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Cobordism group with local coefficients and its application to 4-manifolds

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ABSTRACT. For a pair (X, A) of topological spaces and $w \in H^1(X; \mathbf{Z}_2)$ the cobordism group $\Omega_n(X, A; \mathscr{S}_w)$ with local coefficients is introduced. If X is a CW complex and $\mathscr{S}_w$ is a local system over X determined by w, then we have an Atiyah-Hirzeburch spectral sequence $E^2_{p,q} = H_p(X; \Omega_q \otimes \mathscr{S}_w) \Rightarrow \Omega_{p+q}(X; \mathscr{S}_w)$ which is regular and hence convergent. For a connected CW complex X the map $\mu : \Omega_4(X; \mathscr{S}_w) \to H_4(X; \mathscr{S}_w)$, defined by $\mu([M, f, \varphi]) = f_*(\varphi_*(\sigma))$, is a surjection and its kernel is $\Omega_4 \otimes \mathbf{Z}_2$ if $w \neq 0$, where σ is a fundamental homology class with respect to the orientation sheaf of a manifold M and φ is a local orientation. The closed 4-manifolds with finitely presentable fundamental group π and the first Stiefel-Whitney class induced from w are almost classified modulo connected sums with simply connected manifolds by the quotient $H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)^w_*$, and precisely in the case that π is abelian.

1. Introduction

The oriented cobordism functor $\{\Omega_*(X, A), \varphi_*, \partial\}$ satisfies the first six Eilenberg-Steenrod axioms for the category of pairs of topological spaces and maps [2]. So, for any CW complex X the Atiyah-Hirzeburch spectral sequence

$$E^2_{p,q} = H_p(X; \Omega_q) \Rightarrow \Omega_{p+q}(X)$$

is regular and hence convergent in the sense of [1]. Using this spectral sequence, the classification of oriented closed 4-manifolds having the finitely presentable fundamental group π modulo connected sums with simply connected manifolds is given by the quotient $H_4(B\pi; \mathbf{Z})/(\mathrm{Aut}\,\pi)_*$ [4], [7].

Our goal of this paper is to extend the above result to the non-orientable case. We introduce a cobordism group $\Omega_n(X, A; \mathscr{S}_w)$ for a pair (X, A) of topological spaces and $w \in H^1(X; \mathbf{Z}_2)$, which reduces to $\Omega_n(X, A)$ if $w = 0$. Let $w_1 : BO_r \to K(\mathbf{Z}_2, 1)$ be the map corresponding to the first Stiefel-Whitney class. Consider w to be a map of X to $K(\mathbf{Z}_2, 1)$, and let

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$$\begin{array}{ccc} B_r & \longrightarrow & X \\ {\scriptstyle f_r}\Big\downarrow & & \Big\downarrow{\scriptstyle w} \\ BO_r & \xrightarrow[w_1]{} & K(\mathbf{Z}_2, 1) \end{array}$$

be the pull-back. Then $\Omega_n(X; \mathscr{S}_w)$ coincides with $\Omega_n(B, f)$ given by Stong in [12, p. 17]. We show that this cobordism group has the properties similar to the oriented cobordism group.

For a pair of points $x, y \in X$ we denote by $\Gamma(y, x)$ the set of relative homotopy classes of paths from x to y. Let $\mathscr{S}$ be a family $\{\mathscr{S}(x), \mathscr{S}(\gamma)\}$ satisfying the following conditions, which will be called a local system (of abelian groups) over X:

(1) for each $x \in X$, $\mathscr{S}(x)$ is an abelian group,

(2) for each $\gamma \in \Gamma(y, x)$, $\mathscr{S}(\gamma)$ is an isomorphism of $\mathscr{S}(y)$ to $\mathscr{S}(x)$ and

(3) $\mathscr{S}(\gamma\gamma') = \mathscr{S}(\gamma) \circ \mathscr{S}(\gamma')$ for any $\gamma \in \Gamma(y, x)$ and $\gamma' \in \Gamma(z, y)$.

By the definition we see that $\mathscr{S}$ induces a homomorphism $\overline{\mathscr{S}}_x : \pi_1(X, x) \to \operatorname{Aut} \mathscr{S}(x)$ defined by $\overline{\mathscr{S}}_x(\alpha) = \mathscr{S}(\alpha)$ $(\alpha \in \pi_1(X, x))$ for each $x \in X$. Fix $x_0 \in X$ and choose an element $\alpha_x \in \Gamma(x, x_0)$ for each $x \in X$. Then we see also that

$$\mathscr{S}(\gamma) = \mathscr{S}(\alpha_x)^{-1} \circ \overline{\mathscr{S}}_{x_0}(\alpha_x \gamma \alpha_y^{-1}) \circ \mathscr{S}(\alpha_y)$$

for each $\gamma \in \Gamma(y, x)$. When X is arcwise connected and G is an abelian group, any homomorphism $\rho : \pi_1(X, x_0) \to \operatorname{Aut} G$ induces one and only one local system over X such that $\mathscr{S}(x_0) = G$ and $\overline{\mathscr{S}}_{x_0} = \rho$ [10], which is called a local system determined by ρ.

For $w \in H^1(X; \mathbf{Z}_2)$ let $\mathscr{S}_w$ be a local system over X which satisfies the following conditions.

(1.1) For each $x \in X$, $\mathscr{S}_w(x)$ is isomorphic to the group $\mathbf{Z}$ of integers.

(1.2) $\mathscr{S}_w$ is determined by the homomorphism $\rho_w : \pi_1(X, x_0) \to \operatorname{Aut} \mathbf{Z}$.

Here ρ_w is a composite of the Hurewicz homomorphism $\Xi : \pi_1(X, x_0) \to H_1(X; \mathbf{Z})$ with w considered as a homomorphism from $H_1(X; \mathbf{Z})$ to $\operatorname{Aut} \mathbf{Z} = \mathbf{Z}_2$.

We will prove the following theorem.

THEOREM 1. *Let X be a CW complex and $w \in H^1(X; \mathbf{Z}_2)$. Then we have a spectral sequence*

$$E^2_{p,q} = H_p(X; \Omega_q \otimes \mathscr{S}_w) \Rightarrow \Omega_{p+q}(X; \mathscr{S}_w)$$

which is regular and hence convergent.

For an n-manifold N the orientation sheaf $\mathscr{S}_N$ is defined as follows.

(2.1) $\mathscr{S}_N(u) = H_n(N, N - u; \mathbf{Z})$ for each $u \in \operatorname{Int} N$ and $\mathscr{S}_N(u) = H_{n-1}(\partial N, \partial N - u; \mathbf{Z})$ for each $u \in \partial N$.

(2.2) $\mathscr{S}_N$ is determined by the homomorphism $\rho_N = w_1(N) \circ \Xi$, where Ξ is the Hurewicz homomorphism and $w_1(N)$ is the first Stiefel-Whitney class of N.

Now we define $\Omega_n(X; \mathscr{S}_w)$ assuming the notion of equivalence between local systems. We consider a pair of a closed n-manifold M and a continuous map $f : M \to X$ such that $\mathscr{S}_M$ and the induced local system $f^*\mathscr{S}_w$ are equivalent. Let $\varphi = \{\varphi_u\}_{u \in M}$ denote the family of isomorphisms $\varphi_u : \mathscr{S}_M(u) \to f^*\mathscr{S}_w(u)$ which gives this equivalence (See §2). Let $\mathscr{M}_n(X; \mathscr{S}_w)$ be the set which consists of such triples (M, f, φ). We define the equivalence relation in $\mathscr{M}_n(X; \mathscr{S}_w)$ as follows. $(M_1, f_1, \varphi_1) \sim (M_2, f_2, \varphi_2)$ means that there exist a compact $(n+1)$-manifold W and a map $F : W \to X$ satisfying the following conditions:

(1) $\partial W = M_1 \cup M_2$,

(2) $F|M_j = f_j$ $(j = 1, 2)$,

(3) there exists an equivalence $\Phi : \mathscr{S}_W \to F^*\mathscr{S}_w$ such that $\dot{\Phi} = \Phi \mid \partial W : \mathscr{S}_{\partial W} \to F^*\mathscr{S}_w|\partial W$ satisfies $\dot{\Phi}|M_1 = \varphi_1$ and $\dot{\Phi}|M_2 = -\varphi_2$.

The set of equivalence classes $\mathscr{M}_n(X; \mathscr{S}_w)/\sim$ has a natural group structure and is denoted by $\Omega_n(X; \mathscr{S}_w)$ and called a cobordism group with local coefficients. We use the notation $[M, f, \varphi]$ for the cobordism class in $\Omega_n(X; \mathscr{S}_w)$.

Since φ induces an isomorphism $\varphi_* : H_n(M; \mathscr{S}_M) \to H_n(M; f^*\mathscr{S}_w)$, we can define a homomorphism

$$\mu : \Omega_n(X; \mathscr{S}_w) \to H_n(X; \mathscr{S}_w)$$

by $\mu([M, f, \varphi]) = f_*(\varphi_*(\sigma))$, where σ is the fundamental class in $H_n(M; \mathscr{S}_M)$. We may call φ a local orientation of M associated with f. We have only two local orientations $\pm\varphi$ associated with f provided that M is connected.

Using Theorem 1 we will get the following corollary.

Corollary 2. *Let X be a connected CW complex and $w \in H^1(X; \mathbf{Z}_2)$. The map $\mu : \Omega_4(X; \mathscr{S}_w) \to H_4(X; \mathscr{S}_w)$ is a surjection and the kernel is Ω_4 if $w = 0$, and $\Omega_4 \otimes \mathbf{Z}_2$ if $w \neq 0$.*

Let π be a finitely presentable group, $B\pi = K(\pi, 1)$ be an Eilenberg-MacLane complex and w be an element of $H^1(B\pi; \mathbf{Z}_2)$. We consider the set $\mathscr{M}^4_{\pi, w}$ consisting of the closed connected 4-manifolds M such that $\pi_1(M) = \pi$ and $w_1(M) = w$, or more precisely, there is a map $f : M \to B\pi$ satisfying

(3.1) f induces an isomorphism on π_1, that is, $f_* : \pi_1(M, u) \to \pi_1(B\pi, f(u))$ is isomorphism for any u, and

(3.2) $f^*w = w_1(M) \in H^1(M; \mathbf{Z}_2)$.

By Proposition 15 in §7 $\mathscr{M}^4_{\pi, w}$ is not empty. For every $M \in \mathscr{M}^4_{\pi, w}$ there exists an element (M, f, φ) of $\mathscr{M}_4(B\pi; \mathscr{S}_w)$ by Proposition 11 in §6. For a non-zero

w Proposition 13 in §6 says that $[M, f, \varphi] = [M, f, -\varphi]$ in $\Omega_4(B\pi; \mathscr{S}_w)$ under some condition which is automaticaly satisfied when π is abelian.

We will say that closed connected 4-manifolds M and N are weakly stably equivalent, if there exist closed simply connected 4-manifolds M_0 and N_0 such that $M \sharp M_0$ and $N \sharp N_0$ are diffeomorphic to each other. Let $(\mathrm{Aut}\,\pi)^w$ be the subgroup of $\mathrm{Aut}\,\pi$ consisting of the elements whose corresponding classifying base point preserving maps $\lambda : B\pi \to B\pi$ satisfy $\lambda^* w = w$ on $H^1(B\pi; \mathbf{Z}_2)$.

Then we can extend Theorem 1 in [7] to the non-orientable case at least in the case of abelian fundamental groups.

THEOREM 3. *Let π be a finitely generated abelian group and w be a non-trivial element of $H^1(B\pi; \mathbf{Z}_2)$. Then, the set of weakly stable equivalence classes in $\mathscr{M}^4_{\pi, w}$ is in one-to-one correspondence with the quotient $H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)^w_*$ by the correspondence $(M, f, \varphi) \mapsto f_*(\varphi_*(\sigma))$, where σ is the fundamental homology class of M with local coefficients $\mathscr{S}_M$.*

A more general form of Theorem 3 (Theorem 20 in §7) implies the following theorem which characterizes the Lusternik-Schnirelmann π_1-category of closed connected 4-manifolds including both the orientable and non-orientable cases.

THEOREM 4. *If the Lusternik-Schnirelmann π_1-category of a connected closed 4-manifold M is not 4, then M is weakly stably equivalent to the boundary $\partial N(K^2)$ of the regular neighborhood of an embedded finite 2-complex K^2 in $\mathbf{R}P^4 \times \mathbf{R}$ realizing the fundamental group $\pi = \pi_1(M)$ and $\rho_{w_1(M)} : \pi \to \mathrm{Aut}\,\mathbf{Z}$.*

We recall the notion of equivalence between local systems and define the relative cobordism group with local coefficients in §2, and we describe the properties of cobordism group with local coefficiens in §3. We prove Theorem 1 in §4 and then we compute some cobordism groups with local coefficients and prove Corollary 2 in §5. We discuss the relation of local orientations and cobordism classes in §6 and we prove Theorem 3, its generalized form Theorem 20, and Theorem 4 in §7. Finally we give some calculations of $H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)^w_*$ in §8.

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2. Cobordism group with local coefficients

Let M be a compact n-manifold, and f a map of $(M, \partial M)$ into (X, A). If $A = \phi$ then $\partial M = \phi$. We denote by $f^*\mathscr{S}_w$ the local system over M induced from $\mathscr{S}_w$ by f, that is, $f^*\mathscr{S}_w(u) = \mathscr{S}_w(f(u))$ for $u \in M$ and $f^*\mathscr{S}_w(\gamma) = \mathscr{S}_w(f_*\gamma)$ for $\gamma \in \Gamma(u', u)$.

If the following conditions are satisfied, two local systems $\mathscr{S}, \mathscr{T}$ over M are called equivalent, and denoted by $\varphi : \mathscr{S} \xrightarrow{\sim} \mathscr{T}$.

