx_2, \quad \overset{k_1k_2}{\boldsymbol{\psi}} = \int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+} \boldsymbol{\psi} P_{k_1}(\omega_1) P_{k_2}(\omega_2)\,dx_1 dx_2,$$

where $k_1 = \overline{0, N_1}$, $k_2 = \overline{0, N_2}$. Hence the vector function $\boldsymbol{\Delta}_{N_1N_2} = \boldsymbol{u}_{N_1N_2} - \boldsymbol{w}_{N_1N_2}$ is a solution of the following problem:

$$\begin{aligned}
&\frac{d}{dt}(\boldsymbol{\Delta}'_{N_1N_2}, \boldsymbol{v}_{N_1N_2}) + B^{\Omega}(\boldsymbol{\Delta}_{N_1N_2}, \boldsymbol{v}_{N_1N_2}) \\
&\qquad = -((\gamma''_{N_1N_2}, \boldsymbol{v}_{N_1N_2}) + B^{\Omega}(\gamma_{N_1N_2}, \boldsymbol{v}_{N_1N_2})), \qquad \forall \boldsymbol{v}_{N_1N_2} \in V_{N_1N_2}(\Omega), \\
&\boldsymbol{\Delta}_{N_1N_2}(0) = \boldsymbol{u}_{N_1N_2}(0) - \boldsymbol{\varphi}_{N_1N_2} = \mathbf{0}, \qquad \boldsymbol{\Delta}'_{N_1N_2}(0) = \boldsymbol{u}'_{N_1N_2}(0) - \boldsymbol{\psi}_{N_1N_2} = \mathbf{0}.
\end{aligned}$$

Applying Theorem 3.1 to the last problem, we have

$$
\begin{aligned}
&(\boldsymbol{\Delta}'_{N_1N_2}(t), \boldsymbol{\Delta}'_{N_1N_2}(t)) + B^{\Omega}(\boldsymbol{\Delta}_{N_1N_2}(t), \boldsymbol{\Delta}_{N_1N_2}(t)) \\
&\qquad = -2\int_0^t (\boldsymbol{\gamma}''_{N_1N_2}(\tau), \boldsymbol{\Delta}'_{N_1N_2}(\tau))d\tau - 2B^{\Omega}(\boldsymbol{\gamma}_{N_1N_2}(t), \boldsymbol{\Delta}_{N_1N_2}(t)) \\
&\qquad\quad + 2\int_0^t B^{\Omega}(\boldsymbol{\gamma}'_{N_1N_2}(\tau), \boldsymbol{\Delta}_{N_1N_2}(\tau))d\tau, \qquad 0 \le t \le T.
\end{aligned}
$$

From this equality it follows that for all $t \in [0, T]$,

$$
\begin{aligned}
(3.16) \quad |\boldsymbol{\Delta}'_{N_1N_2}(t)|^2 + \|\boldsymbol{\Delta}_{N_1N_2}(t)\|^2_{E(\Omega)} &\le c_4\bigg(\int_0^t (|\boldsymbol{\Delta}'_{N_1N_2}(\tau)|^2 + \|\boldsymbol{\Delta}_{N_1N_2}(\tau)\|^2_{E(\Omega)})d\tau \\
&\quad + \int_0^t |\boldsymbol{\gamma}''_{N_1N_2}(\tau)|^2 d\tau + \|\boldsymbol{\gamma}_{N_1N_2}(t)\|^2_{E(\Omega)} + \int_0^t \|\boldsymbol{\gamma}'_{N_1N_2}(\tau)\|^2_{E(\Omega)}d\tau\bigg),
\end{aligned}
$$

where $\|\boldsymbol{v}\|^2_{E(\Omega)} = B^{\Omega}(\boldsymbol{v}, \boldsymbol{v})$, for all $\boldsymbol{v} \in V(\Omega)$ and c_4 is independent of $\gamma_{N_1N_2}$, $\Delta_{N_1N_2}$ and Ω. Applying Gronwall's lemma to (3.16), we have

$$
\begin{aligned}
|\boldsymbol{\Delta}'_{N_1N_2}(t)|^2 + \|\boldsymbol{\Delta}_{N_1N_2}(t)\|^2_{E(\Omega)} &\le c_5\bigg(\int_0^t |\boldsymbol{\gamma}''_{N_1N_2}(\tau)|^2 d\tau + \|\boldsymbol{\gamma}_{N_1N_2}(t)\|^2_{E(\Omega)} \\
&\quad + \int_0^t \|\boldsymbol{\gamma}'_{N_1N_2}(\tau)\|^2_{E(\Omega)}d\tau\bigg), \qquad \forall t \in [0, T].
\end{aligned}
$$

