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A boundary uniqueness property for weighted Sobolev functions

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ABSTRACT. The aim of this paper is to discuss a uniqueness property for Sobolev functions with certain condition on area integrals.

1. Introduction and statement of results

In 1950, Tsuji [13] discussed a uniqueness property for analytic functions on the unit disk with certain condition on area integrals. His result has recently been extended in several manners (see Jenkins [5], Koskela [7], Miklyukov-Vuorinen [8] and Mizuta [11]). In this paper we further extend those results in the weighted case.

Let $1 < p < \infty$ and D be an open set in $\mathbf{R}^n$. For a Borel measure μ on D, consider the (p,μ)-capacity $\mathrm{cap}_{p,\mu}(\cdot; D)$ relative to D. When K is a compact subset of D, it is defined by

$$\mathrm{cap}_{p,\mu}(K; D) = \inf \int_D |\nabla u|^p d\mu,$$

where the infimum is taken over all functions $u \in C_c^\infty(D)$ such that $u \geq 1$ on K; here $C_c^\infty(D)$ denotes the space of infinitely differentiable functions with compact support in D. We extend the capacity $\mathrm{cap}_{p,\mu}(\cdot; D)$ in the usual way (see Heinonen-Kilpeläinen-Martio [4]). In case μ is the Lebesgue measure in $\mathbf{R}^n$, (p,μ)-capacity will be called p-capacity. We say that a set $E \subset \mathbf{R}^n$ has (p,μ)-capacity zero if

$$\mathrm{cap}_{p,\mu}(E \cap G; G) = 0$$

for every bounded open set $G \subset \mathbf{R}^n$. In this case we write $\mathrm{cap}_{p,\mu}(E) = 0$. If E is not of (p,μ)-capacity zero, we say that E has positive (p,μ)-capacity and write $\mathrm{cap}_{p,\mu}(E) > 0$.

P. Koskela [7, Theorem A] proved that a continuous ACL^p-function u on

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the unit ball $\mathbf{B}$ in $\mathbf{R}^n$, which approaches zero in the weak sense for a set in $\partial\mathbf{B}$ of positive p-capacity, is identically zero provided that

$$\int_{\mathbf{B}_{u,\varepsilon}} |\nabla u(x)|^p dx \le C\varepsilon^p \left(\log \frac{1}{\varepsilon}\right)^{p-1}$$

for all $0 < \varepsilon < 1/2$, where $\mathbf{B}_{u,\varepsilon} = \{x \in \mathbf{B} : |u(x)| < \varepsilon\}$. Recall that u approaches zero in the weak sense for a set $F \subset \partial\mathbf{B}$ if for each $x \in F$ and all rectifiable curves γ in $\mathbf{B}$ terminating at x there exists a sequence of points in γ for which u tends to zero. In view of [7, Remark (3)], one can replace $\mathbf{B}$ by a bounded domain if one replaces the p-capacity by the p-modulus. Y. Mizuta [11, Theorem 1] replaced $(\log(1/\varepsilon))^{p-1}$ by a positive nonincreasing function φ on the interval $(0,\infty)$ satisfying $(\varphi 1)$ and $(\varphi 2)$ given in Theorem 1 below.

For a family Γ of curves on $\mathbf{R}^n$, we denote by $\mathscr{F}(\Gamma)$ the family of all nonnegative Borel functions ρ on $\mathbf{R}^n$ such that

$$\int_\gamma \rho \, ds \ge 1$$

for each locally rectifiable curve $\gamma \in \Gamma$. For $1 < p < \infty$ and a Borel measure μ on $\mathbf{R}^n$, we define the (p,μ)-modulus of Γ by

$$\mathrm{M}_p(\Gamma;\mu) = \inf_{\rho \in \mathscr{F}(\Gamma)} \int \rho(x)^p d\mu(x);$$

in case $\mathscr{F}(\Gamma) = \emptyset$, we set $\mathrm{M}_p(\Gamma;\mu) = \infty$. For elementary properties of moduli, see Ohtsuka [12], Väisälä [14] and Vuorinen [15].