(4.1) For every $u \in M$, there exists an isomorphism $\varphi_u : \mathscr{S}(u) \to \mathscr{T}(u)$.

(4.2) For every pair of points $u, v \in M$ and every homotopy class γ of path from v to u, the following diagram is commutative.

$$\begin{array}{ccc} \mathscr{S}(u) & \xrightarrow{\varphi_u} & \mathscr{T}(u) \\ {\scriptstyle \mathscr{S}(\gamma)}\downarrow & & \downarrow{\scriptstyle \mathscr{T}(\gamma)} \\ \mathscr{S}(v) & \xrightarrow[\varphi_v]{} & \mathscr{T}(v) \end{array}$$

Now we define a w-singular manifold (M, f, φ) of dimension n in (X, A) by the following three conditions.

(5.1) M is a compact n-manifold.

(5.2) f is a continuous map from $(M, \partial M)$ into (X, A).

(5.3) $\varphi : \mathscr{S}_M \to f^*\mathscr{S}_w$ is an equivalence.

We recall here the definition of the isomorphism $\mathscr{S}_M(\alpha)$ for the relative homotopy class α of any path from u to v. For each point $u \in \operatorname{Int} M$ there exists an open neighborhood U of u with a homeomorphism $h : (U, u) \to (\mathbf{R}^n, 0)$. We put $D(r) = \{x \in \mathbf{R}^n; ||x|| \le r\}$ and $U(r) = h^{-1}(\operatorname{Int} D(r))$ for a positive number r. Then the inclusion $i_u^{U(r)} : (M, M - U(r)) \to (M, M - u)$ induces an isomorphism $i_{u*}^{U(r)} : H_n(M, M - U(r); \mathbf{Z}) \to H_n(M, M - u; \mathbf{Z})$. For another choice of open neghborhood U' of u, a homeomorphism h', and a positive number r' we write $U'(r')$ as above. If $U'(r') \subset U(r)$, then the homomorphism $i_{U'(r')_*}^{U(r)}$ induced by the inclusion $i_{U'(r')}^{U(r)} : (M, M - U(r)) \to (M, M - U'(r'))$ coincides with the isomorphism $(i_{u*}^{U'(r')})^{-1} \circ i_{u*}^{U(r)}$. The set $\mathscr{B}$ consisting of all $U(r)$'s obtained by changing u, U, h, r forms an open basis of M and $\mathscr{B}_u = \{U(r) \in \mathscr{B}; u \in U(r)\}$ is a directed set. Therefore $\{H_n(M, M - U(r); \mathbf{Z}), i_{U'(r')_*}^{U(r)}; u \in U(r)\}$ forms an inductive system over $\mathscr{B}_u$ and we get a canonical isomorphism

$$\lim_{\to} H_n(M, M - U(r); \mathbf{Z}) \cong H_n(M, M - u; \mathbf{Z}).$$

For any two points u, v of $\operatorname{Int} M$ and any embbeded path γ from u to v, we take a Lebesgue number ε of an open covering $\{\gamma^{-1}(U(r))\}$ of [0, 1] and a division $0 = t_0 < t_1 < \cdots < t_l = 1$ of [0, 1] such that $t_j - t_{j-1} < \varepsilon$. We put $\gamma(t_j) = u_j$. For each j $(1 \le j \le l)$ there exists some $U(r)$ which contains $\gamma([t_{j-1}, t_j])$. Denoting such $U(r)$ by $U_j(r_j)$, we define a homomorphism $\gamma_* : H_n(M, M - v; \mathbf{Z}) \to H_n(M, M - u; \mathbf{Z})$ by

$$\gamma_* = i_{u*}^{U_1(r_1)} \circ (i_{u_1*}^{U_1(r_1)})^{-1} \circ i_{u_1*}^{U_2(r_2)} \circ (i_{u_2*}^{U_2(r_2)})^{-1} \circ \cdots \circ i_{u_{l-1}*}^{U_l(r_l)} \circ (i_{u_l*}^{U_l(r_l)})^{-1}.$$

It is known that the homomorphism γ_* depends only on the homotopy class of γ keeping the boundary fixed [6]. When γ is a closed path, $\gamma^* w_1(M)$ is an obstruction to the trivialization of $\gamma^* T(M)$, where $T(M)$ is the tangent bundle of M. So, $\mathscr{S}_M([\gamma])$ is given by γ_* for any path γ connecting two points of $\mathrm{Int}\, M$. If $v \in \mathrm{Int}\, M$ and $u \in \partial M$, we choose a closed neighborhood V of u in M homeomorphic to a closed disk D^n, and choose a point $v_0 \in V \cap \mathrm{Int}\, M$ and an embbeded path δ in V from u to v_0. We can assume $\delta \cap \partial M = \{u\}$. Moreover we put $V_1 = V \cap \partial M$ and $V_2 = \overline{\partial V - V_1}$. Let $\delta_* : H_n(\mathrm{Int}\, M, \mathrm{Int}\, M - v_0; \mathbf{Z}) \to H_{n-1}(\partial M, \partial M - u; \mathbf{Z})$ be a composite of the following maps:

$$
\begin{aligned}
H_n(\mathrm{Int}\, M, \mathrm{Int}\, M - v_0; \mathbf{Z}) &\xrightarrow{i_*} H_n(M, M - v_0; \mathbf{Z}) \\
&\xrightarrow{i_{V*}^{-1}} H_n(V, \partial V; \mathbf{Z}) \xrightarrow{\partial_*} \tilde{H}_{n-1}(\partial V; \mathbf{Z}) \\
&\xrightarrow{j_*} H_{n-1}(\partial V, V_2; \mathbf{Z}) \xrightarrow{k_*^{-1}} H_{n-1}(V_1, \partial V_1; \mathbf{Z}) \\
&\xrightarrow{i_{V_1 *}} H_{n-1}(\partial M, \partial M - u; \mathbf{Z}),
\end{aligned}
$$

where $i_*, i_{V*}, j_*, k_*, i_{V_1 *}$ are isomorphisms induced by the inclusions. Then $\mathscr{S}_M([\delta])$ is given by δ_*. The composition of γ_* and δ_*'s gives the isomorphism $\mathscr{S}_M(\alpha)$ for the relative homotopy class α of any path from u to v with $u, v \in \partial M$. Note that $\mathscr{S}_M | \partial M$ is also determined by $w_1(\partial M)$ and $\mathscr{S}_M | \partial M = \mathscr{S}_{\partial M}$.

Given an equivalence $\varphi : \mathscr{S}_{\mathrm{Int}\, M} \to f^* \mathscr{S}_w | \mathrm{Int}\, M$. Then we can extend it to an equivalence $\varphi : \mathscr{S}_M \to f^* \mathscr{S}_w$ by defining $\varphi_u = \mathscr{S}_w(f_*[\delta]) \circ \varphi_{v_0} \circ \delta_*^{-1}$ for $u \in \partial M$, where δ is a path in M from u to $v_0 \in \mathrm{Int}\, M$. This remark is very useful, especially in the proof of Propositions 5 (3) and 6. We will use the notation $\dot{\varphi} : \mathscr{S}_{\partial M} \to (f|\partial M)^* \mathscr{S}_w$ as a restriction of φ on $\mathscr{S}_M | \partial M = \mathscr{S}_{\partial M}$ hereafter.

Let $\mathscr{M}_n(X, A; \mathscr{S}_w)$ be the set of all w-singular manifolds of dimension n in (X, A). For (M, f, φ), $(N, g, \psi) \in \mathscr{M}_n(X, A; \mathscr{S}_w)$, we define

$$
-(M, f, \varphi) = (M, f, -\varphi), \qquad (M, f, \varphi) + (N, g, \psi) = (M \cup N, f \cup g, \varphi \cup \psi).
$$

We say that (M, f, φ) is null cobordant: $(M, f, \varphi) \sim 0$, if there exists an element $(W, F, \Phi) \in \mathscr{M}_{n+1}(X, X; \mathscr{S}_w)$ such that $\partial(W, F, \Phi) \equiv (M, f, \varphi) \bmod A$, that is,

(6.1) M is a regular submanifold of ∂W,

(6.2) $F|M = f$ and $F(\partial W - M) \subset A$, and

(6.3) $\dot{\Phi}|\mathrm{Int}\, M = \varphi$ by identifying $H_n(\partial W, \partial W - v; \mathbf{Z})$ with $H_n(\mathrm{Int}\, M, \mathrm{Int}\, M - v; \mathbf{Z})$ for any $v \in \mathrm{Int}\, M$.

We define $(M, f, \varphi) \sim (N, g, \psi)$ when $(M, f, \varphi) + (N, g, -\psi) \sim 0$. Then we have the following proposition.

PROPOSITION 5. *The relation* $\sim$ *in* $\mathcal{M}_n(X, A; \mathcal{S}_w)$ *is an equivalence relation.*

PROOF. (1) For $(M, f, \varphi) \in \mathcal{M}_n(X, A; \mathcal{S}_w)$ let $W = M \times I$ and define a map $F : W \to X$ by

$$F(u, t) = f(u) \qquad ((u, t) \in M \times I).$$

For each $v = (u, t)$ we define a path α_v from $(u, 0)$ to v and a path β_v from $(u, 1)$ to v by

$$\alpha_v(s) = (u, st), \qquad \beta_v(s) = (u, 1 - s + st) \quad (s \in I).$$

Note that identifying u with $(u, 0)$ and $(u, 1)$ we get $\varphi_u \circ \alpha_{v*} = -\varphi_u \circ \beta_{v*}$ for $v = (u, t) \in \text{Int } W$ and we define Φ_v by this map. Then, identifying $M \times 0$ and $M \times 1$ with M, it is easy to see

$$\partial(W, F, \Phi) \equiv (M, f, \varphi) + (M, f, -\varphi) \mod A.$$

(2) The reflective law is clear.

(3) Assume that

$$\partial(W_1, F_1, \Phi_1) \equiv (M_1, f_1, \varphi_1) + (M_2, f_2, -\varphi_2) \mod A$$

$$\partial(W_2, F_2, \Phi_2) \equiv (M_2, f_2, \varphi_2) + (M_3, f_3, -\varphi_3) \mod A.$$

We glue W_1 and W_2 by identifying M_2 by a diffeomorphism which reverses the local orientation at each point, and denote the resulting manifold by W. We define a map $F : W \to X$ by $F(v) = F_i(v)$ $(v \in W_i)$ for $i = 1, 2$. For $v \in \text{Int } W_j$ $(j = 1, 2)$ the inclusion $i_j(v) : (\text{Int } W_j, \text{Int } W_j - v) \to (\text{Int } W, \text{Int } W - v)$ induces an isomorphism

$$i_j(v)_* : H_{n+1}(\text{Int } W_j, \text{Int } W_j - v; \mathbf{Z}) \to H_{n+1}(\text{Int } W, \text{Int } W - v; \mathbf{Z}).$$

If $v \in \text{Int } M_2$, we take a neighborhood U of v in $\text{Int } W$ such that $(U, U \cap M_2)$ is homeomorphic to $(\mathbf{R}^{n+1}, \mathbf{R}^n)$. We take further a point $v_j \in U \cap \text{Int } W_j$ and a path $\tilde{\alpha}_j$ from v to v_j in $U \cap W_j$ $(j = 1, 2)$. If we regard $\tilde{\alpha}_j$ as a path in $\text{Int } W$, we rewrite this α_j. Then we have isomorphisms

$$\tilde{\alpha}_{j*} : H_{n+1}(\text{Int } W_j, \text{Int } W_j - v_j; \mathbf{Z}) \to H_n(\text{Int } M_2, \text{Int } M_2 - v; \mathbf{Z}),$$

$$\alpha_{j*} : H_{n+1}(\text{Int } W, \text{Int } W - v_j; \mathbf{Z}) \to H_{n+1}(\text{Int } W, \text{Int } W - v; \mathbf{Z}).$$

Since U is simply connected, from the way of the gluing we get

$$-\tilde{\alpha}_{1*} \circ i_1(v_1)_*^{-1} \circ \alpha_{1*}^{-1} = \tilde{\alpha}_{2*} \circ i_2(v_2)_*^{-1} \circ \alpha_{2*}^{-1}.$$

So, we define Φ by

$$\Phi_v = \begin{cases} (\Phi_j)_v \circ i_j(v)_*^{-1} & (v \in \mathrm{Int}\ W_j, \quad j = 1,2) \\ -(\varphi_2)_v \circ \tilde{\alpha}_{1*} \circ i_1(v_1)_*^{-1} \circ \alpha_{1*}^{-1} & (v \in \mathrm{Int}\ M_2). \end{cases}$$

The definition is independent of the choice of $U, v_j, \tilde{\alpha}_j$. Moreover we have $\mathscr{S}_w(F_*[\alpha_j]) \circ \Phi_{v_j} = \Phi_v \circ (\alpha_j)_*$ for $v \in \mathrm{Int}\ M_2$ $(j = 1,2)$. Let $v_j \in \mathrm{Int}\ W_j$, $v \in \mathrm{Int}\ M_2$ be any point and γ_j be any path from v to v_j in W_j for $j = 1,2$. From the above equality we see that $\mathscr{S}_w(F_*[\gamma_j]) \circ \Phi_{v_j} = \Phi_v \circ (\gamma_j)_*$. This leads to $\mathscr{S}_w(F_*\gamma) \circ \Phi_{v'} = \Phi_v \circ \gamma_*$ for any points $v, v' \in \mathrm{Int}\ W$ and any $\gamma \in \Gamma(v', v)$. Hence we get an equivalence $\Phi : \mathscr{S}_{\mathrm{Int}\ W} \xrightarrow{\sim} F^*\mathscr{S}_w|\mathrm{Int}\ W$. Since this can be extended naturally to $\Phi : \mathscr{S}_W \to F^*\mathscr{S}_w$ as remaked before, we have

$$\partial(W, F, \Phi) \equiv (M_1, f_1, \varphi_1) + (M_3, f_3, -\varphi_3) \mod A. \qquad \square$$

We put $\Omega_n(X, A; \mathscr{S}_w) = \mathscr{M}_n(X, A; \mathscr{S}_w)/\sim$ and denote by $[M, f, \varphi]$ the equivalence class of (M, f, φ). By setting $[M, f, \varphi] + [N, g, \psi] = [M \cup N, f \cup g, \varphi \cup \psi]$, $\Omega_n(X, A; \mathscr{S}_w)$ has a structure of an abelian group. We call this group an n-dimensional cobordism group with local coefficients $\mathscr{S}_w$ of (X, A). If $w = 0$, then M and W are orientable; φ and Φ give the orientation of M and W respectively. Therefore $\Omega_n(X, A; \mathscr{S}_0)$ coincides with $\Omega_n(X, A)$.