Note that $\|\boldsymbol{v}\|^2_{E(\Omega)} \le c_6\|\boldsymbol{v}\|^2_{\mathbf{H}^1(\Omega)}$, for all $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, where $c_6 = 3\max\{3\lambda, \mu\}$, λ, μ are Lamé constants and hence c_6 is independent of $\boldsymbol{v}$ and Ω. Therefore, as in the proof of Theorem 2.2 we can show that

$$
\begin{aligned}
&\int_0^t |\boldsymbol{\gamma}''_{N_1N_2}(\tau)|^2 d\tau \le \left(\frac{1}{N_1^{2s_2}} + \frac{1}{N_2^{2s_2}}\right)\bar{\eta}(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2), \\
(3.17) \qquad &\|\boldsymbol{\gamma}_{N_1N_2}(t)\|^2_{E(\Omega)} \le \left(\frac{1}{N_1^{2s_1-3}} + \frac{1}{N_2^{2s_1-3}}\right)\bar{\eta}(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2), \\
&\int_0^t \|\boldsymbol{\gamma}'_{N_1N_2}(\tau)\|^2_{E(\Omega)}d\tau \le \left(\frac{1}{N_1^{2s_1-3}} + \frac{1}{N_2^{2s_1-3}}\right)\bar{\eta}(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2),
\end{aligned}
$$

where $\bar{\eta}(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2) \to 0$, as $\min\{N_1, N_2\} \to \infty$, $0 \le t \le T$.

Consequently, taking into account coerciveness of the bilinear form $B^{\Omega}(.,.)$, we have that for all $t \in [0, T]$,

$$
|\boldsymbol{\Delta}'_{N_1N_2}(t)|^2 + \|\boldsymbol{\Delta}_{N_1N_2}(t)\|^2 \le \left(\frac{1}{N_1^{2s}} + \frac{1}{N_2^{2s}}\right)\hat{\eta}(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2),
$$

where $\hat{\eta}(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2) \to 0$, as $\min\{N_1, N_2\} \to \infty$, $s = \min\{s_2, s_1 - 3/2\}$.

In addition, since $\boldsymbol{u}' \in C^0([0,T];\mathbf{H}^{s_2,s_2,1}(\Omega))$, we have

$$|\gamma'_{N_1N_2}(t)|^2 \leq \left(\frac{1}{N_1^{2s_2}} + \frac{1}{N_2^{2s_2}}\right)\tilde{\eta}(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2), \qquad \forall t \in [0,T],$$

where $\tilde{\eta}(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2) \to 0$, as $\min\{N_1, N_2\} \to \infty$. Therefore, for all $t \in [0,T]$,

$$|\boldsymbol{u}'(t) - \boldsymbol{w}'_{N_1N_2}(t)|^2 + \|\boldsymbol{u}(t) - \boldsymbol{w}_{N_1N_2}(t)\|^2 \leq \left(\frac{1}{N_1^{2s}} + \frac{1}{N_2^{2s}}\right)\eta(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2),$$

where $\eta(T, h_1^{\pm}, h_2^{\pm}, N_1, N_2) \to 0$, as $\min\{N_1, N_2\} \to \infty$.

If $\|d^k\boldsymbol{u}/dt^k\|_{L^2(0,T;\mathbf{H}^{s_k,s_k,1}(\Omega))} \leq \tilde{c}$, $k = 0,1,2$, where $\tilde{c}$ is independent of h_1, h_2, then instead of (3.17) we have

$$\int_0^t |\gamma''_{N_1N_2}(\tau)|^2 d\tau \leq \left(\frac{h_1^{2s_2}}{N_1^{2s_2}} + \frac{h_2^{2s_2}}{N_2^{2s_2}}\right)\bar{\eta}_1(T, N_1, N_2),$$

$$\|\gamma_{N_1N_2}(t)\|^2_{E(\Omega)} \leq \left(\frac{h_1^{2(s_1-1)}}{N_1^{2s_1-3}} + \frac{h_2^{2(s_1-1)}}{N_2^{2s_1-3}}\right)\bar{\eta}_1(T, N_1, N_2),$$

$$\int_0^t \|\gamma'_{N_1N_2}(\tau)\|^2_{E(\Omega)} d\tau \leq \left(\frac{h_1^{2(s_1-1)}}{N_1^{2s_1-3}} + \frac{h_2^{2(s_1-1)}}{N_2^{2s_1-3}}\right)\bar{\eta}_1(T, N_1, N_2),$$

where $\bar{\eta}_1(T, N_1, N_2) \to 0$, as $\min\{N_1, N_2\} \to \infty$ and hence

$$|\boldsymbol{\Delta}'_{N_1N_2}(t)|^2 + \|\boldsymbol{\Delta}_{N_1N_2}(t)\|^2_{E(\Omega)} \leq \left(\frac{h_1^{2\bar{s}}}{N_1^{2s}} + \frac{h_2^{2\bar{s}}}{N_2^{2s}}\right)\hat{\eta}_1(T, N_1, N_2),$$

where $\hat{\eta}_1(T, N_1, N_2) \to 0$, as $\min\{N_1, N_2\} \to \infty$, $\bar{s} = \min\{s_2, s_1 - 1\}$.

Similarly, for all $t \in [0,T]$,

$$|\gamma'_{N_1N_2}(t)|^2 \leq \left(\frac{h_1^{2s_2}}{N_1^{2s_2}} + \frac{h_2^{2s_2}}{N_2^{2s_2}}\right)\tilde{\eta}_1(T, N_1, N_2),$$

where $\tilde{\eta}_1(T, N_1, N_2) \to 0$, as $\min\{N_1, N_2\} \to \infty$, from which we obtain the second estimate of the theorem. □

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