We say that a property holds (p,μ)-a.e. on a curve family Γ if it holds except on a subfamily Γ' of Γ with $\mathrm{M}_p(\Gamma';\mu) = 0$. Further a function u on D is called (p,μ)-precise if u is absolutely continuous along (p,μ)-a.e. curve in D and the partial derivatives of u are L^p-integrable with respect to μ. When μ is the Lebesgue measure on D, we write $\mathrm{M}_p(\cdot;D)$ and p-precise instead of $\mathrm{M}_p(\cdot;\mu)$ and (p,μ)-precise, respectively. We say that u is called locally p-precise in D if u is p-precise on every relatively compact open subset of D. Note that if u is locally p-precise in D, then u is ACL on D and the partial derivatives of u are Borel measurable (see [12, Theorem 4.6]).

For $E, F \subset \overline{D}$, we denote by $\Lambda_D(E,F)$ the family of all curves $\gamma : [a,b] \to \overline{D}$ such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. For simplicity, set $\Lambda_D(F) = \Lambda_D(D,F)$.

Our aim in this paper is to show the following theorem.

Theorem 1. *Let φ be a positive nonincreasing function on the interval $(0,\infty)$ satisfying*

$(\varphi 1)$ $\qquad A^{-1}\varphi(r) \le \varphi(r^2) \le A\varphi(r) \qquad$ *for all* $r > 0$

with a constant $A \geq 1$ *and*

$$(\varphi 2) \qquad \int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr = \infty.$$

Let ω *be a positive continuous function on a domain* D *and set* $d\mu(x) = \omega(x)dx$. *Suppose* u *is a locally* p*-precise function on* D *satisfying*

$$(1) \qquad \int_{D_{u,\varepsilon}} |\nabla u(x)|^p d\mu(x) \leq \varepsilon^p \varphi(\varepsilon) \qquad \text{for every } \varepsilon > 0$$

where $D_{u,\varepsilon} = \{x \in D : |u(x)| < \varepsilon\}$. *If there exists a set* $F \subset \partial D$ *such that* $M_p(\Lambda_D(F); \mu) > 0$ *and* u *tends to zero along* (p, μ)*-a.e. curve* $\gamma \in \Lambda_D(F)$, *then* $u = 0$ *in* D.

Remark 1. The existence of boundary limits was studied by many authors. Carleson [2] showed the existence of nontangential limits for harmonic functions in weighted Sobolev classes in connection with the convergence property of Fourier series. We know that a locally p-precise function u on D satisfying

$$\int_D |\nabla u(x)|^p d\mu(x) < \infty$$

has a finite limit along (p, μ)-a.e. curve $\gamma \in \Lambda_D(\partial D)$, which is denoted by $u(\gamma)$ (see e.g. Ohtsuka [12], Väisälä [14], Vuorinen [15] and Ziemer [16, 17]). Here we note that u tends to zero along (p, μ)-a.e. curve $\gamma \in \Lambda_D(F)$ if u approaches zero in the weak sense for a bounded set $F \subset \partial D$.

Remark 2. The boundary uniqueness for analytic functions f on the unit disk $U \subset \mathbf{C}$ (complex plane) with $|f'| \in L^2(U)$ was first studied by Tsuji [13]. Mizuta [11] treated p-precise functions $u \in W^{1,p}(\mathbf{B})$, whose extension u^* to $\mathbf{R}^n$ vanishes on a set $F \subset \partial \mathbf{B}$ of positive p-capacity. We see that u tends to zero along p-a.e. rectifiable curve $\gamma \in \Lambda_{\mathbf{B}}(F)$ and $M_p(\Lambda_{\mathbf{B}}(F); \mathbf{B}) > 0$ (cf. Remark 1 and Lemma 7). Recently, Miklyukov-Vuorinen [8] has extended these results to a bounded domain in the non-weighted case.