The relative cobordism group may be also defined by the method of [12, p. 43], but our method makes clear the representatives and able to prove Theorems 1 and 3.

3. Properties of cobordism group with local coefficients

In this section, we study the properties of cobordism group with local coefficients needed to construct the Atiyah-Hirzeburch spectral sequence. Cobordism groups with local coefficients have properties similar to the Eilenberg-Steenrod axioms for the homology theory.

Fix $\eta \in H^1(Y; \mathbf{Z}_2)$ and a continuous map $h : (X, A) \to (Y, B)$. For each $[M, f, \varphi] \in \Omega_n(X, A; h^*\mathscr{S}_\eta)$, we have $\varphi : \mathscr{S}_M \xrightarrow{\sim} (h \circ f)^*\mathscr{S}_\eta$. Hence we define a homomorphism $h_* : \Omega_n(X, A; h^*\mathscr{S}_\eta) \to \Omega_n(Y, B; \mathscr{S}_\eta)$ by

$$h_*([M, f, \varphi]) = [M, h \circ f, \varphi].$$

Let $i : A \to X$ be the inclusion map. We define a boundary operator $\partial : \Omega_n(X, A; \mathscr{S}_w) \to \Omega_{n-1}(A; i^*\mathscr{S}_w)$ by

$$\partial([M, f, \varphi]) = [\partial M, f|\partial M, \dot{\varphi}],$$

where $\dot{\varphi} = \varphi|\mathscr{S}_{\partial M}$.

PROPOSITION 6. *Cobordism groups with local coefficients have the following properties.*

(1) *If* $id : (X, A) \to (X, A)$ *is the identity map, then* $id_* : \Omega_n(X, A; \mathscr{S}_w) \to \Omega_n(X, A; \mathscr{S}_w)$ *is the identity map.*

(2) *Let* $h : (X, A) \to (Y, B)$ *and* $h' : (Y, B) \to (Z, C)$ *be continuous maps and* $\zeta \in H^1(Z; \mathbf{Z}_2)$. *Then* $(h' \circ h)_* : \Omega_n(X, A; (h' \circ h)^* \mathscr{S}_\zeta) \to \Omega_n(Z, C; \mathscr{S}_\zeta)$ *is a composite of* $h_* : \Omega_n(X, A; (h' \circ h)^* \mathscr{S}_\zeta) \to \Omega_n(Y, B; (h')^* \mathscr{S}_\zeta)$ *and* $h'_* : \Omega_n(Y, B; (h')^* \mathscr{S}_\zeta) \to \Omega_n(Z, C; \mathscr{S}_\zeta)$.

(3) *For any* $\eta \in H^1(Y; \mathbf{Z}_2)$ *and any map* $h : (X, A) \to (Y, B)$, *the diagram*

$$\begin{array}{ccc} \Omega_n(X, A; h^* \mathscr{S}_\eta) & \xrightarrow{\partial} & \Omega_{n-1}(A; i^* h^* \mathscr{S}_\eta) \\ {\scriptstyle h_*}\big\downarrow & & \big\downarrow{\scriptstyle (h|A)_*} \\ \Omega_n(Y, B; \mathscr{S}_\eta) & \xrightarrow{\partial} & \Omega_{n-1}(B; i^* \mathscr{S}_\eta) \end{array}$$

is commutative.

(4) *For every pair* (X, A) *and every* $w \in H^1(X; \mathbf{Z}_2)$, *the sequence*

$$\cdots \to \Omega_n(A; i^* \mathscr{S}_w) \xrightarrow{i_*} \Omega_n(X; \mathscr{S}_w) \xrightarrow{j_*} \Omega_n(X, A; \mathscr{S}_w) \xrightarrow{\partial} \Omega_{n-1}(A; i^* \mathscr{S}_w) \to \cdots$$

is exact.

(5) *If there is a homotopy* $h_t : (X, A) \to (Y, B)$, *then* $h_{0*} = h_{1*} : \Omega_n(X, A; \mathscr{S}_w) \to \Omega_n(Y, B; \mathscr{S}_\eta)$ *for* $w = h_0^* \eta = h_1^* \eta$, $\eta \in H^1(Y; \mathbf{Z}_2)$.

(6) *If* $\overline{U} \subset \operatorname{Int} A$, *then the inclusion* $i : (X - U, A - U) \to (X, A)$ *induces an isomorphism* $i_* : \Omega_n(X - U, A - U; i^* \mathscr{S}_w) \to \Omega_n(X, A; \mathscr{S}_w)$.

Proof. (1), (2) and (3) are trivial.

(4) For $[M, f, \varphi] \in \Omega_n(A; i^* \mathscr{S}_w)$ we put $W = M \times I$. We define a map $F : W \to X$ by $F(u, t) = f(u)$ $((u, t) \in M \times I)$ and a path α_v from $(u, 0)$ to $v = (u, t)$ by $\alpha_v(s) = (u, st)$. Moreover define Φ by extending $\Phi_v = \varphi_u \circ \alpha_{v*}$ $(v = (u, t) \in \operatorname{Int} W)$. Then $\partial(W, F, \Phi) \equiv (M, f, \varphi) \bmod A$. Hence we have $j_* i_* = 0$.

Assume that $j_*[M, f, \varphi] = 0$ for $[M, f, \varphi] \in \Omega_n(X; \mathscr{S}_w)$. Then there exists an element $(W, F, \Phi) \in \mathscr{M}_{n+1}(X, X; \mathscr{S}_w)$ such that $\partial(W, F, \Phi) \equiv (M, f, \varphi) \bmod A$. Now we put

$$N = \partial W - M, \qquad g = F|N, \qquad \psi = -\dot{\Phi}|N.$$

Then $[N, g, \psi] \in \Omega_n(A; i^* \mathscr{S}_w)$ and $i_*[N, g, \psi] = [M, f, \varphi]$. Hence we have $\operatorname{Ker} j_* \subset \operatorname{Im} i_*$.

$\partial j_* = 0$ and $i_* \partial = 0$ are trivially verified. Assume that $\partial[M, f, \varphi] = 0$ for $[M, f, \varphi] \in \Omega_n(X, A; \mathscr{S}_w)$. Then there exists an element $(N, g, \psi) \in \mathscr{M}_n(A, A; i^* \mathscr{S}_w)$ such that $\partial(N, g, \psi) \equiv (\partial M, f|\partial M, \dot{\varphi})$. Now we put

$$M' = M \cup_{\partial M} N, \qquad f' = f \cup g, \qquad \varphi' = \varphi \cup \psi$$

and $W = M' \times I$. Define a map $F : W \to X$ by $F(u,t) = f'(u)$. Moreover define Φ by extending $\Phi_v = \varphi'_u \circ \alpha_{v*}$ $(v = (u,t) \in \mathrm{Int}\, W)$. Then it holds

$$\partial(W, F, \Phi) \equiv (M', f', \varphi') + (M, f, -\varphi) \mod A.$$

This implies $j_*[M', f', \varphi'] = [M, f, \varphi]$. Hence we have $\mathrm{Ker}\, \partial \subset \mathrm{Im}\, j_*$.

Assume that $i_*[M, f, \varphi] = 0$ for $[M, f, \varphi] \in \Omega_{n-1}(A; i^*\mathscr{S}_w)$. Then there exists an element $(W, F, \Phi) \in \mathscr{M}_n(X, A; \mathscr{S}_w)$ such that $\partial(W, F, \Phi) \equiv (M, f, \varphi)$. Since $[W, F, \Phi] \in \Omega_n(X, A; w)$, we have $[M, f, \varphi] \in \mathrm{Im}\, \partial$. Hence $\mathrm{Ker}\, i_* \subset \mathrm{Im}\, \partial$.

(5) For $[M, f, \varphi] \in \Omega_n(X, A; \mathscr{S}_w)$ we put $W = M \times I$ and define a map $F : W \to Y$ by $F(u,t) = h_t(f(u))$ $((u,t) \in M \times I)$. Since $h_t^*\eta = w$ for any $t \in I$, we can define Φ just in the same way as in the proof of Proposition 5. Hence we get

$$\partial(W, F, \Phi) \equiv (M, h_0 \circ f, \varphi) + (M, h_1 \circ f, -\varphi) \mod A.$$

(6) We will show that i_* is surjective; the remainder of argument is similar. For $[M, f, \varphi] \in \Omega_n(X, A; \mathscr{S}_w)$, let $P = f^{-1}(X - \mathrm{Int}\, A)$ and $Q = f^{-1}(\overline{U})$. Then there exists a compact submanifold $N \subset M$ such that $P \subset N$ and $Q \cap N = \phi$. We put $g = f|N$ and $\psi|\mathrm{Int}\, N = \varphi|\mathrm{Int}\, N$ by identifying $H_n(\mathrm{Int}\, M, \mathrm{Int}\, M - v; \mathbf{Z})$ with $H_n(\mathrm{Int}\, N, \mathrm{Int}\, N - v; \mathbf{Z})$ for any $v \in \mathrm{Int}\, N$. The equivalence $\psi : \mathscr{S}_N \to g^*\mathscr{S}_w$ is defined as a natural unique extension. Then we have $[N, g, \psi] \in \Omega_n(X - U, A - U; i^*w)$ and $i^*[N, g, \psi] = [M, f, \varphi]$. □

From (1), (2), (3) and (4) of Proposition 6 we see that the following sequence is exact for any triple (X, A, B) and w according to [3].

$$\cdots \to \Omega_n(A, B; i^*\mathscr{S}_w) \xrightarrow{i_*} \Omega_n(X, B; \mathscr{S}_w) \xrightarrow{j_*} \Omega_n(X, A; \mathscr{S}_w)$$

$$\xrightarrow{\partial} \Omega_{n-1}(A, B; i^*\mathscr{S}_w) \to \cdots$$

For $w \in H^1(X; \mathbf{Z}_2)$ and $\eta \in H^1(Y; \mathbf{Z}_2)$ let $\xi = w \otimes 1 + 1 \otimes \eta \in H^1(X \times Y; \mathbf{Z}_2) \cong H^1(X; \mathbf{Z}_2) \otimes H^0(Y; \mathbf{Z}_2) \oplus H^0(X; \mathbf{Z}_2) \otimes H^1(Y; \mathbf{Z}_2)$. Then we can choose a local system $\mathscr{S}_\xi$ equivalent to $\mathscr{S}_w \otimes \mathscr{S}_\eta$ on $X \times Y$. Through this equivalence for $[M, f, \varphi] \in \Omega_m(X, A; \mathscr{S}_w)$ and $[N, g, \psi] \in \Omega_n(Y; \mathscr{S}_\eta)$ we have

$$\varphi \otimes \psi : \mathscr{S}_{M \times N} \xrightarrow{\sim} (f \times g)^*\mathscr{S}_\xi.$$

Then, $\dot{\varphi} \otimes \psi : \mathscr{S}_{\partial M \times N} \xrightarrow{\sim} (f \times g)^*\mathscr{S}_\xi|\partial M \times N$ and hence we can define a homomorphism

$$\Theta : \Omega_m(X, A; \mathscr{S}_w) \otimes \Omega_n(Y; \mathscr{S}_\eta) \to \Omega_{m+n}(X \times Y, A \times Y; \mathscr{S}_\xi)$$

by $\Theta([M, f, \varphi] \otimes [N, g, \psi]) = [M \times N, f \times g, \varphi \otimes \psi]$. In particular, if $Y = pt$ then we get a homomorphism

$$\Theta : \Omega_m(X, A; \mathscr{S}_w) \otimes \Omega_n \to \Omega_{m+n}(X, A; \mathscr{S}_w),$$

where Ω_n is the Thom group ([2], [13]).

Let A be a closed subset of X. We want to use an open subset V of X which contains A and

(7.1) A is a deformation retract of V by a retraction $r : V \to A$, that is, $i_A \circ r : V \to V$ is homotopic to the identity $1_V : V \to V$ for the natural inclusion $i_A : A \to V$.

For a continuous map $f : A \to Y$, let $\bar{f} : (X, A) \to (Y \cup_f X, Y)$ be a map defined by

$$\bar{f}(x) = \begin{cases} f(x) & (x \in A) \\ x & (x \in X - A). \end{cases}$$

We have the following theorem.

Theorem 7 (Cf. [6]). *Let A be a closed subeset of X and $f : A \to Y$ be a continuous map. If there exists an open subset $V \supset A$ satisfying* (7.1), *then $\bar{f}_* : \Omega_n(X, A; \bar{f}^*\mathscr{S}_\eta) \to \Omega_n(Y \cup_f X, Y; \mathscr{S}_\eta)$ is an isomorphism for any $\eta \in H^1(Y \cup_f X; \mathbf{Z}_2)$.*

Proof. We put $Z = Y \cup_f X$ and let $i : (X, A) \to (X, V)$, $j : (Z, Y) \to (Z, Y \cup \bar{f}(V))$ be inclusion maps. Consider the left part of the following commutative diagram:

$$\begin{array}{ccccc}
\Omega_n(X, A; \bar{f}^*\mathscr{S}_\eta) & \xrightarrow{i_*} & \Omega_n(X, V; \bar{f}^*\mathscr{S}_\eta) & \xleftarrow{i'_*} & \Omega_n(X - A, V - A; i'^*\bar{f}^*\mathscr{S}_\eta) \\
{\scriptstyle \bar{f}_*}\big\downarrow & & \big\downarrow{\scriptstyle \bar{f}_*} & & \big\downarrow{\scriptstyle \bar{f}_*} \\
\Omega_n(Z, Y; \mathscr{S}_\eta) & \xrightarrow[j_*]{} & \Omega_n(Z, Y \cup \bar{f}(V); \mathscr{S}_\eta) & \xleftarrow[j'_*]{} & \Omega_n(Z - Y, \bar{f}(V - A); j'^*\mathscr{S}_\eta).
\end{array}$$

For the homotopy $h_t : V \to V$ between $i_A \circ r$ and 1_V given by (7.1), $h_t^* : H^1(V; \mathbf{Z}_2) \to H^1(V; \mathbf{Z}_2)$ is an identity isomorphism for every t. Hence by (1), (2), (3), (4) and (5) of Proposition 6 we have $\Omega_q(V, A; i_V^*\bar{f}^*\mathscr{S}_\eta) = 0$ for the natural inclusion $i_V : V \to X$ and every q. From the exact sequence of triple (X, V, A) we see that i_* is an isomorphism. By a similar argument we see that j_* is also an isomorphism. Next we consider the right part of the above commutative diagram. From (6) of Proposition 6 we see that i'_* and j'_* are isomorphisms for the natural inclusions i' and j'. Since the map $\bar{f} : (X - A, V - A) \to (Z - Y, \bar{f}(V - A))$ is a homeomorphism, $\bar{f}_*$ on the right-hand side is an isomorphism. Hence so is $\bar{f}_*$ on the center. Consequently $\bar{f}_*$ on the left-hand side is an isomorphism. $\square$

Let X be a CW complex and X^p its p-skeleton. Hereafter until the end of § 5, $i : X^p \to X$ denotes the natural inclusion. For each p-cell e_λ of X,

$h_\lambda : (D_\lambda^p, S_\lambda^{p-1}) \to (\bar{e}_\lambda, \dot{e}_\lambda)$ denotes its characteristic map. Then we have the following corollary applying Proposition 7 to $X = \coprod_\lambda D_\lambda^p$, $A = \coprod_\lambda S_\lambda^{p-1}$, $Y = X^{p-1}$ and $\bar{f} = \coprod_\lambda h_\lambda$, because a CW complex has the homotopy extension property.

COROLLARY 8. *The map* $\Sigma h_{\lambda *} : \sum_\lambda \Omega_n(D_\lambda^p, S_\lambda^{p-1}; h_\lambda^* i^* \mathscr{S}_w) \to \Omega_n(X^p, X^{p-1}; i^* \mathscr{S}_w)$ *is an isomorphism.*

Moreover, we have

COROLLARY 9. *The map* $\Theta : \Omega_n(X^n, X^{n-1}; i^* \mathscr{S}_w) \otimes \Omega_q \to \Omega_{n+q}(X^n, X^{n-1}; i^* \mathscr{S}_w)$ *is an isomorphism.*

PROOF. Since D_λ^p is simply connected, the local system $h_\lambda^* i^* \mathscr{S}_w$ is equivalent to $\mathscr{S}_0$. So, the map

$$\Theta_\lambda : \Omega_n(D_\lambda^n, S_\lambda^{n-1}; h_\lambda^* i^* \mathscr{S}_w) \otimes \Omega_q \to \Omega_{n+q}(D_\lambda^n, S_\lambda^{n-1}; h_\lambda^* i^* \mathscr{S}_w)$$

is an isomorphism for every λ by [2]. Furthermore, the following diagram is commutative:

$$\begin{array}{ccc}
\sum_\lambda \Omega_n(D_\lambda^n, S_\lambda^{n-1}; h_\lambda^* i^* \mathscr{S}_w) \otimes \Omega_q & \xrightarrow{\Sigma \Theta_\lambda} & \sum_\lambda \Omega_{n+q}(D_\lambda^n, S_\lambda^{n-1}; h_\lambda^* i^* \mathscr{S}_w) \\
{\scriptstyle \Sigma h_{\lambda *} \otimes 1} \downarrow & & \downarrow {\scriptstyle \Sigma h_{\lambda *}} \\
\Omega_n(X^n, X^{n-1}; i^* \mathscr{S}_w) \otimes \Omega_q & \xrightarrow[\Theta]{} & \Omega_{n+q}(X^n, X^{n-1}; i^* \mathscr{S}_w).
\end{array}$$

Therefore, Corollary 8 implies Corollary 9. □

4. Proof of Theorem 1

For $[M, f, \varphi] \in \Omega_n(X, A; \mathscr{S}_w)$ let $\check{H}_n(\mathrm{Int}\, M; \mathscr{S}_{\mathrm{Int}\, M})$ be a homology group of infinite chains with local coefficients $\mathscr{S}_{\mathrm{Int}\, M}$ and $\varphi_\sharp : \check{H}_n(\mathrm{Int}\, M; \mathscr{S}_{\mathrm{Int}\, M}) \to \check{H}_n(\mathrm{Int}\, M; f^* \mathscr{S}_w)$ be the isomorphism induced by $\varphi | \mathrm{Int}\, M$. We know that there is a natural isomorphism $\imath : \check{H}_n(\mathrm{Int}\, M; f^* \mathscr{S}_w) \to H_n(M, \partial M; f^* \mathscr{S}_w)$ for any compact manifold M (cf. [6]). We put $\varphi_* = \imath \circ \varphi_\sharp$ and define a homomorphism

$$\mu : \Omega_n(X, A; \mathscr{S}_w) \to H_n(X, A; \mathscr{S}_w)$$

by $\mu([M, f, \varphi]) = f_*(\varphi_*(\sigma_M))$, where f_* is an induced homomorphism

$$f_* : H_n(M, \partial M; f^* \mathscr{S}_w) \to H_n(X, A; \mathscr{S}_w)$$

and σ_M is a fundamental class of $\check{H}_n(\mathrm{Int}\, M; \mathscr{S}_{\mathrm{Int}\, M})$. Then, for the any CW complex X we have the following.

Theorem 10. *The map* $\mu : \Omega_n(X^n, X^{n-1}; i^*\mathscr{S}_w) \to H_n(X^n, X^{n-1}; i^*\mathscr{S}_w)$ *is an isomorphism for every* $w \in H^1(X; \mathbf{Z}_2)$.

Proof. We know that the map

$$\mu_\lambda : \Omega_n(D^n_\lambda, S^{n-1}_\lambda; h^*_\lambda i^*\mathscr{S}_w) \to H_n(D^n_\lambda, S^{n-1}_\lambda; h^*_\lambda i^*\mathscr{S}_w)$$

is an isomorphism for every λ by [2], and the following diagram is commutative:

$$\begin{array}{ccc}
\sum_\lambda \Omega_n(D^n_\lambda, S^{n-1}_\lambda; h^*_\lambda i^*\mathscr{S}_w) & \xrightarrow{\Sigma\mu_\lambda} & \sum_\lambda H_n(D^n_\lambda, S^{n-1}_\lambda; h^*_\lambda i^*\mathscr{S}_w) \\
{\scriptstyle \Sigma h_{\lambda *}}\big\downarrow & & \big\downarrow{\scriptstyle \Sigma h_{\lambda *}} \\
\Omega_n(X^n, X^{n-1}; i^*\mathscr{S}_w) & \xrightarrow[\mu]{} & H_n(X^n, X^{n-1}; i^*\mathscr{S}_w).
\end{array}$$

Since the vertical map at the right-hand side is an isomorphism, Corollary 8 implies Theorem 10. □

Proof of Theorem 1. For $w \in H^1(X; \mathbf{Z}_2)$ and each pair of integers (p, q) such that $-\infty \le p \le q \le \infty$, we put $H(p, q) = \sum_n \Omega_n(X^{-p}, X^{-q}; i^*\mathscr{S}_w)$. Then $\{H(p, q)\}$ satisfies the axioms in the theory of spectral sequences [1, Chap. XV, p. 334]. Now let $\overline{H}(p, q) = H(-p, -q)$, $\overline{H}(p) = \overline{H}(p, -\infty)$, $\overline{H} = \overline{H}(\infty, -\infty)$. We define a filtration $F_{p,q}\overline{H}$ of $\overline{H}$ by

$$F_{p,q}\overline{H} = \mathrm{Im}(\overline{H}_{p+q}(p) \to \overline{H}_{p+q}) = \mathrm{Im}(\Omega_{p+q}(X^p; i^*\mathscr{S}_w) \to \Omega_{p+q}(X; \mathscr{S}_w)).$$

We define also

$$\begin{aligned}
Z^r_{p,q} &= \mathrm{Im}(\overline{H}_{p+q}(p, p-r) \to \overline{H}_{p+q}(p, p-1)) \\
&= \mathrm{Im}(\Omega_{p+q}(X^p, X^{p-r}; i^*\mathscr{S}_w) \to \Omega_{p+q}(X^p, X^{p-1}; i^*\mathscr{S}_w)) \\
B^r_{p,q} &= \mathrm{Im}(\overline{H}_{p+q+1}(p+r-1, p) \to \overline{H}_{p+q}(p, p-1)) \\
&= \mathrm{Im}(\Omega_{p+q+1}(X^{p+r-1}, X^p; i^*\mathscr{S}_w) \to \Omega_{p+q}(X^p, X^{p-1}; i^*\mathscr{S}_w)) \\
E^r_{p,q} &= Z^r_{p,q}/B^r_{p,q}
\end{aligned}$$

where $1 \le r \le \infty$, $-\infty < p < \infty$. Since $\overline{H}_n(p) = \Omega_n(X^p; i^*\mathscr{S}_w) = 0$ for every n and $p \le -1$, F is regular and hence convergent in the sense of [1]. Then we have particularly

$$E^1_{p,q} = \Omega_{p+q}(X^p, X^{p-1}; i^*\mathscr{S}_w).$$

By Corollary 9 and Theorem 10 we get

$$\Omega_{p+q}(X^p, X^{p-1}; i^*\mathcal{S}_w) \xleftarrow{\cong} \Omega_p(X^p, X^{p-1}; i^*\mathcal{S}_w) \otimes \Omega_q \xrightarrow{\cong} H_p(X^p, X^{p-1}; i^*\mathcal{S}_w) \otimes \Omega_q.$$

By the universal coefficient theorem for the homology with local coefficients [6] we have

$$H_p(X^p, X^{p-1}; i^*\mathcal{S}_w) \otimes \Omega_q \xleftarrow{\cong} H_p(X^p, X^{p-1}; \Omega_q \otimes i^*\mathcal{S}_w).$$

Moreover, through these isomorphisms, we have the following commutative diagram:

$$\begin{array}{ccc} \Omega_{p+q}(X^p, X^{p-1}; i^*\mathcal{S}_w) & \xrightarrow{\cong} & H_p(X^p, X^{p-1}; \Omega_q \otimes i^*\mathcal{S}_w) \\ \partial \big\downarrow & & \big\downarrow \partial \\ \Omega_{p+q-1}(X^{p-1}, X^{p-2}; i^*\mathcal{S}_w) & \xrightarrow[\cong]{} & H_{p-1}(X^{p-1}, X^{p-2}; \Omega_q \otimes i^*\mathcal{S}_w). \end{array}$$

Therefore the differential $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ is identified with the boundary operator $\partial : H_p(X^p, X^{p-1}; \Omega_q \otimes i^*\mathcal{S}_w) \to H_{p-1}(X^{p-1}, X^{p-2}; \Omega_q \otimes i^*\mathcal{S}_w)$. Hence we have

$$E^2_{p,q} \cong H_p(X; \Omega_q \otimes \mathcal{S}_w).$$

Thus we proved Theorem 1.

5. Some calculations and proof of Corollary 2

Using Theorem 1 we will calculate the cobordism group with local coefficients for some examples and prove Corollary 2.

EXAMPLE 1. Let $X = S^1$ and $w \neq 0$. We have an exact sequence

$$0 \to E^\infty_{0,n} \to \Omega_n(S^1; \mathcal{S}_w) \to E^\infty_{1,n-1} \to 0$$

since $E^2_{m,n-m} = 0$ for $m \neq 0, 1$. From $H_0(S^1; \mathcal{S}_w) = \mathbf{Z}_2$ and $H_1(S^1; \mathcal{S}_w) = 0$, we have $E^\infty_{0,n} \cong H_0(S^1; \Omega_n \otimes \mathcal{S}_w) \cong \Omega_n \otimes \mathbf{Z}_2$ and $E^\infty_{1,n-1} \cong H_1(S^1; \Omega_{n-1} \otimes \mathcal{S}_w) \cong \mathrm{Tor}(\mathbf{Z}_2, \Omega_{n-1})$. It is known that $\Omega_0 \cong \mathbf{Z}$, $\Omega_1 = \Omega_2 = \Omega_3 = 0$, $\Omega_4 \cong \mathbf{Z}$. Hence we have $\Omega_n(S^1; \mathcal{S}_w) \cong \Omega_n \otimes \mathbf{Z}_2$ for $n \leq 5$.