For $d\mu(x) = \omega(x)dx$ with $\omega(x) = |1 - |x||^\alpha dx$, $-1 < \alpha < p - 1$, we consider a locally p-precise function u on $\mathbf{B}$ satisfying

$$\int_{\mathbf{B}} |\nabla u(x)|^p d\mu(x) < \infty.$$

In view of [9], we can find a (p, μ)-precise extension u^* on $\mathbf{R}^n$ such that $u^* = u$ on $\mathbf{B}$ and

$$\int_{\mathbf{R}^n} |\nabla u^*(x)|^p d\mu(x) < \infty.$$

Note that u^* is uniquely determined on $\overline{\mathbf{B}}$ except for (p,μ)-capacity zero.

Corollary 1. *Let φ be as in Theorem* 1 *and* $-1 < \alpha < p-1$. *Let u be a locally p-precise function on* $\mathbf{B}$ *satisfying*

$$(2) \qquad \int_{\mathbf{B}_{u,\varepsilon}} |\nabla u(x)|^p (1-|x|)^\alpha dx \le \varepsilon^p \varphi(\varepsilon) \qquad \text{for every } \varepsilon > 0.$$

If u^* vanishes on a set $E \subset \partial\mathbf{B}$ with $\mathrm{cap}_{p,\mu}(E) > 0$, *then* $u = 0$ *in* $\mathbf{B}$.

Our theorem is sharp, as the following result shows.

Theorem 2. *Let φ be a positive nonincreasing function on the interval* $(0,\infty)$ *satisfying* $(\varphi 1)$ *and*

$$(\varphi 3) \qquad \int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr < \infty.$$

Let ω be a positive continuous function on $\mathbf{B}$ *such that*

$$\omega(x) \le C(1-|x|)^\alpha \qquad \text{for all } x \in \mathbf{B}$$

with a positive constant C and a nonpositive constant α. Then there exists a p-precise and continuous function u on $\mathbf{R}^n$ such that $u > 0$ on $\mathbf{B}$, $u = 0$ *outside* $\mathbf{B}$ *and*

$$(3) \qquad \int_{\mathbf{B}_{u,\varepsilon}} |\nabla u(x)|^p \omega(x) dx \le \varepsilon^p \varphi(\varepsilon) \qquad \text{for every } \varepsilon > 0.$$

2. Proof of Theorem 1

Let D be a domain and μ be a Borel measure on D with a positive continuous density. Consider a nonnegative Borel function h on D which is L^p-integrable with respect to μ. We say that two points x and y in D are h-equivalent if there exists a rectifiable curve $\gamma \in \Lambda_D(\{x\},\{y\})$ such that $\int_\gamma h ds < \infty$. It is clear that this is an equivalence relation in D which partitions D into h-equivalence classes; each class consists of all points which are h-equivalent to a given one. Further note that there exists an h-equivalence class $E_D(h)$ which contains almost all points of D.

First we collect several lemmas from Ohtsuka [12], whose proofs will be given for the reader's convenience.

Let us begin with the following lemma.

Lemma 1. *Let $D \subset \mathbf{R}^n$ be a domain and μ be a Borel measure on D with a positive continuous density. If $\Gamma_\infty(D)$ denotes the family of all curves γ such that the linear measure of $\gamma \cap K$ is infinity for some compact set $K \subset D$, then*

$$\mathrm{M}_p(\Gamma_\infty(D); \mu) = 0.$$

To prove this, for a compact set $K \subset D$, letting $\Gamma_\infty(D; K)$ be the family of all curves γ such that the linear measure of $\gamma \cap K$ is infinity, we have only to see that

$$\mathrm{M}_p(\Gamma_\infty(D; K); \mu) = 0.$$

Lemma 2. *Let $D \subset \mathbf{R}^n$ be a domain and μ be a Borel measure on D with a positive continuous density. Then, for a set $E \subset D$, the following assertions are equivalent:*

(i) $\mathrm{M}_p(\Lambda_D(E); D) = 0$.