EXAMPLE 2. Let X be a real projective plane P^2 and $w \neq 0$. We see that

$$E^2_{m,n-m} = H_m(P^2; \Omega_{n-m} \otimes \mathcal{S}_w) \cong \begin{cases} \Omega_n \otimes \mathbf{Z}_2 & (m = 0) \\ \mathrm{Tor}(\mathbf{Z}_2, \Omega_{n-1}) & (m = 1) \\ \Omega_{n-2} & (m = 2) \\ 0 & (m \geq 3) \end{cases}$$

since $H_0(P^2; \mathcal{S}_w) = \mathbf{Z}_2$, $H_1(P^2; \mathcal{S}_w) = 0$, $H_2(P^2; \mathcal{S}_w) = \mathbf{Z}$. Hence for $n \leq 5$ we have an exact sequence

$$0 \to E^{\infty}_{0,n} \to \Omega_n(P^2; \mathscr{S}_w) \to E^{\infty}_{2,n-2} \to 0.$$

Then we have $\Omega_2(P^2; \mathscr{S}_w) \cong \Omega_0$ and $\Omega_n(P^2; \mathscr{S}_w) \cong \Omega_n \otimes \mathbf{Z}_2$ for $n \neq 2$, $n \leq 5$.

Proof of Crollary 2. Since $\Omega_0 \cong \mathbf{Z}$, $\Omega_1 = \Omega_2 = \Omega_3 = 0$, $\Omega_4 \cong \mathbf{Z}$, we have an exact sequence

$$0 \to E^{\infty}_{0,4} \to \Omega_4(X; \mathscr{S}_w) \to E^{\infty}_{4,0} \to 0.$$

The map μ induces a map μ_* from the Atiyah-Hirzeburch spectral sequence for $\Omega_{p+q}(X; \mathscr{S}_w)$ to the Atiyah-Hirzeburch spectral sequence $\{E'^r_{p,q}\}$ for $H_{p+q}(X; \mathscr{S}_w)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \Omega_4(X; \mathscr{S}_w) & \longrightarrow & E^{\infty}_{4,0} = H_4(X; \Omega_0 \otimes \mathscr{S}_w) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu_*} \\ H_4(X; \mathscr{S}_w) & \xrightarrow{id} & E'^{\infty}_{4,0} = H_4(X; \mathscr{S}_w). \end{array}$$

Since μ_* is an isomorphism, we may identify the map μ with the above map $\Omega_4(X; \mathscr{S}_w) \to E^{\infty}_{4,0}$. Since X is connected, $H_0(X; \mathscr{S}_w) \cong \mathbf{Z}$ if $w = 0$, and $H_0(X; \mathscr{S}_w) \cong \mathbf{Z}_2$ if $w \neq 0$. Therefore we have $E^{\infty}_{0,4} \cong \Omega_4$ if $w = 0$, and $E^{\infty}_{0,4} \cong \Omega_4 \otimes \mathbf{Z}_2$ if $w \neq 0$. Hence we get the conclusion.

6. Local orientations of non-orientable manifolds

At first we prove the following Proposition.

Proposition 11. *Let X be an arcwise connected space and $w \in H^1(X; \mathbf{Z}_2)$. Suppose that M is a connected manifold without boundary. Then for any continuous map $f : M \to X$ the local system $\mathscr{S}_M$ is equivalent to $f^*\mathscr{S}_w$ if and only if $f^*w = w_1(M)$.*

Proof. Assume that $\varphi : \mathscr{S}_M \to f^*\mathscr{S}_w$ is an equivalence. We regard w and $w_1(M)$ as the homomorphisms from $H_1(X; \mathbf{Z})$ to $\operatorname{Aut} \mathbf{Z} = \mathrm{Z}_2$ and $H_1(M; \mathbf{Z})$ to $\operatorname{Aut} \mathbf{Z} = \mathrm{Z}_2$ respectively. We put $\rho_w = w \circ \Xi$ and $\rho_M = w_1(M) \circ \Xi$ for the Hurewicz homomorphism Ξ. For every point $u \in M$ and every element $\gamma \in \pi_1(M, u)$, the following diagram is commutative:

$$\begin{array}{ccc} S_M(u) & \xrightarrow{\varphi_u} & (f^*\mathscr{S}_w)(u) = \mathscr{S}_w(f(u)) \\ {\scriptstyle \mathscr{S}_M(\gamma)}\downarrow & & \downarrow{\scriptstyle \mathscr{S}_w(f_*\gamma)} \\ \mathscr{S}_M(u) & \xrightarrow{\varphi_u} & (f^*\mathscr{S}_w)(u) = \mathscr{S}_w(f(u)). \end{array}$$

So, $(\bar{\mathscr{S}}_M)_u(\gamma) = \varphi_u^{-1} \circ (\bar{\mathscr{S}}_w)_{f(u)}(f_*\gamma) \circ \varphi_u$ as an automorphism of $\mathscr{S}_M(u)$. Because φ_u identifies $\mathscr{S}_M(u) = \mathscr{S}_w(f(u)) \cong \mathbf{Z}$, this means $\rho_M = \rho_w \circ f_*$. Since Ξ is a surjection, we see $f^*w = w_1(M)$ by the following commutative diagram:

$$\begin{array}{ccc} \pi_1(M,u) & \xrightarrow{f_*} & \pi_1(X, f(u)) \\ \Xi \downarrow & & \downarrow \Xi \\ H_1(M;\mathbf{Z}) & \xrightarrow{f_*} & H_1(X;\mathbf{Z}). \end{array}$$

Conversely assume that $f^*w = w_1(M)$. Fix a base point u_0. Then, the local systems $f^*\mathscr{S}_w$ and $\mathscr{S}_M$ have the same associated homomorphism $\rho_M = \rho_w \circ f_* : \pi_1(M,u_0) \to \mathrm{Aut}\,\mathbf{Z}$. We choose an element $\alpha_u \in \Gamma(u,u_0)$ for each point $u \in M$. If we choose an isomorphism $\varphi_{u_0} : \mathscr{S}_M(u_0) \to (f^*\mathscr{S}_w)(u_0)$ for the base point u_0, the isomorphism $\varphi_u : \mathscr{S}_M(u) \to (f^*\mathscr{S}_w)(u)$ is determined by $\varphi_u = \mathscr{S}_w(f_*\alpha_u)^{-1} \circ \varphi_{u_0} \circ \mathscr{S}_M(\alpha_u)$. In fact $\varphi = \{\varphi_u\}$ satisfies

$$\begin{aligned} \varphi_u \circ \mathscr{S}_M(\gamma) &= \mathscr{S}_w(f_*\alpha_u)^{-1} \circ \varphi_{u_0} \circ (\bar{\mathscr{S}}_M)_{u_0}(\alpha_u\gamma\alpha_v^{-1}) \circ \mathscr{S}_M(\alpha_v) \\ &= \mathscr{S}_w(f_*\alpha_u)^{-1} \circ (\bar{\mathscr{S}}_w)_{f(u_0)}(f_*(\alpha_u\gamma\alpha_v^{-1})) \circ \varphi_{u_0} \circ \mathscr{S}_M(\alpha_v) \\ &= \mathscr{S}_w(f_*\gamma) \circ \varphi_v \end{aligned}$$

for every $\gamma \in \Gamma(v,u)$. Hence φ is an equivalence. □

Let M be a closed connected n-manifold, $\pi = \pi_1(M)$ and $f, f' : (M,u_0) \to (B\pi, y_0)$ be two maps which satisfy the conditions (3.1) and (3.2). Moreover let $\varphi : \mathscr{S}_M \to f^*\mathscr{S}_w$ and $\varphi' : \mathscr{S}_M \to f'^*\mathscr{S}_w$ be equivalences. Suppose that f and f' are homotopic by a homotopy $F : M \times I \to B\pi$. For each point $u \in M$ let γ_u be a path from $(u,0)$ to $(u,1)$ in $M \times I$ defined by $\gamma_u(t) = (u,t)$ and define isomorphisms $\delta_u : f'^*\mathscr{S}_w(u) \to f^*\mathscr{S}_w(u)$ and $\kappa_F(u) : \mathscr{S}_M(u) \to \mathscr{S}_M(u)$ by

(8.1) $$\delta_u = \mathscr{S}_w(F_*[\gamma_u]) \qquad \text{and} \qquad \kappa_F(u) = \varphi_u^{-1} \circ \delta_u \circ \varphi'_u.$$

Then we have

$$\kappa_F(u) = \mathscr{S}_M(\alpha)^{-1} \circ \kappa_F(u_0) \circ \mathscr{S}_M(\alpha)$$

for every relative homotopy class α of paths from u_0 to u in M. We may regard κ_F as a map from M to $\mathrm{Aut}\,\mathbf{Z}$. From the above equation we see that κ_F is continuous. We define $\mathrm{sgn}\,\kappa_F$ by

$$\mathrm{sgn}\,\kappa_F = \begin{cases} 1 & \text{if } \kappa_F(u) = id \text{ for any } u \\ -1 & \text{if } \kappa_F(u) = -id \text{ for any } u. \end{cases}$$

We have the following proposition.

Proposition 12. *Let M be a closed connected n-manifold, $\pi = \pi_1(M)$ and $f, f' : (M, u_0) \to (B\pi, y_0)$ be two maps which satisfy the conditions* (3.1) *and* (3.2). *Moreover let $\varphi : \mathscr{S}_M \to f^*\mathscr{S}_w$ and $\varphi' : \mathscr{S}_M \to f'^*\mathscr{S}_w$ be equivalences. Suppose that f and f' are homotopic by a homotopy F. Then it holds $[M, f, \varphi] = [M, f', (\operatorname{sgn} \kappa_F)\varphi']$ in $\Omega_n(B\pi; \mathscr{S}_w)$, where κ_F is a map defined by* (8.1).

Proof. We put $W = M \times I$. For $v = (u, t) \in \operatorname{Int} W$ we define $\Phi_v : \mathscr{S}_W(v) \to F^*\mathscr{S}_w(v)$ by

$$\Phi_v = \mathscr{S}_w(F_*[\alpha_v])^{-1} \circ \varphi_u \circ \alpha_{v*},$$

where α_v is a path from $(u, 0)$ to $v = (u, t)$ defined by $\alpha_v(s) = (u, st)$. Let β_v be a path from $(u, 1)$ to $v = (u, t)$ defined by $\beta_v(s) = (u, 1 - s + st)$. By the definitions of Φ_v and $\kappa_F(v)$ we see that

$$\begin{aligned} \Phi_v &= -\mathscr{S}_w(F_*[\beta_v])^{-1} \circ \varphi'_u \circ \kappa_F(u)^{-1} \circ \beta_{v*} \\ &= -\mathscr{S}_w(F_*[\beta_v])^{-1} \circ (\operatorname{sgn} \kappa_F)\varphi'_u \circ \beta_{v*}. \end{aligned}$$

So, $\dot{\Phi}_v : \mathscr{S}_{\partial W}(v) \to (F|\partial W)^*\mathscr{S}_w(v)$ is written as

$$\dot{\Phi}_v = \begin{cases} \varphi_u & (v = (u, 0)) \\ -(\operatorname{sgn} \kappa_F)\varphi'_u & (v = (u, 1)). \end{cases}$$

Hence we get $(W, F, \Phi) \equiv (M, f, \varphi) + (M, f', -(\operatorname{sgn} \kappa_F)\varphi')$. $\square$

Let g be an element of orthogonal group $O(n-1)$ with $\det g = -1$ and denote by N the quotient space of $\mathbf{R} \times D^{n-1}$ gained by identifying (s, v) and $(s + 1, gv)$ for each $(s, v) \in \mathbf{R} \times D^{n-1}$. Then N is a non-orientable smooth $O(n-1)$ bundle over S^1 with fiber D^{n-1}. We denote by $[s, v]$ the point represented by (s, v) in N.

Let $\delta : [0, 1] \to [0, 1]$ be a monotone and smooth function such that $\delta|[0, \varepsilon] = 1$ and $\delta|[1 - \varepsilon, 1] = 0$ for a positive number ε which is small enough.

For each $t \in I$ we define a map $H_t : N \to N$ by

$$H_t([s, ru]) = [s + t\delta(r), ru],$$

where $s \in \mathbf{R}$, $0 \leq r \leq 1$ and $u \in \partial D^{n-1}$. Then H_t is a diffeomorphism such that $H_t|\partial N = 1_{\partial N}$ for each t and H_1 is homotopic to $H_0 = 1_N$.

Let M be a closed, connected and non-orientable n-manifold and α be a simple closed arc with based point u_0 such that $w_1(M)([\alpha]) \neq 0$. The tubular neighborhood of α is diffeomorphic to the above bundle N for a some $g \in O(n-1)$ with $\det g = -1$. Hence we have a diffeomorphism $h : (M, u_0) \to (M, u_0)$ which satisfies the conditions

(9.1) h is the identity map out of a tubular neighborhood $N(\alpha)$, (9.2) h is homotopic to the identity map 1_M by a homotopy $H : M \times I \to M$ and

(9.3) $H_*[\gamma_{u_0}] = [\alpha]$,

where γ_{u_0} is a path from $(u_0, 0)$ to $(u_0, 1)$ in $M \times I$ defined by $\gamma_{u_0}(t) = (u_0, t)$. We define a family of isomorphisms $\bar{h} = \{\bar{h}_u\} : \mathscr{S}_M \to h^* \mathscr{S}_M$ by

$$\bar{h}_u = \mathscr{S}_M(H_*[\gamma_u])^{-1},$$

where γ_u is a path from $(u, 0)$ to $(u, 1)$ defined by $\gamma_u(t) = (u, t)$. Then $\bar{h}$ is an equivalence. In particular, $\bar{h}_{u_0} = \mathscr{S}_M([\alpha])^{-1} = -id$.

Let $f : (M, u_0) \to (B\pi, y_0)$ be a continuous map which satisfies the conditions (3.1) and (3.2), and $\varphi : \mathscr{S}_M \to f^* \mathscr{S}_w$ be an equivalence. Composing $\bar{h}$ with φ we get an equivalence $h^*\varphi = \{\varphi_{h(u)} \circ \bar{h}_u\} : \mathscr{S}_M \to (f \circ h)^* \mathscr{S}_w$.

Proposition 13. *Let M be a closed, connected and non-orientable n-manifold, α be a simple closed arc with based point u_0 such that $w_1(M)([\alpha]) \neq 0$ and $h : (M, u_0) \to (M, u_0)$ be a diffeomorphism which satisfies the conditions* (9.1), (9.2) *and* (9.3). *Let $f : (M, u_0) \to (B\pi, y_0)$ be a continuous map which satisfies the conditions* (3.1) *and* (3.2), *and $\varphi : \mathscr{S}_M \to f^* \mathscr{S}_w$ be an equivalence. If f and $f \circ h$ are homotopic preserving the base point, then $[M, f, \varphi] = [M, f, -\varphi]$ in $\Omega_n(B\pi; \mathscr{S}w)$. Moreover, the assumption that f and $f \circ h$ are homotopic preserving the base point is always satisfied when π is abelian.*

Proof. At first we show that $[M, f, \varphi] = [M, f \circ h, h^*\varphi]$. We put $F = f \circ H$. Let κ_F be a map defined by (8.1). Since $\delta_{u_0} = \mathscr{S}_w(f_*[\alpha]) = -id$ and $(h^*\varphi)_{u_0} = \varphi_{u_0} \circ \bar{h}_{u_0} = -\varphi_{u_0}$, $\kappa_F(u_0) = \varphi_{u_0}^{-1} \circ \delta_{u_0} \circ (h^*\varphi)_{u_0} = id$. Hence we get $[M, f, \varphi] = [M, f \circ h, h^*\varphi]$ by Proposition 12.

Next we show that $[M, f, \varphi] = [M, f \circ h, -h^*\varphi]$. Let G be a homotopy of f to $f \circ h$ preserving the base point. Since $\delta_{u_0} = \mathscr{S}_w(G_*[\gamma_{u_0}]) = \mathscr{S}_w(1_{y_0}) = id$, we have $\kappa_G(u_0) = -id$. Hence we get $[M, f, \varphi] = [M, f \circ h, -h^*\varphi]$ by Proposition 12.

Assume now that π is abelian and two continuous maps $f, f' : (M, u_0) \to (B\pi, y_0)$ are homotopic by a homotopy $F : M \times I \to B\pi$. We put $X = M \times I$, $A = M \times 0 \cup M \times 1 \cup u_0 \times I$. Then (X, A) can be considered to be a pair of CW complexes by the triangulation theorem of differentiable manifolds. We define a map $G' : (A, (u_0, 0)) \to (B\pi, y_0)$ by

$$G'(a) = \begin{cases} f(u) & (a = (u, 0) \in M \times 0) \\ f'(u) & (a = (u, 1) \in M \times 1) \\ y_0 & (a = (u_0, t) \in u_0 \times I). \end{cases}$$

We regard $\pi_1(M \times 0, (u_0, 0))$ and $\pi_1(M \times 1 \cup u_0 \times I, (u_0, 0))$ as the subgroups of $\pi_1(A, (u_0, 0))$. For any $\gamma \in \pi_1(M \times 0, (u_0, 0))$ we have $G'_*(\gamma) = f_*(\gamma) =$

$F_* \circ i_*(\gamma)$, for the natural inclusion $i: A \to X$. Any element γ' of $\pi_1(M \times 1 \cup u_0 \times I, (u_0, 0))$ is represented by $[\gamma_{u_0}]\gamma[\gamma_{u_0}^{-1}]$, where γ_{u_0} is a path defined by $\gamma_{u_0}(t) = (u_0, t)$. Remark that $F_*\gamma_{u_0}$ is a closed arc with base point y_0. By the assumption that π is abelian we have $F_* \circ i_*(\gamma') = (F_*[\gamma_{u_0}])f'_*(\gamma)(F_*[\gamma_{u_0}])^{-1} = f'_*(\gamma) = G'_*(\gamma')$. Hence G' has an extension $G: (X, (u_0, 0)) \to (B\pi, y_0)$ by the obstruction theory. G gives a homotopy of f to f' preserving the base point.

Since $f \circ h(u_0) = y_0$ and f and $f \circ h$ are homotopic, we may apply the argument to $f' = f \circ h$ and get the result: f and $f \circ h$ are homotopic preserving the base point when π is abelian. $\square$

Let $\lambda: (B\pi, y_0) \to (B\pi, y_0)$ be a classifing map for $\lambda_* \in (\mathrm{Aut}\,\pi)^w$. Since $\lambda^* w = w$, two local systems $\mathscr{S}_w$ and $\lambda^*\mathscr{S}_w$ are equivalent and there is a unique equivalence $\bar{\lambda}: \mathscr{S}_w \to \lambda^*\mathscr{S}_w$ such that $\bar{\lambda}_{y_0} = id$ holds. Then we have a canonical isomorphism $\tilde{\lambda}_*$ associated to λ defined by $\tilde{\lambda}_* = \lambda_* \circ \bar{\lambda}_*: H_n(B\pi; \mathscr{S}_w) \to H_n(B\pi; \mathscr{S}_w)$, where $\lambda_*: H_n(B\pi; \lambda^*\mathscr{S}_w) \to H_n(B\pi; \mathscr{S}_w)$ is a natural isomorphism induced from λ and $\bar{\lambda}_*: H_n(B\pi; \mathscr{S}_w) \to H_n(B\pi; \lambda^*\mathscr{S}_w)$ is an isomorphism induced from $\bar{\lambda}$. We denote by $(\mathrm{Aut}\,\pi)_*^w$ the set consisting of such $\tilde{\lambda}_*$. For a local orientation φ of M associated with $f: M \to B\pi$, we define a local orientation $\bar{\lambda}_*\varphi$ of M associated with $\lambda \circ f$ by $(\bar{\lambda}_*\varphi)_u = \bar{\lambda}_{f(u)} \circ \varphi_u$. Since the diagram

$$\begin{tikzcd}
H_n(M; \mathscr{S}_M) \arrow[r, "(\bar{\lambda}_*\varphi)_*"] \arrow[d, "\varphi_*"'] & H_n(M; (\lambda \circ f)^*\mathscr{S}_w) \arrow[r, "(\lambda\circ f)_*"] & H_n(B\pi; \mathscr{S}_w) \\
H_n(M; f^*\mathscr{S}_w) \arrow[r, "f_*"] & H_n(B\pi; \mathscr{S}_w) \arrow[r, "\bar{\lambda}_*"] & H_n(B\pi; \lambda^*\mathscr{S}_w) \arrow[u, "\lambda_*"']
\end{tikzcd}$$

is commutative, we obtain

$$\tilde{\lambda}_* \circ f_* \circ \varphi_* = (\lambda \circ f)_* \circ (\bar{\lambda}_*\varphi)_*. \tag{10.1}$$

For closed connected n-manifolds M and M' let $h: (M, u_0) \to (M', u'_0)$ be a diffeomorphism. Let $f: (M, u_0) \to (B\pi, y_0)$ and $f': (M', u'_0) \to (B\pi, y_0)$ be continuous maps satisfying the conditions (3.1) and (3.2), and φ, φ' be local orientations associated with f and f' respectively. Since h is a diffeomorphism, we have a natural isomorphism $(h_*)_{u_0}: H_n(M, M - u_0; \mathbf{Z}) \to H_n(M', M' - u'_0; \mathbf{Z})$. As above we take a unique equivalence $\bar{h}: \mathscr{S}_M \to h^*\mathscr{S}_{M'}$ satisfying $\bar{h}_{u_0} = (h_*)_{u_0}$ and define an isomorphism $\tilde{h}_*: H_n(M; \mathscr{S}_M) \to H_n(M'; \mathscr{S}_{M'})$ by $\tilde{h}_* = h_* \circ \bar{h}_*$. Moreover, from φ' we define a local orientation $\bar{h}^*\varphi'$ of M associated with $f' \circ h$ by $(\bar{h}_*\varphi')_u = \varphi'_{h(u)} \circ \bar{h}_u$.

On the other hand, since the isomorphism $(f' \circ h)_* \circ f_*^{-1}: \pi_1(B\pi, y_0) \to \pi_1(B\pi, y_0)$ is an automorphism of π, there is a based point preserving map $\lambda: (B\pi, y_0) \to (B\pi, y_0)$ such that $\lambda \circ f$ is homotopic (not necessarily preserving the base point) to $f' \circ h$. Then, it is easy to see $\lambda_* \in (\mathrm{Aut}\,\pi)^w$. Let F be a homotopy from $\lambda \circ f$ to $f' \circ h$ and define an equivalence $\psi : (\lambda \circ f)^*\mathscr{S}_w \to (f' \circ h)^*\mathscr{S}_w$ by $\psi_u = \mathscr{S}_w(F_*[\gamma_u])^{-1}$, where γ_u is a path in $M \times I$ defined by $\gamma_u(t) = (u, t)$. We consider the following diagram:

$$\begin{tikzcd}
H_n(M;\mathscr{S}_M) \arrow[r, "(\bar{\lambda}_*\varphi)_*"] \arrow[d, "\bar{h}_*"'] \arrow[dr, "(\bar{h}_*\varphi')_*"] & H_n(M;(\lambda\circ f)^*\mathscr{S}_w) \arrow[d, "\psi_*"] \arrow[dr, "(\lambda\circ f)_*"] & \\
H_n(M;h^*\mathscr{S}_{M'}) \arrow[d, "h_*"'] & H_n(M;(f'\circ h)^*\mathscr{S}_w) \arrow[r, "(f'\circ h)_*"] \arrow[d, "h_*"] & H_n(B\pi;\mathscr{S}_w). \\
H_n(M';\mathscr{S}_{M'}) \arrow[r, "\varphi'_*"] & H_n(M';f'^*\mathscr{S}_w) \arrow[ur, "f'_*"'] &
\end{tikzcd}$$

The diagrams is commutative except the upper triangle part including the upper horizontal arrow where the diagram is commutative up to sign, more precisely, it holds $\psi_* \circ (\bar{\lambda}_*\varphi)_* = (\bar{h}_*\varphi')_*$ or $\psi_* \circ (\bar{\lambda}_*\varphi)_* = -(\bar{h}_*\varphi')_*$ according to $\psi_{u_0} = id$ or $\psi_{u_0} = -id$. Hence it holds

$$(11.1) \qquad (\lambda \circ f)_* \circ (\bar{\lambda}_*\varphi)_* = \begin{cases} f'_* \circ \varphi'_* \circ \tilde{h}_* & \text{if } \psi_{u_0} = id \\ -f'_* \circ \varphi'_* \circ \tilde{h}_* & \text{if } \psi_{u_0} = -id. \end{cases}$$

Summarizing and extending the above argument, we will get the following Proposition 14.

PROPOSITION 14. *Let M and M' be mutually diffeomorphic closed connected n-manifolds. Let $f : M \to B\pi$ and $f' : M' \to B\pi$ be continuous maps which satisfy the conditions* (3.1) *and* (3.2). *Moreover, let φ and φ' be local orientations associated with f and f' respectively. Then $[\mu([M, f, \varphi])] = [\mu([M', f', \varphi'])]$ or $[\mu([M, f, \varphi])] = [-\mu([M', f', \varphi'])]$ in $H_n(B\pi;\mathscr{S}_w)/(\mathrm{Aut}\,\pi)^w_*$.*

PROOF. *If* $f : (M, u_0) \to (B\pi, y_0)$, $f' : (M', u_0') \to (B\pi, y_0)$ and there is a diffeomorphism $h : (M, u_0) \to (M', u_0')$, then the above argument implies Proposition 14. So, if $f(M) \cap f'(M') \neq \phi$, we can choose y_0, u_0, u_0' and $h : (M, u_0) \to (M', u_0')$ and then the proposition follows.

In case $f(M) \cap f'(M') = \phi$, we choose y_0, u_0' such that $f' : (M', u_0') \to (B\pi, y_0)$ and choose u_0 so that $h : (M, u_0) \to (M', u_0')$. We put $y_1 = f(u_0)$ and choose a path β from y_1 to y_0. Let $g_t : \{u_0\} \to B\pi$ be a homotopy such that $g_t(u_0) = \beta(t)$. We can consider that (M, u_0) is a pair of CW complexes. By the homotopy extension theorem there exists a homotopy $f_t : M \to B\pi$ such that $f_t(u_0) = g_t(u_0)$ and $f_0 = f$. We see that f_1 satisfies the conditions (3.1) and (3.2). Therefore $f^*\mathscr{S}_w$ and $f_1^*\mathscr{S}_w$ are equivalent. Hence there is a unique equivalence $\bar{\beta} : f^*\mathscr{S}_w \to f_1^*\mathscr{S}_w$ satisfying $\bar{\beta}_{u_0} = \mathscr{S}_w([\beta])^{-1}$. For a local orientation φ associated with f we define a local orientation φ_1 associated with f_1 by $(\varphi_1)_u = \bar{\beta}_u \circ \varphi_u$. Then $[M, f, \varphi] = [M, f_1, \varphi_1]$. In fact, the cobordism is given by (W, F, Φ) defined as follows. We put $W = M \times I$ and define a map $F : W \to B\pi$ by $F(u, t) = f_t(u)$. Furthermore we define a local orientation Φ associated with F by extending

$$\Phi_v = \mathscr{S}_w(F_*[\gamma_v])^{-1} \circ \varphi_u \circ \gamma_{v*}(v = (u, t) \in \mathrm{Int}\ W),$$

where γ_v is a path in W defined by $\gamma_v(s) = (u, st)$. Then $\dot{\Phi}|M \times 0 = \varphi$, $\dot{\Phi}|M \times 1 = -\varphi_1$ and $\partial(W, F, \Phi) \equiv (M, f, \varphi) + (M, f_1, -\varphi_1)$.