(ii) $\mathrm{M}_p(\Lambda_D(E); \mu) = 0$.

(iii) *$E \subset D \backslash E_D(h)$ for some nonnegative Borel function $h \in L^p(D; \mu)$.*

(iv) *E has p-capacity zero.*

Proof. Clearly (i) is equivalent to (ii). Since the equivalence of (i) and (iv) can be carried out in a way similar to that of [16, Theorem 4.3], we have only to check the equivalence of (ii) and (iii). Assume that (ii) holds. Then there exists a nonnegative Borel function $h \in L^p(D; \mu)$ such that $\int_\gamma h\, ds = \infty$ for all $\gamma \in \Lambda_D(E)$. It follows from the definition of $E_D(h)$ that $E \subset D \backslash E_D(h)$.

Conversely, assume that (iii) holds, that is, there exists a nonnegative Borel function $h \in L^p(D; \mu)$ such that $E \subset D \backslash E_D(h)$. Since $D \backslash E_D(h)$ has measure zero, we obtain $\mathrm{M}_p(\Gamma; \mu) = 0$ for the family Γ of curves γ in D satisfying $|\gamma \cap (D \backslash E_D(h))| > 0$ or $\int_\gamma h\, ds = \infty$. It suffices to show that $\Lambda_D(E) \backslash (\Gamma_\infty(D) \cup \Gamma)$ is empty. If $\Lambda_D(E) \backslash (\Gamma_\infty(D) \cup \Gamma)$ has a curve γ terminating at $x \in E$, then there exists a point $y \in E_D(h) \cap \gamma$. Since γ is rectifiable and $\int_\gamma h\, ds < \infty$, two points x and y are h-equivalent, so that $x \in E_D(h)$. This gives a contradiction by (iii).

Lemma 3. *Let $D \subset \mathbf{R}^n$ be a domain and μ be a Borel measure on D with a positive continuous density. If E and F are subsets of D which have positive p-capacity, then*

$$\mathrm{M}_p(\Lambda_D(E, F); \mu) > 0.$$

Proof. Suppose $\mathrm{M}_p(\Lambda_D(E, F); \mu) = 0$ on the contrary. Then there exists a nonnegative Borel function $h \in L^p(D; \mu)$ such that

$$\int_\gamma h\, ds = \infty$$

for each $\gamma \in \Lambda_D(E, F)$. In view of Lemma 2, there are $x \in E \cap E_D(h)$ and $y \in$

$F \cap E_D(h)$. Since x and y are h-equivalent, this gives a contradiction by the definition of $E_D(h)$.

LEMMA 4. *Let $D \subset \mathbf{R}^n$ be a domain, $F \subset \bar{D}$ and μ be a Borel measure on D with a positive continuous density. Suppose $E \subset D$ is a set of positive p-capacity and G is a relatively compact subset of D with $\mathrm{M}_p(\Lambda_D(G,F);\mu) > 0$. Then $\mathrm{M}_p(\Gamma;\mu) > 0$ for the family Γ consisting of $\gamma \in \Lambda_D(E,F)$ intersecting G.*

PROOF. Suppose $\mathrm{M}_p(\Gamma;\mu) = 0$ on the contrary. Then there exists a nonnegative Borel function $h \in L^p(D;\mu)$ such that $\int_\gamma h\,ds = \infty$ for each $\gamma \in \Gamma$. We may assume that h has positive lower bound in a neighborhood U of G, if we replace h by $h+1$ in U. We denote by $G' \subset G$ the set of all points x such that $\int_\gamma h\,ds = \infty$ for all $\gamma \in \Lambda_D(\{x\},E)$. Then we see that $\mathrm{M}_p(\Lambda_D(G',E);\mu) = 0$, which gives $\mathrm{M}_p(\Lambda_D(G');\mu) = 0$ by Lemmas 2 and 3. Since any curve $\gamma \in \Lambda_D(G',F)$ contains a subcurve in $\Lambda_D(G')$, we have