Now we can apply the previous argument to f_1 with φ_1 and the proposition follows. □

7. Generalized form and proof of Theorem 3

In this section we present a generalized form of Theorem 3 as Theorem 20 and using it we prove Theorems 3 and 4.

Let $\bar{\mathscr{M}}_4(B\pi; \mathscr{S}_w)$ be the subset of $\mathscr{M}_4(B\pi; \mathscr{S}_w)$ consisting of triples (M, f, φ) such that f induces an isomorphism on π_1. Proposition 11 together with following proposition guarantees that $\bar{\mathscr{M}}_4(B\pi; \mathscr{S}_w)$ is not empty.

Proposition 15 ([5]). *Let π be a finitely presentable group. For each element w of $H^1(B\pi; \mathbf{Z}_2)$, there exist a connected closed 4-manifold M and a map $f : M \to B\pi$ which induces an isomorphism on π_1 and satisfies $f^*w = w_1(M)$. In fact the zero element of $\Omega_4(B\pi; \mathscr{S}_w)$ is representable by (N_0, g_0, ψ_0), where g_0 induces an isomorphism on π_1 and ψ_0 is a local orientation associated with g_0.*

Proof. Let K^2 be a geometric realization of π by a compact 2-complex. We have a map $g_1 : K^2 \to B\pi$ which induces an isomorphism on π_1. Let $\overline{w} : B\pi \to K(\mathbf{Z}_2, 1) = P^\infty \times \mathbf{R}$ be the map corresponding to w. Here, P^n, $2 \leq n \leq \infty$, denotes the n-dimensional real projective space. Then we find a map $g : K^2 \to P^4 \times \mathbf{R}$ such that g is an embedding approximating $\overline{w} \circ g_1$ and $g^*w_1(P^4) = g_1^*w$. Note that $g^*w_1(P^4) = g^*i^*w_1(P^\infty)$. We regard $K^2 \subset P^4 \times \mathbf{R}$. Let $N(K^2)$ be the regular neighborhood of K^2 and $g_2 : N(K^2) \to K^2$ be the projection. We put $N_0 = \partial N(K^2)$ and $g_0 = (g_1 \circ g_2)|N_0$. Then g_0 induces an isomorphism on the fundamental group and $g_0^*w = (g_2|N_0)^*g_1^*w = (g_2|N_0)^*g^*w_1(P^4) = w_1(N_0)$. Hence the pair (N_0, g_0) is a desired one. Note that $[N_0, g_0, \psi_0] = 0 \in \Omega_4(B\pi; \mathscr{S}_w)$ for any local orientation ψ_0, because (N_0, g_0, ψ_0) bounds $(N(K^2), g_1 \circ g_2, \Phi)$ for some Φ. In fact, Φ is uniquely determined because the natural inclusion $N_0 \to N(K^2)$ induces an isomorphism on π_1. □

For the proof of Theorem 20 we need some lemmas.

Lemma 16. *Let* $[M, f, \varphi] \in \Omega_4(X; \mathscr{S}_w)$ *and* γ *be a nonzero element of* $Ker[f_* : \pi_1(M) \to \pi_1(X)]$. *We can perform the* 1*-dimensional surgery on the embedded circle representing* γ *and get a new triple* $(N, g, \psi) \in \mathscr{M}_4(X; \mathscr{S}_w)$ *which represents the same element* $[M, f, \varphi]$. *Note that* $\pi_1(N) = \pi_1(M)/(\gamma = 1)$.

Proof. For an element $[M, f, \varphi] \in \Omega_4(X; \mathscr{S}_w)$ put $W_1 = M \times I$ and $F_1 : W_1 \to X$ be a map defined by $F_1(u, t) = f(u)$. Then there is an equivalence $\Phi_1 : \mathscr{S}_{W_1} \xrightarrow{\sim} F_1^* \mathscr{S}_w$ such that $\dot{\Phi}_1 | M \times 0 = -\varphi$ and $\dot{\Phi}_1 | M \times 1 = \varphi$. Since φ is an equivalence of $\mathscr{S}_M$ to $f^* \mathscr{S}_w$ and $\gamma \in \operatorname{Ker} f_*$, the normal bundle ν of γ is orientable and hence trivial. Let $\tilde{\gamma} : S^1 \times D^3 \to M \times 1$ be a trivialization of ν. We may assume that $f | \tilde{\gamma}(S^1 \times D^3)(x, y) = f \circ \tilde{\gamma}(x, 0)$ for $(x, y) \in S^1 \times D^3$. Since $f_*(\gamma) = 0$, there exists a map $g_1 : D^2 \times 0 \to X$ such that $g_1 | S^1 \times 0 = f \circ \tilde{\gamma} | S^1 \times 0$. We extend g_1 to $F_2 : D^2 \times D^3 \to X$ by $F_2(x, y) = g_1(x, 0)$. We put $W_2 = D^2 \times D^3$ and $W = W_1 \cup_{\tilde{\gamma}} W_2$ by straightening the angles (cf. [2]). Let $F : W \to X$ be a map defined by $F(z) = F_j(z)$ $(z \in W_j, j = 1, 2)$. Note that there is an equivalence $\Phi_2 : \mathscr{S}_{W_2} \xrightarrow{\sim} F_2^* \mathscr{S}_w$ so that $-\dot{\Phi}_2 | S^1 \times D^3 = \varphi | \tilde{\gamma}(S^1 \times D^3)$. Using Φ_j $(j = 1, 2)$ similarly to the proof of Proposition 5, we get an equivalence $\Phi : \mathscr{S}_W \xrightarrow{\sim} F^* \mathscr{S}_w$ such that $\dot{\Phi} | M \times 0 = -\varphi$. Now we put $N = \partial W - M \times 0$, $g = F|N$, and $\psi = \dot{\Phi}|N$ by identifying $H_4(\partial W, \partial W - u; \mathbf{Z})$ with $H_4(N, N - u; \mathbf{Z})$ for $u \in N$. Then we get $\partial(W, F, \Phi) \equiv (M, f, -\varphi) + (N, g, \psi)$. □

Lemma 17. *Let* π *be a finitely presentable group. Any element of* $\Omega_4(B\pi; \mathscr{S}_w)$ *is representable by a triple* (N, g, ψ) *with an additional property that* $g : N \to B\pi$ *induces an isomorphism on* π_1.

Proof. The null element is already represented by (N_0, g_0, ψ_0) in Proposition 15. Let (M, f, φ) be a representative of a given element of $\Omega_4(B\pi; \mathscr{S}_w)$ and we put $f' = f \sharp g_0$ and $\varphi' = \varphi \sharp \psi_0$. Then $[M \sharp N_0, f', \varphi'] \in \Omega_4(B\pi; \mathscr{S}_w)$ and $f'_* : \pi_1(M \sharp N_0) \to \pi$ is surjective. Let $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_m\}$ be generators of $\pi_1(N_0)$ and $\pi_1(M)$ respectively. They are generators of $\pi_1(M \sharp N_0)$. We put $\bar{\beta}_j = g_{0*}^{-1} f_*(\beta_j)$ $(1 \le j \le m)$. If every $\bar{\beta}_j \beta_j^{-1}$ $(1 \le j \le m)$ anihilates, then any element of $\operatorname{Ker} f'_*$ anihilates. Using Lemma 16 we anihilate the elements $\bar{\beta}_j \beta_j^{-1}$ $(1 \le j \le m)$ of $\pi_1(M \sharp N_0)$ by 1-dimensional surgery on the embedded circles representing these elements. Then we get a new closed 4-manifold N and a map $g : N \to B\pi$ which induces an isomorphism on π_1 such that $[N, g, \psi] = [M \sharp N_0, f', \varphi'] = [M, f, \varphi] + [N_0, g_0, \psi_0]$. The equivalence $\psi : \mathscr{S}_N \to g^* \mathscr{S}_w$ is given as in the proof of Lemma 16. □

Lemma 18. *If two triples* (M, f, φ) *and* (N, g, ψ) *represent the same element of* $\Omega_4(B\pi; \mathscr{S}_w)$ *such that the induced maps on the fundamental group are* *isomorphic, then we have a cobordism* (W, F, Φ) *such that* $\partial(W, F, \Phi) \equiv (M, f, \varphi) + (N, g, -\psi)$ *and both* $M \subset W$ *and* $N \subset W$ *induce isomorphisms on* π_1.

Proof. Let (W', F', Φ') be a cobordism of (M, f, φ) and (N, g, ψ). We see that $F'_* : \pi_1(W') \to \pi$ is surjective. Let $\gamma_1, \dots, \gamma_l$ be the generators of $\pi_1(W')$. We put $\beta_j = f_*^{-1} F'_*(\gamma_j)$ $(j = 1, 2, \dots, l)$. Then $\beta_j \gamma_j^{-1} \in \operatorname{Ker} F'_*$ $(j = 1, 2, \dots, l)$. We can consider that β_j and γ_j are elements of $\pi_1(\operatorname{Int} W')$ and so $\beta_j \gamma_j^{-1} \in \operatorname{Ker}(F' | \operatorname{Int} W')_*$. Since Φ' is an equivalence of $\mathscr{S}_{W'}$ to $F'^* \mathscr{S}_w$, we can anihilate $\beta_j \gamma_j^{-1}$ $(j = 1, 2, \dots, l)$ by 1-dimensional surgery on 5-dimensional manifold W' as in the proof of Lemma 16. If we anihilate the elements $\beta_j \gamma_j^{-1}$ $(j = 1, 2, \dots, l)$, we get a manifold W and a map $F : W \to B\pi$ such that the inclusion $M \subset W$ and F induce isomorphisms on π_1. Since $N \subset W \xrightarrow{F} B\pi$ induces an isomorphism on π_1, $N \subset W$ also induces an isomorphism on π_1. Note that the surgery does not affect the existence of Φ as in the proof of Lemma 16. □

Lemma 19. *Assume that the* 5*-dimensional cobordism* W *between* M *and* N *satisfies the condition that* $\partial W = M \cup N$ *and both* $M \subset W$ *and* $N \subset W$ *induce isomorphisms on* π_1. *Then, there are* M_0 *and* N_0 *which are connected sums of some copies of* $S^2 \times S^2$ *or* $S^2 \tilde{\times} S^2$ *such that* $M \sharp M_0$ *is diffeomorphic to* $N \sharp N_0$.

Proof. We can simplify the handle decomposition of W relative to M so that it has only 2-handles and 3-handles as in the usual proof of s-cobordism theorem in higher dimension, because $M \subset W$ and $N \subset W$ induce isomorphism on π_1. Then the feet of 2-handles are isotopic to the trivial one because it should represent the zero element in π_1 by the assumption. So, the middle level manifold is a connected sum of M and some copies of $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. By thinking from another direction it is also diffeomorphic to a connected sum of N and some copies of $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. □

Theorem 20 (Generalized form of Theorem 3). *Let* π *be a finitely presentable group and* $w \in H^1(B\pi; \mathbf{Z}_2)$. *Then, any equivalence class* $[\xi]$ *of* $H_4(B\pi; \mathscr{S}_w)/(\operatorname{Aut} \pi)_*^w$ *has a representative* (M, f, φ) *in* $\bar{\mathscr{M}}_4(B\pi; \mathscr{S}_w)$ *such that* $\xi = \mu([M, f, \varphi])$, *and for another representative* (M', f', φ') *of the same class* M *and* M' *are weakly stably equivalent. Moreover, the induced map:* $H_4(B\pi; \mathscr{S}_w)/(\operatorname{Aut} \pi)_*^w \to \mathscr{S}\mathscr{M}_{\pi,w}^4$ *is* $1:1$ *or* $2:1$ *according to that* $[\mu([M, f, \varphi])] = [\mu([M, f, -\varphi])]$ *or not, where* $\mathscr{S}\mathscr{M}_{\pi,w}^4$ *is the set of weakly stable equivalence classes in* $\mathscr{M}_{\pi,w}^4$.

Proof. Take any element ξ of $H_4(B\pi; \mathscr{S}_w)$. Then there exists an element ζ of $\Omega_4(B\pi; \mathscr{S}_w)$ such that $\mu(\zeta) = \xi$ by Corollary 2. It comes from a triple (M, f, φ) in $\bar{\mathscr{M}}_4(B\pi; \mathscr{S}_w)$ by Lemma 17. Let (M', f', φ') be another triple in $\bar{\mathscr{M}}_4(B\pi; \mathscr{S}_w)$ such that $\mu([M, f, \varphi]) = \mu([M', f', \varphi'])$. Then we have $[M, f, \varphi] =$

$[M'\sharp mCP^2, f'\sharp\varepsilon, \varphi'\sharp\psi_0]$ for some m by Corollarly 2 and the fact that Ω_4 is generated by the complex projective plane CP^2, where mCP^2 means the connected sum of $|m|$ copies of CP^2 or $\overline{CP^2}$ (the manifold CP^2 with the opposite orientation) according to the signature of m, ε is a map sending CP^2's to one point and ψ_0 is the orientation of mCP^2, that is, an appropriate equivalence of $\mathscr{S}_{mCP^2}$ to $\varepsilon^*\mathscr{S}_w$. Therefore, the manifolds M and M' are weakly stably equivalent by Lemmas 18 and 19. Let $\xi = \mu([M, f, \varphi])$, $\xi' = \mu([M', f', \varphi'])$ for (M, f, φ), $(M', f', \varphi') \in \bar{\mathscr{M}}_4(B\pi; \mathscr{S}_w)$. Assume now that $[\xi] = [\xi']$ in $H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)_*^w$ and not necessarily $\mu([M, f, \varphi]) = \mu([M', f', \varphi'])$. Then there is a classifying map $\lambda : (B\pi, y_0) \to (B\pi, y_0)$ for some element of $(\mathrm{Aut}\,\pi)^w$ such that $\xi' = \tilde{\lambda}_*\xi$, where $\tilde{\lambda}_*$ is an element of $(\mathrm{Aut}\,\pi)_*^w$ defined in §6. Since $\tilde{\lambda}_* \circ f_* \circ \varphi_*(\sigma_M) = (\lambda \circ f)_* \circ (\bar{\lambda}_*\varphi)_*(\sigma_M)$ by (10.1) in §6, we have $\mu([M', f', \varphi']) = \mu([M, \lambda \circ f, \bar{\lambda}_*\varphi])$. By the same argument as before M and M' are weakly stably equivalent. Therefore, we can assign a weakly stable equivalence class of M to $[\xi]$.

On the other hand, let M be an element of $\mathscr{M}^4_{\pi,w}$. Then, there is an element (M, f, φ) of $\bar{\mathscr{M}}_4(B\pi; \mathscr{S}_w)$ by Proposition 11. The triple determines the cobordism class $[M, f, \varphi]$ in $\Omega_4(B\pi; \mathscr{S}_w)$ and then an element $\mu([M, f, \varphi])$ of $H_4(B\pi; \mathscr{S}_w)$. Any weakly stabilized w-singular manifold determines the same element of $H_4(B\pi; \mathscr{S}_w)$ because $\mu([M\sharp M_0, f\sharp f_0, \varphi\sharp\varphi_0]) = f_*(\varphi_*(\sigma))$, where M_0 is a closed simply connected manifold, $f_0 : M_0 \to B\pi$ is a collapsing map to one point, φ_0 is an equivalence of $\mathscr{S}_{M_0}$ to $f_0^*\mathscr{S}_w$ and σ is the fundamental homology class of M with local coefficients $\mathscr{S}_M$. If M and M' are weakly stably equivalent, we may assume that M and M' are already diffeomorphic as far as we consider the element of $H_4(B\pi; \mathscr{S}_w)$. Take another element $(M', f', \varphi') \in \bar{\mathscr{M}}_4(B\pi; \mathscr{S}_w)$. Then, by Proposition 14 we have $[\xi'] = [\xi]$ or $[\xi'] = [-\xi]$ for $\xi = \mu([M, f, \varphi])$ and $\xi' = \mu([M', f', \varphi'])$. This means that there are at most two elements $[\xi]$, $[-\xi] \in H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)_*^w$ corresponding to the weakly stable equivalence class of M. The correspondence is $1:1$ or $2:1$ according to that $[\xi] = [-\xi]$ or not.

Even when $w = 0$, this theorem holds because we did not use Proposition 13 yet. In this case the induced map $H_4(B\pi; \mathbf{Z})/(\mathrm{Aut}\,\pi)_* \to \mathscr{S}\mathscr{M}^4_{\pi,w}$ is just the orientation forgetful map. □

Proof of Theorem 3. Let M be any element of $\mathscr{M}^4_{\pi,w}$. By Theorem 20 there are at most two elements $[\pm\xi] = [\mu([M, f, \pm\varphi])]$ of $H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)_*^w$ corresponding to the weakly stable equivalence class of M. Since π is abelian and $w \neq 0$, it holds $[M, f, \varphi] = [M, f, -\varphi]$ by Proposition 13. This means $[\xi] = [-\xi]$. Hence we get the conclusion. □

Proof of Theorem 4. Let $\pi = \pi_1(M)$ and take a map $f : M \to B\pi$ inducing an isomorphism on π_1. If the Lusternik-Schnirelmann π_1-category of

M is not 4, then $f^* : H^4(B\pi; \mathrm{Hom}(\mathscr{S}_w, \mathbf{Z}_m)) \to H^4(M; \mathrm{Hom}(f^*\mathscr{S}_w, \mathbf{Z}_m))$ is a zero map for any m by [8]. From the universal coefficient theorem for the cohomology with local coefficients [6], we see that $f_* : H_4(M; f^*\mathscr{S}_w) \to H_4(B\pi; \mathscr{S}_w)$ is a zero map. Hence $f_*(\varphi_*(\sigma)) = 0$ for any equivalence $\varphi : \mathscr{S}_M \to f^*\mathscr{S}_w$. On the other hand $[\partial N(K^2), g_0, \psi_0] = 0$ for the example given in the proof of Proposition 15. Especially $\mu([\partial N(K^2), g_0, \psi_0]) = g_{0*}(\sigma_{\partial N(K^2)}) = 0$. So, by Theorem 20 M is weakly stably equivalent to $\partial N(K^2)$. $\square$

If the fundamental group π is a non-trivial free group, then its classifying space is a bouquet of circles and $H_4(\vee S^1; \mathscr{S}_w) = 0$ so that every manifold in $\mathscr{M}^4_{\pi,w}$ with $w \neq 0$ is weakly stably equivalent to $\sharp_k S^1 \times S^3 \sharp_l S^1 \tilde{\times} S^3$ as shown in [9]. We know that its Lusternik-Schnirelmann π_1-category is 1. If π is not a free group, then the Lusternik-Schnirelmann π_1-category of M is 2 for the manifold M which belongs to the weakly stable equivalence class corresponding to the zero element of $H_4(B\pi; \mathscr{S}_w)$ and 4 otherwise by Theorem 4.

8. Some calculations of $H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)^w_*$

In this section we calculate $H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)^w_*$ for some examples. Example 3 contains many non-trivial group cases. Example 4 is a non-abelian group case where $H_4(B\pi; \mathscr{S}_w) = \mathbf{Z}$. Here, P^n, $2 \le n \le \infty$, denotes the n-dimensional real projective space. For convenience sake we put $A = H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)^w_*$.

Example 3. Let $\pi = \pi_1$ (a closed aspherical k-manifold) ($k \le 3$). Then $A = H_4(B\pi; \mathscr{S}_w) = 0$ for any w. Therefore, the weakly stable equivalence classes of closed 4-manifolds in $\mathscr{M}^4_{\pi,w}$ is unique. Moreover the ones of closed 4-manifolds is 1:1 correspondence with the equivalence classes of w modulo automorphisms of π.

Example 4. Let $\pi = \mathbf{Z} \times \mathbf{Z} \times \pi_1(P^2 \sharp P^2)$ and $\mathscr{S}_w = \mathscr{S}_0 \otimes \mathscr{S}_0 \otimes \mathscr{S}_\eta$ with $\eta = w_1(P^2 \sharp P^2) \neq 0$. Take $S^1 \times S^1 \times P^2 \sharp P^2$ as $B\pi$. Then we have $H_4(B\pi; \mathscr{S}_w) = H_2(S^1 \times S^1; \mathscr{S}_0) \otimes H_2(P^2 \sharp P^2; \mathscr{S}_\eta) = \mathbf{Z}$ because $H_2(P^2 \sharp P^2; \mathscr{S}_\eta) = \mathbf{Z}$. Note that $\mathrm{Aut}(\mathbf{Z} \times \mathbf{Z})(\subset (\mathrm{Aut}\,\pi)^w_*)$ contains an element which exchanges the sign of one of two generators of $H_1(S^1 \times S^1; \mathscr{S}_0)$ and hence change the sign of the generator of $H_2(S^1 \times S^1; \mathscr{S}_0)$. Then we get $A = \mathbf{Z}/\{\pm 1\}$. The generator of $H_4(B\pi; \mathscr{S}_w)$ is given by $\mu([B\pi, id, \varphi])$.

Example 5. Let $\pi = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ (n copies) and $w \neq 0$. Take $S^1 \times \cdots \times S^1$ as $B\pi$. Any $\mathscr{S}_w$ is equivalent to $\mathscr{S}_0 \otimes \cdots \otimes \mathscr{S}_0 \otimes \mathscr{S}_\eta$ for a non-trivial element $\eta \in H^1(B\mathbf{Z}; \mathbf{Z}_2) = \mathbf{Z}_2$. Then each canonical generator of $H_4(B\pi; \mathscr{S}_w) = \mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2$ ($\binom{n-1}{4}$ copies of $\mathbf{Z}_2$) corresponds to a 4-element subset in $\{1, 2, \ldots, n-1\}$. Let $C_{4,n-1}$ be the set which consists of subsets of distinct 4-elements in $\{1, 2, \ldots, n-1\}$. Then the symmetric group S_{n-1} operates naturally on $C_{4,n-1}$, which corresponds to the set of non-trivial elements of $H_4(B\pi; \mathscr{S}_w)$ with the operation by $(\mathrm{Aut}\,\pi)_*^w$. Hence we get

$$|A| = \begin{cases} |C_{4,n-1}/S_{n-1}| + 1 \geq 2 & (n \geq 5) \\ 1 & (n \leq 4). \end{cases}$$

When $n = 5, 6$, we see that S_{n-1} operates transitively on each subset of $C_{4,n-1}$ consisting of the same number of 4-element subsets and hence $|A| = \binom{n-1}{4} + 1$. Let N be a closed 4-manifold obtained from T^4 by attaching a non-orientable 1-handle and $\tilde{T}_w^4$ be a manifold with $\pi_1 = \pi$ which is obtained from N by 1-dimensional surgery. In the case $n = 5$ the non-trivial element of A is represented by $\mu([\tilde{T}_w^4, f, \varphi])$. Even when $n \geq 6$, any element of A can be constructed by using several copies of $\tilde{T}_w^4$. Furthermore, in the case $n \leq 4$ the weakly stable class of the closed non-orientable 4-manifold is unique, because any $w \neq 0$ is equivalent modulo automorphisms of π.

EXAMPLE 6. Let $\pi = \mathbf{Z}_2$ and $w \neq 0$. Take P^∞ as $B\pi$. Then we have $A = H_4(B\pi; \mathscr{S}_w) = \mathbf{Z}_2$. The non-trivial element is represented by $\mu([P^4, i, \varphi])$, where $i : P^4 \to P^\infty$ is a natural inclusion.

EXAMPLE 7. Let $\pi = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $w \neq 0$. Take $P^\infty \times P^\infty$ as $B\pi$. Any $\mathscr{S}_w$ is equivalent to $\mathscr{S}_\eta \otimes \mathscr{S}_\eta$ for a non-trivial element η of $H^1(P^\infty; \mathbf{Z}_2)$. Hereafter we distinguish the first P^∞ from the second P^∞. Let $f_1' : N \to P_1^\infty \times P_2^\infty$ be a closed w-singular 4-manifold obtained from $i_1 : P^4 \to P_1^\infty$ by attaching a non-orientable 1-handle and $f_1 : \tilde{P}_w^4 \to P_1^\infty \times P_2^\infty$ be a w-singular manifold with $\pi_1 = \pi$ obtained from $f_1' : N \to P_1^\infty \times P_2^\infty$ by 1-dimensional surgery. Then $H_4(B\pi; \mathscr{S}_w) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\xi_0 = \mu([P^2 \times P^2, i \times i, \varphi])$, $\xi_1 = \mu([\tilde{P}_w^4, f_1, \varphi_1])$ and $\xi_2 = \mu([\tilde{P}_w^4, \lambda \circ f_1, \varphi_2])$, where λ is an automorphism which exchanges P_i^∞ $(i = 1, 2)$. The exchange of the canonical basis of $\pi = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is a non-trivial element of $(\mathrm{Aut}\,\pi)^w$ but the other non-trivial elements of $\mathrm{Aut}\,\pi = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ does not belong to $(\mathrm{Aut}\,\pi)^w$. So, $(\mathrm{Aut}\,\pi)_*^w$ identifies only the generators ξ_1 and ξ_2. Hence we get $|A| = 6$. In fact, A consists of $[0]$, $[\xi_0]$, $[\xi_1]$, $[\xi_0 + \xi_1]$, $[\xi_1 + \xi_2]$ and $[\xi_0 + \xi_1 + \xi_2]$.

References

[1] H. Cartan and S. Eilenberg, Homological Algebra, Princeton Math. Ser. 19, Princeton Univ. Press, Princeton, 1956.

[2] P. E. Conner and E. E. Floyd, Differentiable Periodic Maps, Erg. Math. Gr. 33, Springer-Verlag, Berlin-Göttingen, 1964.

[3] S. Eilenberg and N. E. Steenrod, Foundations of Algebraic Topology, Princeton Univ. Press, Princeton, 1952.

[4] I. Hambelton and M. Kreck, Cancellation results for 2-complexes and 4-manifolds and some applications, Two-dimensional Homotopy and Combinatorial Group Theory, London Mathematical Society Lecture Notes Series **197** (1993), 281–308.

[5] M. Kreck, Some closed 4-manifolds with exotic differentiable structure, Algebraic Topology, Aarhus 1982, 246–262.

[6] A. Hattori, Topology I, II, III (in Japanese), Iwanamishoten, Tokyo, 1977.

[7] I. Kurazono and T. Matumoto, Classification of closed oriented 4-manifolds modulo connected sum with simply connected manifolds, Hiroshima Math. J. **28** (1998), 207–212.

[8] E. Laitinen and T. Matumoto, A gap theorem for Lusternik-Schnirelmann π_1-category, Topology and its Applications **93** (1999), 35–40.

[9] T. Matumoto and A. Katanaga, On 4-dimensional closed manifolds with free fundamental groups, Hiroshima Math. J. **25** (1995), 367–370.

[10] N. E. Steenrod, Homology with local coefficients, Ann. of Math. **44** (1943), 610–627.

[11] ———, The topology of fibre bundles, Princeton University Press, 1951.

[12] R. E. Stong, Notes on Cobordism Theory, Princeton University Press and The University of Tokyo Press, Princeton, 1968.

[13] R. Thom, Quelques propriétés globales des variétés différentiables, Comment. Math. Helv. **28** (1954), 17–86.

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