$$\mathrm{M}_p(\Lambda_D(G',F);\mu) = 0.$$

Since h has positive lower bound in U, if a curve $\gamma \in \Lambda_D(\{x\})$ for $x \in G$ satisfies $\int_\gamma h\,ds < \infty$, then there exists a rectifiable subcurve $\gamma' \in \Lambda_D(\{x\})$ of γ. Hence by the definition of G', we have $\mathrm{M}_p(\Lambda_D(G \backslash G',F);\mu) = 0$, so that $\mathrm{M}_p(\Lambda_D(G,F);\mu) = 0$. This contradicts our assumption. Now our lemma is proved.

LEMMA 5. *Let $D \subset \mathbf{R}^n$ be a domain and μ be a Borel measure on D with a positive continuous density. Suppose $F \subset \partial D$. Then $\mathrm{M}_p(\Lambda_D(F);\mu) = 0$ if and only if there exists a set $E \subset D$ such that $\mathrm{M}_p(\Lambda_D(E);D) > 0$ and $\mathrm{M}_p(\Lambda_D(E,F);\mu) = 0$.*

PROOF. Assume that there exists a subset E of D such that $\mathrm{M}_p(\Lambda_D(E);D) > 0$ and $\mathrm{M}_p(\Lambda_D(E,F);\mu) = 0$. For a proof of $\mathrm{M}_p(\Lambda_D(F);D) = 0$, it suffices to show that $\mathrm{M}_p(\Lambda_D(G,F);\mu) = 0$ for all relatively compact subset G of D. Suppose $\mathrm{M}_p(\Lambda_D(G,F);\mu) > 0$ for some relatively compact subset G of D. It follows from lemma 4 that $\mathrm{M}_p(\Gamma;\mu) > 0$ for the family Γ of curves $\gamma \in \Lambda_D(E,F)$ intersecting G. Hence we have $\mathrm{M}_p(\Lambda_D(E,F);\mu) > 0$, which gives a contradiction.

The converse is evident.

Here we prepare the following technical lemma needed for the proof of Theorem 1.

LEMMA 6 (cf. [11, Lemma 2]). *Let φ be a positive nonincreasing function on the interval $(0,\infty)$ satisfying (φ2). Then there exists a positive nondecreasing function h satisfying*

$$\int_0^1 h(r) r^{-1} dr = \infty \tag{h1}$$

and

$$\text{(h2)} \qquad \int_0^1 h(r)^p \varphi(r) r^{-1} dr < \infty.$$

Now we give a proof of Theorem 1.

Proof of Theorem 1. Set

$$E = \{x \in D : |u(x)| > 0\}$$

and suppose that $M_p(\Lambda_D(E); D) > 0$. Next, we define a function ρ by

$$\rho(x) = \sum_{j=1}^{\infty} 2^j h(2^{-j}) |\nabla u(x)| \chi_{G_j}(x),$$

where h is as in Lemma 6 and χ_{G_j} denotes the characteristic function of $G_j = \{x \in D : 2^{-j} < |u(x)| \le 2^{-j+1}\}$. Then we have by (1) and (h2)

$$\begin{aligned}
\int \rho(x)^p \omega(x) dx &= \sum_{j=1}^{\infty} [2^j h(2^{-j})]^p \int_{G_j} |\nabla u(x)|^p \omega(x) dx \\
&\le \sum_{j=1}^{\infty} [2^j h(2^{-j})]^p 2^{(-j+1)p} \varphi(2^{-j+1}) \\
&= 2^p \sum_{j=1}^{\infty} h(2^{-j})^p \varphi(2^{-j+1}) \\
&\le 2^{p+1} \int_0^1 h(r)^p \varphi(r) r^{-1} dr < \infty.
\end{aligned}$$

In view of Lemma 2, we note that a function v on D is absolutely continuous along p-a.e. curve in D if and only if v is absolutely continuous along (p, μ')-a.e. curve in D for all μ' with a positive continuous density ω' in D. Hence u is absolutely continuous along all curves in D except for a family Γ_1 with $M_p(\Gamma_1; \mu) = 0$. By our assumption, there exists a subfamily $\Gamma_2 \subset \Lambda_D(F)$ such that $M_p(\Gamma_2; \mu) = 0$ and u tends to zero along each $\gamma \in \Lambda_D(F) \setminus \Gamma_2$. Fix a locally rectifiable curve $\gamma \in \Lambda_D(E, F) \setminus (\Gamma_1 \cup \Gamma_2)$. Then for large j $(j \ge j_0)$ there exists a subcurve $\gamma_j \subset G_j$ of γ such that

$$\int_{\gamma_j} |\nabla u| ds \ge 2^{-j}.$$

It follows from (h1) that

$$\int_\gamma \rho\, ds \geq \sum_{j=j_0}^{\infty} 2^j h(2^{-j}) \int_{\gamma_j} |\nabla u| ds$$
$$\geq \sum_{j=j_0}^{\infty} h(2^{-j})$$
$$\geq 2\int_0^{2^{-j_0}} h(r) r^{-1} dr = \infty.$$

Thus we can easily see that $\mathrm{M}_p(\Lambda_D(E,F)\backslash(\Gamma_1 \cup \Gamma_2);\mu) = 0$. Therefore we have

$$\mathrm{M}_p(\Lambda_D(E,F);\mu) = 0,$$

which gives a contradiction by Lemma 5, since $\mathrm{M}_p(\Lambda_D(F);\mu) > 0$.

3. Proof of Corollary 1

The Riesz capacity of index (β, p) is denoted by $\mathrm{C}_{\beta,p}$; for its definition we refer the reader to [10].

Now Corollary 1 is obtained from Theorem 1 and the following lemma.

Lemma 7. *Let* $F \subset \partial\mathbf{B}$ *and* $d\mu(x) = |1 - |x||^\alpha dx$ *with* $-1 < \alpha < p - 1$. *Then the following assertions are equivalent:*
(a) $\mathrm{M}_p(\Lambda_\mathbf{B}(F);\mu) = 0$.
(b) $\mathrm{cap}_{p,\mu}(F) = 0$.
(c) $\mathrm{C}_{1-\alpha/p,p}(F) = 0$.

Proof. It is well known that (b) is equivalent to (c) (see [10, Lemma 8.3.3]). Clearly (b) implies (a). Hence it remains to check that (a) implies (c).

Suppose that $\mathrm{M}_p(\Lambda_\mathbf{B}(F);\mu) = 0$. Then there exists a positive function h in $\mathbf{B}$ such that

$$\int_\mathbf{B} h(x)^p d\mu(x) < \infty \tag{4}$$

and

$$\int_\gamma h\, ds = \infty \tag{5}$$

for each locally rectifiable $\gamma \in \Lambda_\mathbf{B}(F)$. Here we assume that $h = 0$ in $\mathbf{R}^n\backslash\mathbf{B}$ and set

$$K = \left\{ x \in \partial\mathbf{B} : \int_{B(x,1)} |x-y|^{1-n} h(y) dy = \infty \right\}.$$

Note here that

$$C_{1-\alpha/p,p}(K) = 0$$

(see [10, Lemma 8.2.3]).

Fix $x \in F$. It follows from (5) that

$$\int_0^1 h(x + r\zeta)dr = \infty$$

for all ζ with $x + \zeta \in \mathbf{B}$. Hence we have

$$\int_{B(x,1)} |x - y|^{1-n} h(y)dy = \int_{\partial \mathbf{B}} \left(\int_0^1 h(x + r\zeta)dr \right) d\mathscr{H}^{n-1}(\zeta) = \infty.$$

This implies that $F \subset K$ and (c) follows.

Remark 3. Let D be a bounded (ε, δ) domain in $\mathbf{R}^n$ due to Jones [6] and denote by $\rho_D(x)$ the distance of $x \in \mathbf{R}^n$ from the boundary ∂D. In view of Chua [3], if $\rho_D(x)^\alpha$ is in the Muckenhoupt class A_p, then every locally p-precise function u on D satisfying

$$\int_D |\nabla u(x)|^p \rho_D(x)^\alpha dx < \infty$$

can be extended to a function u^* on $\mathbf{R}^n$ such that $u^* = u$ on D and

$$\int_{\mathbf{R}^n} |\nabla u^*(x)|^p \rho_D(x)^\alpha dx < \infty.$$

Hence Corollary 1 is also valid for every bounded (ε, δ) domain in $\mathbf{R}^n$.

4. Proof of Theorem 2

For a proof of Theorem 2, we need the following lemma.

Lemma 8 (cf. [10, Lemma 5.3.1]). *Let φ be a positive nonincreasing function on the interval $(0, \infty)$ satisfying $(\varphi 1)$. If $a > 0$, then there exists a positive constant $M = M(a)$ such that*

$$(\varphi 4) \qquad s^a \varphi(s) \le M t^a \varphi(t) \qquad \textit{whenever } t > s > 0.$$

We are now ready to prove Theorem 2. It suffices to show the case that $\omega(x) = (1 - |x|)^\alpha$ with $\alpha \le 0$. Suppose

$$\int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr < \infty.$$

Set

$$f(t) = C\left(\int_0^t [\varphi(r)]^{-1/(p-1)} r^{-1} dr\right)^{(p-1)/(p-1-\alpha)}$$

where C is a positive constant. We can easily see that f is a positive increasing continuous function on $(0, \infty)$ and

$$f'(t) = C^{s_0} s_0^{-1} f(t)^{\alpha/(p-1)} \varphi(t)^{-1/(p-1)} t^{-1},$$

where $s_0 = (p-1-\alpha)/(p-1)$. Consider the function $u \in C(\mathbf{R}^n)$ given by

$$u(x) = \begin{cases} f^{-1}(1-|x|) & \text{if } x \in \mathbf{B}, \\ 0 & \text{if } x \notin \mathbf{B}. \end{cases}$$

We have only to show that u satisfies (3) if we choose C large enough.

For small $\varepsilon > 0$, we can find $\delta > 0$ such that $\varepsilon = f^{-1}(\delta)$. Then

$$\mathbf{B}_{u,\varepsilon} = \{x \in \mathbf{R}^n : 1 - \delta < |x| < 1\}.$$

Hence we have by Lemma 8

$$\begin{aligned}
\int_{\mathbf{B}_{u,\varepsilon}} |\nabla u(x)|^p (1-|x|)^\alpha dx &= \int_{\{1-\delta<|x|<1\}} |(f^{-1})'(1-|x|)|^p (1-|x|)^\alpha dx \\
&= \sigma_n \int_{1-\delta}^1 |(f^{-1})'(1-r)|^p (1-r)^\alpha r^{n-1} dr \\
&\le \sigma_n \int_0^\delta |(f^{-1})'(s)|^p s^\alpha ds \\
&= \sigma_n \int_0^\varepsilon |f'(t)|^{-p+1} f(t)^\alpha dt \\
&= \sigma_n C^{\alpha-p+1} \int_0^\varepsilon \varphi(t) t^{p-1} dt \\
&\le \sigma_n C^{\alpha-p+1} M \varepsilon^p \varphi(\varepsilon),
\end{aligned}$$

where σ_n denotes the surface area of the unit sphere and M is a positive constant independent of ε. If we take C such that $\sigma_n C^{\alpha-p+1} M = 1$, then u satisfies (3).

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